ON SPECTRAL SYNTHESIS IN SEVERAL VARIABLES

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ABSTRACT. In a recent paper we proposed a possible generalization of L. Schwartz's classical spectral synthesis result for continuous functions in several variables. The idea is based on Gelfand pairs and spherical functions while "translation invariance" is replaced by invariance with respect to the action of affine groups. In this paper we describe the function classes which play the role of the exponential monomials in this setting.

1. INTRODUCTION

Spectral analysis and spectral synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. We consider the space $\mathcal{C}(G)$ of all complex valued continuous functions on a locally compact Abelian group G, which is a locally convex topological linear space with respect to the point-wise linear operations (addition, multiplication with scalars) and to the topology of uniform convergence on compact sets. Continuous homomorphisms of G into the additive topological group of complex numbers and into the multiplicative topological group of nonzero complex numbers, respectively, are called *additive* and *exponential functions*, respectively. A function is a *polynomial* if it belongs to the algebra generated by the continuous additive functions. An *exponential monomial* is the product of a polynomial and an exponential.

It turns out that exponential functions, or more generally, exponential monomials can be considered as basic building bricks of varieties. Here "variety" means

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a linear space of continuous functions which is closed under translations and with respect to uniform convergence on compact sets. We shall define this concept in a more general situation but we underline that it is completely different from the one used in algebraic geometry under the same name. A given variety may or may not contain any exponential function or exponential monomial. If every nonzero subvariety of it contains an exponential function, then we say that *spectral analysis* holds for the variety. An exponential function in a variety can be considered as a kind of *spectral value* and the set of all exponential functions in a variety is called the *spectrum* of the variety. It follows that spectral analysis for a variety means that the spectrum of the variety is nonempty. On the other hand, the set of all exponential monomials contained in a variety is called the *spectral set* of the variety. It turns out that if an exponential monomial belongs to a variety, then the exponential function appearing in the representation of this exponential monomial belongs to the variety, too. Hence, if the spectral set of a variety is nonempty, then also the spectrum of the variety is nonempty and vice versa. There is, however, an even stronger property of some varieties, namely, if the spectral set of the variety spans a dense subspace of the variety. In this case we say that the variety is *synthesizable*. If every subvariety of a variety is synthesizable, then we say that *spectral synthesis* holds for the variety. It follows, that for spectral synthesis for a variety implies spectral analysis. If spectral analysis, respectively, spectral synthesis holds for every variety on an Abelian group, then we say that *spectral analysis*, respectively, *spectral synthesis* holds on the Abelian group. A famous and pioneering result of L. Schwartz [14] exhibits the situation by stating that if the group is the reals with the Euclidean topology, then spectral values do exist, that is, every nonzero variety contains an exponential function. In other words, in this case the spectrum is nonempty, and spectral analysis holds. Furthermore, spectral synthesis also holds in this situation: there are sufficiently many exponential monomials in the variety in the sense that their linear hull is dense in the subspace.

Although the result of L. Schwartz is the strongest general result on non-discrete locally compact Abelian groups, there has been considerable break-through in the theory of spectral synthesis on discrete Abelian groups. Discrete Abelian groups possessing spectral analysis have been characterized in [11], and those having spectral synthesis in [12] (see also [7]). Besides partial generalizations (see e.g. [5, 6, 7]) several attempts to extend Schwartz's result to higher dimensions failed. Finally, in [8] counterexamples have been given proving the impossibility of a direct extension.

In our recent paper (see [17]) we have presented a possible way how to extend Schwartz's result to functions in several variables. Our idea is based on the observation that the basic tools of commutative spectral analysis and synthesis can be adopted in non-commutative situations using the theory of Gelfand pairs and spherical functions (see [9, 3, 18, 10, 4]). Our point was to show that the original version of Schwartz's result can be considered as a "spherical spectral synthesis" result on the affine group $\mathrm{Aff}\, SO(n) = SO(n) \ltimes \mathbb{R}^n$ with $n = 1$, as this group is identical with $\mathbb{R}$. Hence, a proper generalization was obtained in [17] by proving

the corresponding "spherical" result for $n > 1$. From the results of [17] it is clear that in spherical spectral synthesis the role of exponential functions is played by the spherical functions on the affine group Aff $SO(n)$ and those functions can be identified with the normalized radial eigenfunctions of the Laplace–Beltrami operator on $\mathbb{R}^n$. Nevertheless, for better understanding of spherical spectral synthesis a description of the spherical version of exponential monomials would be desirable. In this paper we present a complete characterization of those functions which serve as basic building blocks of spherical spectral synthesis. In particular, we show that these functions are represented by generalized moment functions on certain double coset hypergroups.

