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REPRODUCING PAIRS OF MEASURABLE FUNCTIONS AND PARTIAL INNER PRODUCT SPACES

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Abstract. We continue the analysis of reproducing pairs of weakly measurable functions, which generalize continuous frames. More precisely, we examine the case where the defining measurable functions take their values in a partial inner product space (PIP spaces). Several examples, both discrete and continuous, are presented.

1. Introduction

Frames and their relatives are most often considered in the discrete case, for instance in signal processing [10]. However, continuous frames have also been studied and offer interesting mathematical problems. They have been introduced originally by Ali, Gazeau and one of us [1, 2] and also, independently, by Kaiser [14]. Since then, several papers dealt with various aspects of the concept, see for instance [11], [12], [17] or [18]. However, there may occur situations where it is impossible to satisfy both frame bounds.

Therefore, several generalizations of frames have been introduced. Semi-frames [6, 7], for example, are obtained when functions only satisfy one of the two frame bounds. It turns out that a large portion of frame theory can be extended to this larger framework, in particular the notion of duality.

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More recently, a new generalization of frames was introduced by Balazs and Speckbacher [21], namely, reproducing pairs. Here, given a measure space (X, μ), one considers a couple of weakly measurable functions (ψ, ϕ), instead of a single mapping, and one studies the correlation between the two (a precise definition is given below). This definition also includes the original definition of a continuous frame [1, 2] to which it reduces when $\psi = \phi$. In a previous paper [8], we have analyzed in detail the mathematical structure generated by a reproducing pair and we have described a number of concrete examples. In particular, we have shown that a reproducing pair (ψ, ϕ) determines a couple of Hilbert spaces $V_\psi(X, \mu), V_\phi(X, \mu)$, conjugate duals of each other with respect to the $L^2(X, \mathrm{d}\mu)$ inner product. And this immediately suggests to work in the context of partial inner product spaces (PIP-spaces) [4].

The increase of freedom in choosing the mappings ψ and ϕ, however, leads to the problem of characterizing the range of the analysis operators, which in general need no more be contained in $L^2(X, \mathrm{d}\mu)$, as in the frame case. Therefore, we extend the theory to the case where the weakly measurable functions take their values in a partial inner product space (PIP-space). We discuss first the case of a rigged Hilbert space, then we consider a genuine PIP-space. We conclude with two natural families of examples, namely, Hilbert scales and several PIP-spaces generated by the family $\{L^p(X, \mathrm{d}\mu), 1 \leq p \leq \infty\}$.

We might remark that the increased flexibity afforded by reproducing pairs effectively yields new insights in some physical problems [8] or [22].

2. Preliminaries

Before proceeding, we list our definitions and conventions. The framework is a (separable) Hilbert space $\mathcal{H}$, with the inner product $\langle\cdot|\cdot\rangle$ linear in the first factor. Given an operator A on $\mathcal{H}$, we denote its domain by $D(A)$, its range by $\mathsf{Ran}\,(A)$ and its kernel by $\mathsf{Ker}\,(A)$. $GL(\mathcal{H})$ denotes the set of all invertible bounded operators on $\mathcal{H}$ with bounded inverse. Throughout the paper, we will consider weakly measurable functions $\psi : X \to \mathcal{H}$, where (X, μ) is a locally compact space with a Radon measure μ, that is, $\langle\psi_x|f\rangle$ is $\mu-$measurable for every $f \in \mathcal{H}$. Then the weakly measurable function ψ is a *continuous frame* if there exist constants $0 < \mathsf{m} \leq \mathsf{M} < \infty$ (the frame bounds) such that

$$\mathsf{m}\,\|f\|^2 \leq \int_X |\langle f|\psi_x\rangle|^2 \;\mathrm{d}\mu(x) \leqslant \mathsf{M}\,\|f\|^2\,, \forall\, f \in \mathcal{H}. \tag{2.1}$$

Given the continuous frame ψ, the *analysis* operator $C_\psi : \mathcal{H} \to L^2(X, \mathrm{d}\mu)$ [1] is defined as

$$(C_\psi f)(x) = \langle f|\psi_x\rangle,\;\; f \in \mathcal{H}, \tag{2.2}$$

and the corresponding *synthesis operator* $C_\psi^* : L^2(X, \mathrm{d}\mu) \to \mathcal{H}$ as (the integral being understood in the weak sense, as usual)

$$C_\psi^*\xi = \int_X \xi(x)\,\psi_x\;\,\mathrm{d}\mu(x),\;\text{ for }\;\xi \in L^2(X, \mathrm{d}\mu). \tag{2.3}$$

[1] As usual, we identify a function ξ with its residue class in $L^2(X, \mathrm{d}\mu)$.

We set $S := C_\psi^* C_\psi$, which is self-adjoint.

More generally, the couple of weakly measurable functions (ψ, ϕ) is called a *reproducing pair* if [8]

(a) The sesquilinear form

$$\Omega_{\psi,\phi}(f,g) = \int_X \langle f|\psi_x\rangle \langle \phi_x|g\rangle \, \mathrm{d}\mu(x) \tag{2.4}$$

is well-defined and bounded on $\mathcal{H} \times \mathcal{H}$, that is, $|\Omega_{\psi,\phi}(f,g)| \leq c\,\|f\|\,\|g\|$, for some $c > 0$.

(b) The corresponding bounded (resolution) operator $S_{\psi,\phi}$ belongs to $GL(\mathcal{H})$.

Under these hypotheses, one has

$$S_{\psi,\phi} f = \int_X \langle f|\psi_x\rangle \phi_x \, \mathrm{d}\mu(x), \ \forall\, f \in \mathcal{H}, \tag{2.5}$$

the integral on the r.h.s. being defined in weak sense. If $\psi = \phi$, we recover the notion of continuous frame, so that we have indeed a genuine generalization of the latter. Notice that $S_{\psi,\phi}$ is in general neither positive, nor self-adjoint, since $S_{\psi,\phi}^* = S_{\phi,\psi}$. However, if ψ, ϕ is reproducing pair, then $\psi, S_{\psi,\phi}^{-1}\phi$ is a dual pair, that is, the corresponding resolution operator is the identity. Therefore, there is no restriction of generality to assume that $S_{\phi,\psi} = I$ [21]. The worst that can happen is to replace some norms by equivalent ones.

In [8], it has been shown that each weakly measurable function ϕ generates an intrinsic pre-Hilbert space $V_\phi(X,\mu)$ and, moreover, a reproducing pair (ψ, ϕ) generates two Hilbert spaces, $V_\psi(X,\mu)$ and $V_\phi(X,\mu)$, conjugate dual of each other with respect to the $L^2(X,\mu)$ inner product. Let us briefly sketch that construction, that we will generalize further on.

Given a weakly measurable function ϕ, let us denote by $\mathcal{V}_\phi(X,\mu)$ the space of all measurable functions $\xi : X \to \mathbb{C}$ such that the integral $\int_X \xi(x)\langle \phi_x|g\rangle \, \mathrm{d}\mu(x)$ exists for every $g \in \mathcal{H}$ (in the sense that $\xi\langle \phi.|g\rangle \in L^1(X, \mathrm{d}\mu)$) and defines a bounded conjugate linear functional on $\mathcal{H}$, i.e., $\exists\, c > 0$ such that

$$\left| \int_X \xi(x)\langle \phi_x|g\rangle \, \mathrm{d}\mu(x) \right| \leq c\,\|g\| \,, \ \forall\, g \in \mathcal{H}. \tag{2.6}$$

Clearly, if (ψ, ϕ) is a reproducing pair, all functions $\xi(x) = \langle f|\psi_x\rangle = (C_\psi f)(x)$ belong to $\mathcal{V}_\phi(X,\mu)$.

By the Riesz lemma, we can define a linear map $T_\phi : \mathcal{V}_\phi(X,\mu) \to \mathcal{H}$ by the following weak relation

$$\langle T_\phi \xi|g\rangle = \int_X \xi(x)\langle \phi_x|g\rangle \, \mathrm{d}\mu(x), \ \forall\, \xi \in \mathcal{V}_\phi(X,\mu), g \in \mathcal{H}. \tag{2.7}$$

Next, we define the vector space

$$V_\phi(X,\mu) = \mathcal{V}_\phi(X,\mu)/\mathsf{Ker}\ T_\phi$$

and equip it with the norm

$$\|[\xi]_\phi\|_\phi := \sup_{\|g\|\leq 1} \left| \int_X \xi(x)\langle \phi_x|g\rangle \, \mathrm{d}\mu(x) \right| = \sup_{\|g\|\leq 1} |\langle T_\phi \xi|g\rangle| \,, \tag{2.8}$$

where we have put $[\xi]_\phi = \xi + \mathsf{Ker}\, T_\phi$ for $\xi \in \mathcal{V}_\phi(X,\mu)$. Clearly, $V_\phi(X,\mu)$ is a normed space. However, the norm $\|\cdot\|_\phi$ is in fact Hilbertian, that is, it derives from an inner product, as can be seen as follows. First, it turns out that the map $\widehat{T}_\phi : V_\phi(X,\mu) \to \mathcal{H}$, $\widehat{T}_\phi[\xi]_\phi := T_\phi \xi$ is a well-defined isometry of $V_\phi(X,\mu)$ into $\mathcal{H}$. Next, one may define on $V_\phi(X,\mu)$ an inner product by setting

$$\langle [\xi]_\phi | [\eta]_\phi \rangle_{(\phi)} := \langle \widehat{T}_\phi[\xi]_\phi | \widehat{T}_\phi[\eta]_\phi \rangle, \ [\xi]_\phi, [\eta]_\phi \in V_\phi(X,\mu),$$

and one shows that the norm defined by $\langle\cdot|\cdot\rangle_{(\phi)}$ coincides with the norm $\|\cdot\|_\phi$ defined in (2.8). One has indeed

$$\|[\xi]_\phi\|_{(\phi)} = \left\| \widehat{T}_\phi[\xi]_\phi \right\| = \|T_\phi \xi\| = \sup_{\|g\|\le 1} |\langle T_\phi \xi | g\rangle| = \|[\xi]_\phi\|_\phi .$$

Thus $V_\phi(X,\mu)$ is a pre-Hilbert space.

With these notations, the main result of [8] reads as

Theorem 2.1. *If (ψ,ϕ) is a reproducing pair, the spaces $V_\phi(X,\mu)$ and $V_\psi(X,\mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form*

$$\langle\!\langle \xi|\eta \rangle\!\rangle_\mu := \int_X \xi(x)\overline{\eta(x)}\, \mathrm{d}\mu(x), \tag{2.9}$$

which coincides with the inner product of $L^2(X,\mu)$ whenever the latter makes sense. This is true, in particular, for $\phi = \psi$, since then ψ is a continuous frame and $V_\psi(X,\mu)$ is a closed subspace of $L^2(X,\mu)$.

