

ON MAPS COMPRESSING THE NUMERICAL RANGE BETWEEN C^* -ALGEBRAS

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ABSTRACT. In this paper, we deal with the problem of characterizing linear maps compressing the numerical range. A counterexample is given to show that such a map need not be a Jordan $*$ -homomorphism in general even if the C^* -algebras are commutative. Next, under an auxiliary condition we show that such a map is a Jordan $*$ -homomorphism.

1. INTRODUCTION

Let \mathcal{A} and \mathcal{B} be unital complex Banach algebras. Denote by $\mathbf{1}_{\mathcal{A}}$ and $\mathbf{1}_{\mathcal{B}}$ the units of \mathcal{A} and \mathcal{B} respectively (or simply $\mathbf{1}$ if no confusion can arise). Define the set of normalized states

$$S(\mathcal{A}) = \{f \in \mathcal{A}' : f(\mathbf{1}) = \|f\| = 1\},$$

where \mathcal{A}' denotes the dual space. For any element $a \in \mathcal{A}$, the algebraic *numerical range* $V(a)$ and *numerical radius* $v(a)$ of a are defined by

$$V(a) = \{f(a) : f \in S(\mathcal{A})\} \text{ and } v(a) = \sup_{z \in V(a)} |z|.$$

It is well known that V is a compact and convex set of the complex plane, $v(\cdot)$ is a norm on \mathcal{A} and this norm is equivalent to the usual operator norm. The suggested references on numerical ranges are [2, 10]. A linear map $T : \mathcal{A} \rightarrow \mathcal{B}$ is said to be numerical range (resp. numerical radius) preserving if $V(T(a)) = V(a)$

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(resp. $v(T(a)) = v(a)$) for every $a \in \mathcal{A}$. Also, we shall say that T compresses the numerical range if $V(T(a)) \subset V(a)$ for every $a \in \mathcal{A}$.

There has been considerable interest in studying maps between C^* -algebras leaving invariant the numerical range or the numerical radius. A nice survey of earlier known results relating to the preserving problem can be found in [4, 14]. In 1975, Pellegrini [16] studied numerical range preserving operators on a Banach algebra. Particularly, when \mathcal{A} and \mathcal{B} are two C^* -algebras, it was shown that a linear isomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$ is a Jordan $*$ -isomorphism if and only if it is numerical range preserving. Later, Chan [5] showed that a linear isomorphism $T : \mathcal{A} \rightarrow \mathcal{A}$ is numerical radius preserving if and only if cT is a Jordan $*$ -isomorphism for some central and unitary element $c \in \mathcal{A}$. Surjective nonlinear maps $T : \mathcal{A} \rightarrow \mathcal{B}$ between unital C^* -algebras that satisfy $v(T(a) - T(b)) = v(a - b)$ for all $a, b \in \mathcal{A}$ were characterized in [1] under a mild condition that $T(\mathbf{1}) - T(0)$ belongs to the center of \mathcal{B} . Recently, in [3], the assumption $T(\mathbf{1}) - T(0)$ belongs to the center of \mathcal{B} is successfully removed.

The aim of this paper, is to study maps between C^* -algebras compressing the numerical range. Firstly, we shall give an example showing that such a map need not to be a Jordan $*$ -homomorphism. Next, We will show that under some supplementary condition such a map is a Jordan $*$ -homomorphism.

We close this Introduction with some definitions and properties of the numerical range needed in the sequel. In the case of C^* -algebra, a linear functional $f \in \mathcal{A}'$ is said to be *positive* ($f \geq 0$) if $f(xx^*) \geq 0$ for all $x \in \mathcal{A}$. Note that the set of normalized states $S(\mathcal{A})$ is nothing but

$$S(\mathcal{A}) = \{f \in \mathcal{A}' : f \geq 0 \text{ and } f(\mathbf{1}) = 1\}.$$

Recall also that a positive linear functional f on \mathcal{A} is said to be *pure* if for every positive functional g on \mathcal{A} satisfying $g(xx^*) \leq f(xx^*)$ for all $x \in \mathcal{A}$, there is a scalar $0 \leq \lambda \leq 1$ such that $g = \lambda f$. The set of pure states on \mathcal{A} is denoted by $P(\mathcal{A})$. It is well known that $P(\mathcal{A})$ coincides with the set of all extremal points of $S(\mathcal{A})$.