2. Notation and terminology

The results in this paper are closely related to those in [17] and we use that paper as reference. Nevertheless, a short summary about notation and terminology is given in this paragraph. We denote by $\mathbb{R}$ and the set of real and complex numbers, respectively. In this paper $G = $ Aff $SO(n)$ will denote the *affine group* of $SO(n)$, the *special linear group* over $\mathbb{R}^n$, where n is a positive integer. We recall that Aff $SO(n)$ is the *semidirect product* $SO(n) \ltimes \mathbb{R}^n$, which is a topological group defined in the following way: we consider the set $G = SO(n) \times \mathbb{R}^n$ equipped with the product topology. Then G is a locally compact Hausdorff topological space and $SO(n)$ is a compact subspace of it. The group operation in G imitates the composition of affine mappings (k, u) on $\mathbb{R}^n$ where (k, u) is the *proper Euclidean motion* acting on $\mathbb{R}^n$ by

$$x \mapsto k \cdot x + u,$$

where k is in $SO(n)$ representing a proper rotation and u is in $\mathbb{R}^n$ representing a translation. More exactly, in G we have

$$(k, u) \cdot (l, v) = (k \cdot l, k \cdot v + u).$$

Here $k \cdot l$ is the composition of the rotations k and l, and $k \cdot v$ is the image of the vector v under k. The identity of G is $(id, 0)$, where id is the identity mapping on $\mathbb{R}^n$ and 0 is the zero vector in $\mathbb{R}^n$. The inverse of (k, u) is $(k^{-1}, -k^{-1} \cdot u)$. It is easy to see that G is a locally compact topological group in which the set $K = \{(k, 0) : k \in SO(n)\}$ forms a compact subgroup topologically isomorphic to $SO(n)$, and the set $\{(id, u) : u \in \mathbb{R}^n\}$ forms a normal subgroup topologically isomorphic to $\mathbb{R}^n$. If we identify these subgroups with $SO(n)$, and with $\mathbb{R}^n$, respectively, then we have the topological isomorphism $G/\mathbb{R}^n \cong SO(n)$. We note that in the case $n = 1$, it is clear that $SO(1) = \{id\}$, and consequently $G = $ Aff $SO(1) \cong \mathbb{R}$.

In spherical spectral synthesis we consider varieties of invariant functions. In general, given a locally compact topological group G and a compact subgroup K, a continuous function $f : G \to$ is called *K-invariant*, or simply *invariant* if it is invariant with respect to K in the sense that

$$f(x) = f(kxl)$$

holds for each k, l in K and x in G. The set of all invariant functions is denoted by $\mathcal{C}(G//K)$, and its dual space is $\mathcal{M}_c(G//K)$, the space of *invariant measures* (see

[17]). If $K = \{e\}$ is trivial, then we simply write $\mathcal{C}(G)$ and $\mathcal{M}_c(G)$ for $\mathcal{C}(G//K)$ and $\mathcal{M}_c(G//K)$, respectively. In this paper we assume that $G = \text{Aff } SO(n) = SO(n) \ltimes \mathbb{R}^n$. Clearly, for $n = 1$ every function is invariant. On the other hand, it is shown in [17] (see also [4]) that in the case $n \geq 2$ the continuous complex valued function f is invariant if and only if it is independent of the first variable and it depends on the norm of the second variable, only. In other words, for $n \geq 2$ the function $f : G \to$ is invariant if and only if it has the form

$$f(m, x) = \varphi(\|x\|)$$

for each m in $SO(n)$ and for every x in $\mathbb{R}^n$ with some continuous function $\varphi : \mathbb{R} \to$. Such functions will be called *invariant* or *radial*. The set of all invariant functions will be denoted by $\mathcal{C}_K(\mathbb{R}^n)$. When $\mathcal{C}(G)$ is equipped with the pointwise linear operations and with the topology of compact convergence then it is a locally convex topological vector space and $\mathcal{C}_K(\mathbb{R}^n)$ is a closed subspace. In [17] it is shown that the dual of $\mathcal{C}_K(\mathbb{R}^n)$ can be identified with a weak*-closed subspace of the dual of $\mathcal{C}(G)$. In fact, we have $\mathcal{C}(G)^* = \mathcal{M}_c(G)$, the space of all compactly supported complex Borel measures on G, and $\mathcal{C}_K(\mathbb{R}^n)^*$ is identified with the space of all compactly supported complex measures μ on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} f(x) \, d\mu(x) = \int_{\mathbb{R}^n} f(k \cdot x) \, d\mu(x)$$

for each f in $\mathcal{C}(\mathbb{R}^n)$ and k in $SO(n)$. Such measures are called *invariant*, or *radial measures* and the set of all invariant measures will be denoted by $\mathcal{M}_K(\mathbb{R}^n)$. We note that for the sake of simplicity in the case $n = 1$ we let $\mathcal{C}_K(\mathbb{R}) = \mathcal{C}(\mathbb{R})$ and $\mathcal{M}_K(\mathbb{R}) = \mathcal{M}_c(\mathbb{R})$ according to the fact that in this case every continuous function and every compactly supported Borel measure is invariant.

