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complete characterization for wheels

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# Pairwise compatibility graphs: complete characterization for wheels

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A simple graph  $G$  is a pairwise compatibility graph (PCG) if there exists an edge-weighted tree  $T$  with positive weights and nonnegative numbers  $d_{\min}$  and  $d_{\max}$  such that the leaves of  $T$  are exactly the vertices of  $G$ , and  $uv$  is an edge in  $G$  if and only if the sum of weights of edges on the unique path between  $u$  and  $v$  in  $T$  is at least  $d_{\min}$  and at most  $d_{\max}$ . We show that a wheel on  $n$  vertices is a PCG if and only if  $n \leq 8$ , settling an open problem proposed by Calamoneri and Sinimeri (*SIAM Review* **58:3** (2016), 445–460). Our approach is based on unavoidable binary classifications of the edges in the complement of wheels that are PCGs. (Note: during the review process of our work, we learned that the same result has been obtained independently with an alternative proof.)

## 1. Introduction

Edge-weighted rooted trees are common graph models used in phylogenetics, a branch of biology that studies the evolutionary history and relationships of sets of taxa, i.e., organisms sharing similar characteristics (e.g., species, populations). In such a phylogenetic tree, a leaf represents a taxon, an internal vertex represents a possible common ancestor of its descendant leaves, and the weight of an edge may be interpreted as the length of the evolutionary history separating the species or populations represented by its two incident vertices. One of the first illustrations of a phylogenetic tree appeared in Charles Darwin's groundbreaking work [1859].

In computational biology, the problem of reconstructing an optimal phylogenetic tree from a given set of taxa is complex [Calamoneri and Sinimeri 2016], and so researchers have focused on constrained instances of this problem. For example, since very large and very small distances between pairs of taxa in the evolutionary history may have a negative impact on the performance of reconstruction algorithms, bounding these distances is a natural constraint [Kearney et al. 2003]. In graph-theoretical

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terms, let  $G$  be a graph where each vertex represents a taxon and  $uv$  be an edge in  $G$  if the evolutionary distance between vertices  $u$  and  $v$  is within an acceptable range. One is interested in finding an edge-weighted tree  $T$  with positive weights and nonnegative numbers  $d_{\min}$  and  $d_{\max}$  such that the set of leaves of  $T$  is exactly the set of vertices of  $G$ , and  $uv$  is an edge in  $G$  if and only if the sum of weights of edges on the unique path between  $u$  and  $v$  in  $T$  is at least  $d_{\min}$  and at most  $d_{\max}$ . If such  $T$ ,  $d_{\min}$  and  $d_{\max}$  exist, then we say that  $G$  is a *pairwise compatibility graph* (PCG) with *witness tree  $T$  bounded* by  $d_{\min}$  and  $d_{\max}$ , or simply  $G = \text{PCG}(T, d_{\min}, d_{\max})$ . For any two vertices  $u$  and  $v$  in  $G$  (not necessarily adjacent),  $d(u, v)$  will denote the sum of weights of the edges on the unique path in  $T$  between the leaves  $u$  and  $v$  (for simplicity, we omitted the subscript in  $d_T(u, v)$  which is traditionally used to denote the weighted distance between any pair of vertices  $u$  and  $v$  in  $T$ ).

The literature suggests that the PCG recognition problem is difficult, and it has been conjectured to be NP-hard [Durocher et al. 2015]. Since no complete characterization of PCGs is currently known, a large portion of the existing research has focused on determining whether particular graphs are PCGs or not. The following are some examples of the known classes of PCGs: graphs with at most seven vertices [Calamoneri et al. 2013a; Phillips 2002]; bipartite graphs with at most eight vertices [Mehnaz and Rahman 2013]; cycles, single-chord cycles, cacti, tree power graphs, Steiner and phylogenetic  $k$ -power graphs [Mehnaz and Rahman 2013; Yanhaona et al. 2009]; trees, ladders, triangle-free outerplanar 3-graphs [Salma et al. 2013]; Dilworth 2 graphs [Calamoneri and Petreschi 2014]; split matrogenic graphs and certain superclasses [Calamoneri et al. 2013b]. Some particular graphs that are not PCGs have also been identified: a nonbipartite circular arc graph on 8 vertices, a bipartite graph on 15 vertices, and a planar graph on 20 vertices [Yanhaona et al. 2009; 2010]. Recently, two results involving the complement  $G^c$  of a graph  $G$  provided additional tools in the study of PCGs [Hossain et al. 2017]: if  $G^c$  is acyclic then  $G$  is a PCG; if  $G^c$  contains two vertex-disjoint chordless cycles without an edge simultaneously incident to both cycles, then  $G$  is not a PCG. One instance relevant to our work is the class of  *$k$ -leaf power graphs* which are PCGs, where  $d_{\min} = 0$  and  $d_{\max} = k$ . It is well known that these graphs are strongly chordal, i.e., chordal and sun-free [Farber 1983]; however, the converse is not true [Bibelnicks and Dearing 1993]. In fact, no complete characterization of  $k$ -leaf power graphs is known except when  $k \leq 4$  [Brandstädt and Le 2006; Brandstädt et al. 2008; Dom et al. 2004; 2005; Rautenbach 2006].