For any element $a \in \mathcal{A}$ and any scalars $\alpha, \beta \in \mathbb{C}$, we have: $V(a) \subset \mathbb{R}$ (resp. $V(a) \subset [0, +\infty)$) if and only if $a = a^*$ (resp. $a \geq 0$). Also $V(\alpha\mathbf{1} + \beta a) = \alpha + \beta V(a)$ and $V(a) = \{\alpha\} \iff a = \alpha\mathbf{1}$. The numerical radius v is a norm and satisfies $\frac{1}{e}\|a\| \leq v(a) \leq \|a\|$, where $e = \exp(1)$. See [2] and [11] for further details.

2. MAIN RESULT

Let \mathcal{A} and \mathcal{B} be two unital complex C^* -algebras. Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a linear map. Recall that T is numerical range compressing if

$$V(T(a)) \subset V(a), \quad \forall a \in \mathcal{A}, \tag{2.1}$$

Note that if T compresses the numerical range then $T(\mathbf{1}) = \mathbf{1}$, since the numerical range is a nonempty set of \mathbb{C} and $V(T(\mathbf{1})) \subset V(\mathbf{1}) = \{1\}$. Let us begin by the following example, which shows that a linear map which compresses the numerical range need not to be a Jordan $*$ -homomorphism.

Example 2.1. Consider the C^* -algebra $\mathcal{A} = \mathcal{M}_2(\mathbb{C})$ and define the map $T : \mathcal{A} \longrightarrow \mathcal{A}$ for any matrix $A = (a_{ij})_{1 \leq i, j \leq 2} \in \mathcal{A}$ by

$$T(A) = \frac{1}{2} A + \frac{1}{4} \text{tr}(A) \mathbf{1}$$

where tr denotes the usual function trace. Clearly, we have $f \circ T \in S(\mathcal{A})$ whenever $f \in S(\mathcal{A})$. Hence according to [16, Theorem 2.2], T satisfies condition (2.1).

Consider the two matrices $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$. An easy calculation will convince the reader that B is unitary, but $T(A^2) \neq T(A)^2$ and $T(B)$ is not unitary. This shows that T is not neither a Jordan $*$ -homomorphism nor a unitary preserving map.

At the 4th Seminar on Functional analysis and its applications, which was held in University of Mashhad in March 2016 it is shown that in [9] that if \mathcal{A} and \mathcal{B} are commutative and T is a numerical range compressing, then T is a unital $*$ -homomorphism, see [9, Theorem 2.5 & 2.6]. In fact, in his proof, the author shows that such a map is completely positive and preserves unitary elements. But we remark that this proof is based on the fact that if an element u is unitary in \mathcal{A} , then $|f(u)| = 1$ for any $f \in S(\mathcal{A})$. But this fails to be true even if the C^* -algebras \mathcal{A} and \mathcal{B} are commutative. It is in fact true only when f is a pure state, see for instance [5, Proposition 1]. To see why this, let $\mathcal{A} = C(\mathbb{T})$ be the C^* -algebra of all continuous functions on the unit circle \mathbb{T} and let m be the normalised arc length measure on \mathbb{T} . Then the linear functional φ , defined by $\varphi(f) = \int f dm = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) dt$ is a state of \mathcal{A} . The element $u \in \mathcal{A}$ defined by $u(z) = z, \forall z \in \mathbb{T}$ is unitary but $\varphi(u) = 0 < 1$. Finally observe that φ is not a $*$ -homomorphism although that $V(\varphi(a)) \subset V(a)$ for all $a \in \mathcal{A}$, since φ is a state. Therefore, the main result [9, Theorem 2.5 & 2.6] is wrong in general.

