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On the irreducibility of some induced representations of real reductive Lie groups

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We prove the irreducibility of some standard modules of the metaplectic group $\mathrm{Mp}_{2n}(\mathbb{R})$ and some nonstandard modules of the split odd special orthogonal group $\mathrm{SO}_{2n+1}(\mathbb{R})$.

1. Introduction

This article is a supplement to [Gan and Ichino 2017], in which we establish the Shimura–Waldspurger correspondence for the metaplectic group Mp_{2n} of higher rank. Namely, we describe the tempered part of the automorphic discrete spectrum of Mp_{2n} in terms of that of SO_{2n+1} via theta lifts. In the course of the proof, we use the inductive property of local L- and A-packets and need to show that some induced representations are irreducible. The purpose of this article is to prove this irreducibility in the real case.

We now describe our results. Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$. We say that an irreducible representation ϕ of $W_{\mathbb{R}}$ is almost tempered if the image of $\phi|\cdot|^{-s}$ is bounded for some $s \in \mathbb{R}$ with $|s| < \frac{1}{2}$. We consider two cases and give the details in turn.

In Section 2, we consider some standard modules of the metaplectic group $\mathrm{Mp}_{2n}(\mathbb{R})$ (which is a nonlinear two-fold cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ of rank n). Let $\psi : W_{\mathbb{R}} \to \mathrm{Sp}_{2n}(\mathbb{C})$ be an L-parameter, which we may regard as a $2n$-dimensional symplectic representation of $W_{\mathbb{R}}$. We assume

$$\psi = \phi \oplus \phi^\vee \oplus \psi_0,$$

where

- ϕ is a k-dimensional representation of $W_{\mathbb{R}}$ whose irreducible summands are all nonsymplectic and almost tempered;
- ψ_0 is a $2n_0$-dimensional representation of $W_{\mathbb{R}}$ whose irreducible summands are all symplectic;
- $k + n_0 = n$.

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Let P be a parabolic subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ with Levi component $\mathrm{GL}_k(\mathbb{R}) \times \mathrm{Sp}_{2n_0}(\mathbb{R})$ and $\widetilde{P}$ the preimage of P in $\mathrm{Mp}_{2n}(\mathbb{R})$. Let τ be the irreducible representation of $\mathrm{GL}_k(\mathbb{R})$ associated to ϕ and $\tilde{\tau} = \tau \otimes \chi$ its twist by a fixed genuine quartic character χ of the two-fold cover of $\mathrm{GL}_k(\mathbb{R})$, as in [Gan and Ichino 2014, §2.5 and §2.6]. Then the L-packet $\Pi_\psi(\mathrm{Mp}_{2n}(\mathbb{R}))$ consists of the unique irreducible quotients of

$$\mathrm{Ind}_{\widetilde{P}}^{\mathrm{Mp}_{2n}(\mathbb{R})}(\tilde{\tau} \otimes \pi_0)$$

for all $\pi_0 \in \Pi_{\psi_0}(\mathrm{Mp}_{2n_0}(\mathbb{R}))$, and as stated in [Gan and Ichino 2017, Lemma 5.2], the irreducibility of this induced representation is required. We should mention that the irreducibility of standard modules of real reductive linear Lie groups was studied in [Speh and Vogan 1980] and their result was extended to the nonlinear case in [Miličić 1991] (via the Beilinson–Bernstein localization theorem). Nevertheless, for the convenience of the reader, we give a more direct proof of this irreducibility, following the argument in [Speh and Vogan 1980] but using the machinery of cohomological induction [Knapp and Vogan 1995].

In Section 3, we consider some nonstandard modules of the split odd special orthogonal group $\mathrm{SO}_{2n+1}(\mathbb{R})$ of rank n. Let

$$\psi : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$$

be an A-parameter, which we may regard as a $2n$-dimensional symplectic representation of $W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$. We assume

$$\psi = \phi \oplus \phi^\vee \oplus \psi_0,$$

where

- ϕ is a k-dimensional representation of $W_{\mathbb{R}}$ whose irreducible summands are all nonsymplectic and almost tempered;
- ψ_0 is a $2n_0$-dimensional representation of $W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$ whose irreducible summands are all symplectic;
- $k + n_0 = n$.

Let Q be a parabolic subgroup of $\mathrm{SO}_{2n+1}(\mathbb{R})$ with Levi component $\mathrm{GL}_k(\mathbb{R}) \times \mathrm{SO}_{2n_0+1}(\mathbb{R})$. Let τ be the irreducible representation of $\mathrm{GL}_k(\mathbb{R})$ associated to ϕ. Then the A-packet $\Pi_\psi(\mathrm{SO}_{2n+1}(\mathbb{R}))$ consists of the semisimplifications of

$$\mathrm{Ind}_Q^{\mathrm{SO}_{2n+1}(\mathbb{R})}(\tau \otimes \sigma_0)$$

for all $\sigma_0 \in \Pi_{\psi_0}(\mathrm{SO}_{2n_0+1}(\mathbb{R}))$, and as stated in [Gan and Ichino 2017, Lemma 5.5], the irreducibility of this induced representation is required. To prove this irreducibility, we reduce it to the irreducibility of a standard module

$$\mathrm{Ind}_{Q'}^{\mathrm{SO}_{2k+1}(\mathbb{R}) \times \mathrm{SO}_{2n_0+1}(\mathbb{R})}(\tau \otimes \sigma_0)$$

of an endoscopic group of $\mathrm{SO}_{2n+1}(\mathbb{R})$, where Q' is a parabolic subgroup of $\mathrm{SO}_{2k+1}(\mathbb{R}) \times \mathrm{SO}_{2n_0+1}(\mathbb{R})$ with Levi component $\mathrm{GL}_k(\mathbb{R}) \times \mathrm{SO}_{2n_0+1}(\mathbb{R})$. This reduction relies on the Kazhdan–Lusztig algorithm and is essentially due to Matumoto [2004, §4], but we include it here for the sake of completeness. We also include a more direct proof of this irreducibility given to us by the referee, using normalized intertwining operators and the irreducibility result of [Speh and Vogan 1980].

2. Irreducibility of some standard modules of $\mathrm{Mp}_{2n}(\mathbb{R})$

In this section, we show that some standard modules of $\mathrm{Mp}_{2n}(\mathbb{R})$ are irreducible (see Proposition 2.3 below), which finishes the proof of [Gan and Ichino 2017, Lemma 5.2] in the real case.

2A. ***Notation.*** Let $G = \mathrm{Mp}_{2n}(\mathbb{R})$ be the metaplectic two-fold cover of $\mathrm{Sp}_{2n}(\mathbb{R})$, which we realize as

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2n}(\mathbb{R}) \;\middle|\; g \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix} {}^t g = \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix} \right\}.$$

We define a maximal compact subgroup K of G as the preimage in G of

$$\{ g \in \mathrm{Sp}_{2n}(\mathbb{R}) \mid {}^t g^{-1} = g \}.$$

Let θ be the Cartan involution of G corresponding to K. Let $\mathfrak{g}_0 = \mathrm{Lie}\, G$ be the Lie algebra of G and $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ its complexification; analogous notation is used for other groups.

For any nonnegative integers k, l, m such that $k+2l+m=n$, we define a θ-stable Cartan subalgebra $\mathfrak{h}_0^{k,l,m}$ of $\mathfrak{g}_0$ as follows. For $a = (a_1, \dots, a_k) \in \mathbb{R}^k$, put

$$h^{k,0,0}(a) = \begin{pmatrix} \boldsymbol{a} & \\ & -\boldsymbol{a} \end{pmatrix} \in \mathfrak{sp}_{2k}(\mathbb{R}),$$

where $\boldsymbol{a} = \mathrm{diag}(a_1, \dots, a_k)$. For $z = (z_1, \dots, z_l) \in \mathbb{C}^l$ with $z_i = x_i + \sqrt{-1}\, y_i$, put

$$h^{0,l,0}(z) = \begin{pmatrix} & \boldsymbol{x} & \boldsymbol{y} & \\ \boldsymbol{x} & & & -\boldsymbol{y} \\ -\boldsymbol{y} & & & -\boldsymbol{x} \\ & \boldsymbol{y} & -\boldsymbol{x} & \end{pmatrix} \in \mathfrak{sp}_{4l}(\mathbb{R}),$$

where $\boldsymbol{x} = \mathrm{diag}(x_1, \dots, x_l)$ and $\boldsymbol{y} = \mathrm{diag}(y_1, \dots, y_l)$. For $\vartheta = (\vartheta_1, \dots, \vartheta_m) \in \mathbb{R}^m$, put

$$h^{0,0,m}(\vartheta) = \begin{pmatrix} & \boldsymbol{\vartheta} \\ -\boldsymbol{\vartheta} & \end{pmatrix} \in \mathfrak{sp}_{2m}(\mathbb{R}),$$

where $\boldsymbol{\vartheta} = \mathrm{diag}(\vartheta_1, \dots, \vartheta_m)$. Let $h^{k,l,m}(a, z, \vartheta)$ be the image of

$$(h^{k,0,0}(a), h^{0,l,0}(z), h^{0,0,m}(\vartheta))$$

under the natural embedding

$$\mathfrak{sp}_{2k}(\mathbb{R}) \oplus \mathfrak{sp}_{4l}(\mathbb{R}) \oplus \mathfrak{sp}_{2m}(\mathbb{R}) \hookrightarrow \mathfrak{sp}_{2n}(\mathbb{R}).$$

Then we set

$$\mathfrak{h}_0^{k,l,m} = \{h^{k,l,m}(a, z, \vartheta) \mid a \in \mathbb{R}^k,\ z \in \mathbb{C}^l,\ \vartheta \in \mathbb{R}^m\}.$$

These $\mathfrak{h}_0^{k,l,m}$ with $k+2l+m=n$ form a set of representatives for the G-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$.

Fix such k, l, m and write $\mathfrak{h}_0 = \mathfrak{h}_0^{k,l,m}$. We define a basis $e_1, \dots, e_n$ of $\mathfrak{h}^*$ by

$$\begin{aligned} e_i(h) &= a_i && (1 \le i \le k), \\ e_{k+2i-1}(h) &= x_i + \sqrt{-1}y_i && (1 \le i \le l), \\ e_{k+2i}(h) &= x_i - \sqrt{-1}y_i && (1 \le i \le l), \\ e_{k+2l+i}(h) &= \sqrt{-1}\vartheta_i && (1 \le i \le m) \end{aligned}$$

for $h = h^{k,l,m}(a, z, \vartheta)$. Note that

$$\begin{aligned} \theta(e_i) &= -e_i && (1 \le i \le k), \\ \theta(e_{k+2i-1}) &= -e_{k+2i} && (1 \le i \le l), \\ \theta(e_{k+2l+i}) &= e_{k+2l+i} && (1 \le i \le m). \end{aligned}$$

Using the above basis, we identify $\mathfrak{h}^*$ with $\mathbb{C}^n$. Let $\langle \cdot , \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be the standard bilinear form:

$$\langle \lambda, \mu \rangle = \lambda_1\mu_1 + \cdots + \lambda_n\mu_n$$

for $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_n) \in \mathfrak{h}^* \cong \mathbb{C}^n$. We denote by Δ the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$:

$$\Delta = \{\pm e_i \pm e_j \mid 1 \le i < j \le n\} \cup \{\pm 2e_i \mid 1 \le i \le n\}.$$

For any subspace $\mathfrak{f}$ of $\mathfrak{g}$ stable under the adjoint action of $\mathfrak{h}$, we denote by $\Delta(\mathfrak{f})$ the set of roots of $\mathfrak{h}$ in $\mathfrak{f}$ and put

$$\rho(\mathfrak{f}) = \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{f})} \alpha.$$

2B. ***Discrete series.*** As in [Adams and Barbasch 1998, §3], genuine (limits of) discrete series representations of G are classified as follows. Suppose $\mathfrak{h}_0 = \mathfrak{h}_0^{0,0,n}$. Let Δ_c be the set of compact roots and take the positive system

$$\Delta_c^+ = \{e_i - e_j \mid 1 \le i < j \le n\}.$$

Then a genuine discrete series representation of G is parametrized by its Harish-Chandra parameter $\lambda \in \sqrt{-1}\mathfrak{h}_0^*$ of the form

$$\lambda = (a_1, \dots, a_r, -b_1, \dots, -b_s),$$

where

- $a_i, b_j \in \mathbb{Z} + \frac{1}{2}$;
- $a_1 > \cdots > a_r > 0$ and $0 < b_1 < \cdots < b_s$;
- $a_i \neq b_j$ for all i, j.

More generally, a genuine limit of discrete series representation of G is parametrized by a pair (λ, Ψ) consisting of $\lambda \in \sqrt{-1}\mathfrak{h}_0^*$ of the form

$$\lambda = (\underbrace{a_1, \ldots, a_1}_{m_1}, \ldots, \underbrace{a_r, \ldots, a_r}_{m_r}, \underbrace{-a_r, \ldots, -a_r}_{n_r}, \ldots, \underbrace{-a_1, \ldots, -a_1}_{n_1}),$$

where

- $a_i \in \mathbb{Z} + \frac{1}{2}$;
- $a_1 > \cdots > a_r > 0$;
- $m_i, n_j \geq 0$;
- $m_i + n_i > 0$ and $|m_i - n_i| \leq 1$ for all i,

and a positive system Ψ of Δ such that

- $\Delta_c^+ \subset \Psi$;
- $\langle \alpha, \lambda \rangle \geq 0$ for all $\alpha \in \Psi$;
- if α is a simple root in Ψ such that $\langle \alpha, \lambda \rangle = 0$, then α is noncompact; see the condition (F-1) in [Vogan 1984].

Note that, given such λ, there are precisely 2^t positive systems Ψ satisfying the above conditions, where t is the number of indices i such that $m_i = n_i > 0$.

Remark 2.1. The L-parameter of the representation associated to (λ, Ψ) is

$$\bigoplus_{i=1}^{r} (m_i + n_i)\mathcal{D}_{a_i},$$

where for $a \in \frac{1}{2}\mathbb{Z}$, we denote by $\mathcal{D}_a$ the 2-dimensional representation of $W_\mathbb{R}$ induced from the character $z \mapsto (z/\bar{z})^a$ of $W_\mathbb{C} = \mathbb{C}^\times$. Note that

- $\mathcal{D}_{-a} = \mathcal{D}_a$;
- $\mathcal{D}_a$ is irreducible if and only if $a \neq 0$;
- $\mathcal{D}_a$ is symplectic if and only if $a \in \mathbb{Z} + \frac{1}{2}$.

In particular, any irreducible summand of the above L-parameter is symplectic and the associated L-packet consists of 2^r representations.