In this paper, we will consider reproducing pairs in the context of PIP-spaces. The motivation is the following. Let (ψ,ϕ) be a reproducing pair. By definition,

$$\langle S_{\psi,\phi} f|g\rangle = \int_X \langle f|\psi_x\rangle\langle \phi_x|g\rangle\, \mathrm{d}\mu(x) = \int_X C_\psi f(x)\, \overline{C_\phi g(x)}\, \mathrm{d}\mu(x) \tag{2.10}$$

is well defined for all $f,g \in \mathcal{H}$. The r.h.s. coincides with the sesquilinear form (2.9), that is, the L^2 inner product, but generalized, since in general $C_\psi f, C_\phi g$ need not belong to $L^2(X, \mathrm{d}\mu)$. If, following [21], we make the innocuous assumption that ψ is bounded, i.e., $\sup_{x\in X} \|\psi_x\|_{\mathcal{H}} \le c$ for some $c > 0$ (often $\|\psi_x\|_{\mathcal{H}} = \text{const.}$, e.g. for wavelets or coherent states), then $(C_\psi f)(x) = \langle f|\psi_x\rangle \in L^\infty(X, \mathrm{d}\mu)$.

These two facts suggest to take $\mathsf{Ran}\, C_\psi$ within some PIP-space of measurable functions, possibly related to the L^p spaces. We shall present several possibilities in that direction in Section 6.

3. Reproducing pairs and RHS

We begin with the simplest example of a PIP-space, namely, a rigged Hilbert space (RHS). Let indeed $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a RHS with $\mathcal{D}[t]$ reflexive (so that t and $t^\times$ coincide with the respective Mackey topologies). Given a measure space (X,μ), we denote by $\langle\cdot,\cdot\rangle$ the sesquilinear form expressing the duality between $\mathcal{D}$ and $\mathcal{D}^\times$. As usual, we suppose that this sesquilinear form extends the inner product of $\mathcal{D}$ (and $\mathcal{H}$). This allows to build the triplet above. Let $x \in X \mapsto \psi_x$, $x \in X \mapsto \phi_x$ be weakly measurable functions from X into $\mathcal{D}^\times$.

Instead of (2.4), we consider he sesquilinear form

$$\Omega^{\mathcal{D}}_{\psi,\phi}(f,g) = \int_X \langle f, \psi_x\rangle\langle \phi_x, g\rangle \,\mathrm{d}\mu(x),\ \ f,g \in \mathcal{D}. \tag{3.1}$$

For short we put $\Omega^{\mathcal{D}} := \Omega^{\mathcal{D}}_{\psi,\phi}$ and we assume that it is jointly continuous on $\mathcal{D}\times\mathcal{D}$, that is, $\Omega^{\mathcal{D}} \in \mathsf{B}(\mathcal{D},\mathcal{D})$ in the notation of [3, Sec.10.2]. Writing

$$\langle S_{\psi,\phi} f, g\rangle := \int_X \langle f, \psi_x\rangle\langle \phi_x, g\rangle \,\mathrm{d}\mu(x),\ \forall\, f,g \in \mathcal{D}, \tag{3.2}$$

we see that the operator $S_{\psi,\phi}$ belongs to $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$, the space of all continuous linear maps from D into $\mathcal{D}^\times$.

3.1. **A Hilbertian approach.** We first assume that the sesquilinear form $\Omega^{\mathcal{D}}$ is well-defined and bounded on $\mathcal{D}\times\mathcal{D}$ in the topology of $\mathcal{H}$. Then $\Omega^{\mathcal{D}}_{\psi,\phi}$ extends to a bounded sesquilinear form on $\mathcal{H}\times\mathcal{H}$, denoted by the same symbol.

The definition of the space $\mathcal{V}_\phi(X,\mu)$ must be modified as follows. Instead of (2.6), we suppose that the integral below exists and defines a conjugate linear functional on $\mathcal{D}$, bounded in the topology of $\mathcal{H}$, i.e.,

$$\left| \int_X \xi(x)\langle \phi_x, g\rangle \,\mathrm{d}\mu(x) \right| \leq c\,\|g\|\,,\ \forall\, g \in \mathcal{D}. \tag{3.3}$$

Then the functional extends to a bounded conjugate linear functional on $\mathcal{H}$, since $\mathcal{D}$ is dense in $\mathcal{H}$. Hence, for every $\xi \in \mathcal{V}_\phi(X,\mu)$, there exists a unique vector $h_{\phi,\xi} \in \mathcal{H}$ such that

$$\int_X \xi(x)\langle \phi_x, g\rangle \,\mathrm{d}\mu(x) = \langle h_{\phi,\xi}|g\rangle, \quad \forall g \in \mathcal{D}.$$

It is worth remarking that this interplay between the two topologies on $\mathcal{D}$ is similar to the approach of Werner [23], who treats L^2 functions as distributions, thus identifies the L^2 space as the dual of $\mathcal{D} = \mathcal{C}_0^\infty$ with respect to the norm topology. And, of course, this is fully in the spirit of PIP-spaces.

Then, we can define a linear map $T_\phi : \mathcal{V}_\phi(X,\mu) \to \mathcal{H}$ by

$$T_\phi \xi = h_{\phi,\xi} \in \mathcal{H},\ \forall\, \xi \in \mathcal{V}_\phi(X,\mu), \tag{3.4}$$

in the following weak sense

$$\langle T_\phi \xi | g\rangle = \int_X \xi(x)\langle \phi_x, g\rangle \,\mathrm{d}\mu(x),\ \ g \in \mathcal{D}, \xi \in \mathcal{V}_\phi(X,\mu).$$

In other words we are *imposing* that $\int_X \xi(x)\phi_x \,\mathrm{d}\mu(x)$ converge weakly to an element of $\mathcal{H}$.

The rest proceeds as before. We consider the space $V_\phi(X,\mu) = \mathcal{V}_\phi(X,\mu)/\mathsf{Ker}\ T_\phi$, with the norm $\|[\xi]_\phi\|_\phi = \|T_\phi \xi\|$, where, for $\xi \in V_\phi(X,\mu)$, we have put $[\xi]_\phi = \xi + \mathsf{Ker}\ T_\phi$. Then $V_\phi(X,\mu)$ is a pre-Hilbert space for that norm.

Note that ϕ was called in [8] *μ-independent* whenever $\mathsf{Ker}\ T_\phi = \{0\}$. In that case, of course, $V_\phi = \mathcal{V}_\phi$.

Assume, in addition, that the corresponding bounded operator $S_{\psi,\phi}$ is an element of $GL(\mathcal{H})$. Then (ψ,ϕ) is a reproducing pair and Theorem 3.14 of [8] remains true, that is,

Theorem 3.1. *If (ψ, ϕ) is a reproducing pair, the spaces $V_\phi(X,\mu)$ and $V_\psi(X,\mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form*

$$\langle [\xi]_\phi | [\eta]_\psi \rangle = \int_X \xi(x)\overline{\eta(x)}\, \mathrm{d}\mu(x),\ \forall\, \xi \in \mathcal{V}_\phi(X,\mu),\ \eta \in \mathcal{V}_\psi(X,\mu). \tag{3.5}$$

Example 3.2. To give a trivial example, consider the Schwartz rigged Hilbert space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, \mathrm{d}x) \subset \mathcal{S}^\times(\mathbb{R})$, $(X,\mu) = (\mathbb{R}, \mathrm{d}x)$, $\psi_x(t) = \phi_x(t) = \frac{1}{\sqrt{2\pi}} e^{ixt}$. Then $C_\phi f = \widehat{f}$, the Fourier transform, so that $\langle f | \phi(\cdot)\rangle \in L^2(\mathbb{R}, \mathrm{d}x)$.

In this case

$$\Omega_{\psi,\phi}(f,g) = \int_{\mathbb{R}} \langle f, \psi_x\rangle\langle \phi_x, g\rangle\, \mathrm{d}x = \langle \widehat{f} | \widehat{g}\rangle = \langle f|g\rangle,\ \forall f, g \in \mathcal{S}(\mathbb{R}),$$

and $V_\phi(\mathbb{R}, \mathrm{d}x) = L^2(\mathbb{R}, \mathrm{d}x)$.

3.2. **The general case.** In the general case, we only assume that the form Ω is jointly continuous on $\mathcal{D} \times \mathcal{D}$, with no other regularity requirement. In that case, the vector space $\mathcal{V}_\phi(X,\mu)$ must be defined differently. Let the topology of $\mathcal{D}$ be given by a directed family $\mathfrak{P}$ of seminorms. Given a weakly measurable function ϕ, we denote again by $\mathcal{V}_\phi(X,\mu)$ the space of all measurable functions $\xi : X \to \mathbb{C}$ such that the integral $\int_X \xi(x)\langle \phi_x, g\rangle\, \mathrm{d}\mu(x)$ exists for every $g \in \mathcal{D}$ and defines a continuous conjugate linear functional on $\mathcal{D}$, namely, there exists $c > 0$ and a seminorm $\mathsf{p} \in \mathfrak{P}$ such that

$$\left| \int_X \xi(x)\langle \phi_x, g\rangle\, \mathrm{d}\mu(x) \right| \leq c\, \mathsf{p}(g).$$

This in turn determines a linear map $T_\phi : \mathcal{V}_\phi(X,\mu) \to \mathcal{D}^\times$ by the following relation

$$\langle T_\phi \xi, g\rangle = \int_X \xi(x)\langle \phi_x, g\rangle\, \mathrm{d}\mu(x),\ \forall\, \xi \in \mathcal{V}_\phi(X,\mu), g \in \mathcal{D}. \tag{3.6}$$

Next, we define as before the vector space

$$V_\phi(X,\mu) = \mathcal{V}_\phi(X,\mu)/\mathsf{Ker}\ T_\phi,$$

and we put again $[\xi]_\phi = \xi + \mathsf{Ker}\ T_\phi$ for $\xi \in \mathcal{V}_\phi(X,\mu)$.