From the references above and from our recent experience, we have learned that many of the existing results concerning the PCG recognition problem required determination and clever, nontrivial approaches to generate witness trees or to show that none exist. Nevertheless, the efforts behind these approaches have not been readily apparent since they often describe witness trees without providing a clear

discussion of what drives their particular structures. Perhaps for these reasons there are still many open problems in the area, as mentioned in the comprehensive survey [Calamoneri and Sinimeri 2016], including the following:

**Open Problem 1.** Find other graph classes that do not belong to the PCG class.

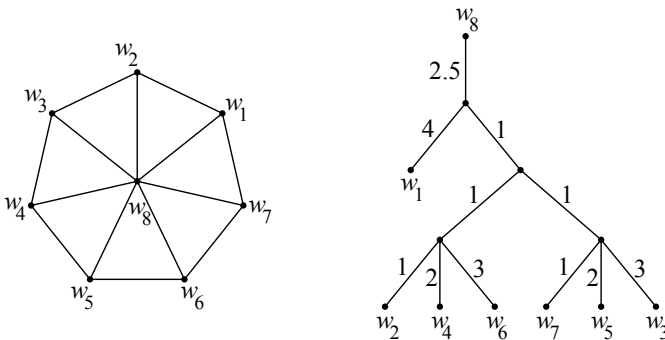
**Open Problem 2.** It is not known whether or not wheels on at least eight nodes are PCGs.

We add one more class for **Open Problem 1** while settling **Open Problem 2** in our main result:

**Theorem 1.1.** *Wheels on  $n$  vertices are PCGs if and only if  $n \leq 8$ .*

We will be using the following notation throughout this work. The *wheel*  $W_n$  with order  $n \geq 4$  has vertices  $w_1, w_2, \dots, w_n$ , edges  $w_i w_n$  for  $i = 1, 2, \dots, n - 1$ , edges  $w_i w_{i+1}$  for  $i = 1, 2, \dots, n - 2$ , and edge  $w_1 w_{n-1}$ . The cycle induced by the vertices  $w_1, w_2, \dots, w_{n-1}$  is called the *rim* of the wheel.

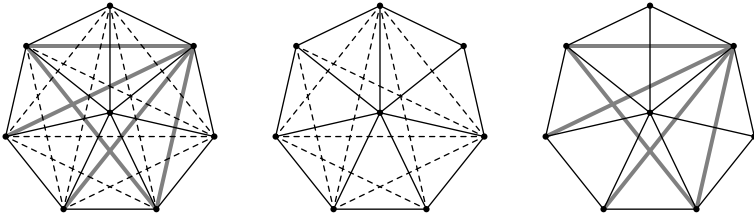
**Figure 1** shows the wheel  $W_8$  and a witness tree  $T$  bounded by  $d_{\min} = 5.5$  and  $d_{\max} = 7.5$ , that is,  $W_8 = \text{PCG}(T, 5.5, 7.5)$ . This claim can be easily verified using the information in **Table 1**, where for each entry  $(i, j)$ , the corresponding column header is  $d(w_i, w_j)$  for  $T$  in **Figure 1** (pairs in bold correspond to the edges in  $W_8$ ).



**Figure 1.** Wheel  $W_8$  on the left and a witness tree  $T$  on the right with  $W_8 = \text{PCG}(T, 5.5, 7.5)$ .

3	4	5	5.5	6	6.5	7	7.5	8	9
(2,4)	(2,6)	(2,5)	<b>(2,8)</b>	<b>(2,3)</b>	<b>(1,8)</b>	<b>(1,2)</b>	<b>(3,8)</b>	(1,4)	(1,3)
(5,7)	(2,7)	(3,5)	<b>(7,8)</b>	<b>(4,5)</b>	<b>(4,8)</b>	<b>(1,7)</b>	<b>(6,8)</b>	(1,5)	(1,6)
	(3,7)	(4,6)		<b>(6,7)</b>	<b>(5,8)</b>	<b>(3,4)</b>		(3,6)	
		(4,7)			<b>(5,6)</b>				

**Table 1.** For each entry  $(i, j)$ , the corresponding column header is  $d(w_i, w_j)$  for  $T$  in **Figure 1** (pairs in bold correspond to the edges in  $W_8$ ).