2C. *Standard modules.* We will use Vogan's version [1984] of the Langlands classification for real reductive Lie groups in Harish-Chandra's class. Suppose again that $\mathfrak{h}_0 = \mathfrak{h}_0^{k,l,m}$ is arbitrary. Let H be the centralizer of $\mathfrak{h}_0$ in G. Then H is the preimage in G of a Cartan subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ isomorphic to

$$(\mathbb{R}^\times)^k \times (\mathbb{C}^\times)^l \times (S^1)^m.$$

Let $\mathfrak{t}_0$ and $\mathfrak{a}_0$ be the $+1$ and -1 eigenspaces of θ in $\mathfrak{h}_0$, respectively. Put $T = H \cap K$ and $A = \exp(\mathfrak{a}_0)$, so that

$$H = T \times A.$$

Let M be the centralizer of $\mathfrak{a}_0$ in G. Then M is the preimage in G of a Levi subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ isomorphic to

$$\mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times \mathrm{Sp}_{2m}(\mathbb{R}).$$

For the inducing data of a standard module, we take an irreducible representation of M as follows. Let $\widetilde{\mathrm{GL}}_d(\mathbb{R})$ be the two-fold cover of $\mathrm{GL}_d(\mathbb{R})$ given in [Gan and Ichino 2014, §2.5]. Let χ_ψ be the genuine quartic character of $\widetilde{\mathrm{GL}}_1(\mathbb{R})$ given in §2.6 of the same paper, relative to a fixed nontrivial additive character ψ of $\mathbb{R}$. For $1 \le i \le k$, let χ_i be a character of $\widetilde{\mathrm{GL}}_1(\mathbb{R})$ of the form

$$\chi_i = \mathrm{sgn}^{\delta_i} \otimes \chi_\psi \otimes |\cdot|^{\nu_i}$$

for some $\delta_i \in \{0, 1\}$ and some $\nu_i \in \mathbb{C}$. For $1 \le i \le l$, let τ_i be an irreducible representation of $\widetilde{\mathrm{GL}}_2(\mathbb{R})$ of the form

$$\tau_i = D_{\kappa_i} \otimes (\chi_\psi \circ \widetilde{\det}) \otimes |\det|^{\nu'_i}$$

for some $\kappa_i \in \frac{1}{2}\mathbb{Z}$ and some $\nu'_i \in \mathbb{C}$, where D_{κ_i} is the relative (limit of) discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $2|\kappa_i| + 1$ with central character trivial on $\mathbb{R}^\times_+$ and $\widetilde{\det}$ is the natural lift of the determinant map given in [Gan and Ichino 2014, §2.6]:

$$\begin{array}{ccc} \widetilde{\mathrm{GL}}_2(\mathbb{R}) & \xrightarrow{\widetilde{\det}} & \widetilde{\mathrm{GL}}_1(\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathrm{GL}_2(\mathbb{R}) & \xrightarrow{\det} & \mathrm{GL}_1(\mathbb{R}) \end{array}$$

Note that τ_i does not depend on the choice of ψ since $D_{\kappa_i} \otimes (\mathrm{sgn} \circ \det) \cong D_{\kappa_i}$. Let π' be a genuine (limit of) discrete series representation of $\mathrm{Mp}_{2m}(\mathbb{R})$ associated to (λ', Ψ') as in Section 2B. Then

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi'$$

descends to an irreducible representation of M.

Put

$$\gamma = (\lambda, \nu) \in \mathfrak{h}^* = \mathfrak{t}^* \oplus \mathfrak{a}^*, \tag{2-1}$$

where

$$\lambda = (\underbrace{0, \dots, 0}_{k}, \kappa_1, -\kappa_1, \dots, \kappa_l, -\kappa_l, \lambda'_1, \dots, \lambda'_m),$$
$$\nu = (\nu_1, \dots, \nu_k, \nu'_1, \nu'_1, \dots, \nu'_l, \nu'_l, \underbrace{0, \dots, 0}_{m}).$$

Assume that the condition (F-2) in [Vogan 1984], which is explicated in [Adams and Barbasch 1998, Lemma 4.3], holds:

(i) If $\nu_i = \pm\nu_j$, then $\delta_i = \delta_j$.

(ii) If $\nu'_i = 0$, then $\kappa_i \in \mathbb{Z}$.

Choose a parabolic subgroup $P = MN$ of G with Levi component M and unipotent radical N such that

$$\operatorname{Re}\langle \alpha, \nu\rangle \geq 0$$

for all roots α of $\mathfrak{a}$ in $\mathfrak{n}$. Then, by [Vogan 1984, Proposition 2.6], the normalized parabolic induction

$$\operatorname{Ind}_P^G(\pi)$$

has a unique irreducible quotient $\mathrm{J}_P^G(\pi)$. Note that $\mathrm{J}_P^G(\pi)$ is tempered if and only if $\operatorname{Re}\nu_i = \operatorname{Re}\nu'_j = 0$ for all i, j, in which case $\operatorname{Ind}_P^G(\pi)$ is irreducible. Moreover, every irreducible genuine representation of G arises in this way; see [Vogan 1984, Theorem 2.9].

Remark 2.2. The L-parameter of $\mathrm{J}_P^G(\pi)$ is

$$\phi \oplus \phi^\vee \oplus \phi',$$

where ϕ is given by

$$\phi = \left(\bigoplus_{i=1}^{k} \operatorname{sgn}^{\delta_i} |\cdot|^{\nu_i}\right) \oplus \left(\bigoplus_{j=1}^{l} \mathcal{D}_{\kappa_j} |\cdot|^{\nu'_j}\right)$$

and ϕ' is the L-parameter of π' (see Remark 2.1). Note that any irreducible summand of ϕ is nonsymplectic by the above conditions (i), (ii).

Finally, for any real root $\alpha \in \Delta$, we consider the following "parity conditions":

- If $\alpha = \pm(e_i - e_j)$ with $1 \leq i < j \leq k$, then either

$$\begin{cases} \delta_i = \delta_j \text{ and } \nu_i - \nu_j \in 2\mathbb{Z}+1; \text{ or} \\ \delta_i \neq \delta_j \text{ and } \nu_i - \nu_j \in 2\mathbb{Z}. \end{cases}$$

- If $\alpha = \pm(e_i + e_j)$ with $1 \leq i < j \leq k$, then either

$$\begin{cases} \delta_i = \delta_j \text{ and } \nu_i + \nu_j \in 2\mathbb{Z}+1; \text{ or} \\ \delta_i \neq \delta_j \text{ and } \nu_i + \nu_j \in 2\mathbb{Z}. \end{cases}$$

- If $\alpha = \pm 2e_i$ with $1 \le i \le k$, then $\nu_i \in \mathbb{Z} + \frac{1}{2}$.
- If $\alpha = \pm(e_{k+2i-1} + e_{k+2i})$ with $1 \le i \le l$, then either

$$\begin{cases} \kappa_i \in \mathbb{Z} \text{ and } \nu'_i \in \mathbb{Z} + \frac{1}{2}; \text{ or} \\ \kappa_i \in \mathbb{Z} + \frac{1}{2} \text{ and } \nu'_i \in \mathbb{Z}. \end{cases}$$

With the above notation, we now state the main result of this section.

Proposition 2.3. *Assume that there exists no root $\alpha \in \Delta$ such that either*

(i) *α is complex and satisfies $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle \in \mathbb{Z}$, $\langle\alpha, \gamma\rangle > 0$, and $\langle\theta\alpha, \gamma\rangle < 0$; or*

(ii) *α is real and satisfies the parity condition.*

Then $\mathrm{Ind}_P^G(\pi)$ is irreducible. In particular, if $|\mathrm{Re}\,\nu_i|, |\mathrm{Re}\,\nu'_j| < \frac{1}{2}$ for all i, j, then $\mathrm{Ind}_P^G(\pi)$ is irreducible.

2D. ***Proof of Proposition 2.3.*** We first express the standard module $\mathrm{Ind}_P^G(\pi)$ as cohomological induction from a principal series representation. By [Knapp and Vogan 1995, §XI.8], combined with Lemma 11.202 of the same work, we may write

$$\pi = ({}^{\mathrm{u}}\mathcal{R}_{\mathfrak{b},T}^{\mathfrak{m},M\cap K})^{\dim \mathfrak{v}\cap\mathfrak{k}}(\zeta \otimes \chi_{\rho(\mathfrak{v})}),$$

where

- $({}^{\mathrm{u}}\mathcal{R}_{\mathfrak{b},T}^{\mathfrak{m},M\cap K})^i$ is the functor defined by [Knapp and Vogan 1995, (11.71d)];
- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{v}$ is a θ-stable Borel subalgebra of $\mathfrak{m}$ with Levi component $\mathfrak{h}$ and nilpotent radical $\mathfrak{v}$ such that
$$\langle\alpha, \lambda\rangle \ge 0$$
for all $\alpha \in \Delta(\mathfrak{v})$;
- ζ is the character of H given by
$$\zeta = \chi_1 \otimes \cdots \otimes \chi_k \otimes \xi_1 \otimes \cdots \otimes \xi_l \otimes \eta_1 \otimes \cdots \otimes \eta_m,$$
where
  - ξ_i is the character of $\mathbb{C}^\times \times \{\pm 1\}$ given by $\xi_i(z, \epsilon) = \epsilon \cdot (z/\bar{z})^{\kappa_i} \cdot (z\bar{z})^{\nu'_i}$;
  - η_i is the genuine character of the nonsplit two-fold cover of S^1 whose square descends to the character $z \mapsto z^{2\lambda'_i}$ of S^1;
- $\chi_{\rho(\mathfrak{v})}$ is the character of H such that
  - $\chi_{\rho(\mathfrak{v})}$ factors through the image of H in $\mathrm{Sp}_{2n}(\mathbb{R})$;
  - $\chi_{\rho(\mathfrak{v})}$ is trivial on $(\mathbb{R}^\times)^k$;
  - the differential of $\chi_{\rho(\mathfrak{v})}$ is $\rho(\mathfrak{v})$ (which is analytically integral).

Let L be the centralizer of $\mathfrak{t}_0$ in G. Then L is the preimage in G of a Levi subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ isomorphic to

$$\mathrm{Sp}_{2k}(\mathbb{R}) \times \mathrm{U}(1,1)^l \times \mathrm{U}(1)^m.$$

Choose a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ with Levi component $\mathfrak{l}$ and nilpotent radical $\mathfrak{u}$ such that $\mathfrak{v} \subset \mathfrak{u}$ and such that

$$\langle \alpha, \lambda \rangle \geq 0$$

for all $\alpha \in \Delta(\mathfrak{u})$. Then, by [Knapp and Vogan 1995, Theorem 11.225], we have

$$\mathrm{Ind}_P^G(\pi) \cong \mathcal{L}_{\dim \mathfrak{u} \cap \mathfrak{k}}(\mathrm{Ind}_{P \cap L}^L(\zeta \otimes \chi_{\rho(\mathfrak{u})}^{-1})),$$

where

- $\mathcal{L}_i$ is the functor defined by [Knapp and Vogan 1995, (5.3a)];
- $P \cap L = H(N \cap L)$ is a Borel subgroup of L with Levi component H and unipotent radical $N \cap L$;
- $\chi_{\rho(\mathfrak{u})}$ is the character of H such that
  - $\chi_{\rho(\mathfrak{u})}$ factors through the image of H in $\mathrm{Sp}_{2n}(\mathbb{R})$;
  - $\chi_{\rho(\mathfrak{u})}$ is trivial on $(\mathbb{R}^\times)^k$;
  - the differential of $\chi_{\rho(\mathfrak{u})}$ is $\rho(\mathfrak{u})$ (which is analytically integral).

Assume for a moment that

$$|\mathrm{Re}\, \langle \alpha, \nu \rangle| \leq \langle \alpha, \lambda \rangle \tag{2-2}$$

for all $\alpha \in \Delta(\mathfrak{u})$. Then, by [Knapp and Vogan 1995, Corollary 11.227], $\mathrm{Ind}_P^G(\pi)$ is irreducible if $\mathrm{Ind}_{P \cap L}^L(\zeta \otimes \chi_{\rho(\mathfrak{u})}^{-1})$ is irreducible. Hence, noting that

$$\chi_{\rho(\mathfrak{u})} = \chi_1' \otimes \cdots \otimes \chi_k' \otimes \xi_1' \otimes \cdots \otimes \xi_l' \otimes \eta_1' \otimes \cdots \otimes \eta_m',$$

where

- χ_i' is the trivial character of $\mathbb{R}^\times$;
- ξ_i' is a character of $\mathbb{C}^\times$ of the form $\xi_i'(z) = (z/\bar{z})^{a_i}$ for some $a_i \in \mathbb{Z}$;
- η_i' is a character of S^1 of the form $\eta_i'(z) = z^{b_i}$ for some $b_i \in \mathbb{Z}$,

we are reduced to the following irreducibility:

- The principal series representation of $\mathrm{Mp}_{2k}(\mathbb{R})$ induced from $\chi_1 \otimes \cdots \otimes \chi_k$ is irreducible. Indeed, as in [Vogan 1981, Theorem 4.2.25], this can be deduced from the following:
  - The principal series representation of $\mathrm{GL}_d(\mathbb{R})$ induced from any unitary character is irreducible; see, e.g., [Mœglin 1997].
  - The principal series representation of $\mathrm{Mp}_{2d}(\mathbb{R})$ induced from any genuine unitary character is irreducible; see the proof of [Gan and Ichino 2017, Lemma 5.2].
  - For $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $s_1, s_2 \in \mathbb{C}$, the principal series representation of $\mathrm{GL}_2(\mathbb{R})$ induced from $\mathrm{sgn}^{\epsilon_1} |\cdot|^{s_1} \otimes \mathrm{sgn}^{\epsilon_2} |\cdot|^{s_2}$ is irreducible if and only if either

$$\begin{cases} \epsilon_1 = \epsilon_2 \text{ and } s_1 - s_2 \notin 2\mathbb{Z} + 1; \text{ or} \\ \epsilon_1 \neq \epsilon_2 \text{ and } s_1 - s_2 \notin 2\mathbb{Z} \smallsetminus \{0\}. \end{cases}$$

- For $\epsilon \in \{0, 1\}$ and $s \in \mathbb{C}$, the principal series representation of $\mathrm{Mp}_2(\mathbb{R})$ induced from $\mathrm{sgn}^\epsilon \chi_\psi |\cdot|^s$ is irreducible if and only if $s \notin \mathbb{Z} + \frac{1}{2}$.