In the space $\mathcal{M}_c(G)$ *convolution* is defined in the usual manner for the measures μ, ν by

$$\langle \mu * \nu, f \rangle = \int_G \int_G f[(k, x) \cdot (l, y)] \, d\mu(k, x) \, d\nu(l, y)$$

whenever f is in $\mathcal{C}(G)$. Then $\mu * \nu$ is in $\mathcal{M}_c(G)$ and $\mathcal{M}_c(G)$ is a topological algebra with unit $\delta_{(id,0)}$, the point mass at $(id, 0)$. The set $\mathcal{M}_K(\mathbb{R}^n)$ of all invariant measures forms a weak*-subalgebra in $\mathcal{M}_c(G)$. It has been proved in [17] that $\mathcal{M}_K(\mathbb{R}^n)$ is commutative, which is equivalent to the basic fact that (G, K) is a *Gelfand pair*. More generally, we define

$$\mu * f(k, x) = \int_G f[(k, x) \cdot (l, y)^{-1}] \, d\mu(l, y)$$

whenever μ is in $\mathcal{M}_c(G)$, f is in $\mathcal{C}(G)$ and (k, x) is in G. Then $\mu * f$ is in $\mathcal{C}(G)$ and $\mathcal{C}(G)$ is a topological left module over the algebra $\mathcal{M}_c(G)$. Also $\mathcal{C}(G)$ is a topological left module over $\mathcal{M}_K(\mathbb{R}^n)$ and the set $\mathcal{C}_K(\mathbb{R}^n)$ of all invariant functions forms a closed submodule of this module. The dual pair $\mathcal{C}_K(\mathbb{R}^n), \mathcal{M}_K(\mathbb{R}^n)$ together with the module structure on $\mathcal{C}_K(\mathbb{R}^n)$ plays the most important role in spherical spectral synthesis.

Closed submodules of $\mathcal{C}_K(\mathbb{R}^n)$ are called *K-varieties* or simply *invariant varieties*. Clearly, in the case $n = 1$ the concept of invariant variety reduces to the

classical concept of variety on $\mathbb{R}^n$ (see e.g. [16]). The simplest nonzero invariant varieties are the one-dimensional ones: they have been characterized in [17] with the property that they are exactly the linear spans of K-spherical functions. The K-invariant function $s : G \to$ is called K-*spherical function* or simply *spherical function* if it is nonzero and it satisfies

$$\int_G s[(k,x) \cdot (m,u) \cdot (l,y)] \, d\omega(m) = s[(k,x) \cdot (l,y)]$$

for each $(k,x), (l,y)$ in G. As we identify invariant functions with radial functions we can simply write $s(x)$ for $s(k,x)$ and then the above equation can be rewritten as

$$\int_{SO(n)} s(x + k \cdot y) \, d\omega(k) = s(x)s(y)$$

for each x, y in $\mathbb{R}^n$. Clearly, in the case $n = 1$ spherical functions are exactly the exponential functions of the form $x \mapsto \exp \lambda x$ on $\mathbb{R}$, where λ is any complex number.

Invariant varieties have been introduced in [17] using K-translations. First we define K-projections on $\mathcal{C}(G)$ in the following way: for each f in $\mathcal{C}(G)$ we let

$$f^{\#}(x) = \int_{SO(n)} f(k, k \cdot x) \, d\omega(k)$$

whenever x is in $\mathbb{R}^n$. Then $f^{\#}$ is in $\mathcal{C}_K(\mathbb{R}^n)$ and it is called the K-*projection* or simply the *projection* of f. The mapping $f \mapsto f^{\#}$ is a continuous linear mapping of $\mathcal{C}(G)$ onto $\mathcal{C}_K(\mathbb{R}^n)$ and the continuous function f is invariant if and only if $f = f^{\#}$. Similarly, for each μ in $\mathcal{M}_c(G)$ we define $\mu^{\#}$ by the formula

$$\langle \mu^{\#}, f \rangle = \langle \mu, f^{\#} \rangle$$

whenever f is in $\mathcal{C}(G)$. Then $\mu^{\#}$ is an invariant measure and the mapping $\mu \mapsto \mu^{\#}$ is a continuous linear mapping of $\mathcal{M}_c(G)$ onto $\mathcal{M}_K(\mathbb{R}^n)$. In particular, for each (l,y) in G and for every f in $\mathcal{C}(G)$ we have

$$\langle \delta_{(l,y)}^{\#}, f \rangle = \langle \delta_{(l,y)}, f^{\#} \rangle$$

$$= \int_G \int_{SO(n)} f(k, k \cdot x) \, d\omega(k) d\delta_{(l,y)}(m, x) = \int_{SO(n)} f(k, k \cdot y) \, d\omega(k).$$

As $\delta_{(l,y)}^{\#}$ is invariant, it is independent of l, hence we may simply write $\delta_y^{\#}$. For each f in $\mathcal{C}_K(\mathbb{R}^n)$ and y in $\mathbb{R}^n$ we define the K-*translate* or *invariant translate* of f by y as

$$\tau_y f(x) = \delta_{-y}^{\#} * f(x) = \int_{SO(n)} f(x + k \cdot y) \, d\omega(k)$$

whenever x is in $\mathbb{R}^n$. It is proved in [17] that invariant varieties are exactly those closed linear subspaces of $\mathcal{M}_K(\mathbb{R}^n)$ which are invariant with respect to all invariant translations. The smallest invariant variety generated by a given invariant function f is denoted by $\tau_K(f)$.

