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**A new explicit formula for Bernoulli numbers involving the Euler number**

Sumit Kumar Jha



# A new explicit formula for Bernoulli numbers involving the Euler number

Sumit Kumar Jha

We derive a new explicit formula for Bernoulli numbers in terms of the Stirling numbers of the second kind and the Euler numbers. As a corollary of our result, we obtain an explicit formula for the even Euler numbers in terms of the Stirling numbers of the second kind.

**Definition 1.** The *Bernoulli numbers*  $B_n$  can be defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n t^n}{n!},$$

where  $|t| < 2\pi$ .

**Definition 2.** A *Stirling number of the second kind*, denoted by  $S(n, m)$ , is the number of ways of partitioning a set of  $n$  elements into  $m$  nonempty sets.

There are many known explicit formulas known for the Bernoulli numbers [Gould 1972; Jha 2019]. The following formulas express the Bernoulli numbers explicitly in terms of the Stirling numbers of the second kind:

$$B_r = \sum_{k=1}^r (-1)^k \cdot k! \frac{S(r, k)}{k+1},$$

$$(-1)^{r-1} B_r = \sum_{k=1}^r (-1)^k \frac{S(r, k)}{k+1} \cdot (k-1)!,$$

$$B_{r+1} = \frac{(-1)^r \cdot (r+1) \cdot 2^r}{2^{r+1} - 1} \sum_{k=1}^r \frac{S(r, k)}{k+1} (-1)^k 2^{-2k} \frac{(2k-1)!}{(k-1)!}.$$

**Definition 3.** The *Euler numbers* are a sequence of integers, denoted by  $E_n$ , which can be defined by the Taylor series expansion

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n,$$

where  $\cosh t$  is the hyperbolic cosine.

We prove the following.

MSC2010: 11B68.

Keywords: Bernoulli numbers, Stirling numbers of the second kind, Euler numbers, polylogarithm function.

**Theorem 4.** *We have*

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))} \left( \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} + 4^{-r} E_r \right), \tag{1}$$

where  $S(r, k)$  denotes the Stirling numbers of the second kind,  $x^{(n)} = (x)(x+1) \cdots (x+n-1)$  denotes the rising factorial, and  $E_r$  denotes the Euler number.

*Proof.* We begin with the result

$$\frac{\sin n\pi}{\pi} \int_0^\infty x^{n-1} \frac{\text{Li}_s(-x)}{1+x} dx = \zeta(s) - \zeta(s, 1-n),$$

where  $\text{Li}_s(-x)$  denotes the polylogarithm function,  $\zeta(s)$  is the Riemann zeta function, and  $\zeta(s, 1-n)$  is the Hurwitz zeta function. The integral above is valid for all  $s \in \mathbb{C} \setminus \{1\}$  and  $0 < n < 1$ . This integral can be obtained from formula 3.2.1.6 in [Brychkov et al. 2019].

Plugging  $n = \frac{3}{4}$  and  $s = -r$ , a negative integer, into the integral above we get

$$\int_0^\infty x^{-1/4} \frac{\text{Li}_{-r}(-x)}{1+x} dx = \sqrt{2}\pi \left( \frac{B_{r+1}(\frac{1}{4}) - B_{r+1}}{r+1} \right).$$

Now, we use the representation from [Landsburg 2009]

$$\text{Li}_{-r}(-x) = \sum_{k=1}^r k! S(r, k) \left( \frac{1}{1+x} \right)^{k+1} (-x)^k,$$

which can be easily proved using induction on  $r$ .

As a result, we have

$$\begin{aligned} \int_0^\infty x^{-1/4} \frac{\text{Li}_{-r}(-x)}{1+x} dx &= \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \int_0^\infty \frac{x^{k-1/4}}{(1+x)^{k+2}} dx \\ &= \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \cdot \frac{\Gamma(k + \frac{3}{4}) \Gamma(\frac{5}{4})}{\Gamma(k+2)} \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \Gamma(k + \frac{3}{4}) \Gamma(\frac{5}{4}) \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}+1\right) \cdots \left(\frac{3}{4}+k-1\right) \Gamma(\frac{3}{4}) \Gamma(\frac{5}{4}) \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}+1\right) \cdots \left(\frac{3}{4}+k-1\right) \frac{1}{2\sqrt{2}}\pi \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} \frac{1}{2\sqrt{2}}\pi, \end{aligned}$$

where  $\Gamma(\cdot)$  is the Gamma function.

But, from [Weisstein], we have

$$\frac{B_{r+1}\left(\frac{1}{4}\right) - B_{r+1}}{r+1} = \frac{(-2^{-(r+1)}(1 - 2^{-r})B_{r+1} - 4^{-(r+1)}(r+1)E_r - B_{r+1})}{r+1}.$$

Thus, we have

$$B_{r+1} = -\frac{r+1}{4(1 + 2^{-(r+1)}(1 - 2^{-r}))} \left( \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} + 4^{-r} E_r \right).$$

□

If we let  $r = 2l$ , an even integer, in (1) we immediately obtain:

**Corollary 5.** 
$$E_{2l} = -4^{2l} \sum_{k=1}^{2l} (-1)^k \cdot \frac{S(2l, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)}.$$

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Paramodular forms of level 16 and supercuspidal representations	289
CRIS POOR, RALF SCHMIDT and DAVID S. YUEN	
Generalized Beatty sequences and complementary triples	325
JEAN-PAUL ALLOUCHE and F. MICHEL DEKKING	
Counting formulas for CM-types	343
MASANARI KIDA	
On polynomial-time solvable linear Diophantine problems	357
ISKANDER ALIEV	
Discrete analogues of John's theorem	367
SÖREN LENNART BERG and MARTIN HENK	
On the domination number of a graph defined by containment	379
PETER FRANKL	
A new explicit formula for Bernoulli numbers involving the Euler number	385
SUMIT KUMAR JHA	
Correction to the article "Intersection theorems for $(0, \pm 1)$ -vectors and $s$ -cross-intersecting families"	389
PETER FRANKL and ANDREY KUPAVSKII	