Now we need to introduce a topology on $V_\phi(X,\mu)$. We proceed as follows. Let $\mathcal{M}$ be a bounded subset of $\mathcal{D}[t]$. Then we define

$$\widehat{\mathsf{p}}_{\mathcal{M}}([\xi]_\phi) := \sup_{g \in \mathcal{M}} |\langle T_\phi \xi, g\rangle| \,. \tag{3.7}$$

That is, we are defining the topology of $V_\phi(X,\mu)$ by means of the strong dual topology $t^\times$ of $\mathcal{D}^\times$ which we recall is defined by the seminorms

$$\|F\|_{\mathcal{M}} = \sup_{g \in \mathcal{M}} |\langle F|g\rangle, \quad F \in \mathcal{D}^\times,$$

where $\mathcal{M}$ runs over the family of bounded subsets of $\mathcal{D}[t]$. As said above, the reflexivity of $\mathcal{D}$ entails that $t^\times$ is equal to the Mackey topology $\tau(\mathcal{D}^\times, \mathcal{D})$. More precisely,

Lemma 3.3. *The map* $\widehat{T}_\phi : V_\phi(X,\mu) \to \mathcal{D}^\times$, $\widehat{T}_\phi[\xi]_\phi := T_\phi \xi$ *is a well-defined linear map of* $V_\phi(X,\mu)$ *into* $\mathcal{D}^\times$ *and, for every bounded subset* $\mathcal{M}$ *of* $\mathcal{D}[t]$*, one has*

$$\widehat{\mathsf{p}}_{\mathcal{M}}([\xi]_\phi) = \|T_\phi \xi\|_{\mathcal{M}}, \quad \forall \xi \in \mathcal{V}_\phi(X,\mu)$$

The latter equality obviously implies the continuity of T_ϕ.

Next we investigate the dual $V_\phi(X,\mu)^*$ of the space $V_\phi(X,\mu)$, that is, the set of continuous linear functionals on $V_\phi(X,\mu)$. First, we have to choose a topology for $V_\phi(X,\mu)^*$. As usual we take the strong dual topology. This is defined by the family of seminorms

$$\mathsf{q}_{\mathcal{R}}(F) := \sup_{[\xi]_\phi \in \mathcal{R}} |F([\xi]_\phi)|,$$

where $\mathcal{R}$ runs over the bounded subsets of $V_\phi(X,\mu)$.

Theorem 3.4. *Assume that* $\mathcal{D}[t]$ *is a reflexive space and let* ϕ *be a weakly measurable function. If* F *is a continuous linear functional on* $V_\phi(X,\mu)$*, then there exists a unique* $g \in \mathcal{D}$ *such that*

$$F([\xi]_\phi) = \int_X \xi(x)\langle \phi_x, g\rangle \,\mathrm{d}\mu(x), \ \forall\, \xi \in V_\phi(X,\mu) \tag{3.8}$$

Moreover, every $g \in \mathcal{H}$ *defines a continuous functional* F *on* $V_\phi(X,\mu)$ *with* $\|F\|_{\phi^*} \le \|g\|$*, by* (3.8).

Proof. Let $F \in V_\phi(X,\mu)^*$. Then, there exists a bounded subset $\mathcal{M}$ of $\mathcal{D}[t]$ such that

$$|F([\xi]_\phi)| \le \widehat{\mathsf{p}}_{\mathcal{M}}([\xi]_\phi) = \|T_\phi \xi\|_{\mathcal{M}}, \ \forall\, \xi \in \mathcal{V}_\phi(X,\mu).$$

Let $\mathsf{M}_\phi := \{T_\phi \xi : \xi \in \mathcal{V}_\phi(X,\mu)\} = \mathsf{Ran}\, \widehat{T}_\phi$. Then M_ϕ is a vector subspace of $\mathcal{D}^\times$.

Let $\widetilde{F}$ be the functional defined on M_ϕ by

$$\widetilde{F}(T_\phi \xi) := F([\xi]_\phi), \ \xi \in \mathcal{V}_\phi(X,\mu).$$

We notice that $\widetilde{F}$ is well-defined. Indeed, if $T_\phi \xi = T_\phi \xi'$, then $\xi - \xi' \in \mathsf{Ker}\, T_\phi$. Hence, $[\xi]_\phi = [\xi']_\phi$ and $F([\xi]_\phi) = F([\xi']_\phi)$

Hence, $\widetilde{F}$ is a continuous linear functional on M_ϕ which can be extended (by the Hahn-Banach theorem) to a continuous linear functional on $\mathcal{D}^\times$. Thus, in virtue of the reflexivity of $\mathcal{D}$, there exists a vector $g \in \mathcal{D}$ such that

$$\widetilde{F}(T_\phi \xi) = \langle \widehat{T}_\phi [\xi]_\phi, g\rangle = \langle T_\phi \xi, g\rangle = \int_X \xi(x)\langle \phi_x, g\rangle \,\mathrm{d}\mu(x).$$

In conclusion,

$$F([\xi]_\phi) = \int_X \xi(x)\langle \phi_x, g\rangle \,\mathrm{d}\mu(x), \ \forall\, \xi \in \mathcal{V}_\phi(X,\mu).$$

Moreover, every $g \in \mathcal{D}$ obviously defines a continuous linear functional F on $V_\phi(X,\mu)$ by (3.8). In addition, if $\mathcal{R}$ is a bounded subset of $V_\phi(X,\mu)$, we have

$$\begin{aligned}\mathsf{q}_{\mathcal{R}}(F) &= \sup_{[\xi]_\phi \in \mathcal{R}} |F([\xi]_\phi)| = \sup_{[\xi]_\phi \in \mathcal{R}} \left| \int_X \xi(x)\langle \phi_x, g\rangle \,\mathrm{d}\mu(x) \right| \\ &= \sup_{[\xi]_\phi \in \mathcal{R}} |\langle T_\phi \xi, g\rangle| \le \sup_{[\xi]_\phi \in \mathcal{R}} \widehat{\mathsf{p}}_{\mathcal{M}}([\xi]_\phi),\end{aligned}$$

for any bounded subset $\mathcal{M}$ of $\mathcal{D}$ containing g. $\square$

In the present context, the analysis operator C_ϕ is defined in the usual way, given in (2.2). Then, particularizing the discussion of Theorem 3.4 to the functional $\langle C_\phi g, \cdot\rangle$, one can interpret the analysis operator C_ϕ as a continuous operator from $\mathcal{D}$ to $V_\phi(X,\mu)^*$. As in the case of frames or semi-frames, one may characterize the synthesis operator in terms of the analysis operator.

Proposition 3.5. *Let ϕ be weakly measurable, then $\widehat{T}_\phi \subseteq C_\phi^*$. If, in addition, $V_\phi(X,\mu)$ is reflexive, then $\widehat{T}_\phi^* = C_\phi$. Moreover, ϕ is μ-total (i.e. $\mathsf{Ker}\, C_\phi = \{0\}$) if and only if $\mathsf{Ran}\, \widehat{T}_\phi$ is dense in $\mathcal{D}^\times$.*

Proof. As $C_\phi : \mathcal{D} \to V_\phi(X,\mu)^*$ is a continuous operator, it has a continuous adjoint $C_\phi^* : V_\phi(X,\mu)^{**} \to \mathcal{H}$ [20, Sec.IV.7.4]. Let $C_\phi^\sharp := C_\phi^* \upharpoonright V_\phi(X,\mu)$. Then $C_\phi^\sharp = \widehat{T}_\phi$ since, for every $f \in \mathcal{D}$, $[\xi]_\phi \in V_\phi(X,\mu)$,

$$\langle C_\phi f, [\xi]_\phi\rangle = \int_X \langle f, \phi_x\rangle \overline{\xi(x)}\, \mathrm{d}\mu(x) = \langle f, \widehat{T}_\phi [\xi]_\phi\rangle. \tag{3.9}$$

If $V_\phi(X,\mu)$ is reflexive, we have, of course, $C_\phi^\sharp = C_\phi^* = \widehat{T}_\phi$.

If ϕ is not μ-total, then there exists $f \in \mathcal{D}$, $f \neq 0$ such that $(C_\phi f)(x) = 0$ for a.e. $x \in X$. Hence, $f \in (\mathsf{Ran}\, \widehat{T}_\phi)^\perp := \{f \in \mathcal{D} : \langle F|f\rangle = 0, \forall F \in \mathsf{Ran}\, \widehat{T}_\phi\}$ by (3.9). Conversely, if ϕ is μ-total, as $(\mathsf{Ran}\, \widehat{T}_\phi)^\perp = \mathsf{Ker}\, C_\phi = \{0\}$, by the reflexivity of $\mathcal{D}$ and $\mathcal{D}^\times$, it follows that $\mathsf{Ran}\, \widehat{T}_\phi$ is dense in $\mathcal{D}^\times$. $\square$

In a way similar to what we have done above, we can define the space $V_\psi(X,\mu)$, its topology, the residue classes $[\eta]_\psi$, the operator T_ψ, etc, replacing ϕ by ψ. Then, $V_\psi(X,\mu)$ is a locally convex space.

Theorem 3.6. *Under the condition (3.1), every bounded linear functional F on $V_\phi(X,\mu)$, i.e., $F \in V_\phi(X,\mu)^*$, can be represented as*

$$F([\xi]_\phi) = \int_X \xi(x)\overline{\eta(x)}\, \mathrm{d}\mu(x),\ \forall\, [\xi]_\phi \in V_\phi(X,\mu), \tag{3.10}$$

with $\eta \in \mathcal{V}_\psi(X,\mu)$. The residue class $[\eta]_\psi \in V_\psi(X,\mu)$ is uniquely determined.

Proof. By Theorem 3.4, we have the representation

$$F(\xi) = \int_X \xi(x)\langle \phi_x, g\rangle\, \mathrm{d}\mu(x).$$

It is easily seen that $\eta(x) = \langle g, \phi_x\rangle \in \mathcal{V}_\psi(X,\mu)$.

It remains to prove uniqueness. Suppose that

$$F(\xi) = \int_X \xi(x)\overline{\eta'(x)}\, \mathrm{d}\mu(x).$$

Then

$$\int_X \xi(x)(\overline{\eta'(x)} - \overline{\eta(x)})\, \mathrm{d}\mu(x) = 0.$$

Now the function $\xi(x)$ is arbitrary. Hence, taking in particular for $\xi(x)$ the functions $\langle f, \psi_x\rangle$, $f \in \mathcal{D}$, we get $[\eta]_\psi = [\eta']_\psi$. $\square$

The lesson of the previous statements is that the map

$$j : F \in V_\phi(X, \mu)^* \mapsto [\eta]_\psi \in V_\psi(X, \mu) \tag{3.11}$$

is well-defined and conjugate linear. On the other hand, $j(F) = j(F')$ implies easily $F = F'$. Therefore $V_\phi(X, \mu)^*$ can be identified with a closed subspace of $\overline{V_\psi}(X, \mu) := \{\overline{[\xi]_\psi} : \xi \in \mathcal{V}_\psi(X, \mu)\}$. Working in the framework of Hilbert spaces, as in Section 3.1, we proved in [8] that the spaces $V_\phi(X, \mu)^*$ and $\overline{V_\psi}(X, \mu)$ can be identified. The conclusion was that if (ψ, ϕ) is a reproducing pair, the spaces $V_\phi(X, \mu)$ and $V_\psi(X, \mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form (3.5). And if ϕ and ψ are also μ-total, then the converse statement holds true.