Similarly to modified differences introduced in [16] we define *modified K-differences* or *modified invariant difference* as

$$D_{s;y} = \delta^{\#}_{-y} - s(y)\delta_0$$

in $\mathcal{M}_K(\mathbb{R}^n)$ for each spherical function s and element y in $\mathbb{R}^n$. Then we have for each f in $\mathcal{C}_K(\mathbb{R}^n)$ and for each x, y in $\mathbb{R}^n$

$$D_{s;y} * f(x) = \delta^{\#}_{-y} * f(x) - s(y)f(x) = \tau_y f(x) - s(y)f(x)$$

$$= \int_{SO(n)} f(x + k \cdot y)\, d\omega(k) - s(y)f(x).$$

The closed ideal in $\mathcal{M}_K(\mathbb{R}^n)$ generated by all measures $D_{s;y}$ for y in $\mathbb{R}^n$ is denoted by M_s and in [17] it has been proved that M_s is a maximal ideal with $\mathcal{M}_K(\mathbb{R}^n)/M_s \cong$. Maximal ideals in any ring with this property are called *exponential maximal ideals* (see [16]). In particular, $M_s = \operatorname{Ann} \tau_K(s)$, the annihilator of the invariant variety of the spherical function s in the algebra $\mathcal{M}_K(\mathbb{R}^n)$.

Now we are in the position to formulate the basic problem of spherical spectral analysis: we say that *K-spectral analysis* or *spherical spectral analysis* holds for an invariant variety if every nonzero invariant subvariety of it contains a spherical function. In the case $n = 1$ this is equivalent to ordinary spectral analysis for a variety (see e.g. [16]).

In order to introduce *K*-spectral synthesis we define *K*-spherical monomials in $\mathbb{R}^n$ as the functions in $\mathcal{C}_K(\mathbb{R}^n)$ which are annihilated by some power of the maximal ideal M_s for some K-spherical function s (see [17]). More exactly, the invariant function f in $\mathcal{C}_K(\mathbb{R}^n)$ is called a *K-monomial* or a *spherical monomial* if there exists a spherical function s and a natural number d such that

$$D_{s;y_1,y_2,\ldots,y_{d+1}} * f(x) = \Pi^{d+1}_{j=1} D_{s;y_j} * f(x) = 0$$

holds for each $x, y_1, y_2, \ldots, y_{d+1}$ in $\mathbb{R}^n$. Here Π denotes convolution product. It has been shown in [17] that if $f \neq 0$ then s is unique, hence we may call f a *spherical s-monomial*, and the smallest d with the above property is called the *degree* of f. The identically zero function is a spherical s-monomial for every spherical function s, however, we do not define its degree. It has been shown in [17] that the nonzero function f in $\mathcal{C}_K(\mathbb{R}^n)$ is a spherical s-monomial of degree at most d if and only if $M_s^{d+1} \subseteq \operatorname{Ann} \tau(f)$. Clearly, every spherical function s is a spherical s-monomial of degree 0. We say that an invariant variety is *K-synthesizable* or *spherically synthesizable* if it has a dense subspace spanned by spherical monomials. We say that *K-spectral synthesis* or *spherical spectral synthesis* holds for an invariant variety if every invariant subvariety of it is *K*-synthesizable. It is easy to show (see [17]) that spherical spectral synthesis for an invariant variety implies spherical spectral analysis for it. Obviously, in the case $n = 1$ spherical spectral synthesis coincides with ordinary spectral synthesis (see [16]). Based on this observation the following theorem in [17] can be considered as a proper extension of L. Schwartz's classical spectral synthesis result on $\mathbb{R}$:

Theorem 2.1. *Spherical spectral synthesis holds for every invariant variety on* $\mathbb{R}^n$ *for each natural number* n.

By definition, this theorem says that given radial radial function on $\mathbb{R}^n$ it is the uniform limit on compact sets of a sequence of spherical monomials which are included in the invariant variety of the function, hence they are uniform limits on compact sets of linear combinations of invariant translates of the function. To extract more information about this type of approximation of radial functions a detailed description of spherical monomials is desirable. The purpose of this paper is to present a complete description of spherical monomials on $\mathbb{R}^n$, where we shall always suppose that $n \geq 2$. The following theorem in [17] (see also [4]) is of basic importance.

Theorem 2.2. *The radial function* $\varphi : \mathbb{R}^n \to$ *is a spherical function if and only if it is* $\mathcal{C}^\infty$, *it is an eigenfunction of the Laplacian, and* $\varphi(0) = 1$.