Based on the aforesaid a natural question arises. Namely, what additional condition on a linear map T compressing the numerical range which forces T to be a Jordan $*$ -homomorphism? To that end, we shall impose the following additional requirement on the map T .

Assumption 2.2. For any $a, b \in \mathcal{A}^+$ such that $ab = 0$, we have $T(a) \geq T(b)$ implies that $T(a)T(b) = 0$.

We establish the following.

Theorem 2.3. *Let \mathcal{A} and \mathcal{B} be two unital C^* -algebras. Any surjective linear map $T : \mathcal{A} \longrightarrow \mathcal{B}$ compressing the numerical range and satisfying Assumption 2.2 is a unital Jordan $*$ -homomorphism.*

Before turning to the proof of Theorem 2.3, few remarks can be made.

Remark 2.4. If T preserves the numerical range then Assumption 2.2 is already satisfied. Indeed, let $a, b \in \mathcal{A}^+$ such that $ab = 0$ and $T(a) \geq T(b)$. Since $V(T(a-b)) = V(a-b)$ and $T(a-b) \geq 0$, then $a-b \geq 0$. By [15, Theorem 2.2.5], $0 \leq b^3 \leq bab = 0$. Accordingly $b^3 = b = 0$. Therefore $T(a) = T(b) = 0$.

Remark 2.5. Conditions (2.1) and Assumption 2.2 do not imply in general that T is linear as the following example quoted from [12] shows. Let $\mathcal{A} = \mathcal{B} = \mathcal{M}_2(\mathbb{C})$. Consider the mapping $T : \mathcal{A} \rightarrow \mathcal{A}$ defined as

$$T(A) = \begin{cases} A & \text{if } A \text{ is invertible} \\ 0 & \text{otherwise.} \end{cases}$$

Straightforward computations show that T satisfies assumptions (2.1) and Assumption 2.2 but is not additive.

Remark 2.6. In [6], it was been shown that if $T : \mathcal{A} \rightarrow \mathcal{B}$ is a bounded linear map between unital C^* -algebras preserving the zero products of self-adjoint elements in \mathcal{A} then $T = T(\mathbf{1})J$ for a Jordan $*$ -homomorphism J from \mathcal{A} into the bidual \mathcal{B}^{**} of \mathcal{B} . Note that Assumption 2.2 does not imply in general that T preserves the zero product of self-adjoint elements or a Jordan $*$ -homomorphism. To see why this consider the C^* -algebra $\mathcal{A} = C([0, 1])$ and the map $T : \mathcal{A} \rightarrow \mathcal{A}$ given by $T(f) = 2f - f(1)$. Clearly, T is surjective and unital. But then $(T(f))^2 - T(f^2) = 2f^2 - 4f(1)f + 2f(1)^2$ is not always zero. Hence T is not a Jordan $*$ -homomorphism. Next, let $f, g \in \mathcal{A}^+$ be such that $fg = 0$ and $T(f) \geq T(g)$. Since $T(f) \geq T(g)$, then $f(1) \geq g(1)$ and $f(x) \geq g(x) + \frac{1}{2}(f(1) - g(1))$, for any $x \in [0, 1]$. This together with the fact $fg = 0$ yields that $g = 0$. Therefore $T(f)T(g) = 0$. Accordingly T satisfies Assumption 2.2. On the other hand, one can check easily that T does not preserve the zero product of self-adjoint elements.

3. PROOF OF THEOREM 2.3:

We present now the proof of Theorem 2.3. Our arguments are influenced by ideas from the proof of [7, Theorem 5] but by using properties of the numerical range. We divide the proof into three steps.

Step 1. T is unital and positive. Moreover, for each $b \geq 0$ in \mathcal{B} there is an $a \geq 0$ in \mathcal{A} such that $T(a) = b$.