In the present situation, however, a result of this kind cannot be proved with techniques similar to those adopted in [8], which are specific of Hilbert spaces. In particular, the condition (b), $S_{\psi,\phi} \in GL(\mathcal{H})$, which was essential in the proof of [8, Lemma 3.11], is now missing, and it is not clear by what regularity condition it should replaced.

However, *assume* that $\mathsf{Ran}\, \widehat{C}_{\psi,\phi}[\|\cdot\|_\phi] = V_\phi(X, \mu)[\| \cdot \|_\phi]$ and $\mathsf{Ran}\, \widehat{C}_{\phi,\psi}[\|\cdot\|_\psi] = V_\psi(X, \mu)[\| \cdot \|_\psi]$, where we have defined the operator $\widehat{C}_{\phi,\psi} : \mathcal{H} \to V_\psi(X, \mu)$ by $\widehat{C}_{\phi,\psi} f := [C_\phi f]_\psi$ and similarly for $\widehat{C}_{\psi,\phi}$. Then the proof of [8, Theorem 3.14] works and the same result may be obtained. This is, however, a strong and non-intuitive assumption.

4. Reproducing pairs and genuine pip-spaces

In this section, we will consider the case where our measurable functions take their values in a genuine PIP-space. However, for simplicity, we will restrict ourselves to a lattice of Banach spaces (LBS) or a lattice of Hilbert spaces (LHS) [4]. For the convenience of the reader, we have summarized in the Appendix the basic notions concerning LBSs and LHSs.

Let (X, μ) be a locally compact, σ-compact measure space. Let $V_J = \{V_p, p \in J\}$ be a LBS or a LHS of measurable functions with the property

$$\xi \in V_p,\ \eta \in V_{\overline{p}} \implies \xi\overline{\eta} \in L^1(X, \mu) \qquad \text{and} \qquad \left| \int_X \xi(x)\overline{\eta(x)}\, \mathrm{d}\mu(x) \right| \leq \|\xi\|_p \, \|\eta\|_{\overline{p}}. \tag{4.1}$$

Thus the central Hilbert space is $\mathcal{H} := V_o = L^2(X, \mu)$ and the spaces $V_p, V_{\overline{p}}$ are dual of each other with respect to the L^2 inner product. The partial inner product, which extends that of $L^2(X, \mu)$, is denoted again by $\langle\cdot|\cdot\rangle$. As usual we put $V = \sum_{p \in J} V_p$ and $V^\# = \bigcap_{p \in J} V_p$. According to the general theory of PIP-spaces [4], V is the algebraic inductive limit of the V_p's (see the Appendix). Thus $\psi : X \to V$ means that $\psi : X \to V_p$ for some $p \in J$.

Example 4.1. A typical example is the lattice generated by the Lebesgue spaces $L^p(\mathbb{R}, \mathrm{d}x)$, $1 \leq p \leq \infty$, with $\frac{1}{p} + \frac{1}{\overline{p}} = 1$ [4]. We shall discuss it in detail in Section 6.

We will envisage two approaches, depending whether the functions ψ_x themselves belong to V or rather the scalar functions $C_\psi f$.

4.1. **Vector-valued measurable functions ψ_x.** This approach is the exact generalization of the one used in the RHS case. Let $x \in X \mapsto \psi_x$, $x \in X \mapsto \phi_x$ weakly measurable functions from X into V, where the latter is equipped with the weak topology $\sigma(V, V^{\#})$. More precisely, assume that $\psi : X \to V_p$ for some $p \in J$ and $\phi : X \to V_q$ for some $q \in J$, both weakly measurable. In that case, the analysis of Section 3.1 may be repeated *verbatim*, simply replacing $\mathcal{D}$ by $V^{\#}$, thus defining reproducing pairs. The problem with this approach is that, in fact, it does not exploit the PIP-space structure, only the RHS $V^{\#} \subset \mathcal{H} \subset V$! Clearly, this approach yields no benefit, so we turn to a different strategy.

4.2. **Scalar measurable functions $C_\psi f$.** Let ψ, ϕ be weakly measurable functions from X into $\mathcal{H}$. In view of (2.10), (4.1) and the definition of V, we assume that the following condition holds:

(p) $\exists\, p \in J$ such that $C_\psi f = \langle f|\psi_\cdot\rangle \in V_p$ and $C_\phi g = \langle g|\phi_\cdot\rangle \in V_{\overline{p}}, \forall\, f, g \in \mathcal{H}$.

We recall that $V_{\overline{p}}$ is the conjugate dual of V_p. In this case, then

$$\Omega_{\psi,\phi}(f, g) := \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \,\mathrm{d}\mu(x),\ \ f, g \in \mathcal{H},$$

defines a sesquilinear form on $\mathcal{H} \times \mathcal{H}$ and one has

$$|\Omega_{\psi,\phi}(f, g)| \leq \|C_\psi f\|_p \, \|C_\phi g\|_{\overline{p}}\,,\ \forall\, f, g \in \mathcal{H}. \tag{4.2}$$

If $\Omega_{\psi,\phi}$ is bounded as a form on $\mathcal{H} \times \mathcal{H}$ (this is not automatic, see Corollary 4.4), there exists a bounded operator $S_{\psi,\phi}$ in $\mathcal{H}$ such that

$$\int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \,\mathrm{d}\mu(x) = \langle S_{\psi,\phi} f|g\rangle,\ \forall\, f, g \in \mathcal{H}. \tag{4.3}$$

Then (ψ, ϕ) is a *reproducing pair* if $S_{\psi,\phi} \in GL(\mathcal{H})$.

Let us suppose that the spaces V_p have the following property

(k) If $\xi_n \to \xi$ in V_p, then, for every compact subset $K \subset X$, there exists a subsequence $\{\xi_n^K\}$ of $\{\xi_n\}$ which converges to ξ almost everywhere in K.

We note that condition (k) is satisfied by L^p-spaces [19].

As seen before, $C_\psi : \mathcal{H} \to V$, in general. This means, given $f \in \mathcal{H}$, there exists $p \in J$ such that $C_\psi f = \langle f|\psi_\cdot\rangle \in V_p$. We define

$$D_r(C_\psi) = \{f \in \mathcal{H} : C_\psi f \in V_r\},\ r \in J.$$

In particular, $D_r(C_\psi) = \mathcal{H}$ means $C_\psi(\mathcal{H}) \subset V_r$.

Proposition 4.2. *Assume that* (k) *holds. Then $C_\psi : D_r(C_\psi) \to V_r$ is a closed linear map.*

Proof. Let $f_n \to f$ in $\mathcal{H}$ and $\{C_\psi f_n\}$ be Cauchy in V_r. Since V_r is complete, there exists $\xi \in V_r$ such that $\|C_\psi f_n - \xi\|_r \to 0$. By (k), for every compact subset $K \subset X$, there exists a subsequence $\{f_n^K\}$ of $\{f_n\}$ such that $(C_\psi f_n^K)(x) \to \xi(x)$ a.e. in K. On the other hand, since $f_n \to f$ in $\mathcal{H}$, we get

$$\langle f_n|\psi_x\rangle \to \langle f|\psi_x\rangle, \quad \forall x \in X,$$

and the same holds true, of course, for $\{f_n^K\}$. From this we conclude that $\xi(x) = \langle f|\psi_x\rangle$ almost everywhere. Thus, $f \in D_r(C_\psi)$ and $\xi = C_\psi f$. □

By a simple application of the closed graph theorem we obtain

Corollary 4.3. *Assume that* (k) *holds. If for some* $r \in J$, $C_\psi(\mathcal{H}) \subset V_r$, *then* $C_\psi : \mathcal{H} \to V_r$ *is continuous.*

Combining Corollary 4.3 with (4.2), we get

Corollary 4.4. *Assume that* (k) *holds. If* $C_\psi(\mathcal{H}) \subset V_p$ *and* $C_\phi(\mathcal{H}) \subset V_{\overline{p}}$, *the form* Ω *is bounded on* $\mathcal{H} \times \mathcal{H}$, *that is,* $|\Omega_{\psi,\phi}(f,g)| \leq c \|f\| \, \|g\|$.

Hence, if condition (k) holds, $C_\psi(\mathcal{H}) \subset V_r$ implies that $C_\psi : \mathcal{H} \to V_r$ is continuous. If we don't know whether the condition holds, we will have to assume explicitly that $C_\psi : \mathcal{H} \to V_r$ is continuous.

If $C_\psi : \mathcal{H} \to V_r$ continuously, then $C_\psi^* : V_{\overline{r}} \to \mathcal{H}$ exists and it is continuous. By definition, if $\xi \in V_{\overline{r}}$,

$$\langle C_\psi f | \xi \rangle = \int_X \langle f | \psi_x \rangle \overline{\xi(x)} \, \mathrm{d}\mu(x), \ \forall f \in \mathcal{H}. \tag{4.4}$$

The relation (4.4) then implies that

$$\int_X \langle f | \psi_x \rangle \overline{\xi(x)} \, \mathrm{d}\mu(x) = \langle f | \int_X \psi_x \xi(x) \, \mathrm{d}\mu(x) \rangle, \ \forall f \in \mathcal{H}.$$

Thus,

$$C_\psi^* \xi = \int_X \psi_x \xi(x) \, \mathrm{d}\mu(x).$$

Of course, what we have said about C_ψ holds in the very same way for C_ϕ. Assume now that for some $p \in J, C_\psi : \mathcal{H} \to V_p$ and $C_\phi : \mathcal{H} \to V_{\overline{p}}$ continuously. Then, $C_\phi^* : V_p \to \mathcal{H}$ so that $C_\phi^* C_\psi$ is a well-defined bounded operator in $\mathcal{H}$. As before, we have

$$C_\phi^* \eta = \int_X \eta(x) \phi_x \, \mathrm{d}\mu(x), \ \forall \eta \in V_p.$$

Hence,

$$C_\phi^* C_\psi f = \int_X \langle f | \psi_x \rangle \phi_x \, \mathrm{d}\mu(x) = S_{\psi,\phi} f, \quad \forall f \in \mathcal{H},$$

the last equality following also from (4.3) and Corollary 4.4. Of course, this does not yet imply that $S_{\psi,\phi} \in GL(\mathcal{H})$, thus we don't know whether (ψ, ϕ) is a reproducing pair.