Let φ be a $\mathcal{C}^\infty$ radial function on $\mathbb{R}^n$, which is a solution of $\Delta\varphi = \lambda\varphi$ for some complex number λ. Let $\widetilde{\varphi}$ be defined for $r \geq 0$ as

$$\widetilde{\varphi}(r) = \varphi(r, 0, 0, \dots, 0)$$

with r in $\mathbb{R}$. Then $\varphi(x) = \widetilde{\varphi}(\|x\|)$ holds for every x in $\mathbb{R}^n$, and $\widetilde{\varphi}$ is a regular even solution of the differential equation

$$\frac{d^2\widetilde{\varphi}}{dr^2} + \frac{n-1}{r}\frac{d\widetilde{\varphi}}{dr} = \lambda\widetilde{\varphi},$$

hence it is proportional to the Bessel function J_λ (see [10, 4]) defined by

$$J_\lambda(r) = \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!\,\Gamma(k + \frac{n}{2})} \left(\frac{r}{2}\right)^{2k}.$$

If, in addition, $\varphi(0) = 1$, then $\varphi(x) = J_\lambda(\|x\|)$ for each x in $\mathbb{R}^n$. For the rest of this paper we introduce the notation

$$j_\lambda(x) = J_\lambda(\|x\|)$$

for each x in $\mathbb{R}^n$ whenever $n \geq 2$ and $j_\lambda(x) = \exp \lambda x$ for $n = 1$. Then the family $(j_\lambda)_{\lambda \in}$ is equal to the set of all spherical functions on $\mathbb{R}^n$ for each natural number n. For instance, for $n = 3$ we have

$$j_\lambda(x) = \frac{\sinh \lambda\|x\|}{\lambda\|x\|}$$

for $x \neq 0$ and $\lambda \neq 0$, and $j_\lambda(0) = 1$, $j_0(x) = 1$(see [4]).

3. Double coset hypergroups

In order to give the complete description of all spherical monomials we show in the subsequent paragraphs how K-spherical functions and K-monomials relate to generalized moment functions on some double coset hypergroups. By a *hypergroup* we mean a locally compact hypergroup. The identity element of the hypergroup H will be denoted by e. For basics about hypergroups see the monographs [2, 15].

Let G be a locally compact group with identity e and K a compact subgroup with Haar measure ω. For each x in G one defines the *double coset* (see [13, 1, 2]) of x as the set

$$KxK = \{kxl : k, l \in K\}.$$

A hypergroup structure is introduced on the set $H_{G,K} = G//K$ of all double cosets in the following manner: the topology of $H_{G,K}$ is the factor topology, which is locally compact. The identity is the coset $K = KeK$ itself and the involution is defined by

$$(KxK)^{\vee} = Kx^{-1}K.$$

Finally, the convolution of δ_{KxK} and δ_{KyK} is defined by

$$\delta_{KxK} * \delta_{KyK} = \int_K \delta_{KxkyK}\, d\omega(k).$$

It is known that this gives a hypergroup structure on $H_{G,K}$ (see [2], p. 12.), which is non-commutative, in general. The hypergroup $H_{G,K}$ is called the *double coset hypergroup* with respect to K.

In our setting $G = \mathrm{Aff}\, SO(n)$ and K is the compact subgroup of G of the form $K = \{(k,0) : k \in SO(n)\}$, as above.

Theorem 3.1. *Given G, K as above the space of all continuous invariant functions on G is topologically isomorphic to the space of all continuous complex functions on the double coset hypergroup $H_{G,K}$.*

Proof. Let $\Phi : G \to H_{G,K}$ denote the natural mapping defined by $\Phi(x) = KxK$ for each x in G. Obviously, Φ is continuous and open. Then it is easy to check that the mapping $f \mapsto f \circ \Phi$ is a topological isomorphism of the topological vector space $\mathcal{C}(H_{G,K})$ onto the topological vector space $\mathcal{C}(G//K)$. $\square$

In other words, we have the topological isomorphism $\mathcal{C}(H_{G,K}) \cong \mathcal{C}_K(\mathbb{R}^n)$.

Theorem 3.2. *Given G, K as above the algebras $\mathcal{M}_c(G//K)$ and $\mathcal{M}_c(H_{G,K})$ are topologically isomorphic, the isomorphism being provided by the adjoint of the mapping $f \mapsto f \circ \Phi$.*

Proof. We have to prove only that the convolution is preserved by the adjoint of the mapping $f \mapsto f \circ \Phi$. For each μ in $\mathcal{M}_c(G//K)$ we denote by μ_Φ the measure defined by

$$\langle \mu_\Phi, f \rangle = \langle \mu, f \circ \Phi \rangle$$

for each f in $\mathcal{C}(H_{G,K})$. Clearly, μ_Φ is in $\mathcal{M}_c(H_{G,K})$, in fact, $\mu \mapsto \mu_\Phi$ is the adjoint of the linear mapping $f \mapsto f \circ \Phi$. We have to show that