- For $\kappa \in \frac{1}{2}\mathbb{Z}$ and $s \in \mathbb{C}$, the principal series representation of $\mathrm{U}(1, 1)$ induced from the character $z \mapsto (z/\bar{z})^\kappa \cdot (z\bar{z})^s$ of $\mathbb{C}^\times$ is irreducible if and only if either

$$\begin{cases} \kappa \in \mathbb{Z} \text{ and } s \notin \mathbb{Z} + \frac{1}{2}; \text{ or} \\ \kappa \in \mathbb{Z} + \frac{1}{2} \text{ and } s \notin \mathbb{Z}. \end{cases}$$

Thus, in view of condition (ii) in Proposition 2.3, we have shown that $\mathrm{Ind}_P^G(\pi)$ is irreducible under the assumption (2-2).

We now consider the general case. We reduce it to the case where γ as in (2-1) satisfies the condition (2-2) by using the translation functor. Fix a positive system $\Delta^+(\mathfrak{l})$ of $\Delta(\mathfrak{l})$ such that

$$\mathrm{Re}\, \langle \alpha, \gamma \rangle \geq 0$$

for all $\alpha \in \Delta^+(\mathfrak{l})$. Then $\Delta^+ = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u})$ is a positive system of Δ. We denote by $\Delta(\gamma)$ the set of integral roots defined by γ:

$$\Delta(\gamma) = \left\{ \alpha \in \Delta \,\middle|\, 2\frac{\langle \alpha, \gamma \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \right\}.$$

Put $\Delta^+(\gamma) = \Delta(\gamma) \cap \Delta^+$. Then we have

$$\langle \alpha, \gamma \rangle \geq 0$$

for all $\alpha \in \Delta^+(\gamma)$. Indeed, if $\langle \alpha, \gamma \rangle < 0$ for some $\alpha \in \Delta(\gamma) \cap \Delta(\mathfrak{u})$, then since $\langle \alpha, \lambda \rangle \geq 0$, we have $\langle \alpha, \nu \rangle < 0$ and hence

$$\langle \theta\alpha, \gamma \rangle = \langle \alpha, \lambda \rangle - \langle \alpha, \nu \rangle > 0.$$

Namely, $-\alpha$ satisfies condition (i) in Proposition 2.3, which contradicts the assumption. Let $\mu \in \mathfrak{h}^*$ be an integral weight; i.e., $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_i \in \mathbb{Z}$. Then we have $\Delta(\gamma + \mu) = \Delta(\gamma)$. Recall that the translation functor $\psi^\gamma_{\gamma+\mu}$ for G is defined by

$$\psi^\gamma_{\gamma+\mu}(X) = P_\gamma(P_{\gamma+\mu}(X) \otimes F_{-\mu})$$

for any $(\mathfrak{g}, K)$-module X of finite length, where P_γ is the projection to the γ-primary component and $F_{-\mu}$ is the (nongenuine) finite-dimensional irreducible $(\mathfrak{g}, K)$-module with extreme weight $-\mu$. The translation functor for M is defined similarly and is also denoted by $\psi^\gamma_{\gamma+\mu}$; see [Knapp 1986, §XIV.12]. We now take μ of the form $\mu = (t\rho(\mathfrak{u}), \mu')$ for some positive integer t and some integral weight $\mu' \in \mathfrak{a}^*$ such that

- $\langle \alpha, \mu' \rangle > 0$ for all $\alpha \in \Delta^+(\mathfrak{l})$;
- $|\mathrm{Re}\, \langle \alpha, \nu + \mu' \rangle| < \langle \alpha, \lambda + t\rho(\mathfrak{u}) \rangle$ for all $\alpha \in \Delta(\mathfrak{u})$.

Then we have:

- $\gamma+\mu$ is regular;
- $\gamma+\mu$ satisfies (2-2);
- $\Delta^+(\gamma)=\{\alpha\in\Delta(\gamma)\mid\langle\alpha,\gamma+\mu\rangle>0\}$.

Moreover, if $\tilde\pi$ is the irreducible representation of M associated to

$$\begin{aligned}
&\tilde\delta_i\equiv\delta_i+\mu_i \bmod 2, && \tilde\nu_i=\nu_i+\mu_i,\\
&\tilde\kappa_i=\kappa_i+\tfrac12(\mu_{k+2i-1}-\mu_{k+2i}), && \tilde\nu_i'=\nu_i'+\tfrac12(\mu_{k+2i-1}+\mu_{k+2i}),\\
&\tilde\lambda_i'=\lambda_i'+\mu_{k+2l+i}, && \widetilde{\Psi}'=\Psi',
\end{aligned}$$

then we have shown that $\mathrm{Ind}_P^G(\tilde\pi)$ is irreducible. On the other hand, by [Knapp and Vogan 1995, Theorem 7.237], we have

$$\psi^{\gamma}_{\gamma+\mu}(\tilde\pi)=\pi.$$

Hence it follows from the argument in the proof of [Knapp 1986, Theorem 14.67] combined with [Vogan 1981, Lemma 7.2.18] that

$$\psi^{\gamma}_{\gamma+\mu}(\mathrm{Ind}_P^G(\tilde\pi))=\mathrm{Ind}_P^G(\psi^{\gamma}_{\gamma+\mu}(\tilde\pi))=\mathrm{Ind}_P^G(\pi).$$

From this and [Knapp and Vogan 1995, Theorem 7.229] (which asserts that under the integral dominance condition, the translation functor sends an irreducible $(\mathfrak g,K)$-module to either an irreducible $(\mathfrak g,K)$-module or zero), we deduce that $\mathrm{Ind}_P^G(\pi)$ is irreducible. This completes the proof.

3. Irreducibility of some nonstandard modules of $\mathrm{SO}_{2n+1}(\mathbb R)$

In this section, we show that some nonstandard modules of $\mathrm{SO}_{2n+1}(\mathbb R)$ are irreducible (see Proposition 3.4 below), which finishes the proof of [Gan and Ichino 2017, Lemma 5.5] in the real case.

3A. ***Notation.*** Let G be a real reductive linear Lie group with abelian Cartan subgroups. Let $\mathfrak g_0=\mathrm{Lie}\,G$ be the Lie algebra of G and fix a Cartan involution θ of $\mathfrak g_0$. We denote by K the maximal compact subgroup of G associated to θ. Then we have a Cartan decomposition $\mathfrak g_0=\mathfrak k_0\oplus\mathfrak p_0$, where $\mathfrak k_0=\mathrm{Lie}\,K$ and $\mathfrak p_0$ are the $+1$ and -1 eigenspaces of θ in $\mathfrak g_0$, respectively. Fix a nondegenerate invariant symmetric bilinear form

$$\langle\cdot,\cdot\rangle:\mathfrak g_0\times\mathfrak g_0\to\mathbb R \tag{3-1}$$

such that

- $\langle\cdot,\cdot\rangle$ is preserved by θ;
- $\langle\cdot,\cdot\rangle$ is negative definite on $\mathfrak k_0$ and positive definite on $\mathfrak p_0$.

Let $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}_0$ and $Z(\mathfrak{g})$ the center of the universal enveloping algebra of $\mathfrak{g}$. Let $\mathrm{Ad}(\mathfrak{g})$ be the identity component of the automorphism group of $\mathfrak{g}$.

Let H be a θ-stable Cartan subgroup of G. Let $\mathfrak{h}_0 = \mathrm{Lie}\, H$ be the corresponding Cartan subalgebra of $\mathfrak{g}_0$ (so that H is the centralizer of $\mathfrak{h}_0$ in G) and $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathfrak{h}_0$. Let $\langle \cdot , \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be the bilinear form induced by (3-1). We denote by $\Delta(\mathfrak{g}, \mathfrak{h})$ the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$. Let $W(\mathfrak{g}, \mathfrak{h}) = W(\Delta(\mathfrak{g}, \mathfrak{h}))$ be the associated Weyl group and put $W(G,H) = N(G,H)/H$, where $N(G,H)$ is the normalizer of H in G. Then we may regard $W(G,H)$ as a subgroup of $W(\mathfrak{g}, \mathfrak{h})$. For any regular element $\gamma \in \mathfrak{h}^*$, we denote by $\Delta(\gamma)$ the set of integral roots defined by γ:

$$\Delta(\gamma) = \left\{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \;\middle|\; 2\frac{\langle \alpha, \gamma \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \right\}.$$

Then $\Delta(\gamma)$ is a root system. Let $W(\gamma) = W(\Delta(\gamma))$ be the associated Weyl group. We may define a positive system $\Delta^+(\gamma)$ of $\Delta(\gamma)$ by

$$\Delta^+(\gamma) = \{\alpha \in \Delta(\gamma) \mid \langle \alpha, \gamma \rangle > 0\}.$$

Let $\Pi(\gamma)$ be the set of simple roots in $\Delta^+(\gamma)$. We define a homomorphism $\chi_\gamma : Z(\mathfrak{g}) \to \mathbb{C}$ as the composition of the Harish-Chandra isomorphism $Z(\mathfrak{g}) \cong S(\mathfrak{h})^{W(\mathfrak{g},\mathfrak{h})}$ with evaluation at γ.

Fix a θ-stable maximally split Cartan subgroup H^s of G and write $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^s)$. Fix a regular element $\xi \in (\mathfrak{h}^s)^*$. For any $\gamma \in \mathfrak{h}^*$ such that $\chi_\gamma = \chi_\xi$, there exists an isomorphism $i_\gamma : (\mathfrak{h}^s)^* \to \mathfrak{h}^*$ such that

- $i_\gamma(\xi) = \gamma$;
- i_γ is induced by some element $g \in \mathrm{Ad}(\mathfrak{g})$.

Since ξ is regular, i_γ does not depend on the choice of g. We define an automorphism θ_γ of $(\mathfrak{h}^s)^*$ by

$$\theta_\gamma = i_\gamma^{-1} \circ \theta \circ i_\gamma,$$

which depends only on the K-conjugacy class of γ. For $\alpha \in \Delta(\xi)$ and $w \in W(\xi)$, put

$$\alpha_\gamma = i_\gamma(\alpha) \in \Delta(\gamma), \tag{3-2}$$

$$w_\gamma = i_\gamma(w) \in W(\gamma). \tag{3-3}$$

Let $\Lambda = \Lambda^G$ be the subgroup of $\widehat{H^s}$ (where $\widehat{H^s}$ is the group of continuous characters of H^s) consisting of weights of finite-dimensional representations of G. For any $\lambda \in \Lambda$, we denote by $\bar{\lambda} \in (\mathfrak{h}^s)^*$ the differential of λ. Then the homomorphism $\lambda \mapsto \bar{\lambda}$ splits over the root lattice $\mathbb{Z}\Delta$ canonically; see [Vogan 1981, Lemma 0.4.5]. For any $\xi \in (\mathfrak{h}^s)^*$, we denote by $\xi + \Lambda$ the set of formal symbols $\xi + \lambda$ with $\lambda \in \Lambda$. Note that $W(\xi)$ acts on $\xi + \Lambda$; see [Vogan 1981, Definition 7.2.21].

We denote by $\mathrm{R}(\mathfrak{g}, K)$ the Grothendieck group of the category of $(\mathfrak{g}, K)$-modules of finite length. For any $(\mathfrak{g}, K)$-module X of finite length, we denote by $[X]$ the image of X in $\mathrm{R}(\mathfrak{g}, K)$.

3B. ***Regular characters.*** Following [Vogan 1984, Definition 2.2], we call a triple $\gamma = (H, \Gamma, \bar{\gamma})$ a regular character for G if

- H is a θ-stable Cartan subgroup of G;
- Γ is a continuous character of H;
- $\bar{\gamma} \in \mathfrak{h}^*$ is an element such that
  - if $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is an imaginary root, then $\langle \alpha, \bar{\gamma} \rangle$ is a nonzero real number;
  - the differential of Γ is
    $$\bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi),$$
    where Ψ is the positive system of imaginary roots such that
    $$\langle \alpha, \bar{\gamma} \rangle > 0$$
    for all $\alpha \in \Psi$, $\rho(\Psi)$ is half the sum of the roots in Ψ, and $\rho_c(\Psi)$ is half the sum of the compact roots in Ψ.

If further $\bar{\gamma}$ is regular, we define the length $\ell(\gamma) = \ell^G(\gamma)$ of γ by

$$\ell(\gamma) = \tfrac{1}{2}\left|\{\alpha \in \Delta^+(\bar{\gamma}) \mid \theta\alpha \notin \Delta^+(\bar{\gamma})\}\right| + \tfrac{1}{2}\dim \mathfrak{a}_0 \in \tfrac{1}{2}\mathbb{Z},$$

where $\mathfrak{a}_0$ is the -1 eigenspace of θ in $\mathfrak{h}_0$.

To any regular character $\gamma = (H, \Gamma, \bar{\gamma})$ for G such that $\bar{\gamma}$ is regular, we may associate a $(\mathfrak{g}, K)$-module $X(\gamma) = X^G(\gamma)$ of finite length with infinitesimal character $\bar{\gamma}$ as follows; see [Vogan 1984, Definition 2.3]. Let M be the centralizer of $\mathfrak{a}_0$ in G. Then there exists a unique relative discrete series $(\mathfrak{m}, M \cap K)$-module $X^M(\gamma)$ such that

- $X^M(\gamma)$ has infinitesimal character $\bar{\gamma}$;
- $X^M(\gamma)$ has a lowest $(M \cap K)$-type of highest weight $\Gamma|_{H \cap K}$.

Choose a parabolic subgroup $P = MN$ of G with Levi component M and unipotent radical N such that

$$\mathrm{Re}\,\langle \alpha, \bar{\gamma} \rangle \leq 0$$

for all roots α of $\mathfrak{a}$ in $\mathfrak{n}$. Then $X(\gamma)$ is given by

$$X(\gamma) = \mathrm{Ind}_P^G(X^M(\gamma)).$$

We recall some properties of $X(\gamma)$:

- $[X(\gamma)]$ depends only on the K-conjugacy class of γ.

- $X(\gamma)$ has a unique irreducible $(\mathfrak{g}, K)$-submodule $\overline{X}(\gamma)$.
- $\overline{X}(\gamma)$ depends only on the K-conjugacy class of γ.
- For any irreducible $(\mathfrak{g}, K)$-module X with regular infinitesimal character, we have $X \cong \overline{X}(\gamma)$ for some γ.

For any θ-stable Cartan subgroup H of G and any regular element $\xi \in (\mathfrak{h}^s)^*$, we denote by $\mathcal{R}^G(H, \xi)$ the set of regular characters $\gamma = (H, \Gamma, \bar{\gamma})$ for G such that $\chi_{\bar{\gamma}} = \chi_\xi$. Put

$$\mathcal{R}^G(\xi) = \bigcup_H \mathcal{R}^G(H, \xi),$$

where the union runs over θ-stable Cartan subgroups H of G. Later, we also need the following notion.

Definition 3.1. We say that H is ξ-integral if $\mathcal{R}^G(H, \xi) \neq \varnothing$.