Let us now return to the pre-Hilbert space $\mathcal{V}_\phi(X,\mu)$. First, the defining relation (3.3) of [8] must be written as

$$\xi \in \mathcal{V}_\phi(X,\mu) \ \Leftrightarrow \ \left| \int_X \xi(x) \overline{(C_\phi g)(x)} \, \mathrm{d}\mu(x) \right| \leq c \, \|g\| \, , \ \forall g \in \mathcal{H}.$$

Since $C_\phi : \mathcal{H} \to V_{\overline{p}}$, the integral is well defined for all $\xi \in V_p$. This means, the inner product on the r.h.s. is in fact the partial inner product of V, which coincides with the L^2 inner product whenever the latter makes sense. We may rewrite the r.h.s. as

$$|\langle \xi | C_\phi g \rangle| \leq c \, \|g\| \, , \forall g \in \mathcal{H}, \ \xi \in V_p.$$

where $\langle\cdot|\cdot\rangle$ denotes the partial inner product. Next, by (4.1), one has, for $\xi \in V_p, g \in \mathcal{H}$,

$$|\langle\xi|C_\phi g\rangle| \leq \|\xi\|_p \, \|C_\phi g\|_{\overline{p}} \leq c\,\|\xi\|_p \, \|g\| \,,$$

where the last inequality follows from Corollary 4.3 or the assumption of continuity of C_ϕ. Hence indeed $\xi \in \mathcal{V}_\phi(X,\mu)$, so that $V_p \subset \mathcal{V}_\phi(X,\mu)$.

As for the adjoint operator, we have $C_\phi^* : V_p \to \mathcal{H}$. Then we may write, for $\xi \in V_p, g \in \mathcal{H}$, $\langle\xi|C_\phi g\rangle = \langle T_\phi \xi|g\rangle$, thus C_ϕ^* is the restriction from $\mathcal{V}_\phi(X,\mu)$ to V_p of the operator $T_\phi : \mathcal{V}_\phi \to \mathcal{H}$ introduced in Section 2, which reads now as

$$\langle T_\phi \xi|g\rangle = \int_X \xi(x)\langle\phi_x|g\rangle \,\mathrm{d}\mu(x), \;\forall\, \xi \in V_p, g \in \mathcal{H}. \tag{4.5}$$

Thus $C_\phi^* \subset T_\phi$.

Next, the construction proceeds as in Section 3.

The space $V_\phi(X,\mu) = \mathcal{V}_\phi(X,\mu)/\mathsf{Ker}\, T_\phi$, with the norm $\|[\xi]_\phi\|_\phi = \|T_\phi\xi\|$, is a pre-Hilbert space. Then Theorem 3.14 and the other results from Section 3 of [8] remain true. In particular, we have:

Theorem 4.5. *If (ψ,ϕ) is a reproducing pair, the spaces $V_\phi(X,\mu)$ and $V_\psi(X,\mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form (2.9), namely,*

$$\langle\!\langle\xi|\eta\rangle\!\rangle_\mu := \int_X \xi(x)\overline{\eta(x)}\,\mathrm{d}\mu(x).$$

Note the form (2.9) coincides with the inner product of $L^2(X,\mu)$ whenever the latter makes sense.

Let (ψ,ϕ) is a reproducing pair. Assume again that $C_\phi : \mathcal{H} \to V_{\overline{p}}$ continuously, which me may write $\widehat{C}_{\phi,\psi} : \mathcal{H} \to V_{\overline{p}}/\mathsf{Ker}\, T_\psi$, where $\widehat{C}_{\phi,\psi} : \mathcal{H} \to V_\psi(X,\mu)$ is the operator defined by $\widehat{C}_{\phi,\psi} f := [C_\phi f]_\psi$, already introduced at the end of Section 3.2. In addition, by [8, Theorem 3.13], one has $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}[\|\cdot\|_\phi] = V_\phi(X,\mu)[\|\cdot\|_\phi]$ and $\mathsf{Ran}\,\widehat{C}_{\phi,\psi}[\|\cdot\|_\psi] = V_\psi(X,\mu)[\|\cdot\|_\psi]$.

Putting everything together, we get

Corollary 4.6. *Let (ψ,ϕ) be a reproducing pair. Then, if $C_\psi : \mathcal{H} \to V_p$ and $C_\phi : \mathcal{H} \to V_{\overline{p}}$ continuously, one has*

$$\widehat{C}_{\phi,\psi} : \mathcal{H} \to V_{\overline{p}}/\mathsf{Ker}\, T_\psi = V_\psi(X,\mu) \simeq \overline{V_\phi}(X,\mu)^*, \tag{4.6}$$

$$\widehat{C}_{\psi,\phi} : \mathcal{H} \to V_p/\mathsf{Ker}\, T_\phi = V_\phi(X,\mu) \simeq \overline{V_\psi}(X,\mu)^*. \tag{4.7}$$

In these relations, the equality sign means an isomorphism of vector spaces, whereas $\simeq$ denotes an isomorphism of Hilbert spaces.

Proof. On one hand, we have $\mathsf{Ran}\,\widehat{C}_{\phi,\psi} = V_\psi(X,\mu)$. On the other hand, under the assumption $C_\phi(\mathcal{H}) \subset V_{\overline{p}}$, one has $V_{\overline{p}} \subset \mathcal{V}_\psi(X,\mu)$, hence $V_{\overline{p}}/\mathsf{Ker}\, T_\psi = \{\xi + \mathsf{Ker}\, T_\psi,\ \xi \in V_{\overline{p}}\} \subset V_\psi(X,\mu)$. Thus we get $V_\psi(X,\mu) = V_{\overline{p}}/\mathsf{Ker}\, T_\psi$ as vector spaces. Similarly $V_\phi(X,\mu) = V_p/\mathsf{Ker}\, T_\phi$. ∎

Notice that, in Condition (p), the index p cannot depend on f, g. We need some uniformity, in the form $C_\psi(\mathcal{H}) \subset V_p$ and $C_\phi(\mathcal{H}) \subset V_{\overline{p}}$. This is fully in line with the philosophy of PIP-spaces: the building blocks are the (assaying) subspaces V_p, not individual vectors.

5. The case of a Hilbert triplet or a Hilbert scale

5.1. **The general construction.** We have derived in the previous section the relations $V_p \subset \mathcal{V}_\phi(X, \mu), V_{\overline{p}} \subset \mathcal{V}_\psi(X, \mu)$, and their equivalent ones (4.6), (4.7). Then, since $V_\psi(X, \mu)$ and $V_\phi(X, \mu)$ are both Hilbert spaces, it seems natural to take for $V_p, V_{\overline{p}}$ Hilbert spaces as well, that is, take for V a LHS. The simplest case is then a Hilbert chain, for instance, the scale (A.3) $\{\mathcal{H}_k, k \in \mathbb{Z}\}$ built on the powers of a self-adjoint operator $A > I$. This situation is quite interesting, since in that case one may get results about spectral properties of symmetric operators (in the sense of PIP-space operators) [9].

Thus, let (ψ, ϕ) be a reproducing pair. For simplicity, we assume that $S_{\psi,\phi} = I$, that is, ψ, ϕ are dual to each other.

If ψ and ϕ are both frames, there is nothing to say, since then $C_\psi(\mathcal{H}), C_\phi(\mathcal{H}) \subset L^2(X, \mu) = \mathcal{H}_o$, so that there is no need for a Hilbert scale. Thus we assume that ψ is an upper semi-frame and ϕ is a lower semi-frame, dual to each other. It follows that $C_\psi(\mathcal{H}) \subset L^2(X, \mu)$. Hence Condition (p) becomes: There is an index $k \geq 1$ such that $C_\psi : \mathcal{H} \to \mathcal{H}_k$ and $C_\phi : \mathcal{H} \to \mathcal{H}_{\overline{k}}$ continuously, thus $V_p \equiv \mathcal{H}_k$ and $V_{\overline{p}} \equiv \mathcal{H}_{\overline{k}}$. This means we are working in the Hilbert triplet

$$V_p \equiv \mathcal{H}_k \subset \mathcal{H}_o = L^2(X, \mu) \subset \mathcal{H}_{\overline{k}} \equiv V_{\overline{p}}\,. \tag{5.1}$$

Next, according to Corollary 4.6, we have $V_\psi(X, \mu) = \mathcal{H}_{\overline{k}}/\mathsf{Ker}\, T_\psi$ and $V_\phi(X, \mu) = \mathcal{H}_k/\mathsf{Ker}\, T_\phi$, as vector spaces.

In addition, since ϕ is a lower semi-frame, [6, Lemma 2.1] tells us that C_ϕ has closed range in $L^2(X, \mu)$ and is injective. However its domain

$$D(C_\phi) := \{f \in \mathcal{H} : \int_X |\langle f|\phi_x\rangle|^2 \,\mathrm{d}\nu(x) < \infty\}$$

need not be dense, it could be $\{0\}$. Thus C_ϕ maps its domain $D(C_\phi)$ onto a closed subspace of $L^2(X, \mu)$, possibly trivial, and the whole of $\mathcal{H}$ into the larger space $\mathcal{H}_{\overline{k}}$.

5.2. **Examples.** As for concrete examples of such Hilbert scales, we might mention two. First the Sobolev spaces $H^k(\mathbb{R})$, $k \in \mathbb{Z}$, in $\mathcal{H}_0 = L^2(\mathbb{R}, dx)$, which is the scale generated by the powers of the self-adjoint operator $A^{1/2}$, where $A := 1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$. The other one corresponds to the quantum harmonic oscillator, with Hamiltonian $A_{\rm osc} := x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$. The spectrum of $A_{\rm osc}$ is $\{2n+1, n = 0, 1, 2, \ldots\}$ and it gets diagonalized on the basis of Hermite functions. It follows that $A_{\rm osc}^{-1}$, which maps every $\mathcal{H}_k$ onto $\mathcal{H}_{k-1}$, is a Hilbert-Schmidt operator. Therefore, the end space of the scale $\mathcal{D}^\infty(A_{\rm osc}) := \bigcap_k \mathcal{H}_k$, which is simply Schwartz' space $\mathcal{S}$ of C^∞ functions of fast decrease, is a nuclear space.