$$\mu_\Phi * \nu_\Phi = (\mu * \nu)_\Phi$$

holds for each μ, ν in $\mathcal{M}_c(G//K)$. As the linear combinations of the measures $\delta_y^{\#}$ with y in G form a dense subset in $\mathcal{M}_c(G//K)$, by [17, Theorem 3], we need to verify the above equation for such measures, only. With the notation $\mu = \delta_y^{\#}$ we have for each y in G and f in $\mathcal{C}(H_{G,K})$

$$\langle \mu_\Phi, f \rangle = \langle \delta_y^{\#}, f \circ \Phi \rangle = \int_G f\big(\Phi(x)\big)\, d\delta_y^{\#}(x) =$$

$$\int_G \int_K \int_K f\big(\Phi(kxl)\big)\, d\omega(k)\, d\omega(l)\, d\delta_y(x) = \int_K \int_K f\big(\Phi(kyl)\big)\, d\omega(k)\, d\omega(l) =$$

$$\int_K \int_K (f \circ \Phi)(kyl)\, d\omega(k)\, d\omega(l) = \int_K \int_K (f \circ \Phi)(y)\, d\omega(k)\, d\omega(l) =$$

$$(f \circ \Phi)(y) = f(KyK) = \langle \delta_{KyK}, f \rangle.$$

Hence, with x, y in G, $\mu = \delta_x^{\#}$, $\nu = \delta_y^{\#}$, and f in $\mathcal{C}(H_{G,K})$ we obtain

$$\langle (\mu * \nu)_{\Phi}, f \rangle = \langle \delta_x^{\#} * \delta_y^{\#}, f \circ \Phi \rangle = \int_G \int_G (f \circ \Phi)(uv) \, d\delta_x^{\#}(u) \, d\delta_y^{\#}(v) =$$

$$\int_G \int_G \int_K \int_K \int_K \int_K (f \circ \Phi)(kulk'vl') \, d\omega(k) \, d\omega(l) \, d\omega(k') \, d\omega(l') \, d\delta_x^{\#}(u) \, d\delta_y^{\#}(v) =$$

$$\int_K (f \circ \Phi)(xly) \, d\omega(l) = \int_K f\big(\Phi(xly)\big) \, d\omega(l) = \int_K f(KxlyK) \, d\omega(l) =$$

$$f(\delta_{KxK} * \delta_{KyK}) = \langle \delta_{KxK} * \delta_{KyK}, f \rangle = \langle \mu_{\Phi} * \nu_{\Phi}, f \rangle,$$

by the definition of the convolution in $\mathcal{M}_c(H_{G,K})$. $\qquad\square$

Hence we have the topological isomorphism $\mathcal{M}_c(H_{G,K}) \cong \mathcal{M}_K(\mathbb{R}^n)$.

By the above theorems, the basic problems on K-invariant functions and measures can be studied in the hypergroup setting. Although the spaces $\mathcal{C}(G//K)$ and $\mathcal{M}_c(G//K)$ cannot be considered, in general, as spaces of functions and measures on some group, but a hypergroup structure is still available. Based on this observation the theory of hypergroups can be successfully utilized. In particular, given G, K as above, (G, K) is a Gelfand pair is equivalent to the fact that the hypergroup $H_{G,K}$ is commutative. Another easy consequence is the following:

Theorem 3.3. *Given G, K as above the function φ is a K-spherical function on G if and only if $\varphi \circ \Phi$ is an exponential on the hypergroup $H_{G,K}$.*

Proof. We have seen in the proof of Theorem 3.2 that for each x, y in G and for each f in $\mathcal{C}(H_{G,K})$ we have

$$f(\delta_{KxK} * \delta_{KyK}) = \int_K (f \circ \Phi)(xky) \, d\omega(k).$$

It follows that the equations

$$f(\delta_{KxK} * \delta_{KyK}) = f(\delta_{KxK})f(\delta_{KyK})$$

and

$$\int_K (f \circ \Phi)(xky) \, d\omega(k) = (f \circ \Phi)(x)(f \circ \Phi)(y)$$

are equivalent, which proves our statement. $\qquad\square$

Now we describe spherical monomials in the case of the affine group of $SO(n)$ over $\mathbb{R}^n$. We shall use the following notation: for each complex number λ we denote by s_λ the spherical function which is the eigenfunction of the Laplacian corresponding to the eigenvalue λ, by Theorem 2.2. Clearly, for each x in $\mathbb{R}^n$ the function $\lambda \mapsto s_\lambda(x)$ is an entire function. Then for each natural number k we denote by $\partial_\lambda^k s_\lambda$ the k-th derivative of this entire function. In other words, $\partial_\lambda^k s_\lambda$ denotes the k-th derivative of s_λ with respect to the parameter λ. In terms of the double coset hypergroup $H_{G,K} = G//K$, discussed in Section 3, we use the notation $\Phi(KxK, \lambda) = s_\lambda(x)$ for each x in $\mathbb{R}^n$ and λ in , and then $\Phi : \mathbb{R}^n \times \to$ is an *exponential family* for the hypergroup $H_{G,K}$: this means that