Firstly, note that $T(\mathbf{1}) = \mathbf{1}$, since $V(T(\mathbf{1})) \subset V(\mathbf{1}) = \{1\}$. Now, let $a \in \mathcal{A}^+$. Then $V(a) \subset [0, \infty)$. Since $V(T(a)) \subset V(a)$ we infer that $T(a) \in \mathcal{B}^+$ and in particular T is self adjoint, that is $T(a)^* = T(a), \forall a = a^* \in \mathcal{A}$. Now, let $b \geq 0$ and $a \in \mathcal{A}$ such that $T(a) = b$. Without loss of generality we may assume that $a = a^*$ (otherwise take $\frac{a+a^*}{2}$ instead of a). By [8, Proposition 12.5], there exist $a_+, a_- \geq 0$ such that $a = a_+ - a_-$ and $a_+a_- = a_-a_+ = 0$. Then $b = T(a_+) - T(a_-)$ with $T(a_+) \geq 0$ and $T(a_-) \geq 0$. Assumption 2.2 entails that $T(a_+)T(a_-) = T(a_-)T(a_+) = 0$. Since every self adjoint element in a C^* -algebra can be uniquely written as the difference of two positive elements with zero product, we infer that $T(a_-) = 0$. This completes the proof of the first step.

Step 2. The kernel of T is a closed ideal of \mathcal{A} .

Firstly observe that by the proof of Step 1, we have that if $T(a) = 0$ and $a = a_+ - a_-$ with $a_+a_- = a_-a_+ = 0$ and $a_{\pm} \geq 0$, then $T(a_+) = T(a_-) = 0$. Thus each element in $\ker T$ is a linear combination of positive elements in $\ker T$. Now, Lemma 5.1 of [17] can be used to deduce that $\ker T$ is a two sided ideal. However,

for the sake of completeness we sketch a different proof of this fact. To that end it suffices to show that $a \geq 0$ and $T(a) = 0$ imply that $T(ax) = T(xa) = 0$ for all positive element $x \in \mathcal{A}$. Fix such an $a \in \mathcal{A}$, a similar reasoning to that of [7, Theorem 5] entails that $T(ax)^* = T(xa) = -T(ax)$. By keeping in mind that $T(x) \geq 0$ for any $x \in \mathcal{A}^+$, we infer that the linear functional $f \circ T$ is positive and unital, for any $f \in S(\mathcal{B})$. Accordingly $f \circ T \in S(\mathcal{A})$. Hence, applying the Cauchy-Schwarz inequality to $f \circ T$ yields

$$|f \circ T(ax)|^2 = |f \circ T(a^{\frac{1}{2}}a^{\frac{1}{2}}x)|^2 \leq f \circ T(a) f \circ T(xax) = 0.$$

Accordingly, $f(T(ax)) = 0$, for all $f \in S(\mathcal{B})$ and so $T(ax) = T(xa) = 0$ as desired. The kernel of T is therefore an ideal. Since $v(T(a)) \leq v(a)$, $\forall a \in \mathcal{A}$, v and $\|\cdot\|$ are two equivalent norms, then T is bounded and so the kernel of T is closed.

Step 3. T is a Jordan $*$ -homomorphism.

Firstly, note that by Step 1 we have $T(\mathbf{1}) = \mathbf{1}$ and T is positive. By Step 2, $\ker T$ is a closed ideal of \mathcal{A} . Then T induces the unital and positive bijective linear map $\tilde{T} : \mathcal{A}/\ker T \rightarrow \mathcal{B}$ defined by $\tilde{T}(a + \ker T) = T(a)$. Again Step 1, entails that \tilde{T}^{-1} is also positive. So, by [13, Corollary 5] we have \tilde{T} is a Jordan $*$ -isomorphism. Thus T , the composition of the natural quotient map and \tilde{T} , is a Jordan $*$ -homomorphism.

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