Actually one may give an explicit example, using a Sobolev-type scale. Let $\mathcal{H}_K$ be a reproducing kernel Hilbert space (RKHS) of (nice) functions on a measure space (X,μ), with kernel function $k_x, x \in X$, that is, $f(x) = \langle f|k_x\rangle_K$, $\forall f \in \mathcal{H}_K$. The corresponding reproducing kernel is $K(x,y) = k_y(x) = \langle k_y|k_x\rangle_K$. Choose the weight function $m(x) > 1$, the analog of the weight $(1+|x|^2)$ considered in the Sobolev case. Define the Hilbert scale $\mathcal{H}_k$, $k \in \mathbb{Z}$, determined by the multiplication operator $Af(x) = m(x)f(x)$, $\forall x \in X$. Hence, for each $l \geq 1$,

$$\mathcal{H}_l \subset \mathcal{H}_0 \equiv \mathcal{H}_K \subset \mathcal{H}_{\overline{l}}\,.$$

Then, for some $n \geq 1$, define the measurable functions $\phi_x = k_x m^n(x), \psi_x = k_x m^{-n}(x)$, so that $C_\psi : \mathcal{H}_K \to \mathcal{H}_n$, $C_\phi : \mathcal{H}_K \to \mathcal{H}_{\overline{n}}$ continuously, and they are dual of each other. One has indeed $\langle \phi_x|g\rangle_K = \langle k_x m^n(x)|g\rangle_K = \langle k_x|g\, m^n(x)\rangle_K = \overline{g(x)}\, m^n(x) \in \mathcal{H}_{\overline{n}}$ and $\langle \psi_x|g\rangle_K = \overline{g(x)}\, m^{-n}(x) \in \mathcal{H}_{\overline{n}}$, which implies duality. Thus (ψ,ϕ) is a reproducing pair with $S_{\psi,\phi} = I$, where ψ is an upper semi-frame and ϕ a lower semi-frame.

In this case, one can compute the operators T_ψ, T_ϕ explicitly. The definition (4.5) reads as, for all $\xi \in \mathcal{H}_n, g \in \mathcal{H}_K$,

$$\langle T_\phi \xi|g\rangle_K = \int_X \xi(x)\langle \phi_x|g\rangle_K \,\mathrm{d}\mu(x), \qquad = \int_X \xi(x)\, \overline{g(x)}\, m^n(x)\,\mathrm{d}\mu,$$

that is, $(T_\phi\xi)(x) = \xi(x)\, m^n(x)$ or $T_\phi\xi = \xi\, m^n$. However, since the weight $m(x) > 1$ is invertible, $\overline{g}\, m^n$ runs over the whole of $\mathcal{H}_{\overline{n}}$ whenever g runs over H_K. Hence $\xi \in \mathsf{Ker}\ T_\phi \subset \mathcal{H}_n$ means that $\langle T_\phi\xi|g\rangle_K = 0$, $\forall g \in H_K$, which implies $\xi = 0$, since the duality between $\mathcal{H}_n$ and $\mathcal{H}_{\overline{n}}$ is separating. The same reasoning yields $\mathsf{Ker}\ T_\psi = \{0\}$. Therefore $V_\phi(X,\mu) = \mathcal{H}_n$ and $V_\psi(X,\mu) = \mathcal{H}_{\overline{n}}$.

A more general situation may be derived from the discrete example of Section 6.1.3 of [8]. Take a sequence of weights $m := \{|m_n|\}_{n\in\mathbb{N}} \in c_0, m_n \neq 0$, and consider the space ℓ^2_m with norm $\|\xi\|_{\ell^2_m} := \sum_{n\in\mathbb{N}} |m_n\xi_n|^2$. Then we have the following triplet replacing (5.1)

$$\ell^2_{1/m} \subset \ell^2 \subset \ell^2_m. \tag{5.2}$$

Next, for each $n \in \mathbb{N}$, define $\psi_n = m_n\theta_n$, where θ is a frame or an orthonormal basis in ℓ^2. Then ψ is an upper semi-frame. Moreover, $\phi := \{(1/\overline{m_n})\theta_n\}_{n\in\mathbb{N}}$ is a lower semi-frame, dual to ψ, thus (ψ,ϕ) is a reproducing pair. Hence, by [8, Theorem 3.13], $V_\psi \simeq \mathsf{Ran}\, C_\phi = M_{1/m}(V_\theta(\mathbb{N})) = \ell^2_m$ and $V_\phi \simeq \mathsf{Ran}\, C_\psi = M_m(V_\theta(\mathbb{N})) = \ell^2_{1/m}$ (here we take for granted that $\mathsf{Ker}\ T_\psi = \mathsf{Ker}\ T_\phi = \{0\}$).

For making contact with the situation of (5.1), consider in ℓ^2 the diagonal operator $A := \mathrm{diag}[n], n \in \mathbb{N}$ (the number operator), that is $(A\xi)_n = n\,\xi_n, n \in \mathbb{N}$, which is obviously self-adjoint and larger than 1. Then $\mathcal{H}_k = D(A^k)$ with norm $\|\xi\|_k = \left\|A^k\xi\right\| \equiv \ell^2_{r^{(k)}}$, where $(r^{(k)})_n = n^k$ (note that $1/r^{(k)} \in c_0$). Hence we have

$$\mathcal{H}_k = \ell^2_{r^{(k)}} \subset \mathcal{H}_o = \ell^2 \subset \mathcal{H}_{\overline{k}} = \ell^2_{1/r^{(k)}}\,, \tag{5.3}$$

where $(1/r^{(k)})_n = n^{-k}$. In addition, as in the continuous case discussed above, the end space of the scale, $\mathcal{D}^\infty(A) := \bigcap_k \mathcal{H}_k$, is simply Schwartz' space s of fast decreasing sequences, with dual $\mathcal{D}_{\overline{\infty}}(A) := \bigcup_k \mathcal{H}_k = s'$, the space of slowly increasing sequences. Here too, this construction shows that the space s is nuclear, since every embedding $A^{-1} : \mathcal{H}_{k+1} \to \mathcal{H}_k$ is a Hilbert-Schmidt operator.

However, the construction described above yields a much more general family of examples, since the weight sequences m are not ordered.

6. The case of L^p spaces

Following the suggestion made at the end of Section 2, we present now several possibilities of taking $\mathsf{Ran}\, C_\psi$ in the context of the Lebesgue spaces $L^p(\mathbb{R}, \mathrm{d}x)$.

As it is well-known, these spaces don't form a chain, since two of them are never comparable. We have only

$$L^p \cap L^q \subset L^s, \text{ for all } s \text{ such that } \ p < s < q.$$

Take the lattice $\mathcal{J}$ generated by $\mathcal{I} = \{L^p(\mathbb{R}, \mathrm{d}x), 1 \leq p \leq \infty\}$, with lattice operations [4, Sec.4.1.2]:

- $L^p \wedge L^q = L^p \cap L^q$ is a Banach space for the projective norm $\|f\|_{p\wedge q} = \|f\|_p + \|f\|_q$
- $L^p \vee L^q = L^p + L^q$ is a Banach space for the inductive norm
$$\|f\|_{p\vee q} = \inf_{f=g+h} \{\|g\|_p + \|h\|_q; g \in L^p,\, h \in L^q\}$$
- For $1 < p, q < \infty$, both spaces $L^p \wedge L^q$ and $L^p \vee L^q$ are reflexive and $(L^p \wedge L^q)^\times = L^{\overline{p}} \vee L^{\overline{q}}$.

Moreover, no additional spaces are obtained by iterating the lattice operations to any finite order. Thus we obtain an involutive lattice and a LBS, denoted by V_{J}.

It is convenient to introduce a unified notation:

$$L^{(p,q)} = \begin{cases} L^p \wedge L^q = L^p \cap L^q, & \text{if } \ p \geq q, \\ L^p \vee L^q = L^p + L^q, & \text{if } \ p \leq q. \end{cases}$$

Following [4, Sec.4.1.2], we represent the space $L^{(p,q)}$ by the point $(1/p, 1/q)$ of the unit square $\mathrm{J} = [0,1] \times [0,1]$. In this representation, the spaces L^p are on the main diagonal, intersections $L^p \cap L^q$ above it and sums $L^p + L^q$ below, the duality is $[L^{(p,q)}]^\times = L^{(\overline{p},\overline{q})}$, that is, symmetry with respect to L^2. Hence, $L^{(p,q)} \subset L^{(p',q')}$ if $(1/p, 1/q)$ is on the left and/or above $(1/p', 1/q')$ The extreme spaces are

$$V_{\mathrm{J}}^{\#} = L^{(\infty,1)} = L^\infty \cap L^1, \quad \text{ and } \quad V_{\mathrm{J}} = L^{(1,\infty)} = L^1 + L^\infty.$$

For a full picture, see [4, Fig.4.1].

There are three possibilities for using the L^p lattice for controlling reproducing pairs

(1) Exploit the *full lattice* $\mathcal{J}$, that is, find (p,q) such that, $\forall f, g \in \mathcal{H}$, $C_\psi f \,\#\, C_\phi g$ in the PIP-space V_{J}, that is, $C_\psi f \in L^{(p,q)}$ and $C_\phi g \in L^{(\overline{p},\overline{q})}$.

(2) Select in V_{J} a self-dual *Banach chain* V_{I}, centered around L^2, symbolically.

$$\ldots L^{(s)} \subset \ldots \subset L^2 \subset \ldots \subset L^{(\overline{s})} \ldots, \tag{6.1}$$

such that $C_\psi f \in L^{(s)}$ and $C_\phi g \in L^{(\overline{s})}$ (or vice-versa). Here are three examples of such Banach chains.

- The *diagonal* chain : $q = \overline{p}$

$$L^\infty \cap L^1 \subset \ldots \subset L^{\overline{q}} \cap L^q \subset \ldots \subset L^2 \subset \ldots \subset L^q + L^{\overline{q}} = (L^{\overline{q}} \cap L^q)^\times \subset \ldots \subset L^1 + L^\infty.$$

The space $L^1 + L^\infty$ has been considered by Gould [13].

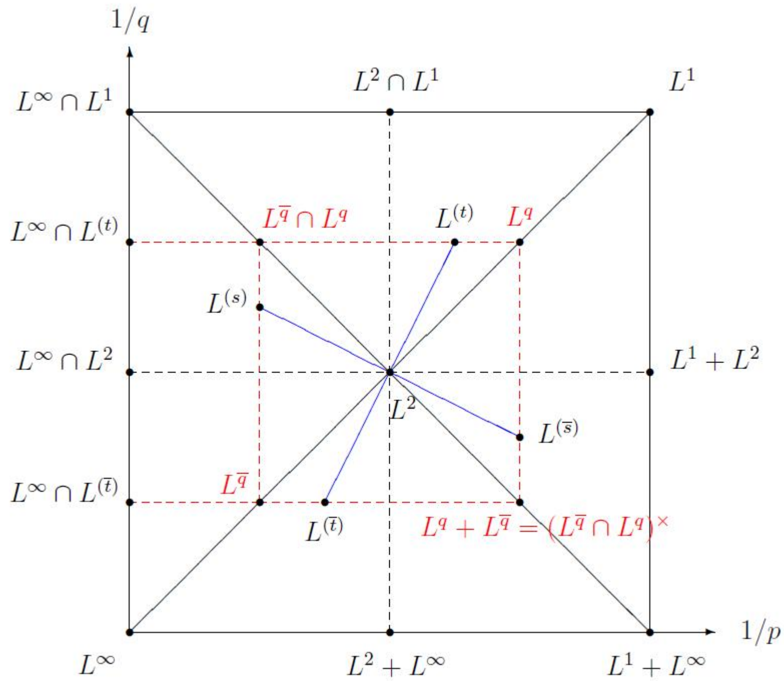

FIGURE 1. (i) The pair $L^{(s)}, L^{(\overline{s})}$ for s in the second quadrant; (ii) The pair $L^{(t)}, L^{(\overline{t})}$ for t in the first quadrant.