$$\Phi(KxK * KyK, \lambda) = \Phi(KxK, \lambda)\Phi(KyK, \lambda)$$

holds for each x, y in $\mathbb{R}^n$ and λ in . It follows

$$\int_K \partial_\lambda^k s_\lambda(x + ky) \, d\omega(k) = \sum_{j=0}^k \binom{k}{j} \partial_\lambda^j s_\lambda(x) \partial_\lambda^{k-j} s_\lambda(y),$$

or, with the notation $f_{j,\lambda}(x) = \partial_\lambda^j s_\lambda(x)$

$$\tau_y f_{k,\lambda}(x) = \sum_{j=0}^k \binom{k}{j} f_{j,\lambda}(x) f_{k-j,\lambda}(y) \tag{3.1}$$

holds for each x, y in $\mathbb{R}^n$ and $k = 0, 1, \ldots$. Here $f_{0,\lambda} = s_\lambda$. This property can be expressed by saying that for each complex number λ the sequence $(f_{k,\lambda})_{k \in \mathbb{N}}$ forms a *generalized moment sequence* associated with the exponential function s_λ (see [15]). Equation (3.1) expresses the fact that the invariant variety of $f_{k,\lambda}$ is spanned by the functions $f_{j,\lambda}$ for $j = 0, 1, \ldots, k$. It is easy to see that $f_{k,\lambda}$ is a spherical monomial of degree k associated with the spherical function s_λ for each $k = 0, 1, \ldots$, as the following theorem shows.

Theorem 3.4. *Let $f_k : \mathbb{R}^n \to$ be a generalized moment function sequence of continuous invariant functions associated with the exponential function s_λ ($k = 0, 1, \ldots$), that is, we have*

$$\tau_y f_k(x) = \sum_{j=0}^k \binom{k}{j} f_j(x) f_{k-j}(y)$$

for each x, y in $\mathbb{R}^n$ and $k = 0, 1, \ldots$. Then f_k is zero, or a it is a spherical monomial of degree at most k associated with s_λ.

Proof. We have to show that

$$D_{s_\lambda; y_1, y_2, \ldots, y_{k+1}} * f_k(x) = 0$$

for each $y_1, y_2, \ldots, y_{k+1}$ in $\mathbb{R}^n$. We prove by induction on k. For $k = 0$, by assumption, we have $f_0 = s_\lambda$, hence our statement is obvious. Suppose that we have proved our statement for $k = 0, 1, \ldots, l$ and now we prove it for $k = l + 1$. We can proceed as follows

$$D_{s_\lambda; y_1, y_2, \ldots, y_{l+1}, y_{l+2}} * f_k(x) = D_{s_\lambda; y_1, y_2, \ldots, y_{l+1}} * [\delta_{-y_{l+2}}^\# * f_{l+1} - s_\lambda(y_{l+2}) f_{l+1}](x) =$$

$$D_{s_\lambda; y_1, y_2, \ldots, y_{l+1}} * [\tau_{y_{l+2}} f_{l+1} - s_\lambda(y_{l+2}) f_{l+1}](x) =$$

$$\sum_{j=0}^l \binom{l+1}{j} f_{l+1-j}(y_{l+2}) D_{s_\lambda; y_1, y_2, \ldots, y_{l+1}} * f_j(x) = 0,$$

which proves our statement. $\qquad\square$

The following result shows that the generalized moment functions $s_\lambda^{(j)}$ are linearly independent for $j = 0, 1, \ldots$.

Proposition 3.5. *Let λ be a complex number. Then for every positive integer k the function $\partial_\lambda^k s_\lambda$ is not a linear combination of the functions $\partial_\lambda^j s_\lambda$ for $j = 0, 1, \ldots, k - 1$.*

Proof. Let $f_k = \partial_\lambda^k s_\lambda$ for $k = 0, 1, \ldots$. We prove by induction of k. Suppose that $f_1 = cf_0 = s_\lambda$ with some complex number c, then $c \neq 0$, and, by (3.1), we have

$$\int_K f_1(x + ky)\, d\omega(k) = \tau_y f_1(x) = f_1(x) s_\lambda(y) + f_1(y) s_\lambda(x)$$

for each x, y in $\mathbb{R}^n$. This implies

$$c \int_K s_\lambda(x + ky)\, d\omega(k) = 2cs_\lambda(x) s_\lambda(y),$$

that is

$$2 s_\lambda(x) s_\lambda(y) = s_\lambda(x) s_\lambda(y),$$

a contradiction. Suppose that we have proved the statement for $k = 1, 2, \ldots, l$ with $l \geq 1$ we prove it for $k = l + 1$. Assume that f_{l+1} is the linear combination of the previous functions in the sequence, that is

$$f_{l+1}(x) = \sum_{j=0}^{l} c_j f_j(x)$$

holds for each x in $\mathbb{R}^n$ with some complex numbers c_j, $j = 0, 1, \ldots, l$. Applying τ_y on both sides we have