3C. ***Coherent families.*** In this subsection, we recall some properties of coherent families.

Fix a regular element $\xi \in (\mathfrak{h}^s)^*$. Following [Vogan 1981, Definition 7.2.5], we call a map

$$\Theta : \xi + \Lambda \to \mathrm{R}(\mathfrak{g}, K)$$

a coherent family on $\xi + \Lambda$ if

- $\Theta(\xi + \lambda)$ has infinitesimal character $\xi + \bar{\lambda}$;
- for any finite-dimensional representation F of G, we have

$$\Theta(\xi + \lambda) \otimes F = \sum_{\mu \in \Delta(F)} \Theta(\xi + \lambda + \mu),$$

  where $\Delta(F)$ is the multiset of weights of H^s in F (counted with multiplicity).

Then the following properties hold:

- For any coherent family Θ on $\xi + \Lambda$ and any $\lambda \in \Lambda$ such that $\xi + \bar{\lambda}$ is dominant for $\Delta^+(\xi)$ (but possibly singular), we have

$$\Theta(\xi + \lambda) = \psi_\xi^{\xi+\lambda}(\Theta(\xi)) \tag{3-4}$$

  by [Vogan 1981, Proposition 7.2.22], where $\psi_\xi^{\xi+\lambda}$ is the translation functor; see Definition 4.5.7 of the same work.

- For any $(\mathfrak{g}, K)$-module X of finite length with infinitesimal character ξ, there exists a unique coherent family Θ_X on $\xi + \Lambda$ such that

$$\Theta_X(\xi) = [X]$$

  by [Vogan 1981, Theorem 7.2.7 and Corollary 7.2.27].

We denote by $\mathcal{C}(\xi + \Lambda)$ the free $\mathbb{Z}$-module of coherent families on $\xi + \Lambda$. Then we may define a representation $W(\xi)$ on $\mathcal{C}(\xi + \Lambda)$ by

$$(w\Theta)(\xi + \lambda) = \Theta(w^{-1}(\xi + \lambda))$$

for $w \in W(\xi)$ and $\Theta \in \mathcal{C}(\xi + \Lambda)$, which we call the coherent continuation representation; see [Vogan 1981, Definition 7.2.28].

For any $\gamma \in \mathcal{R}^G(\xi)$, we define coherent families $\Theta_\gamma = \Theta_\gamma^G$ and $\overline{\Theta}_\gamma = \overline{\Theta}_\gamma^G$ on $\xi + \Lambda$ by

$$\Theta_\gamma = \Theta_{X(\gamma)}, \quad \overline{\Theta}_\gamma = \Theta_{\bar{X}(\gamma)}.$$

Put

$$\mathrm{Std}(G, \xi) = \{\Theta_\gamma \mid \gamma \in \mathcal{R}^G(\xi)\}, \quad \mathrm{Irr}(G, \xi) = \{\overline{\Theta}_\gamma \mid \gamma \in \mathcal{R}^G(\xi)\}.$$

Then both $\mathrm{Std}(G, \xi)$ and $\mathrm{Irr}(G, \xi)$ are bases of $\mathcal{C}(\xi + \Lambda)$, so that we may define a bijection $\Theta \mapsto \overline{\Theta}$ from $\mathrm{Std}(G, \xi)$ to $\mathrm{Irr}(G, \xi)$ by $\Theta_\gamma \mapsto \overline{\Theta}_\gamma$ for $\gamma \in \mathcal{R}^G(\xi)$. Moreover, we may write

$$\overline{\Theta}_\gamma = \sum_{\Theta \in \mathrm{Std}(G,\xi)} M(\Theta, \overline{\Theta}_\gamma)\Theta \tag{3-5}$$

for some $M(\Theta, \overline{\Theta}_\gamma) \in \mathbb{Z}$.

Let P be a parabolic subgroup of G with Levi component M such that $H^s \subset M$. In particular, M is θ-stable and $\Lambda^G \subset \Lambda^M$. Also, the parabolic induction functor Ind_P^G induces a homomorphism

$$\mathrm{Ind}_M^G : \mathrm{R}(\mathfrak{m}, M \cap K) \to \mathrm{R}(\mathfrak{g}, K),$$

which depends only on M. For any coherent family Θ^M on $\xi + \Lambda^M$, we may define a coherent family $\mathrm{Ind}_M^G(\Theta^M)$ on $\xi + \Lambda^G$ by

$$\mathrm{Ind}_M^G(\Theta^M)(\xi + \lambda) = \mathrm{Ind}_M^G(\Theta^M(\xi + \lambda))$$

for $\lambda \in \Lambda^G$; see [Speh and Vogan 1980, Lemma 5.8]. Then we have

$$\mathrm{Ind}_M^G(\Theta_\gamma^M) = \Theta_\gamma^G$$

for $\gamma \in \mathcal{R}^M(\xi)$, noting that $\mathcal{R}^M(\xi) \subset \mathcal{R}^G(\xi)$.

3D. *The Kazhdan–Lusztig algorithm.* In this subsection, we recall the Kazhdan–Lusztig algorithm for real reductive Lie groups, which determines the coefficients $M(\Theta, \overline{\Theta}_\gamma)$ in (3-5).

Fix a regular element $\xi \in (\mathfrak{h}^s)^*$. Recall the cross action of $W(\xi)$ on $\mathcal{R}^G(\xi)$:

$$w \times \gamma = (H, w_{\bar{\gamma}}^{-1} \times \Gamma, w_{\bar{\gamma}}^{-1}\bar{\gamma})$$

for $w \in W(\xi)$ and $\gamma = (H, \Gamma, \bar{\gamma}) \in \mathcal{R}^G(\xi)$, where $w_{\bar{\gamma}}$ is as in (3-3) and $w_{\bar{\gamma}}^{-1} \times \Gamma$ is the cross product given in [Vogan 1981, Definition 8.3.1]. This descends to an action of $W(\xi)$ on $\mathrm{Std}(G,\xi)$ such that $w \times \Theta_\gamma = \Theta_{w\times\gamma}$ for $w \in W(\xi)$ and $\gamma \in \mathcal{R}^G(\xi)$.

Let $\alpha \in \Pi(\xi)$ and $\gamma = (H, \Gamma, \bar{\gamma}) \in \mathcal{R}^G(\xi)$. If the root $\alpha_{\bar{\gamma}}$ as in (3-2) either is noncompact imaginary, or is real and satisfies the parity condition [Vogan 1981, Definition 8.3.11], then we have the Cayley transform of Θ_γ through α (which is a subset of $\mathrm{Std}(G,\xi)$). We recall some details in turn.

- Suppose first that $\alpha_{\bar{\gamma}}$ is noncompact imaginary. Following [Vogan 1981, Definition 8.3.4], we say that $\alpha_{\bar{\gamma}}$ is type I (resp. type II) if the reflection in $W(\mathfrak{g},\mathfrak{h})$ with respect to $\alpha_{\bar{\gamma}}$ does not belong to (resp. belongs to) $W(G,H)$. Let $c^\alpha(\gamma)$ be the Cayley transform of γ through α; i.e., $c^\alpha(\gamma)$ is the subset of $\mathcal{R}^G(\xi)$ given in [Vogan 1981, Definition 8.3.6] of the form

$$c^\alpha(\gamma) = \{\gamma^\alpha\}, \quad \gamma^\alpha = (H^\alpha, \Gamma^\alpha, \bar{\gamma}^\alpha)$$

if $\alpha_{\bar{\gamma}}$ is type I, and

$$c^\alpha(\gamma) = \{\gamma^\alpha_+, \gamma^\alpha_-\}, \quad \gamma^\alpha_\pm = (H^\alpha, \Gamma^\alpha_\pm, \bar{\gamma}^\alpha)$$

if $\alpha_{\bar{\gamma}}$ is type II, where H^α is the θ-stable Cartan subgroup of G given in [Vogan 1981, Definition 8.3.4]. Then the subset

$$c^\alpha(\Theta_\gamma) = \{\Theta_{\gamma'} \mid \gamma' \in c^\alpha(\gamma)\}$$

of $\mathrm{Std}(G,\xi)$ depends only on the K-conjugacy class of γ.

- Suppose next that $\alpha_{\bar{\gamma}}$ is real and satisfies the parity condition [Vogan 1981, Definition 8.3.11]. Following Definition 8.3.8 of the same work, we say that $\alpha_{\bar{\gamma}}$ is type I (resp. type II) if $\alpha_{\bar{\gamma}} : H \cap K \to \{\pm 1\}$ is not surjective (resp. is surjective). Let $c_\alpha(\gamma)$ be the Cayley transform of γ through α; i.e., $c_\alpha(\gamma)$ is the subset of $\mathcal{R}^G(\xi)$ given in [Vogan 1981, Definitions 8.3.14 and 8.3.16] of the form

$$c_\alpha(\gamma) = \{\gamma_\alpha^+, \gamma_\alpha^-\}, \quad \gamma_\alpha^\pm = (H_\alpha, \Gamma_\alpha^\pm, \bar{\gamma}_\alpha^\pm)$$

if $\alpha_{\bar{\gamma}}$ is type I, and

$$c_\alpha(\gamma) = \{\gamma_\alpha\}, \quad \gamma_\alpha = (H_\alpha, \Gamma_\alpha, \bar{\gamma}_\alpha)$$

if $\alpha_{\bar{\gamma}}$ is type II, where H_α is the θ-stable Cartan subgroup of G given in [Vogan 1981, Definition 8.3.8]. Then the subset

$$c_\alpha(\Theta_\gamma) = \{\Theta_{\gamma'} \mid \gamma' \in c_\alpha(\gamma)\}$$

of $\mathrm{Std}(G,\xi)$ depends only on the K-conjugacy class of γ.

Let $\mathcal{H}(W(\xi))$ be the Hecke algebra of $W(\xi)$ over $\mathbb{Z}[q]$, where q is an indeterminate. Note that the specialization at $q=1$ gives a surjection $\mathcal{H}(W(\xi)) \to \mathbb{Z}[W(\xi)]$.

Then, by [Vogan 1983b, Definition 5.2], see also [Vogan 1982, Definition 12.3 and Proposition 12.5], there exists an action of $\mathcal{H}(W(\xi))$ on

$$\mathcal{C}(\xi+\Lambda)_q = \mathcal{C}(\xi+\Lambda)\otimes_{\mathbb{Z}}\mathbb{Z}[q]$$

determined by the cross action and the Cayley transforms. Moreover, by [Vogan 1982, Lemma 14.5], the specialization of $\mathcal{C}(\xi+\Lambda)_q$ at $q=1$ is isomorphic to the coherent continuation representation tensored with the sign representation of $W(\xi)$. More explicitly, this isomorphism is induced by the surjection

$$\epsilon : \mathcal{C}(\xi+\Lambda)_q \to \mathcal{C}(\xi+\Lambda)$$

given by

$$\epsilon(q^i\Theta_\gamma) = (-1)^{\ell^I(\gamma)}\Theta_\gamma \tag{3-6}$$

for $i \geq 0$ and $\gamma \in \mathcal{R}^G(\xi)$, where the integral length $\ell^I(\gamma)$ of γ is given by

$$\ell^I(\gamma) = \ell(\gamma) - c_0(G) \tag{3-7}$$

for some choice of $c_0(G) \in \frac{1}{2}\mathbb{Z}$ such that $\ell^I(\gamma) \in \mathbb{Z}$ for all $\gamma \in \mathcal{R}^G(\xi)$; see [Vogan 1982, Definition 12.1].

Finally, we recall the Kazhdan–Lusztig algorithm for real reductive Lie groups.

Theorem 3.2 [Vogan 1983a; Adams et al. 1992, Theorem 16.22]. *For any $\gamma, \delta \in \mathcal{R}^G(\xi)$, we have*

$$M(\Theta_\gamma, \overline{\Theta}_\delta) = (-1)^{\ell^I(\gamma)-\ell^I(\delta)} P_{\gamma,\delta}(1),$$

where $M(\Theta_\gamma, \overline{\Theta}_\delta)$ is the integer defined by (3-5) and $P_{\gamma,\delta}(q)$ is the Kazhdan–Lusztig–Vogan polynomial defined in terms of the $\mathcal{H}(W(\xi))$-module $\mathcal{C}(\xi+\Lambda)_q$. In particular, $M(\Theta_\gamma, \overline{\Theta}_\delta)$ can be computed by an algorithm which depends only on the $\mathcal{H}(W(\xi))$-module structure on $\mathcal{C}(\xi+\Lambda)_q$.

3E. ***Comparison of Hecke algebra module structures.*** Let G_1 and G_2 be two real reductive linear Lie groups with abelian Cartan subgroups. For $i=1,2$, fix a Cartan involution θ_i of $(\mathfrak{g}_i)_0 = \operatorname{Lie} G_i$ and let K_i be the maximal compact subgroup of G_i associated to θ_i. Fix a θ_i-stable maximally split Cartan subgroup H_i^s of G_i and a regular element $\xi_1 \in (\mathfrak{h}_1^s)^*$.

We now assume that the following conditions hold:

(i) There exists an isomorphism

$$H_1^s \cong H_2^s.$$

(ii) Let $f : \widehat{H}_1^s \to \widehat{H}_2^s$ be the isomorphism induced by the isomorphism in (i). Then we have

$$f(\Lambda^{G_1}) \subset \Lambda^{G_2}.$$

(iii) Let $f : (\mathfrak{h}_1^s)^* \to (\mathfrak{h}_2^s)^*$ be the isomorphism induced by the isomorphism in (i) and put $\xi_2 = f(\xi_1)$. Then ξ_2 is regular.

(iv) The isomorphism in (iii) induces an isomorphism

$$f : \Delta(\xi_1) \to \Delta(\xi_2)$$

of root systems. This induces an isomorphism

$$f : W(\xi_1) \to W(\xi_2)$$

of the associated Weyl groups.

(v) There exists a bijection

$$\varphi : \mathrm{Std}(G_1, \xi_1) \to \mathrm{Std}(G_2, \xi_2).$$

(vi) Let $\gamma_i \in \mathcal{R}^{G_i}(\xi_i)$ be such that $\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}$. Then we have

$$f \circ \theta_{\bar{\gamma}_1} = \theta_{\bar{\gamma}_2} \circ f.$$

This implies that

$$\ell^{G_1}(\gamma_1) = \ell^{G_2}(\gamma_2),$$

and that for any $\alpha \in \Delta(\xi_1)$, $\alpha_{\bar{\gamma}_1}$ is imaginary (resp. real, resp. complex) if and only if $f(\alpha)_{\bar{\gamma}_2}$ is imaginary (resp. real, resp. complex).