- The horizontal chain $q = 2$:

$$L^\infty \cap L^2 \subset \ldots \subset L^2 \subset \ldots \subset L^1 + L^2.$$

- The vertical chain $p = 2$:

$$L^2 \cap L^1 \subset \ldots \subset L^2 \subset \ldots \subset L^2 + L^\infty.$$

All three chains are presented in Figure 1. In this case, the full chain belongs to the second and fourth quadrants (top left and bottom right). A typical point is then $s = (p, q)$ with, $2 \leq p \leq \infty, 1 \leq q \leq 2$, so that one has the situation depicted in (6.1), that is, the spaces $L^{(s)}, L^{(\overline{s})}$ to which $C_\psi f$, resp. $C_\phi g$, belong, are necessarily comparable to each other and to L^2. In particular, one of them is necessarily contained in L^2. Note the extreme spaces of that type are $L^2, L^\infty \cap L^2, L^\infty \cap L^1$ and $L^2 \cap L^1$ (see Figure 1).

(3) Choose a dual pair in the first and third quadrant (top right, bottom left). A typical point is then $t = (p', q')$, with $1 < p', q' < 2$, so that the spaces $L^{(t)}, L^{(\overline{t})}$ are never comparable to each other, nor to L^2.

Let us now add the boundedness condition mentioned at the end of Section 2, $\sup_{x \in X} \|\psi_x\|_{\mathcal{H}} \leq c$ and $\sup_{x \in X} \|\phi_x\|_{\mathcal{H}} \leq c'$ for some $c, c' > 0$. Then $C_\psi f(x) = \langle f|\psi_x \rangle \in L^\infty(X, \mathrm{d}\mu)$ and $C_\phi f(x) = \langle f|\phi_x \rangle \in L^\infty(X, \mathrm{d}\mu)$. Therefore, the third case reduces to the second one, since we have now (in the situation of Figure 1).

$$L^\infty \cap L^{(t)} \subset L^\infty \cap L^2 \subset L^\infty \cap L^{(\overline{t})}.$$

Following the pattern of Hilbert scales, we choose a (Gel'fand) triplet of Banach spaces. One could have, for instance, a triplet of reflexive Banach spaces such as

$$L^{(s)} \subset \ldots \subset L^2 \subset \ldots \subset L^{(\overline{s})}, \tag{6.2}$$

corresponding to a point s inside of the second quadrant, as shown in Figure 1. In this case, according to (4.6) and (4.7), $V_\psi = L^{(\overline{s})}/\mathsf{Ker}\, T_\psi$ and $V_\phi = L^{(s)}/\mathsf{Ker}\, T_\phi$.

On the contrary, if we choose a point t in the second quadrant, case (3) above, it seems that no triplet arises. However, if (ψ, ϕ) is a nontrivial reproducing pair, with $S_{\psi,\phi} = I$, that is, ψ, ϕ are dual to each other, one of them, say ψ, is an upper semi-frame and then necessarily ϕ is a lower semi-frame [6, Prop.2.6]. Therefore $C_\psi(\mathcal{H}) \subset L^2(X, \mu)$, that is, case (3) cannot be realized.

Inserting the boundedness condition, we get a triplet where the extreme spaces are no longer reflexive, such as

$$L^\infty \cap L^{(t)} \subset L^\infty \cap L^2 \subset L^\infty \cap L^{(\overline{t})},$$

and then $V_\psi = (L^\infty \cap L^{(t)})/\mathsf{Ker}\, T_\psi$ and $V_\phi = (L^\infty \cap L^{(\overline{t})})/\mathsf{Ker}\, T_\phi$.

In conclusion, the only acceptable solution is the triplet (6.2), with s strictly inside of the second quadrant, that is, $s = (p, q)$ with, $2 \leq p < \infty, 1 < q \leq 2$.

A word of explanation is in order here, concerning the relations $V_\psi = L^{(\overline{s})}/\mathsf{Ker}\, T_\psi$ and $V_\phi = L^{(s)}/\mathsf{Ker}\, T_\phi$. On the l.h.s., $L^{(s)}$ and $L^{(\overline{s})}$ are reflexive Banach spaces, with their usual norm, and so are the quotients by T_ψ, resp. T_ϕ. On the other hand, $V_\psi(X, \mu)[\|\cdot\|_\psi]$ and $V_\phi(X, \mu)[\|\cdot\|_\phi]$ are Hilbert spaces. However, there is no contradiction, since the equality sign $=$ denotes an isomorphism of vector spaces only, without reference to any topology. Moreover, the two norms, Banach and Hilbert, *cannot* be comparable, unless they are equivalent [16, Coroll. 1.6.8], which is impossible in the case of $L^p, p \neq 2$. The same is true for any LBS where the spaces V_p are not Hilbert spaces.

Although we don't have an explicit example of a reproducing pair, we indicate a possible construction towards one. Let $\theta^{(1)} : \mathbb{R} \to L^2$ be a measurable function such that $\langle h|\theta_x^{(1)}\rangle \in L^q$, $\forall\, h \in L^2, 1 < q < 2$ and let $\theta^{(2)} : \mathbb{R} \to L^2$ be a measurable function such that $\langle h|\theta_x^{(2)}\rangle \in L^{\overline{q}}$, $\forall\, h \in L^2$. Define $\psi_x := \min(\theta_x^{(1)}, \theta_x^{(2)}) \equiv \theta_x^{(1)} \wedge \theta_x^{(2)}$ and $\phi_x := \max(\theta_x^{(1)}, \theta_x^{(2)}) \equiv \theta_x^{(1)} \vee \theta_x^{(2)}$. Then we have

$$(C_\psi h)(x) = \langle h|\psi_x\rangle \in L^q \cap L^{\overline{q}}, \forall\, h \in L^2$$
$$(C_\phi h)(x) = \langle h|\phi_x\rangle \in L^q + L^{\overline{q}}, \forall\, h \in L^2$$

and we have indeed $L^q \cap L^{\overline{q}} \subset L^2 \subset L^q + L^{\overline{q}}$. It remains to guarantee that ψ and ϕ are dual to each other, that is,

$$\int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \, \mathrm{d}\mu(x) = \int_X C_\psi f(x)\, \overline{C_\phi g(x)} \, \mathrm{d}\mu(x) = \langle f|g\rangle, \forall\, f, g \in L^2.$$

7. Outcome

We have seen in [8] that the notion of reproducing pair is quite rich. It generates a whole mathematical structure, which ultimately leads to a pair of Hilbert spaces, conjugate dual to each other with respect to the $L^2(X, \mu)$ inner product. This suggests that one should make more precise the best assumptions on the measurable functions or, more precisely, on the nature of the range of the analysis operators C_ψ, C_ϕ. This in turn suggests to analyze the whole structure in the language of PIP-spaces, which is the topic of the present paper. In particular, a natural choice is a scale, or simply a triplet, of Hilbert spaces, the two extreme spaces being conjugate duals of each other with respect to the $L^2(X, \mu)$ inner product. Another possibility consists of exploiting the lattice of all $L^p(\mathbb{R}, \mathrm{d}x)$ spaces, or a subset thereof, in particular a (Gel'fand) triplet of Banach spaces. Some examples have been described above, but clearly more work along these lines is in order.

Appendix A. Lattices of Banach or Hilbert spaces and operators on them

A.1. **Lattices of Banach or Hilbert spaces.** For the convenience of the reader, we summarize in this Appendix the basic facts concerning PIP-spaces and operators on them. However, we will restrict the discussion to the simpler case of a lattice of Banach (LBS) or Hilbert spaces (LHS). Further information may be found in our monograph [4] or our review paper [5].

Let thus $\mathcal{J} = \{V_p, p \in I\}$ be a family of Hilbert spaces or reflexive Banach spaces, partially ordered by inclusion. Then $\mathcal{I}$ generates an involutive lattice $\mathcal{J}$, indexed by J, through the operations ($p, q, r \in I$):

. involution: $V_r \leftrightarrow V_{\overline{r}} = V_r^\times$, the conjugate dual of V_r
. infimum: $V_{p\wedge q} := V_p \wedge V_q = V_p \cap V_q$
. supremum: $V_{p\vee q} := V_p \vee V_q = V_p + V_q$.

It turns out that both $V_{p\wedge q}$ and $V_{p\vee q}$ are Hilbert spaces, resp. reflexive Banach spaces, under appropriate norms (the so-called projective, resp. inductive norms). Assume that the following conditions are satisfied:

(i) $\mathcal{I}$ contains a unique self-dual, Hilbert subspace $V_o = V_{\overline{o}}$.
(ii) for every $V_r \in \mathcal{I}$, the norm $\|\cdot\|_{\overline{r}}$ on $V_{\overline{r}} = V_r^\times$ is the conjugate of the norm $\|\cdot\|_r$ on V_r.

In addition to the family $\mathcal{J} = \{V_r, r \in J\}$, it is convenient to consider the two spaces $V^\#$ and V defined as

$$V = \sum_{q\in I} V_q, \quad V^\# = \bigcap_{q\in I} V_q. \tag{A.1}$$

These two spaces themselves usually do *not* belong to $\mathcal{I}$. According to the general theory of PIP-spaces [4], V is the algebraic inductive limit of the V_p's and $V^\#$ is the projective limit of the V_p's .

We say that two vectors $f, g \in V$ are *compatible* if there exists $r \in J$ such that $f \in V_r, g \in V_{\overline{r}}$. Then a *partial inner product* on V is a Hermitian form $\langle\cdot|\cdot\rangle$ defined exactly on compatible pairs of vectors. In particular, the partial inner product $\langle\cdot|\cdot\rangle$ coincides with the inner product of V_o on the latter. A *partial inner product space* (PIP-space) is a vector space V equipped with a partial inner product. Clearly LBSs and LHSs are particular cases of PIP-spaces.