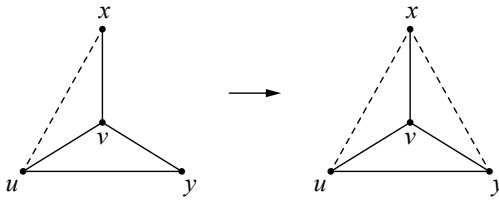


**Figure 2.** Edges in  $W_8$  are solid black, light edges in  $(W_8)^c$  are dashed and heavy edges in  $(W_8)^c$  are thick gray.

Generating this witness tree for  $W_8$  was far from trivial. A brute-force computation approach was infeasible due to the large number of trees with eight leaves and the infinite number of choices for their edge-weights and bounds. We relied on potential binary classifications of the edges in the complement  $(W_8)^c$  of  $W_8$ , more specifically, which edge  $uv$  in  $(W_8)^c$  could be *light*, i.e.,  $d(u, v) < d_{\min}$ , and which could be *heavy*, i.e.,  $d(u, v) > d_{\max}$ . Using general results that do not require the knowledge of an exact witness tree and bounds, we generated the configuration of light and heavy edges given on the left-most graph in Figure 2, where edges in  $W_8$  are solid black, light edges in  $(W_8)^c$  are dashed, and heavy edges in  $(W_8)^c$  are thick gray. The center and right-most graphs in this figure are provided for clarity and show  $W_8$  together with only light and with only heavy edges, respectively. The exact steps to obtain this configuration are omitted, as they are similar to the steps presented in the proof of Theorem 2.6 in Section 2. From this configuration, we were able to obtain the witness tree  $T$  and bounds in Figure 1 by inspection.

Recall that all graphs with at most seven vertices are PCGs. Theorem 1.1 will follow, given that we have shown here that  $W_8$  is also a PCG and will show in Section 2 that no  $W_n$  for  $n \geq 9$  is a PCG.

During the review process of our work, we learned that Theorem 1.1 has been verified independently in the arXiv manuscript [Baiocchi et al. 2017] which was later presented as the conference extended abstract [Baiocchi et al. 2018]. In [Baiocchi et al. 2018], the edges of a PCG are colored black, and edges in the complement are colored red if they are light and white if they are heavy. Several forbidden tricolored structures are identified. The general approach assumes that  $W_n$  for  $n \geq 9$  is a PCG and these forbidden structures are used in an exhaustive case discussion to reach a contradiction. Our approach is similar in the sense that it focuses on certain unavoidable binary configurations of edges and, indeed, one of the forbidden structures identified in [Baiocchi et al. 2018] (namely  $\mathbf{f-c}(2K_2)a$ , coincides with the configuration  $H_5$  described in our Lemma 2.4). Nevertheless, we believe our proof streamlines the case discussion by generating a sequence of unavoidable light edges until the forbidden configuration  $H_5$  is achieved.



**Figure 3.** Configuration  $H_1$  (left), where  $xy$  is an edge in  $G^c$  and  $d(u, v) \geq d(v, x)$ , implies  $H_2$  (right), as shown in [Lemma 2.2](#).

### 2. Wheels with more than eight vertices are not PCGs

Key to our discussion is the following useful result that allows for distance comparisons between certain pairs of leaves in general edge-weighted trees.

**Result 2.1** [[Yanhaona et al. 2010](#)]. Let  $T$  be an edge-weighted tree and let  $u, v, x$  be three leaves in  $T$  such that  $d(u, v) = \max\{d(u, v), d(v, x), d(x, u)\}$ . If  $y$  is a leaf other than  $u, v, x$ , then  $d(x, y) \leq d(u, y)$  or  $d(x, y) \leq d(v, y)$ .  $\square$

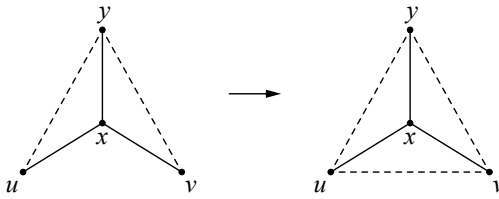
We will apply [Result 2.1](#) to the witness trees of certain PCGs in [Lemmas 2.2, 2.3](#) and [2.4](#). These lemmas will be vital tools used to show that  $W_n$  is not a PCG when  $n \geq 9$  in [Theorem 2.6](#). We first extend the definitions of light and heavy edges mentioned in [Section 1](#) to general PCGs; that is, given  $G = \text{PCG}(T, d_{\min}, d_{\max})$ , we say that an edge  $uv$  in  $G^c$  is *light* if  $d(u, v) < d_{\min}$  and is *heavy* if  $d(u, v) > d_{\max}$ . Any future figures will continue using the conventions given in [Figure 2](#): edges in  $G$  are solid black, light edges in  $G^c$  are dashed, and heavy edges in  $G^c$  are thick gray.