(vii) Let $\gamma_i \in \mathcal{R}^{G_i}(\xi_i)$ and $\alpha \in \Delta(\xi_1)$ be such that $\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}$ and such that $\alpha_{\bar{\gamma}_1}$ is imaginary (and hence so is $f(\alpha)_{\bar{\gamma}_2}$). Then $\alpha_{\bar{\gamma}_1}$ is noncompact if and only if $f(\alpha)_{\bar{\gamma}_2}$ is noncompact, in which case $\alpha_{\bar{\gamma}_1}$ is type I (resp. type II) if and only if $f(\alpha)_{\bar{\gamma}_2}$ is type I (resp. type II).

(viii) Let $\gamma_i \in \mathcal{R}^{G_i}(\xi_i)$ and $\alpha \in \Delta(\xi_1)$ be such that $\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}$ and such that $\alpha_{\bar{\gamma}_1}$ is real (and hence so is $f(\alpha)_{\bar{\gamma}_2}$). Then $\alpha_{\bar{\gamma}_1}$ satisfies the parity condition if and only if $f(\alpha)_{\bar{\gamma}_2}$ satisfies the parity condition, in which case $\alpha_{\bar{\gamma}_1}$ is type I (resp. type II) if and only if $f(\alpha)_{\bar{\gamma}_2}$ is type I (resp. type II).

(ix) The bijection in (v) is compatible with the cross action: for $w \in W(\xi_1)$ and $\gamma \in \mathcal{R}^{G_1}(\xi_1)$, we have

$$\varphi(w \times \Theta_\gamma) = f(w) \times \varphi(\Theta_\gamma).$$

(x) The bijection in (v) is compatible with the Cayley transforms: for $\alpha \in \Pi(\xi_1)$ and $\gamma \in \mathcal{R}^{G_1}(\xi_1)$, we have

$$\varphi(c^\alpha(\Theta_\gamma)) = c^{f(\alpha)}(\varphi(\Theta_\gamma))$$

if $\alpha_{\bar{\gamma}}$ is noncompact imaginary, and

$$\varphi(c_\alpha(\Theta_\gamma)) = c_{f(\alpha)}(\varphi(\Theta_\gamma))$$

if $\alpha_{\bar{\gamma}}$ is real and satisfies the parity condition.

The bijection in (v) induces isomorphisms

$$\varphi : \mathcal{C}(\xi_1 + \Lambda^{G_1}) \to \mathcal{C}(\xi_2 + \Lambda^{G_2}),$$
$$\varphi_q : \mathcal{C}(\xi_1 + \Lambda^{G_1})_q \to \mathcal{C}(\xi_2 + \Lambda^{G_2})_q$$

of $\mathbb{Z}$-modules and $\mathbb{Z}[q]$-modules, respectively. By the definition of the $\mathcal{H}(W(\xi_i))$-module structure on $\mathcal{C}(\xi_i + \Lambda^{G_i})_q$, the above conditions imply φ_q is equivariant under the action of $\mathcal{H}(W(\xi_1)) \cong \mathcal{H}(W(\xi_2))$. From this and the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(\xi_1 + \Lambda^{G_1})_q & \xrightarrow{\varphi_q} & \mathcal{C}(\xi_2 + \Lambda^{G_2})_q \\ \downarrow & & \downarrow \\ \mathcal{C}(\xi_1 + \Lambda^{G_1}) & \xrightarrow{\varphi} & \mathcal{C}(\xi_2 + \Lambda^{G_2}) \end{array}$$

induced by the specialization at $q = 1$ defined by (3-6) (with a suitable choice of $c_0(G_i)$ in the definition of the integral length; see (3-7)), we can deduce that φ is an isomorphism of the coherent continuation representations of $W(\xi_1) \cong W(\xi_2)$. Moreover, by Theorem 3.2, we have

$$M\big(\varphi(\Theta_\gamma), \overline{\varphi(\Theta_\delta)}\big) = M(\Theta_\gamma, \overline{\Theta}_\delta)$$

for all $\gamma, \delta \in \mathcal{R}^{G_1}(\xi_1)$ and hence

$$\overline{\varphi(\Theta)} = \varphi(\overline{\Theta})$$

for all $\Theta \in \mathrm{Std}(G_1, \xi_1)$. In particular, φ induces a bijection from $\mathrm{Irr}(G_1, \xi_1)$ to $\mathrm{Irr}(G_2, \xi_2)$.

Lemma 3.3. *For $i = 1, 2$, let $\Xi_i \in \mathcal{C}(\xi_i + \Lambda^{G_i})$ and $\lambda_i \in \Lambda^{G_i}$ be such that $\varphi(\Xi_1) = \Xi_2$ and $f(\lambda_1) = \lambda_2$. Assume there exists an irreducible $(\mathfrak{g}_2, K_2)$-module X_2 such that*

$$\Xi_2(\xi_2 + \lambda_2) = [X_2].$$

Then there exists an irreducible $(\mathfrak{g}_1, K_1)$-module X_1 such that

$$\Xi_1(\xi_1 + \lambda_1) = [X_1].$$

Proof. The assertion was proved by Matumoto [2004, Lemma 4.1.3] when Cartan subgroups of G_i are all connected, but the argument works in the general case. We include the proof for the convenience of the reader.

Choose $w_1 \in W(\xi_1)$ such that $w_1(\xi_1 + \bar{\lambda}_1)$ is dominant for $\Delta^+(\xi_1)$ and write

$$w_1 \Xi_1 = \sum_{\overline{\Theta} \in \mathrm{Irr}(G_1, \xi_1)} a_{\overline{\Theta}} \overline{\Theta}$$

for some $a_{\overline{\Theta}} \in \mathbb{Z}$. Put $w_2 = f(w_1) \in W(\xi_2)$, so that $\varphi(w_1 \Xi_1) = w_2 \Xi_2$. Then

$$\begin{aligned} \sum_{\overline{\Theta} \in \mathrm{Irr}(G_1, \xi_1)} a_{\overline{\Theta}} \varphi(\overline{\Theta})(w_2(\xi_2 + \lambda_2)) &= \varphi(w_1 \Xi_1)(w_2(\xi_2 + \lambda_2)) \\ &= (w_2 \Xi_2)(w_2(\xi_2 + \lambda_2)) = \Xi_2(\xi_2 + \lambda_2) = [X_2]. \end{aligned}$$

On the other hand, since $w_2(\xi_2+\bar{\lambda}_2)$ is dominant for $\Delta^+(\xi_2)$, we deduce from (3-4) and [Vogan 1983b, Theorem 7.6], see also [Speh and Vogan 1980, Theorem 6.18], that for any $\overline{\Upsilon} \in \mathrm{Irr}(G_2,\xi_2)$, $\overline{\Upsilon}(w_2(\xi_2+\lambda_2))$ is either $[X]$ for some irreducible $(\mathfrak{g}_2, K_2)$-module X or zero, and that there exists a unique $\overline{\Upsilon}_0 \in \mathrm{Irr}(G_2,\xi_2)$ such that

$$\overline{\Upsilon}_0(w_2(\xi_2+\lambda_2)) = [X_2].$$

Hence, noting that $\varphi(\overline{\Theta}) \in \mathrm{Irr}(G_2,\xi_2)$ for $\overline{\Theta} \in \mathrm{Irr}(G_1,\xi_1)$, we have

$$a_{\overline{\Theta}_0} = 1$$

for $\overline{\Theta}_0 = \varphi^{-1}(\overline{\Upsilon}_0)$, and either $a_{\overline{\Theta}} = 0$ or $\varphi(\overline{\Theta})(w_2(\xi_2+\lambda_2)) = 0$ for $\overline{\Theta} \neq \overline{\Theta}_0$. Moreover, recalling the definition of τ-invariants, see [Vogan 1983b, Definition 5.3], we can also deduce from (3-4) and [Vogan 1983b, Theorem 7.6] that

$$\varphi(\overline{\Theta})(w_2(\xi_2+\lambda_2)) = 0 \iff \overline{\Theta}(w_1(\xi_1+\lambda_1)) = 0$$

for all $\overline{\Theta} \in \mathrm{Irr}(G_1,\xi_1)$. Thus, we obtain

$$\begin{aligned}\Xi_1(\xi_1+\lambda_1) &= (w_1\Xi_1)(w_1(\xi_1+\lambda_1)) \\ &= \sum_{\overline{\Theta}\in\mathrm{Irr}(G_1,\xi_1)} a_{\overline{\Theta}}\overline{\Theta}(w_1(\xi_1+\lambda_1)) = \overline{\Theta}_0(w_1(\xi_1+\lambda_1)) = [X_1]\end{aligned}$$

for some irreducible $(\mathfrak{g}_1, K_1)$-module X_1. □

3F. ***Some nonstandard modules of*** $\mathbf{SO_{2n+1}}(\mathbb{R})$**.** Let $G = \mathrm{SO}_{2n+1}(\mathbb{R})$ be the split odd special orthogonal group, which we realize as

$$\mathrm{SO}_{2n+1}(\mathbb{R}) = \left\{ g \in \mathrm{SL}_{2n+1}(\mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} \mathbf{1}_{n+1} & \\ & -\mathbf{1}_n \end{pmatrix} g = \begin{pmatrix} \mathbf{1}_{n+1} & \\ & -\mathbf{1}_n \end{pmatrix} \right\}.$$

We define a Cartan involution θ of G by

$$\theta(g) = {}^t g^{-1}.$$

Let K be the maximal compact subgroup of G associated to θ. We define the bilinear form

$$\langle\cdot,\cdot\rangle : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{R}$$

as in (3-1) by

$$\langle X, Y\rangle = \tfrac{1}{2}\,\mathrm{tr}(XY).$$

For any nonnegative integers k, l, m such that $k+2l+m = n$, we define a θ-stable Cartan subalgebra $\mathfrak{h}_0^{k,l,m}$ of $\mathfrak{g}_0$ as follows. For $a = (a_1, \dots, a_k) \in \mathbb{R}^k$, put

$$h^{k,0,0}(a) = \begin{pmatrix} & \boldsymbol{a} \\ \boldsymbol{a} & \end{pmatrix} \in \mathfrak{so}_{2k}(\mathbb{R}),$$

where $\boldsymbol{a} = \operatorname{diag}(a_1, \dots, a_k)$. For $z = (z_1, \dots, z_l) \in \mathbb{C}^l$ with $z_i = x_i + \sqrt{-1}y_i$, put

$$h^{0,l,0}(z) = \begin{pmatrix} & \boldsymbol{y} & & \boldsymbol{x} \\ -\boldsymbol{y} & & \boldsymbol{x} & \\ & \boldsymbol{x} & & -\boldsymbol{y} \\ \boldsymbol{x} & & \boldsymbol{y} & \end{pmatrix} \in \mathfrak{so}_{4l}(\mathbb{R}),$$

where $\boldsymbol{x} = \operatorname{diag}(x_1, \dots, x_l)$ and $\boldsymbol{y} = \operatorname{diag}(y_1, \dots, y_l)$. For $\vartheta = (\vartheta_1, \dots, \vartheta_m) \in \mathbb{R}^m$, put

$$h^{0,0,m}(\vartheta) = \operatorname{diag}(\boldsymbol{\vartheta}_1, \dots, \boldsymbol{\vartheta}_{m_1}, 0, -\boldsymbol{\vartheta}_{m_1+1}, \dots, -\boldsymbol{\vartheta}_m) \in \mathfrak{so}_{2m+1}(\mathbb{R}),$$

where

$$\boldsymbol{\vartheta}_i = \begin{pmatrix} & \vartheta_i \\ -\vartheta_i & \end{pmatrix}$$

and $m_1 = [(m+1)/2]$. Let $h^{k,l,m}(a, z, \vartheta)$ be the image of

$$(h^{k,0,0}(a), h^{0,l,0}(z), h^{0,0,m}(\vartheta))$$

under the natural embedding

$$\mathfrak{so}_{2k}(\mathbb{R}) \oplus \mathfrak{so}_{4l}(\mathbb{R}) \oplus \mathfrak{so}_{2m+1}(\mathbb{R}) \hookrightarrow \mathfrak{so}_{2n+1}(\mathbb{R}).$$

Then we set

$$\mathfrak{h}_0^{k,l,m} = \{h^{k,l,m}(a, z, \vartheta) \mid a \in \mathbb{R}^k,\ z \in \mathbb{C}^l,\ \vartheta \in \mathbb{R}^m\}.$$

These $\mathfrak{h}_0^{k,l,m}$ with $k + 2l + m = n$ form a set of representatives for the G-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$. Let $H^{k,l,m}$ be the centralizer of $\mathfrak{h}_0^{k,l,m}$ in G. Then $H^{k,l,m}$ is a θ-stable Cartan subgroup of G isomorphic to

$$(\mathbb{R}^\times)^k \times (\mathbb{C}^\times)^l \times (S^1)^m.$$

Note that $W(\mathfrak{g}, \mathfrak{h}^{k,l,m}) \cong W(B_n)$ and

$$W(G, H^{k,l,m}) \cong W(B_k) \times (\mathfrak{S}_l \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^l) \times W(B_{m_1}) \times W(B_{m_2}), \quad \text{(3-8)}$$

where $\mathfrak{S}_d$ is the symmetric group of degree d, $W(B_d) = \mathfrak{S}_d \ltimes (\mathbb{Z}/2\mathbb{Z})^d$ is the Weyl group of type B_d, $m_1 = [(m+1)/2]$, and $m_2 = [m/2]$; see, e.g., [Vogan 1982, Proposition 4.16].

Fix nonnegative integers k, l, m such that $k + 2l + m = n$ and write $\mathfrak{h}_0 = \mathfrak{h}_0^{k,l,m}$. Let M be the centralizer of $\mathfrak{a}_0$ in G, where $\mathfrak{a}_0$ is the -1 eigenspace of θ in $\mathfrak{h}_0$. Then M is a Levi subgroup of G isomorphic to

$$\mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times \mathrm{SO}_{2m+1}(\mathbb{R}).$$

We consider an irreducible representation π of M of the form

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi',$$

where

- χ_i is a character of $\mathrm{GL}_1(\mathbb{R})$ of the form
$$\chi_i = \mathrm{sgn}^{\delta_i} \otimes |\cdot|^{\nu_i}$$
for some $\delta_i \in \{0, 1\}$ and some $\nu_i \in \mathbb{C}$;
- τ_i is an irreducible representation of $\mathrm{GL}_2(\mathbb{R})$ of the form
$$\tau_i = D_{\kappa_i} \otimes |\det|^{\nu'_i}$$
for some $\kappa_i \in \frac{1}{2}\mathbb{Z}$ and some $\nu'_i \in \mathbb{C}$, where D_{κ_i} is the relative (limit of) discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $2|\kappa_i| + 1$ with central character trivial on $\mathbb{R}^\times_+$;
- π' is an irreducible representation of $\mathrm{SO}_{2m+1}(\mathbb{R})$ with infinitesimal character
$$\lambda' = (\lambda'_1, \dots, \lambda'_m) \in (\mathfrak{h}^{m,0,0})^* \cong \mathbb{C}^m$$
(with the identification given in Section 3G below).