$$\sum_{i=0}^{l+1} \binom{l+1}{i} f_i(x) f_{l+1-i}(y) = \sum_{j=0}^{l} c_j \tau_y f_j(x)$$

for each x, y in $\mathbb{R}^n$. We can continue as follows

$$f_{l+1}(x) f_0(y) + \sum_{i=0}^{l} \binom{l+1}{i} f_i(x) f_{l+1-i}(y) = \sum_{j=0}^{l} \sum_{i=0}^{j} c_j \binom{j}{i} f_i(x) f_{j-i}(y) =$$

$$\sum_{i=0}^{l} \sum_{j=i}^{l} c_j \binom{j}{i} f_i(x) f_{j-i}(y) = c_l f_l(x) f_0(y) + \sum_{i=0}^{l-1} \sum_{j=i}^{l} c_j \binom{j}{i} f_i(x) f_{j-i}(y),$$

which implies

$$\sum_{j=0}^{l} c_j f_j(x) f_0(y) + \sum_{i=0}^{l} \binom{l+1}{i} f_i(x) f_{l+1-i}(y) =$$

$$c_l f_l(x) f_0(y) + \sum_{i=0}^{l-1} \sum_{j=i}^{l} c_j \binom{j}{i} f_i(x) f_{j-i}(y),$$

which can be written in the form

$$f_l(x)(l+1) f_1(y) =$$

$$\sum_{i=0}^{l-1} \sum_{j=i}^{l} c_j \binom{j}{i} f_i(x) f_{j-i}(y) - \sum_{j=0}^{l-1} c_j f_j(x) f_0(y) - \sum_{i=0}^{l-1} \binom{l+1}{i} f_i(x) f_{l+1-i}(y),$$

a contradiction, as the right hand side is a linear combination of the functions $f_0, f_1, \ldots, f_{l-1}$, and $(l+1) f_1$ is nonzero. $\qquad\square$

Corollary 3.6. *For each complex number λ the functions $\partial_\lambda^j s_\lambda$ $(j = 0, 1, \dots)$ are linearly independent.*

Corollary 3.7. *For each complex number λ and natural number k any spherical monomial associated with s_λ of degree at most k is a linear combination of the functions $\partial_\lambda^j s_\lambda$ with $j = 0, 1, \dots, k$.*

Proof. By the previous results, it is enough to show that the dimension of the set $\text{Ann}\, M_{s_\lambda}^{k+1}$ of all spherical monomials of degree at most k associated with the spherical function s_λ has dimension at most $k+1$. Replacing s_λ by $\check{s}_\lambda$, it is obvious, by [17, Lemma 1], that $\text{Ann}\, M_{s_\lambda}^{k+1}$ and $(M_{s_\lambda}^{k+1})^\perp$ have the same dimension. On the other hand, $\mathcal{M}_K(\mathbb{R}^n)/M_{s_\lambda}^{k+1}$ is isomorphic to the algebraic dual of $(M_{s_\lambda}^{k+1})^\perp$, and they are finite dimensional, by [17, Theorem 19], hence they also have the same dimension. If m is any exponential on $\mathbb{R}$, then the dimension of $\text{Ann}\, M_m^{k+1}$ is $k+1$, and for each complex number λ there exists an exponential m on $\mathbb{R}$ such that $\mathcal{M}_c(\mathbb{R})/M_m^{k+1}$ is mapped onto $\mathcal{M}_K(\mathbb{R}^n)/M_{s_\lambda}^{k+1}$ by the epimorphism $\mu \mapsto \mu_K$. It follows that the dimension of $\mathcal{M}_K(\mathbb{R}^n)/M_{s_\lambda}^{k+1}$ is at most $k+1$. We infer that the dimension of $(M_{s_\lambda}^{k+1})^\perp$, hence also the dimension of $\text{Ann}\, M_{s_\lambda}^{k+1}$ is at most $k+1$, too. Our statement is proved. $\square$

4. Schwartz's theorem in several dimensions

Now we can formulate the refinement of L. Schwartz's spectral synthesis theorem in [14]. First we formulate the corresponding spherical spectral analysis theorem.

Theorem 4.1. *Let $\mathcal{M}$ be a set of invariant measures on $\mathbb{R}^n$ whose annihilator is not $\{0\}$. Then there is a complex number λ such that the function j_λ satisfies the system of functional equations*

$$\mu * j_\lambda = 0$$

for each μ in $\mathcal{M}$.

Theorem 4.2. *Let $f : \mathbb{R}^n \to$ be an invariant function whose invariant variety is proper. Then f is the uniform limit on compact sets of a sequence of functions which are linear combinations of functions of the form $\partial_\lambda^k j_\lambda$ where j_λ is in the invariant variety of f.*

Corollary 4.3. *Let $\mathcal{M}$ be a set of invariant measures on $\mathbb{R}^n$ whose invariant variety is proper. Then the solution space of the system of functional equations*

$$\mu * f = 0$$

where μ is in $\mathcal{M}$ has a dense subspace spanned by solutions of the form $\partial_\lambda^k j_\lambda$.

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