**Lemma 2.2.** *Let  $G = \text{PCG}(T, d_{\min}, d_{\max})$ . If  $G$  and  $G^c$  contain the edges in the configuration  $H_1$  in [Figure 3](#) (left), where  $xy$  is an edge in  $G^c$  and  $d(u, v) \geq d(v, x)$ , then  $xy$  must be light as indicated in the configuration  $H_2$  in [Figure 3](#) (right).*

*Proof.* Since  $d(u, v) \geq d(v, x)$  and  $xu$  is light, we have  $d(u, v) = \max\{d(u, v), d(v, x), d(x, u)\}$ . By [Result 2.1](#),  $d(x, y) \leq d(u, y)$  or  $d(x, y) \leq d(v, y)$ . But  $d(u, y) \leq d_{\max}$  and  $d(v, y) \leq d_{\max}$  because  $uy$  and  $vy$  are edges in  $G$ , therefore  $d(x, y) \leq d_{\max}$ . This latter inequality combined with the fact that  $xy$  is an edge in  $G^c$  implies  $d(x, y) < d_{\min}$  and therefore  $xy$  is light.  $\square$

**Lemma 2.3.** *Let  $G = \text{PCG}(T, d_{\min}, d_{\max})$ . If  $G$  and  $G^c$  contain the edges in the configuration  $H_3$  in [Figure 4](#) (left), where  $uv$  is an edge in  $G^c$ , then  $uv$  must be light as indicated in the configuration  $H_4$  in [Figure 4](#) (right).*

*Proof.* Suppose by contradiction that  $uv$  is heavy. Since  $xu$  and  $vx$  are edges in  $G$ , we must have  $d(x, u) \leq d_{\max}$  and  $d(v, x) \leq d_{\max}$ ; hence  $d(u, v) = \max\{d(u, v), d(v, x), d(x, u)\}$  and by [Result 2.1](#),  $d(x, y) \leq d(u, y)$  or  $d(x, y) \leq d(v, y)$ . But  $uy$



**Figure 4.** Configuration  $H_3$  (left), where  $uv$  is an edge in  $G^c$ , implies  $H_4$  (right), as shown in Lemma 2.2.

and  $vy$  are light, that is,  $d(u, y) < d_{\min}$  and  $d(v, y) < d_{\min}$ , which would imply  $d(x, y) < d_{\min}$ , contradicting the fact that  $d(x, y) \geq d_{\min}$  as  $xy$  is an edge in  $G$ .  $\square$

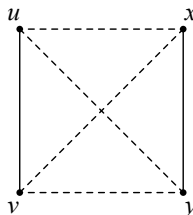
For later discussions, it is important to note the differences between the vertex labels in Figures 3 and 4 (e.g.,  $v$  and  $x$  are the only vertices of degree 3 in each respective figure). These labels were chosen so that Result 2.1 could be readily applied in the proofs of Lemmas 2.2 and 2.3, respectively.

**Lemma 2.4.** *Let  $G = \text{PCG}(T, d_{\min}, d_{\max})$ .  $G$  and  $G^c$  cannot contain the configuration  $H_5$  of Figure 5.*

*Proof.* Suppose by contradiction that  $G$  and  $G^c$  contain the configuration  $H_5$ . Since  $xu$  and  $vx$  are light, we must have  $d(x, u) < d_{\min}$  and  $d(v, x) < d_{\min}$ . But  $uv$  is an edge in  $G$ , so we have  $d(u, v) \geq d_{\min}$ . Hence  $d(u, v) = \max\{d(u, v), d(v, x), d(x, u)\}$  and by Result 2.1,  $d(x, y) \leq d(u, y)$  or  $d(x, y) \leq d(v, y)$ . But  $uy$  and  $vy$  are light, that is,  $d(u, y) < d_{\min}$  and  $d(v, y) < d_{\min}$ , which would imply  $d(x, y) < d_{\min}$ , contradicting the fact that  $d(x, y) \geq d_{\min}$  as  $xy$  is an edge in  $G$ .  $\square$

In the proof of Theorem 2.6, we will assume by contradiction that  $W_n$  is a PCG for some  $n \geq 9$  and apply Lemmas 2.2 and 2.3 repeatedly until a contradiction to Lemma 2.4 is reached. To be able to set this argument in motion, we need to verify the existence of a particular light edge. For each  $p = 2, 3, \dots, n - 3$ , we define a  $p$ -light edge in  $(W_n)^c$  to be a light edge with ends connected by a path on the rim of  $W_n$  with exactly  $p$  edges (note that a  $p$ -light edge is also an  $(n - p - 1)$ -light edge).