Choose a parabolic subgroup P of G with Levi component M.

We now state the main result of this section.

Proposition 3.4. *Assume that*

- *if* $\nu_i = \pm\nu_j$, *then* $\delta_i = \delta_j$;
- *if* $\nu'_i = 0$, *then* $\kappa_i \in \mathbb{Z}$;
- $|\mathrm{Re}\, \nu_i|, |\mathrm{Re}\, \nu'_j| < \frac{1}{2}$ *for all* i, j;
- $\lambda'_i \in \mathbb{Z} + \frac{1}{2}$ *for all* i.

Then the normalized parabolic induction $\mathrm{Ind}_P^G(\pi)$ *is irreducible.*

3G. ***Proof of Proposition 3.4.*** Put
$$G_1 = \mathrm{SO}_{2n+1}(\mathbb{R}), \qquad G_2 = \mathrm{SO}_{2(n-m)+1}(\mathbb{R}) \times \mathrm{SO}_{2m+1}(\mathbb{R}).$$
We define embeddings $\iota : \mathrm{SO}_{2(n-m)+1}(\mathbb{R}) \hookrightarrow G_1$ and $\iota' : \mathrm{SO}_{2m+1}(\mathbb{R}) \hookrightarrow G_1$ by
$$\iota\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & & b & \\ & \mathbf{1}_m & & \\ c & & d & \\ & & & \mathbf{1}_m \end{pmatrix}, \qquad \begin{aligned} &a \in \mathrm{M}_{n-m,n-m}(\mathbb{R}), && b \in \mathrm{M}_{n-m,n-m+1}(\mathbb{R}), \\ &c \in \mathrm{M}_{n-m+1,n-m}(\mathbb{R}), && d \in \mathrm{M}_{n-m+1,n-m+1}(\mathbb{R}), \end{aligned}$$
$$\iota'\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{n-m} & & & \\ & a' & & b' \\ & & \mathbf{1}_{n-m} & \\ & c' & & d' \end{pmatrix}, \quad \begin{aligned} &a' \in \mathrm{M}_{m+1,m+1}(\mathbb{R}), && b' \in \mathrm{M}_{m+1,m}(\mathbb{R}), \\ &c' \in \mathrm{M}_{m,m+1}(\mathbb{R}), && d' \in \mathrm{M}_{m,m}(\mathbb{R}). \end{aligned}$$
For $i = 1, 2$, let θ_i be the Cartan involution of G_i as in Section 3F and K_i the maximal compact subgroup of G_i associated to θ_i. We take a θ_i-stable maximally

split Cartan subgroup H_i^s of G_i given by

$$H_1^s = H^{n,0,0}, \quad H_2^s = H^{n-m,0,0} \times H^{m,0,0}.$$

Then we have an isomorphism $H_2^s \to H_1^s$ given by

$$(h, h') \mapsto \iota(h)\iota'(h'). \tag{3-9}$$

This induces an isomorphism $f : \widehat{H}_1^s \to \widehat{H}_2^s$.

Lemma 3.5. *We have* $f(\Lambda^{G_1}) \subset \Lambda^{G_2}$.

Proof. Let $\mu \in \Lambda^{G_1}$, so that μ occurs in some finite-dimensional representation F of G_1. Then $f(\mu)$ occurs in the representation $\iota^* F \otimes (\iota')^* F$ of G_2. Hence $f(\mu) \in \Lambda^{G_2}$. □

Also, the isomorphism (3-9) induces an isomorphism

$$f : (\mathfrak{h}_1^s)^* \to (\mathfrak{h}_2^s)^*. \tag{3-10}$$

We define a basis $e_1^s, \dots, e_n^s$ of $(\mathfrak{h}_1^s)^* = (\mathfrak{h}^{n,0,0})^*$ by

$$e_i^s(h^{n,0,0}(a)) = a_i.$$

Fix a regular element $\xi_1 = (x_1, \dots, x_n) \in (\mathfrak{h}_1^s)^* \cong \mathbb{C}^n$ (with the identification using the above basis) such that

$$\begin{aligned} x_i &\notin \mathbb{Z} + \tfrac{1}{2} \quad (1 \le i \le n-m), \\ x_i &\in \mathbb{Z} + \tfrac{1}{2} \quad (n-m < i \le n). \end{aligned} \tag{3-11}$$

Put $\xi_2 = f(\xi_1)$. Since $f(\Delta(\mathfrak{g}_1, \mathfrak{h}_1^s)) \supset \Delta(\mathfrak{g}_2, \mathfrak{h}_2^s)$, we know ξ_2 is regular.

Lemma 3.6. *The isomorphism* (3-10) *induces an isomorphism* $f : \Delta(\xi_1) \to \Delta(\xi_2)$ *of root systems.*

Proof. Since

$$\Delta(\mathfrak{g}_1, \mathfrak{h}_1^s) \setminus f^{-1}(\Delta(\mathfrak{g}_2, \mathfrak{h}_2^s)) = \{\pm e_i^s \pm e_j^s \mid 1 \le i \le n-m < j \le n\},$$

it follows from (3-11) that

$$2\frac{\langle \alpha, \xi_1 \rangle}{\langle \alpha, \alpha \rangle} \notin \mathbb{Z}$$

for all $\alpha \in \Delta(\mathfrak{g}_1, \mathfrak{h}_1^s) \setminus f^{-1}(\Delta(\mathfrak{g}_2, \mathfrak{h}_2^s))$. This implies the assertion. □

Recall that

$$\begin{aligned} &H^{k',l',m'} && (k' + 2l' + m' = n), \\ &H^{p,q,r} \times H^{p',q',r'} && (p + 2q + r = n-m,\ p' + 2q' + r' = m) \end{aligned}$$

form a set of representatives for the K_i-conjugacy classes of θ_i-stable Cartan subgroups of G_i for $i = 1, 2$, respectively.

Lemma 3.7. (i) *If the θ_1-stable Cartan subgroup $H^{k',l',m'}$ of G_1 is ξ_1-integral, then $m' \le m$.*

(ii) *If the θ_2-stable Cartan subgroup $H^{p,q,r} \times H^{p',q',r'}$ of G_2 is ξ_2-integral, then $r = 0$.*

Proof. We only prove (i); the proof of (ii) is similar. Put $H_1 = H^{k',l',m'}$ and $\mathfrak{h}_1 = \mathfrak{h}^{k',l',m'}$. We define a basis $e_1, \dots, e_n$ of $\mathfrak{h}_1^*$ by

$$\begin{aligned} e_i(h) &= a_i && (1 \le i \le k'), \\ e_{k'+2i-1}(h) &= x_i + \sqrt{-1}y_i && (1 \le i \le l'), \\ e_{k'+2i}(h) &= -x_i + \sqrt{-1}y_i && (1 \le i \le l'), \\ e_{k'+2l'+i}(h) &= \sqrt{-1}\vartheta_i && (1 \le i \le m') \end{aligned}$$

for $h = h^{k',l',m'}(a, z, \vartheta)$. Note that

$$\begin{aligned} \theta(e_i) &= -e_i && (1 \le i \le k'), \\ \theta(e_{k'+2i-1}) &= e_{k'+2i} && (1 \le i \le l'), \\ \theta(e_{k'+2l'+i}) &= e_{k'+2l'+i} && (1 \le i \le m'). \end{aligned}$$

Then there exists a unique isomorphism $j : (\mathfrak{h}_1^s)^* \to \mathfrak{h}_1^*$ such that

- $j(e_i^s) = e_i$ for all i;
- j is induced by some element in $\mathrm{Ad}(\mathfrak{g}_1)$.

Let $\gamma_1 = (H_1, \Gamma_1, \bar{\gamma}_1) \in \mathcal{R}^{G_1}(\xi_1)$. Then j is $W(\mathfrak{g}_1, \mathfrak{h}_1)$-conjugate to $i_{\bar{\gamma}_1}$. Under the identification $\mathfrak{h}_1^* \cong \mathbb{C}^n$ using the above basis, we write

$$\bar{\gamma}_1 = (u_1, \dots, u_n), \quad \rho(\Psi) - 2\rho_c(\Psi) = (v_1, \dots, v_n),$$

where Ψ is the positive system of imaginary roots as in Section 3B. Then we have $v_i \in \mathbb{Z} + \frac{1}{2}$ for all $k' < i \le n$. Since $\bar{\gamma}_1 + \rho(\Psi) - 2\rho_c(\Psi)$ is the differential of a character of $H_1 \cong (\mathbb{R}^\times)^{k'} \times (\mathbb{C}^\times)^{l'} \times (S^1)^{m'}$, we must have

$$\begin{aligned} u_{k'+2i-1} + v_{k'+2i-1} + u_{k'+2i} + v_{k'+2i} &\in \mathbb{Z} && (1 \le i \le l'), \\ u_{k'+2l'+i} + v_{k'+2l'+i} &\in \mathbb{Z} && (1 \le i \le m'), \end{aligned}$$

so that

$$\begin{aligned} u_{k'+2i-1} + u_{k'+2i} &\in \mathbb{Z} && (1 \le i \le l'), \\ u_{k'+2l'+i} &\in \mathbb{Z} + \tfrac{1}{2} && (1 \le i \le m'). \end{aligned}$$

Hence, noting that $j(\xi_1)$ is $W(\mathfrak{g}_1, \mathfrak{h}_1)$-conjugate to $i_{\bar{\gamma}_1}(\xi_1) = \bar{\gamma}_1$, we deduce from (3-11) that $m' \le m$. □

We now define the map

$$\varphi' : \mathrm{Std}(G_2, \xi_2) \to \mathrm{Std}(G_1, \xi_1)$$

as follows. Let $\gamma_2 = (H_2, \Gamma_2, \bar{\gamma}_2) \in \mathcal{R}^{G_2}(\xi_2)$. Replacing γ_2 by a K_2-conjugate if necessary, we may assume that

$$H_2 = H^{p,q,r} \times H^{p',q',r'}$$

with $p+2q+r = n-m$ and $p'+2q'+r' = m$. By Lemma 3.7, we have $r = 0$. Put

$$H_1 = \{\iota(h)\iota'(h') \mid h \in H^{p,q,0},\ h' \in H^{p',q',r'}\}.$$

Then H_1 is a θ_1-stable Cartan subgroup of G_1 and is K_1-conjugate to $H^{p+p',q+q',r'}$. Moreover, we have an isomorphism $H_2 \to H_1$ given by $(h, h') \mapsto \iota(h)\iota'(h')$. This induces isomorphisms $\phi : \widehat{H}_1 \to \widehat{H}_2$ and $\phi : \mathfrak{h}_1^* \to \mathfrak{h}_2^*$, which in turn induces an embedding

$$W(\mathfrak{g}_2, \mathfrak{h}_2) \hookrightarrow W(\mathfrak{g}_1, \mathfrak{h}_1).$$

We identify $W(\mathfrak{g}_2, \mathfrak{h}_2)$ with its image in $W(\mathfrak{g}_1, \mathfrak{h}_1)$.

Lemma 3.8. *We have*

$$W(G_2, H_2) = W(\mathfrak{g}_2, \mathfrak{h}_2) \cap W(G_1, H_1).$$

Proof. The assertion follows from (3-8). □

Put $\gamma_1 = (H_1, \Gamma_1, \bar{\gamma}_1)$, where

$$\Gamma_1 = \phi^{-1}(\Gamma_2), \quad \bar{\gamma}_1 = \phi^{-1}(\bar{\gamma}_2).$$

Then we have $\gamma_1 \in \mathcal{R}^{G_1}(\xi_1)$, and by Lemma 3.8, the K_1-conjugacy class of γ_1 is uniquely determined by the K_2-conjugacy class of γ_2. Hence we may define φ' by

$$\varphi'(\Theta_{\gamma_2}) = \Theta_{\gamma_1}.$$

We also define the map

$$\varphi : \mathrm{Std}(G_1, \xi_1) \to \mathrm{Std}(G_2, \xi_2)$$

as follows. Let $\gamma_1 = (H_1, \Gamma_1, \bar{\gamma}_1) \in \mathcal{R}^{G_1}(\xi_1)$. Replacing γ_1 by a K_1-conjugate if necessary, we may assume that

$$H_1 = H^{k',l',m'}$$

with $k'+2l'+m' = n$. Write $\bar{\gamma}_1 = (u_1, \dots, u_n)$ as in the proof of Lemma 3.7 and put

$$\begin{aligned} p' &= \left|\left\{1 \le i \le k' \mid u_i \in \mathbb{Z} + \tfrac{1}{2}\right\}\right|, \\ q' &= \tfrac{1}{2}\left|\left\{1 \le i \le 2l' \mid u_{k'+i} \in \mathbb{Z} + \tfrac{1}{2}\right\}\right|, \\ r' &= \left|\left\{1 \le i \le m' \mid u_{k'+2l'+i} \in \mathbb{Z} + \tfrac{1}{2}\right\}\right|. \end{aligned}$$

Then it follows from the proof of Lemma 3.7 that

$$q' \in \mathbb{Z}, \quad r' = m', \quad p' + 2q' + r' = m.$$

Put

$$H_2 = H^{p,q,0} \times H^{p',q',r'},$$

where $p = k' - p'$ and $q = l' - q'$. Then H_2 is a θ_2-stable Cartan subgroup of G_2. Replacing γ_1 by a K_1-conjugate again, we may now assume that

$$H_1 = \{\iota(h)\iota'(h') \mid h \in H^{p,q,0},\ h' \in H^{p',q',r'}\}.$$

Let $\phi : \widehat{H}_1 \to \widehat{H}_2$ and $\phi : \mathfrak{h}_1^* \to \mathfrak{h}_2^*$ be the isomorphisms induced by the isomorphism $H_2 \to H_1$ given by $(h, h') \mapsto \iota(h)\iota'(h')$. Put $\gamma_2 = (H_2, \Gamma_2, \bar{\gamma}_2)$, where

$$\Gamma_2 = \phi(\Gamma_1), \quad \bar{\gamma}_2 = \phi(\bar{\gamma}_1).$$

Replacing γ_1 by a $W(G_1, H_1)$-conjugate if necessary, we may further assume that $\chi_{\bar{\gamma}_2} = \chi_{\xi_2}$. Then we have $\gamma_2 \in \mathcal{R}^{G_2}(\xi_2)$, and by Lemma 3.8, the K_2-conjugacy class of γ_2 is uniquely determined by the K_1-conjugacy class of γ_1. Hence we may define φ by

$$\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}.$$

By construction, we have:

Lemma 3.9. *The two maps φ and φ' are inverses of each other. Moreover, the conditions (i)–(x) in Section 3E hold.*

Finally, as in Section 3F, we define a Levi subgroup M_i of G_i with respect to the θ_i-stable Cartan subgroup

$$H^{k,l,m}, \quad H^{k,l,0} \times H^{0,0,m}$$

of G_i for $i = 1, 2$, respectively. Then we have $H_i^s \subset M_i$ and

$$M_i \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times \mathrm{SO}_{2m+1}(\mathbb{R}).$$

Since $M_2 = M_3 \times \mathrm{SO}_{2m+1}(\mathbb{R})$ for some Levi subgroup $M_3 \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l$ of $\mathrm{SO}_{2(n-m)+1}(\mathbb{R})$, we may identify M_2 with M_1 via the isomorphism $M_2 \to M_1$ given by $(h, h') \mapsto \iota(h)\iota'(h')$. Let P_i be a parabolic subgroup of G_i with Levi component M_i. Note that $P_2 = P_3 \times \mathrm{SO}_{2m+1}(\mathbb{R})$ for some parabolic subgroup P_3 of $\mathrm{SO}_{2(n-m)+1}(\mathbb{R})$ with Levi component M_3. Recall the irreducible representation

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi'$$

of M_1 as in Section 3F. Put

$$\xi_1' = (\nu_1, \ldots, \nu_k, \kappa_1 + \nu_1', \kappa_1 - \nu_1', \ldots, \kappa_l + \nu_l', \kappa_l - \nu_l', \lambda_1', \ldots, \lambda_m') \in (\mathfrak{h}_1^s)^* \cong \mathbb{C}^n,$$

so that $\mathrm{Ind}_{P_1}^{G_1}(\pi)$ has infinitesimal character ξ_1'. Fix a positive system Δ^+ of $\Delta(\mathfrak{g}_1, \mathfrak{h}_1^s)$ such that

$$\mathrm{Re}\,\langle \alpha, \xi_1' \rangle \geq 0$$

for all $\alpha \in \Delta^+$ and let $\rho(\Delta^+)$ be half the sum of the roots in Δ^+. Choose a sufficiently large positive integer t such that

$$\xi_1 = \xi_1' + 2t\rho(\Delta^+)$$

is regular. Then we have $\Delta^+(\xi_1) = \Delta(\xi_1) \cap \Delta^+$, and by the assumption on π, ξ_1 satisfies (3-11). By construction, we have

$$\varphi(\Theta_\gamma^{G_1}) = \Theta_\gamma^{G_2}$$

for all $\gamma \in \mathcal{R}^{M_1}(\xi_1) = \mathcal{R}^{M_2}(\xi_2)$. Since $\Theta_\gamma^{G_i} = \mathrm{Ind}_{M_i}^{G_i}(\Theta_\gamma^{M_i})$ and $\mathrm{Ind}_{M_i}^{G_i}$ is additive, we have

$$\varphi(\mathrm{Ind}_{M_1}^{G_1}(\overline{\Theta})) = \mathrm{Ind}_{M_2}^{G_2}(\overline{\Theta})$$

for all $\overline{\Theta} \in \mathrm{Irr}(M_1, \xi_1) = \mathrm{Irr}(M_2, \xi_2)$. On the other hand, by (3-4) and [Vogan 1983b, Theorem 7.6], there exists $\overline{\Theta} \in \mathrm{Irr}(M_1, \xi_1)$ such that

$$\overline{\Theta}(\xi_1 + \lambda_1) = [\pi],$$

where $\lambda_1 \in \Lambda^{G_1}$ with $\bar{\lambda}_1 = -2t\rho(\Delta^+)$. Put $\Xi_i = \mathrm{Ind}_{M_i}^{G_i}(\overline{\Theta})$ and $\lambda_2 = f(\lambda_1)$, so that

$$\Xi_i(\xi_i + \lambda_i) = [\mathrm{Ind}_{P_i}^{G_i}(\pi)].$$

Then, applying Lemma 3.3 to Ξ_i and λ_i, we can reduce the irreducibility of $\mathrm{Ind}_{P_1}^{G_1}(\pi)$ to that of

$$\mathrm{Ind}_{P_2}^{G_2}(\pi) = \mathrm{Ind}_{P_3}^{\mathrm{SO}_{2(n-m)+1}(\mathbb{R})}(\chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l) \otimes \pi'.$$

Since $\mathrm{Ind}_{P_2}^{G_2}(\pi)$ is a standard module (with a suitable choice of P_2), its irreducibility follows from [Speh and Vogan 1980, Theorem 6.19] (see also Section 3J below) and the assumption on π. This completes the proof.

3H. ***Normalized intertwining operators.*** In the rest of this section, we will give another proof of Proposition 3.4 given to us by the referee, using normalized intertwining operators and the irreducibility result of [Speh and Vogan 1980].

We need to introduce more notation. Let G be a connected reductive linear algebraic group over $\mathbb{R}$. We confuse G with the group of $\mathbb{R}$-rational points of G. Let $P = MN$ be a parabolic subgroup of G with Levi component M and unipotent radical N. We denote by $\overline{P} = M\overline{N}$ the parabolic subgroup of G opposite to P. Let A_M be the split component of the center of M and put

$$\mathfrak{a}_M^* = \mathrm{Rat}(A_M) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(A_M), \mathbb{R}),$$

where $\mathrm{Rat}(A_M)$ is the group of algebraic characters of A_M defined over $\mathbb{R}$. Let $\mathfrak{a}^*_{M,\mathbb{C}} = \mathfrak{a}^*_M \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{a}^*_M$. Put

$$\mathfrak{a}^{*+}_P = \{\lambda \in \mathfrak{a}^*_M \mid \langle \lambda, \check{\alpha} \rangle > 0 \text{ for all } \alpha \in \Sigma(P)\},$$
$$\bar{\mathfrak{a}}^{*+}_P = \{\lambda \in \mathfrak{a}^*_M \mid \langle \lambda, \check{\alpha} \rangle \geq 0 \text{ for all } \alpha \in \Sigma(P)\},$$

where $\langle \cdot, \cdot \rangle : \mathfrak{a}^*_M \times \mathfrak{a}_M \to \mathbb{R}$ is the natural pairing, $\Sigma(P) \subset \mathfrak{a}^*_M$ is the set of reduced roots of A_M in P, and $\check{\alpha} \in \mathfrak{a}_M$ is the coroot corresponding to α. Noting that $\mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{Rat}(A_M) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathfrak{a}^*_M$, we may define a homomorphism $H_M : M \to \mathfrak{a}_M$ by

$$\langle \chi, H_M(m) \rangle = \log |\chi(m)|$$

for $\chi \in \mathrm{Rat}(M)$ and $m \in M$. For any continuous character ω of A_M, we define $\mathrm{Re}\,\omega \in \mathfrak{a}^*_M$ by

$$\langle \mathrm{Re}\,\omega, H_M(a) \rangle = \log |\omega(a)|$$

for $a \in A_M$.

Let M be a Levi subgroup of G. Let π be an irreducible representation of M with central character ω_π on A_M. Put $\pi_\lambda(m) = \pi(m) e^{\langle \lambda, H_M(m) \rangle}$ for $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ and $m \in M$. Let P and P' be two parabolic subgroups of G with common Levi component M. Then we define an intertwining operator

$$J_{P'|P}(\pi_\lambda) : \mathrm{Ind}^G_P(\pi_\lambda) \to \mathrm{Ind}^G_{P'}(\pi_\lambda)$$

by

$$(J_{P'|P}(\pi_\lambda) f)(g) = \int_{N \cap N' \backslash N'} f(n'g)\, dn'$$

for $f \in \mathrm{Ind}^G_P(\pi_\lambda)$ and $g \in G$, where N and N' are the unipotent radicals of P and P', respectively. Note that this integral converges absolutely if $\mathrm{Re}\,\lambda$ lies in some cone and admits a meromorphic continuation to $\mathfrak{a}^*_{M,\mathbb{C}}$. Moreover, by [Arthur 1989], there exists a meromorphic function $r_{P'|P}(\pi_\lambda)$ on $\mathfrak{a}^*_{M,\mathbb{C}}$ such that the normalized intertwining operator

$$R_{P'|P}(\pi_\lambda) = r_{P'|P}(\pi_\lambda)^{-1} J_{P'|P}(\pi_\lambda)$$

satisfies the following properties:

- If π is tempered, then $R_{P'|P}(\pi_\lambda)$ is holomorphic for $\mathrm{Re}\,\lambda \in \bar{\mathfrak{a}}^{*+}_P$.
- If P, P', and P'' are three parabolic subgroups of G with common Levi component M, then
$$R_{P''|P}(\pi_\lambda) = R_{P''|P'}(\pi_\lambda) R_{P'|P}(\pi_\lambda).$$
- Let L be a Levi subgroup of G containing M. Let Q and Q' be two parabolic subgroups of G with common Levi component L. Let S and S' be two parabolic

subgroups of L with common Levi component M. Let $Q(S)$, $Q'(S)$, and $Q(S')$ be the unique parabolic subgroups of G with common Levi component M such that

$$\begin{aligned} Q(S) &\subset Q, & Q(S)\cap L &= S,\\ Q'(S) &\subset Q', & Q'(S)\cap L &= S,\\ Q(S') &\subset Q, & Q(S')\cap L &= S',\end{aligned}$$

respectively. Then we have

$$\begin{aligned} R_{Q'(S)|Q(S)}(\pi_\lambda) &= R_{Q'|Q}(\mathrm{Ind}_S^L(\pi_\lambda)),\\ R_{Q(S')|Q(S)}(\pi_\lambda) &= \mathrm{Ind}_Q^G(R_{S'|S}(\pi_\lambda)).\end{aligned}$$

3I. ***Another proof of Proposition 3.4.*** We now return to the setting of Section 3F, so that $G = \mathrm{SO}_{2n+1}(\mathbb{R})$. Recall that P is a parabolic subgroup of G with Levi component

$$M \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times \mathrm{SO}_{2m+1}(\mathbb{R}),$$

where $k+2l+m=n$. Recall also that π is an irreducible representation of M of the form

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi',$$

where

- $\chi_i = \mathrm{sgn}^{\delta_i} \otimes |\cdot|^{\nu_i}$ for some $\delta_i \in \{0, 1\}$ and some $\nu_i \in \mathbb{C}$ with $|\mathrm{Re}\,\nu_i| < \frac{1}{2}$ such that if $\nu_i = \pm\nu_j$, then $\delta_i = \delta_j$;
- $\tau_i = D_{\kappa_i} \otimes |\det|^{\nu_i'}$ for some $\kappa_i \in \frac{1}{2}\mathbb{Z}$ and some $\nu_i' \in \mathbb{C}$ with $|\mathrm{Re}\,\nu_i'| < \frac{1}{2}$ such that if $\nu_i' = 0$, then $\kappa_i \in \mathbb{Z}$;
- π' is an irreducible representation of $\mathrm{SO}_{2m+1}(\mathbb{R})$ with infinitesimal character $\lambda' = (\lambda_1', \ldots, \lambda_m')$ such that $\lambda_i' \in \mathbb{Z} + \frac{1}{2}$ for all i.

We will show that $\mathrm{Ind}_P^G(\pi)$ is irreducible.

By the Langlands classification and the condition on π', there exist a parabolic subgroup P' of $\mathrm{SO}_{2m+1}(\mathbb{R})$ with Levi component

$$M' \cong \mathrm{GL}_1(\mathbb{R})^p \times \mathrm{GL}_2(\mathbb{R})^q \times \mathrm{SO}_{2r+1}(\mathbb{R}),$$

where $p+2q+r=m$, and an irreducible representation π_0' of M' of the form

$$\pi_0' = \chi_1' \otimes \cdots \otimes \chi_p' \otimes \tau_1' \otimes \cdots \otimes \tau_q' \otimes \pi'',$$

where

- $\chi_i' = \mathrm{sgn}^{\delta_i'} \otimes |\cdot|^{\mu_i}$ for some $\delta_i' \in \{0, 1\}$ and some $\mu_i \in \mathbb{Z} + \frac{1}{2}$ such that if $\mu_i = \pm\mu_j$, then $\delta_i' = \delta_j'$;
- $\tau_i' = D_{\kappa_i'} \otimes |\det|^{\mu_i'}$ for some $\kappa_i' \in \frac{1}{2}\mathbb{Z}$ and some $\mu_i' \in \mathbb{Z} + \kappa_i' + \frac{1}{2}$ with $\mu_i' \neq 0$;
- π'' is a (limit of) discrete series representation of $\mathrm{SO}_{2r+1}(\mathbb{R})$

such that π' is a unique irreducible quotient of $\mathrm{Ind}_{P'}^{\mathrm{SO}_{2m+1}(\mathbb{R})}(\pi_0')$. Then π' is the image of $R_{\bar{P}'|P'}(\pi_0')$. Let S_0 be a parabolic subgroup of M with Levi component M_0 such that

$$S_0 \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times P',$$

$$M_0 \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times M'.$$

We define an irreducible representation π_0 of M_0 by

$$\pi_0 = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi_0'.$$

Then π is the image of $R_{\bar{S}_0|S_0}(\pi_0)$.

Let P_1 and P_2 be the unique parabolic subgroups of G with common Levi component M_0 such that

$$P_1 \subset P, \quad P_1 \cap M = S_0,$$

$$P_2 \subset P, \quad P_2 \cap M = \bar{S}_0,$$

respectively. Then we have

$$R_{P_2|P_1}(\pi_0) = \mathrm{Ind}_P^G(R_{\bar{S}_0|S_0}(\pi_0)),$$

so that $\mathrm{Ind}_P^G(\pi)$ is the image of $R_{P_2|P_1}(\pi_0)$. On the other hand, if we take a parabolic subgroup P_0 of G with Levi component M_0 such that

$$P_0 \cap M = S_0,$$

$$\mathrm{Re}\,\omega_{\pi_0} \in \bar{\mathfrak{a}}_{P_0}^{*+},$$

then

$$R_{P_2|P_1}(\pi_0) = R_{P_2|\bar{P}_0}(\pi_0) R_{\bar{P}_0|P_0}(\pi_0) R_{P_0|P_1}(\pi_0).$$

Lemma 3.10. *The normalized intertwining operators $R_{P_0|P_1}(\pi_0)$ and $R_{P_2|\bar{P}_0}(\pi_0)$ are isomorphisms.*

Proof. We only prove the assertion for $R_{P_0|P_1}(\pi_0)$; the proof for $R_{P_2|\bar{P}_0}(\pi_0)$ is similar. Put $R_0 = P_1$ and write

$$\Sigma(P_1) \cap \Sigma(\bar{P}_0) = \{\alpha_1, \ldots, \alpha_t\}.$$

For $1 \le i \le t$, let R_i be the parabolic subgroup of G with Levi component M_0 such that

$$\Sigma(R_{i-1}) \cap \Sigma(\bar{R}_i) = \{\alpha_i\}.$$

Then we have $R_t = P_0$ and hence

$$R_{P_0|P_1}(\pi_0) = R_{R_t|R_{t-1}}(\pi_0) \cdots R_{R_1|R_0}(\pi_0).$$

Thus, it remains to show that $R_{R_i|R_{i-1}}(\pi_0)$ is an isomorphism for all $1 \le i \le t$.