From now on, we will assume that our PIP-space $(V, \langle\cdot|\cdot\rangle)$ is *nondegenerate*, that is, $\langle f|g\rangle = 0$ for all $f \in V^\#$ implies $g = 0$. As a consequence, $(V^\#, V)$ and every couple $(V_r, V_{\overline{r}})$, $r \in J$, are a dual pair in the sense of topological vector spaces [15]. In particular, the original norm topology on V_r coincides with its Mackey topology $\tau(V_r, V_{\overline{r}})$, so that indeed its conjugate dual is $(V_r)^\times = V_{\overline{r}}$, $\forall\, r \in J$. Then, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^\#$ is dense in every V_r. In the sequel, we also assume the partial inner product to be positive definite, $\langle f|f\rangle > 0$ whenever $f \neq 0$.

A standard, albeit trivial, example is that of a Rigged Hilbert space (RHS) $\Phi \subset \mathcal{H} \subset \Phi^\#$ (it is trivial because the lattice $\mathcal{I}$ contains only three elements).

Familiar concrete examples of PIP-spaces are sequence spaces, with $V = \omega$ the space of *all* complex sequences $x = (x_n)$, and spaces of locally integrable functions with $V = L^1_{\rm loc}(\mathbb{R}, {\rm d}x)$, the space of Lebesgue measurable functions, integrable over compact subsets.

Among LBSs, the simplest example is that of a chain of reflexive Banach spaces. The prototype is the chain $\mathcal{I} = \{L^p := L^p([0,1]; dx),\ 1 < p < \infty\}$ of Lebesgue spaces over the interval [0, 1].

$$L^\infty \subset \ldots \subset L^{\overline{q}} \subset L^{\overline{r}} \subset \ldots \subset L^2 \subset \ldots \subset L^r \subset L^q \subset \ldots \subset L^1, \quad \text{(A.2)}$$

where $1 < q < r < 2$ (of course, L^∞ and L^1 are not reflexive). Here L^q and $L^{\overline{q}}$ are dual to each other $(1/q + 1/\overline{q} = 1)$, and similarly $L^r, L^{\overline{r}}$ $(1/r + 1/\overline{r} = 1)$.

As for a LHS, the simplest example is the Hilbert scale generated by a self-adjoint operator $A > I$ in a Hilbert space$\mathcal{H}_o$. Let $\mathcal{H}_n$ be $D(A^n)$, the domain of A^n, equipped with the graph norm $\|f\|_n = \|A^n f\|$, $f \in D(A^n)$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathcal{H}_{\overline{n}} := \mathcal{H}_{-n} = \mathcal{H}_n^\times$ (conjugate dual):

$$\mathcal{D}^\infty(A) := \bigcap_n \mathcal{H}_n \subset \ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 \ \subset \mathcal{H}_{\overline{1}} \subset \mathcal{H}_{\overline{2}} \ldots \subset \mathcal{D}_{\overline{\infty}}(A) := \bigcup_n \mathcal{H}_n. \quad \text{(A.3)}$$

Note that here the index n may be integer or real, the link between the two cases being established by the spectral theorem for self-adjoint operators. Here again the inner product of $\mathcal{H}_0$ extends to each pair $\mathcal{H}_n, \mathcal{H}_{-n}$, but on $\mathcal{D}_{\overline{\infty}}(A)$ it yields only a *partial* inner product. A standard example is the scale of Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{Z}$, in $\mathcal{H}_0 = L^2(\mathbb{R}, dx)$.

A.2. **Operators on LBSs and LHSs.** Let V_J be a LHS or a LBS. Then an *operator* on V_J is a map from a subset $\mathcal{D}(A) \subset V$ into V, such that

(i) $\mathcal{D}(A) = \bigcup_{q \in \mathsf{d}(A)} V_q$, where $\mathsf{d}(A)$ is a nonempty subset of J;

(ii) For every $q \in \mathsf{d}(A)$, there exists $p \in J$ such that the restriction of A to V_q is a continuous linear map into V_p (we denote this restriction by A_{pq});

(iii) A has no proper extension satisfying (i) and (ii).

We denote by ${\rm Op}(V_J,)$ the set of all operators on V_J . The continuous linear operator $A_{pq} : V_q \to V_p$ is called a *representative* of A. The properties of A are conveniently described by the set $\mathsf{j}(A)$ of all pairs $(q, p) \in J \times J$ such that A maps V_q continuously into V_p Thus the operator A may be identified with the collection of its representatives,

$$A \simeq \{A_{pq} : V_q \to V_p : (q, p) \in \mathsf{j}(A)\}. \quad \text{(A.4)}$$

It is important to notice that an operator is uniquely determined by *any* of its representatives, in virtue of Property (iii): there are no extensions for PIP-space operators.

We will also need the following sets:

$$\mathsf{d}(A) = \{q \in J : \text{there is a } p \text{ such that } A_{pq} \text{ exists}\},$$
$$\mathsf{i}(A) = \{p \in J : \text{there is a } q \text{ such that } A_{pq} \text{ exists}\}.$$

The following properties are immediate:

- $\mathsf{d}(A)$ is an initial subset of J: if $q \in \mathsf{d}(A)$ and $q' < q$, then $q' \in \mathsf{d}(A)$, and $A_{pq'} = A_{pq}E_{qq'}$, where $E_{qq'}$ is a representative of the unit operator.
- $\mathsf{i}(A)$ is a final subset of J: if $p \in \mathsf{i}(A)$ and $p' > p$, then $p' \in \mathsf{i}(A)$ and $A_{p'q} = E_{p'p}A_{pq}$.

Although an operator may be identified with a separately continuous sesquilinear form on $V^{\#} \times V^{\#}$, or a conjugate linear continuous map $V^{\#}$ into V, it is more useful to keep also the *algebraic operations* on operators, namely:

(i) *Adjoint:* every $A \in \mathrm{Op}(V_J)$ has a unique adjoint $A^{\times} \in \mathrm{Op}V_J)$, defined by

$$\langle A^{\times} y | x\rangle = \langle y | Ax\rangle, \text{ for } x \in V_q,\ q \in \mathsf{d}(A) \text{ and } y \in V_{\overline{p}},\ p \in \mathsf{i}(A), \tag{A.5}$$

that is, $(A^{\times})_{\overline{qp}} = (A_{pq})'$, where $(A_{pq})' : V_{\overline{p}} \to V_{\overline{q}}$ is the adjoint map of A_{pq}. Furthermore, one has $A^{\times\times} = A$, for every $A \in \mathrm{Op}(V_J)$: no extension is allowed, by the maximality condition (iii) of the definition.

(ii) *Partial multiplication:* Let $A, B \in \mathrm{Op}(V_J)$. We say that the product BA is defined if and only if there is a $r \in \mathsf{i}(A) \cap \mathsf{d}(B)$, that is, if and only if there is a continuous factorization through some V_r:

$$V_q \xrightarrow{A} V_r \xrightarrow{B} V_p, \quad \text{i.e.,} \quad (BA)_{pq} = B_{pr}A_{rq}, \text{ for some } q \in \mathsf{d}(A), p \in \mathsf{i}(B). \tag{A.6}$$

Of particular interest are *symmetric* operators, defined as those operators satisfying the relation $A^{\times} = A$, since these are the ones that could generate self-adjoint operators in the central Hilbert space, for instance by the celebrated KLMN theorem, suitably generalized to the PIP-space environment [4, Section 3.3].

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References

1. S. T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Square integrability of group representations on homogeneous spaces I. Reproducing triples and frames*, Ann. Inst. H. Poincaré **55** (1991), 829–856.
2. S. T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Continuous frames in Hilbert space*, Annals of Phys. **222** (1993), 1–37.

3. J.-P. Antoine, A. Inoue, and C. Trapani, *Partial ∗-algebras and their operator realizations*, Mathematics and Its Applications, vol. 553, Kluwer, Dordrecht, NL, 2002.
4. J.-P. Antoine and C. Trapani , *Partial inner product spaces: theory and applications*, Lecture Notes in Mathematics, vol. 1986, Springer-Verlag, Berlin, 2009.
5. J.-P. Antoine and C. Trapani, *The partial inner product space method: A quick overview*, Adv. Math. Phys. **2010**, Art. ID 457635, 37 pp.; Erratum, Adv. Math. Phys. 2011, Art. ID 272703, 1 p.
6. J.-P. Antoine and P. Balazs, *Frames and semi-frames*, J. Phys. A: Math. Theor. **44** (2011), 205–201; Corrigendum, *ibid.* **44** (2011), 479–501.
7. J.-P. Antoine and P. Balazs, *Frames, semi-frames, and Hilbert scales*, Numer. Funct. Anal. Optimiz. **33** (2012), 736–769.
8. J.-P. Antoine, M. Speckbacher and C. Trapani, *Reproducing pairs of measurable functions*, Acta Appl. Math. (2017), DOI 10.1007/s10440-017-0095-4.
9. J.-P. Antoine and C. Trapani, *Operators on partial inner product spaces: Towards a spectral analysis*, Mediterr. J. Math. **13** (2016), 323–351
10. O. Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, Boston, MA, 2003
11. M. Fornasier and H. Rauhut, *Continuous frames, function spaces, and the discretization problem*, J. Fourier Anal. Appl. **11** (2005), 245–287
12. J.-P. Gabardo and D. Han, *Frames associated with measurable spaces*, Adv. Comput. Math. **18** (2003), 127–147
13. G. G. Gould, *On a class of integration spaces*, J. London Math. Soc. **34** (1959), 161–172
14. G. Kaiser *A Friendly guide to wavelets*, Birkhäuser, Boston, 1994
15. G. Köthe, *Topological vector spaces. I.*, Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159 Springer-Verlag New York Inc., New York 1969.
16. R. E. Megginson, *An introduction to Banach space theory*, Springer-Verlag, New York-Heidelberg-Berlin, 1998.
17. A. Rahimi, A. Najati, and Y. N. Dehghan, *Continuous frames in Hilbert spaces*, Methods Funct. Anal. Topol. **12** (2006), 170–182
18. A. Rahimi, B. Daraby, and Z. Darvishi, *Construction of continuous frames in Hilbert spaces*, arXiv 1606.08981.
19. W. Rudin, *Real and complex analysis*, Int. Edition, McGraw Hill, New York et al., 1987.
20. H. H. Schaefer, *Topological vector spaces*, Springer-Verlag, New York-Heidelberg-Berlin, 1971.
21. M. Speckbacher and P. Balazs, *Reproducing pairs and the continuous nonstationary Gabor transform on LCA groups*, J. Phys. A **48** (2015), no. 39, 395201, 16 pp.
22. M. Speckbacher and P. Balazs, *Reproducing pairs and Gabor systems at critical density*, arXiv:1610.06697.
23. P. Werner, *A distributional-theoretical approach to certain Lebesgue and Sobolev spaces*, J. Math. Anal. Appl. **29** (1970), 18–78

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