**Lemma 2.5.** *If  $n \geq 5$  and  $W_n = \text{PCG}(T, d_{\min}, d_{\max})$ , then there exists a  $p$ -light edge for each  $p = 2, 3, \dots, n - 3$ .*



**Figure 5.** Forbidden configuration  $H_5$  in Lemma 2.4.

*Proof.* Let  $W_n = \text{PCG}(T, d_{\min}, d_{\max})$  with  $n \geq 5$ . Since  $p$ -light edges are  $(n-p-1)$ -light edges, it is enough to verify the lemma for  $p = 2, 3, \dots, \lfloor (n-1)/2 \rfloor$ . We will proceed by induction on  $p$ .

The rim of  $W_n$  is a chordless cycle; hence  $W_n$  is not chordal and consequently not strongly chordal. Recall from [Section 1](#) that  $k$ -leaf power graphs are strongly chordal so  $W_n$  is not a  $k$ -leaf power graph; that is,  $d_{\min} > 0$  and there exists at least one light edge (if there are no light edges, then  $uv$  would be an edge in  $G$  if and only if  $0 \leq d(u, v) \leq d_{\max}$ , and hence  $W_n$  would be a  $k$ -leaf power graph). Choose a light edge with ends that minimize the distance on the rim of  $W_n$  (i.e., the number of edges on the shortest path between these ends using only edges on the rim) over all light edges, and let  $m$  be this smallest distance. We may assume without loss of generality that  $w_1 w_{m+1}$  is this selected light edge and  $d(w_{m+1}, w_n) \geq d(w_1, w_n)$  (if not, rotate and/or reverse the labels on the rim). Clearly,  $2 \leq m \leq \lfloor (n-1)/2 \rfloor$ . If  $m > 2$ , since  $w_1 w_m$  is an edge in  $(W_n)^c$  and  $d(w_{m+1}, w_n) \geq d(w_1, w_n)$ , then applying [Lemma 2.2](#) with  $u = w_{m+1}, v = w_n, x = w_1, y = w_m$  would imply  $xy = w_1 w_m$  is light with ends connected by a path on the rim with  $m-1$  edges, which contradicts the minimality of  $m$ . Hence  $m = 2$ ; that is,  $w_1 w_3$  is light with ends connected by the path  $w_1 w_2 w_3$  on the rim. Thus, there is a 2-light edge in  $W_n$ , and the basis of the induction has been established.

Assume for  $2 \leq p < \lfloor (n-1)/2 \rfloor$  that there exists a  $p$ -light edge and we will show that there exists a  $(p+1)$ -light edge, concluding our inductive argument. Rotate and/or reverse the labels on the rim so that  $w_1 w_{p+1}$  is this  $p$ -light edge and  $d(w_{p+1}, w_n) \geq d(w_1, w_n)$ . Note that since  $n \geq 5$  and  $p < \lfloor (n-1)/2 \rfloor$ , we have  $p+2 < \lfloor (n-1)/2 \rfloor + 2 \leq n-1$  so  $w_1 w_{p+2}$  is an edge in  $(W_n)^c$ . Applying [Lemma 2.2](#) with  $u = w_{p+1}, v = w_n, x = w_1, y = w_{p+2}$  we conclude that  $xy = w_1 w_{p+2}$  is a  $(p+1)$ -light edge, and so our induction is complete.  $\square$

We can confirm that this lemma holds in the instance of  $W_8$  presented in [Figure 2](#); for example,  $w_1 w_3$  is a 2- and 5-light edge, and  $w_1 w_4$  is a 3- and 4-light edge.