Let L_i be the centralizer of A_{α_i} in G, where A_{α_i} is the identity component of the kernel of α_i in A_{M_0}. Put $S_i = R_{i-1} \cap L_i$. Then L_i is a Levi subgroup of G

containing M_0 and S_i is a maximal parabolic subgroup of L_i with Levi component M_0. Moreover, we have $\bar{S}_i = R_i \cap L_i$ and hence

$$R_{R_i|R_{i-1}}(\pi_0) = \mathrm{Ind}_{Q_i}^{G}(R_{\bar{S}_i|S_i}(\pi_0)),$$

where Q_i is the parabolic subgroup of G with Levi component L_i such that $R_{i-1} \subset Q_i$. Since α_i is not a root in M, it follows from [Speh and Vogan 1980, Theorem 6.19] (see also Section 3J below) and the condition on π_0 that $\mathrm{Ind}_{S_i}^{L_i}(\pi_0)$ is irreducible. Hence $R_{\bar{S}_i|S_i}(\pi_0)$ is an isomorphism, and so is $R_{R_i|R_{i-1}}(\pi_0)$. □

Hence, to prove the irreducibility of $\mathrm{Ind}_P^G(\pi)$, it suffices to show that the image of $R_{\bar{P}_0|P_0}(\pi_0)$ is irreducible. There exists a unique parabolic subgroup Q of G with Levi component L such that

$$P_0 \subset Q, \quad M_0 \subset L, \quad \mathrm{Re}\,\omega_{\pi_0} \in \mathfrak{a}_Q^{*+}.$$

Put $S = P_0 \cap L$, so that S is a parabolic subgroup of L with Levi component M_0. Then we have $P_0 = Q(S)$ and hence

$$R_{\bar{P}_0|P_0}(\pi_0) = R_{\bar{Q}(\bar{S})|Q(S)}(\pi_0) = R_{\bar{Q}(\bar{S})|Q(\bar{S})}(\pi_0) R_{Q(\bar{S})|Q(S)}(\pi_0).$$

Since $R_{\bar{S}|S}(\pi_0)$ is an isomorphism, so is

$$R_{Q(\bar{S})|Q(S)}(\pi_0) = \mathrm{Ind}_Q^G(R_{\bar{S}|S}(\pi_0)).$$

Also, we have

$$R_{\bar{Q}(\bar{S})|Q(\bar{S})}(\pi_0) = R_{\bar{Q}|Q}(\mathrm{Ind}_{\bar{S}}^{L}(\pi_0)).$$

By [Speh and Vogan 1980, Theorem 6.19] (see also Section 3J below) and the condition on π_0, $\mathrm{Ind}_{\bar{S}}^{L}(\pi_0)$ is irreducible, so that $\mathrm{Ind}_Q^G(\mathrm{Ind}_{\bar{S}}^{L}(\pi_0))$ is a standard module. (We remark that the irreducibility of $\mathrm{Ind}_{\bar{S}}^{L}(\pi_0)$ also follows from a result of Knapp and Zuckerman [1982a; 1982b].) Hence the image of $R_{\bar{Q}(\bar{S})|Q(\bar{S})}(\pi_0)$ is irreducible, and so is of $R_{\bar{P}_0|P_0}(\pi_0)$. This completes the proof.

3J. ***Explicit form of the irreducibility results.*** Finally, for the convenience of the reader, we explicate the irreducibility results which are used in the proof of Proposition 3.4. For $\kappa \in \frac{1}{2}\mathbb{Z}$, we have denoted by D_κ the relative (limit of) discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $2|\kappa| + 1$ with central character trivial on $\mathbb{R}_+^\times$.

We first recall the following irreducibility criterion due to Speh; see [Mœglin 1997, Theorem 10b]:

- For $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $s_1, s_2 \in \mathbb{C}$, the representation of $\mathrm{GL}_2(\mathbb{R})$ parabolically induced from

$$\mathrm{sgn}^{\epsilon_1}|\cdot|^{s_1} \otimes \mathrm{sgn}^{\epsilon_2}|\cdot|^{s_2}$$

is irreducible if and only if either

$$\begin{cases} \epsilon_1 = \epsilon_2 \text{ and } s_1 - s_2 \notin 2\mathbb{Z}+1; \text{ or} \\ \epsilon_1 \neq \epsilon_2 \text{ and } s_1 - s_2 \notin 2\mathbb{Z} \setminus \{0\}. \end{cases}$$

- For $\epsilon \in \{0, 1\}$, $\kappa \in \frac{1}{2}\mathbb{Z}$, and $s_1, s_2 \in \mathbb{C}$, the representation of $\mathrm{GL}_3(\mathbb{R})$ parabolically induced from

$$\mathrm{sgn}^{\epsilon} |\cdot|^{s_1} \otimes D_{\kappa} |\det|^{s_2}$$

is irreducible if and only if either

$$\begin{cases} s_1 - s_2 \notin \mathbb{Z}+\kappa; \text{ or} \\ s_1 - s_2 \in \mathbb{Z}+\kappa \text{ and } |s_1 - s_2| \leq |\kappa|. \end{cases}$$

- For $\kappa_1, \kappa_2 \in \frac{1}{2}\mathbb{Z}$ and $s_1, s_2 \in \mathbb{C}$, the representation of $\mathrm{GL}_4(\mathbb{R})$ parabolically induced from

$$D_{\kappa_1} |\det|^{s_1} \otimes D_{\kappa_2} |\det|^{s_2}$$

is irreducible if and only if either

$$\begin{cases} s_1 - s_2 \notin \mathbb{Z}+\kappa_1+\kappa_2; \text{ or} \\ s_1 - s_2 \in \mathbb{Z}+\kappa_1+\kappa_2 \text{ and } |s_1 - s_2| \leq \min(|\kappa_1+\kappa_2|, |\kappa_1-\kappa_2|). \end{cases}$$

We next recall the irreducibility result of [Speh and Vogan 1980] for $G = \mathrm{SO}_{2n+1}(\mathbb{R})$. We retain the notation of Section 3F, so that P is a parabolic subgroup of G with Levi component

$$M \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times \mathrm{SO}_{2m+1}(\mathbb{R}),$$

where $k+2l+m=n$. Put $\mathfrak{h} = \mathfrak{h}^{k,l,m}$. We define a basis $e_1, \ldots, e_n$ of $\mathfrak{h}^*$ by

$$\begin{aligned} e_i(h) &= a_i & &(1 \leq i \leq k), \\ e_{k+2i-1}(h) &= x_i + \sqrt{-1} y_i & &(1 \leq i \leq l), \\ e_{k+2i}(h) &= -x_i + \sqrt{-1} y_i & &(1 \leq i \leq l), \\ e_{k+2l+i}(h) &= \sqrt{-1}\vartheta_i & &(1 \leq i \leq m) \end{aligned}$$

for $h = h^{k,l,m}(a, z, \vartheta)$. Using the above basis, we identify $\mathfrak{h}^*$ with $\mathbb{C}^n$. We denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$:

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}.$$

We also denote by Δ_i, Δ_r, and Δ_{cx} the sets of imaginary, real, and complex roots, respectively:

$$\begin{aligned} \Delta_i &= \{\alpha \in \Delta \mid \theta\alpha = \alpha\}, \\ \Delta_r &= \{\alpha \in \Delta \mid \theta\alpha = -\alpha\}, \\ \Delta_{cx} &= \{\alpha \in \Delta \mid \theta\alpha \neq \pm\alpha\}, \end{aligned}$$

where θ is the Cartan involution of $\mathfrak{g}$. Let π be an irreducible representation of M of the form

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi',$$

where

- $\chi_i = \operatorname{sgn}^{\delta_i} \otimes |\cdot|^{\nu_i}$ for some $\delta_i \in \{0, 1\}$ and some $\nu_i \in \mathbb{C}$;
- $\tau_i = D_{\kappa_i} \otimes |\det|^{\nu'_i}$ for some $\kappa_i \in \frac{1}{2}\mathbb{Z}$ and some $\nu'_i \in \mathbb{C}$;
- π' is a (limit of) discrete series representation of $\mathrm{SO}_{2m+1}(\mathbb{R})$ with Harish-Chandra parameter
$$\lambda' = (\lambda'_1, \ldots, \lambda'_m) \in (\mathfrak{h}^{0,0,m})^* \cong \mathbb{C}^m,$$
where $\lambda'_i \in \mathbb{Z} + \frac{1}{2}$ for all i.

Put

$$\gamma = (\nu_1, \ldots, \nu_k, \kappa_1 + \nu'_1, \kappa_1 - \nu'_1, \ldots, \kappa_l + \nu'_l, \kappa_l - \nu'_l, \lambda'_1, \ldots, \lambda'_m) \in \mathfrak{h}^* \cong \mathbb{C}^n.$$

Then, by [Speh and Vogan 1980, Theorem 6.19], $\operatorname{Ind}_P^G(\pi)$ is irreducible if

(i) there exists no complex root $\alpha \in \Delta_{cx}$ satisfying $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle \in \mathbb{Z}$, $\langle\alpha, \gamma\rangle > 0$, and $\langle\theta\alpha, \gamma\rangle < 0$; and

(ii) there exists no real root $\alpha \in \Delta_r$ satisfying the parity condition [Vogan 1981, Definition 8.3.11].

We now explicate the conditions (i) and (ii). We start with the following special cases:

- Suppose that $k = 1$, $l = 0$, and $m = n - 1$. In this case, we have

$$\begin{aligned} \Delta_i &= \{\pm e_i \pm e_j \mid 2 \le i < j \le n\} \cup \{\pm e_i \mid 2 \le i \le n\}, \\ \Delta_r &= \{\pm e_1\}, \\ \Delta_{cx} &= \{\pm e_1 \pm e_j \mid 2 \le j \le n\}. \end{aligned}$$

Hence the conditions (i) and (ii) are equivalent to the following conditions, respectively:

(i′) $\begin{cases} \nu_1 \notin \mathbb{Z} + \frac{1}{2}; \text{ or} \\ \nu_1 \in \mathbb{Z} + \frac{1}{2} \text{ and } |\nu_1| \le |\lambda'_i| \text{ for all } 1 \le i \le m; \end{cases}$

(ii′) $\nu_1 \notin \mathbb{Z} + \frac{1}{2}$.

- Suppose that $k = 0$, $l = 1$, and $m = n - 2$. In this case, we have

$$\begin{aligned} \Delta_i &= \{\pm(e_1 + e_2)\} \cup \{\pm e_i \pm e_j \mid 3 \le i < j \le n\} \cup \{\pm e_i \mid 3 \le i \le n\}, \\ \Delta_r &= \{\pm(e_1 - e_2)\}, \\ \Delta_{cx} &= \{\pm e_i \pm e_j \mid 1 \le i \le 2 < j \le n\} \cup \{\pm e_i \mid 1 \le i \le 2\}. \end{aligned}$$

Hence the conditions (i) and (ii) are equivalent to the following conditions, respectively:

(i′) $\begin{cases} \nu_1' \notin \frac{1}{2}\mathbb{Z}; \text{ or} \\ \nu_1' \in \mathbb{Z}+\kappa_1 \text{ and } |\nu_1'| \le |\kappa_1|; \text{ or} \\ \nu_1' \in \mathbb{Z}+\kappa_1+\frac{1}{2},\ |\nu_1'| \le |\kappa_1|, \text{ and } |\nu_1'| \le \min(|\kappa_1+\lambda_i'|, |\kappa_1-\lambda_i'|) \text{ for all } 1 \le i \le m; \end{cases}$

(ii′) $\nu_1' \notin \mathbb{Z}+\kappa_1+\frac{1}{2}$.

Similarly, in the general case, we can show that the conditions (i) and (ii) hold if and only if

- for $1 \le i < j \le k$, $\delta_i = \delta_j$ and $\nu_i - \nu_j \notin 2\mathbb{Z}+1$, or $\delta_i \ne \delta_j$ and $\nu_i - \nu_j \notin 2\mathbb{Z}$;
- for $1 \le i < j \le k$, $\delta_i = \delta_j$ and $\nu_i + \nu_j \notin 2\mathbb{Z}+1$, or $\delta_i \ne \delta_j$ and $\nu_i + \nu_j \notin 2\mathbb{Z}$;
- for $1 \le i \le k$ and $1 \le j \le l$, $\nu_i - \nu_j' \notin \mathbb{Z}+\kappa_j$, or $\nu_i - \nu_j' \in \mathbb{Z}+\kappa_j$ and $|\nu_i - \nu_j'| \le |\kappa_j|$;
- for $1 \le i \le k$ and $1 \le j \le l$, $\nu_i + \nu_j' \notin \mathbb{Z}+\kappa_j$, or $\nu_i + \nu_j' \in \mathbb{Z}+\kappa_j$ and $|\nu_i + \nu_j'| \le |\kappa_j|$;
- for $1 \le i < j \le l$, $\nu_i' - \nu_j' \notin \mathbb{Z}+\kappa_i+\kappa_j$, or $\nu_i' - \nu_j' \in \mathbb{Z}+\kappa_i+\kappa_j$ and $|\nu_i' - \nu_j'| \le \min(|\kappa_i+\kappa_j|, |\kappa_i-\kappa_j|)$;
- for $1 \le i < j \le l$, $\nu_i' + \nu_j' \notin \mathbb{Z}+\kappa_i+\kappa_j$, or $\nu_i' + \nu_j' \in \mathbb{Z}+\kappa_i+\kappa_j$ and $|\nu_i' + \nu_j'| \le \min(|\kappa_i+\kappa_j|, |\kappa_i-\kappa_j|)$;
- for $1 \le i \le k$, $\nu_i \notin \mathbb{Z}+\frac{1}{2}$;
- for $1 \le i \le l$, $\nu_i' \notin \frac{1}{2}\mathbb{Z}$, or $\nu_i' \in \mathbb{Z}+\kappa_i$ and $|\nu_i'| \le |\kappa_i|$.

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WEE TECK GAN:

matgwt@nus.edu.sg
Department of Mathematics, National University of Singapore, Singapore

ATSUSHI ICHINO:

ichino@math.kyoto-u.ac.jp
Department of Mathematics, Kyoto University, Kyoto, Japan

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