Applications of [Lemma 2.2](#) similar to the two discussed in the proof of [Lemma 2.5](#) will occur multiple times in the proof of [Theorem 2.6](#), and so we will use the abbreviated notation  $(i, j, k) \xrightarrow{2,2} (j, k)$  to indicate that  $w_j w_k$  is an edge in  $(W_n)^c$ ,  $d(w_i, w_n) \geq d(w_j, w_n)$ , and setting  $u = w_i, v = w_n, x = w_j, y = w_k$  we have the configuration  $H_1$  in [Figure 3](#) (left); therefore applying [Lemma 2.2](#) implies  $xy = w_j w_k$  is light. With this notation, the two applications of [Lemma 2.2](#) in the proof of [Lemma 2.5](#) would simply read  $(m+1, 1, m) \xrightarrow{2,2} (1, m)$  and  $(p+1, 1, p+2) \xrightarrow{2,2} (1, p+2)$ , respectively. In the same spirit, we also define the abbreviated notation  $(i, j, k) \xrightarrow{2,3} (i, j)$  to indicate that  $w_i w_j$  is an edge in  $(W_n)^c$  and setting  $u = w_i, v = w_j, x = w_n, y = w_k$  we have the configuration  $H_3$  in [Figure 4](#) (left); therefore applying [Lemma 2.3](#) implies  $uv = w_i w_j$  is light.



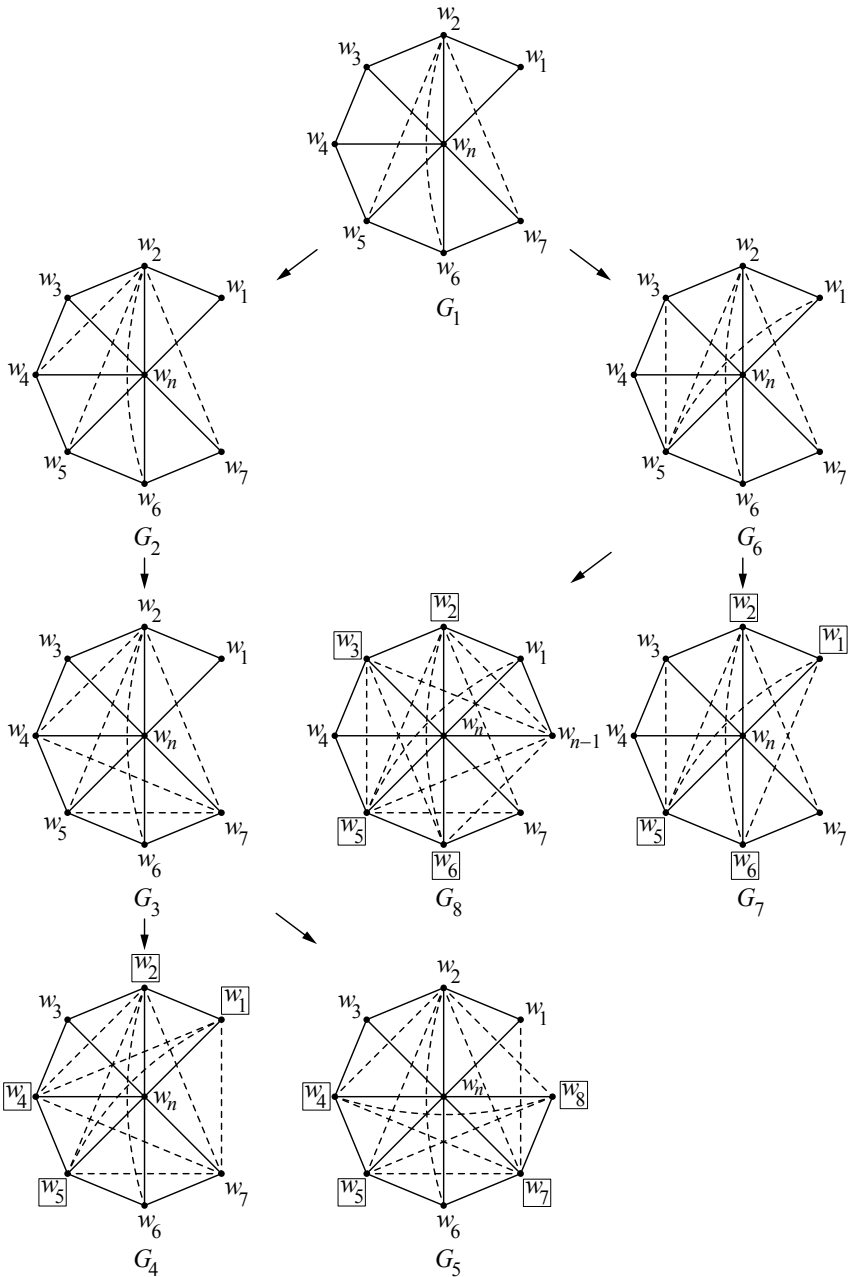
**Theorem 2.6.** *If  $n \geq 9$ , then  $W_n$  is not a PCG.*

*Proof.* Let  $n \geq 9$  and suppose by contradiction that  $W_n = \text{PCG}(T, d_{\min}, d_{\max})$ . From Lemma 2.5, there exists a 4-light edge. We may assume without loss of generality that  $w_2w_6$  is a 4-light edge and that  $d(w_6, w_n) \geq d(w_2, w_n)$  (if not, rotate and/or reverse the labels on the rim). The proof proceeds by adding light edges forced by Lemmas 2.2 and 2.3 until we reach the configuration  $H_5$  featured in Figure 5, which would contradict Lemma 2.4. We begin by observing that  $(6, 2, 5) \xrightarrow{2.2} (2, 5)$  and  $(6, 2, 7) \xrightarrow{2.2} (2, 7)$ . The three current light edges are shown in the configuration  $G_1$  of Figure 6. We split the discussion into two cases:

Case 1: Suppose  $d(w_5, w_n) \geq d(w_2, w_n)$ . Hence  $(5, 2, 4) \xrightarrow{2.2} (2, 4)$ , with current light edges shown in the configuration  $G_2$  of Figure 6. In addition,  $(4, 7, 2) \xrightarrow{2.3} (4, 7)$  and  $(5, 7, 2) \xrightarrow{2.3} (5, 7)$ , with current light edges shown in the configuration  $G_3$  of Figure 6. Let us first examine the subcase where  $d(w_7, w_n) < d(w_2, w_n)$ . Since  $n \geq 9$ , we have that  $w_1w_7$  is an edge in  $(W_n)^c$  and is in fact a light edge, since  $(2, 7, 1) \xrightarrow{2.2} (7, 1)$ . We then have  $(1, 4, 7) \xrightarrow{2.3} (1, 4)$  and  $(1, 5, 7) \xrightarrow{2.3} (1, 5)$ . The current light edges are shown in the configuration  $G_4$  of Figure 6 and therefore we reached the configuration  $H_5$  with  $u = w_1$ ,  $v = w_2$ ,  $x = w_4$ ,  $y = w_5$  (boxed vertices), a contradiction. We now focus on the remaining subcase where  $d(w_7, w_n) \geq d(w_2, w_n)$  and reset our current light edges to those shown in configuration  $G_3$  of Figure 6. First observe that  $(7, 2, 8) \xrightarrow{2.2} (2, 8)$ . We then have  $(4, 8, 2) \xrightarrow{2.3} (4, 8)$  and  $(5, 8, 2) \xrightarrow{2.3} (5, 8)$ . The current light edges are shown in the configuration  $G_5$  of Figure 6 and therefore we reached the configuration  $H_5$  with  $u = w_4$ ,  $v = w_5$ ,  $x = w_7$ ,  $y = w_8$  (boxed vertices), a contradiction.

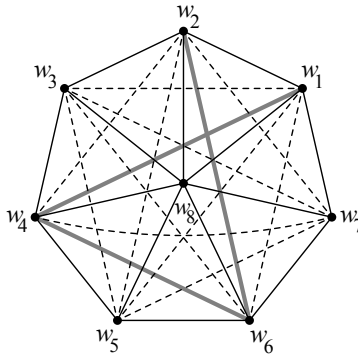
Case 2: Suppose  $d(w_5, w_n) < d(w_2, w_n)$  and reset our current light edges to those shown in configuration  $G_1$  of Figure 6. Hence  $(2, 5, 1) \xrightarrow{2.2} (5, 1)$  and  $(2, 5, 3) \xrightarrow{2.2} (5, 3)$  with current light edges shown in the configuration  $G_6$  of Figure 6. Let us first examine the subcase where  $d(w_5, w_n) \geq d(w_1, w_n)$ , thus  $(5, 1, 6) \xrightarrow{2.2} (1, 6)$ . The current light edges are shown in the configuration  $G_7$  of Figure 6 and therefore we reached the configuration  $H_5$  with  $u = w_1$ ,  $v = w_2$ ,  $x = w_5$ ,  $y = w_6$  (boxed vertices), a contradiction. We now focus on the remaining subcase where  $d(w_5, w_n) < d(w_1, w_n)$  and reset our current light edges to those shown in configuration  $G_6$  of Figure 6. First observe that  $(1, 5, n-1) \xrightarrow{2.2} (5, n-1)$ . We then have  $(2, n-1, 5) \xrightarrow{2.3} (2, n-1)$  and  $(3, n-1, 5) \xrightarrow{2.3} (3, n-1)$ . Now we have  $(6, n-1, 2) \xrightarrow{2.3} (6, n-1)$  (note that  $w_6w_{n-1}$  is an edge in  $(W_n)^c$  since  $n \geq 9$ ) and can finally conclude that  $(3, 6, n-1) \xrightarrow{2.3} (3, 6)$ . The current light edges are shown in the configuration  $G_8$  of Figure 6 and therefore we reached the configuration  $H_5$  with  $u = w_2$ ,  $v = w_3$ ,  $x = w_5$ ,  $y = w_6$  (boxed vertices), a contradiction.

Since contradictions were reached in all possible cases, the theorem holds.  $\square$



**Figure 6.** Configurations from the proof of [Theorem 2.6](#).

A series of steps based on [Lemmas 2.2, 2.3, and 2.4](#), similar to the ones described in the proof of [Theorem 2.6](#), could be applied to  $W_8$  to construct complete configurations of light and heavy edges that do not contain  $H_5$  of [Figure 5](#). After



**Figure 7.** Invalid configuration for  $W_8$  and  $(W_8)^c$  in Lemma 2.7.

exhaustive case discussions (omitted for the sake of brevity), we found only two of these configurations, namely the configurations in Figures 2 and 7. The former allowed us to prove that  $W_8$  is a PCG as shown in Section 1. Interestingly, the latter is not a valid configuration for  $W_8$  and  $(W_8)^c$  as verified in Lemma 2.7.

**Lemma 2.7.** *If  $W_8 = \text{PCG}(T, d_{\min}, d_{\max})$ , then its corresponding light and heavy edges cannot be described by the configuration in Figure 7 (up to rotating and/or reversing the vertex labels on the rim).*

*Proof.* Suppose the lemma does not hold. We examine three cases:

Case 1:  $d(w_1, w_2) = \max\{d(w_1, w_2), d(w_2, w_8), d(w_8, w_1)\}$ . Apply Result 2.1 with  $u = w_1, v = w_2, x = w_8, y = w_5$  to conclude  $d(w_8, w_5) \leq d(w_1, w_5)$  or  $d(w_5, w_8) \leq d(w_2, w_5)$ . But  $w_1w_5$  and  $w_2w_5$  are light which would imply  $d(w_8, w_5) < d_{\min}$ , contradicting the fact that  $w_8w_5$  is an edge in  $W_8$ .

Case 2:  $d(w_2, w_8) = \max\{d(w_1, w_2), d(w_2, w_8), d(w_8, w_1)\}$ . Apply Result 2.1 with  $u = w_2, v = w_8, x = w_1, y = w_4$  to conclude  $d(w_1, w_4) \leq d(w_2, w_4)$  or  $d(w_1, w_4) \leq d(w_8, w_4)$ . If  $d(w_1, w_4) \leq d(w_2, w_4)$ , then  $d(w_1, w_4) < d_{\min}$  since  $w_2w_4$  is light; if  $d(w_1, w_4) \leq d(w_8, w_4)$ , then  $d(w_1, w_4) \leq d_{\max}$  since  $w_8w_4$  is an edge in  $W_8$ ; both options contradict the fact that  $w_1w_4$  is heavy.

Case 3:  $d(w_8, w_1) = \max\{d(w_1, w_2), d(w_2, w_8), d(w_8, w_1)\}$ . Given the symmetry of the configuration in Figure 7, this case can be verified as in Case 2 if we rotate the vertex labels around the rim one unit counterclockwise and then reverse their order clockwise.

Since contradictions were reached in all possible cases, the lemma holds. □

### 3. Closing remarks

We proved that  $W_8$  is a PCG, but  $W_n$  for  $n \geq 9$  are not PCGs, settling an open problem proposed in [Calamoneri and Sinimeri 2016]. The difficulty in showing

$W_8$  is a PCG stemmed from the many degrees of freedom one has in constructing potential witness trees — as both the tree’s structure and its edge weights must be specified, the collection of candidate witness trees is both very large and highly varied. A natural direction for future work would be to ask whether some subfamilies of trees could be conclusively ruled out as witness trees. Our results followed from a series of lemmas concerning light and heavy edges. While considerably distanced from the properties of any underlying witness tree, this layer of abstraction is nonetheless extremely useful. We have presented here a collection of general tools concerning configurations of heavy and/or light edges, but this set is by no means exhaustive — indeed, [Lemma 2.7](#) hints at other families of forbidden subgraphs. We hope to see expanded results, both in terms of composition and complexity of such configurations, in the months and years to come.

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