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Solutions of boundary value problems at resonance with periodic and antiperiodic boundary conditions

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We study the existence of solutions of the second-order boundary value problem at resonance $u'' = f(t, u, u')$ satisfying the boundary conditions $u(0)+u(1)=0$, $u'(0)-u'(1)=0$, or $u(0)-u(1)=0$, $u'(0)+u'(1)=0$. We employ a shift method, making a substitution for the nonlinear term in the differential equation so that these problems are no longer at resonance. Existence of solutions of equivalent boundary value problems is obtained, and these solutions give the existence of solutions of the original boundary value problems.

1. Introduction

Consider the second-order boundary value problem

$$u'' = f(t, u, u'), \quad t \in (0, 1), \tag{1-1}$$

satisfying a combination of antiperiodic and periodic boundary conditions; either

$$u(0)+u(1)=0, \quad u'(0)-u'(1)=0. \tag{1-2}$$

or

$$u(0)-u(1)=0, \quad u'(0)+u'(1)=0. \tag{1-3}$$

Here we assume $f(t, x, y) : [0, 1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous in each of its variables.

Since the boundary value problem $u'' = 0$, (1-2) has the nontrivial solution $u(t) = t - \frac{1}{2}$, the problem (1-1), (1-2) is said to be at resonance. Similarly, since $u'' = 0$, (1-3) has the nontrivial solution $u(t) \equiv 1$, the problem (1-1), (1-3) is also at resonance. Hence, standard methods employing Green's functions cannot be used to show the existence of solutions of these boundary value problems directly. Thus, we consider a shifted boundary value problem so that Green's functions can be employed.

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Han [2007] employed a shift argument when studying a three-point boundary value problem

$$x''(t) = f(t, x(t)), \quad t \in (0, 1),$$
$$x'(0) = 0, \quad x(\eta) = x(1).$$

Here it was assumed $g(t, x) = f(t, x) + \beta^2 x$ and the equivalent boundary value problem

$$x''(t) + \beta^2 x(t) = g(t, x(t)), \quad x'(0) = 0, \ x(\eta) = x(1),$$

was studied using the Krasnosel'skii–Guo fixed point theorem [Krasnosel'skii 1964].

Infante, Pietramala, and Tojo [Infante et al. 2016] also employed a shift argument when studying Neumann boundary value problems at resonance

$$u''(t) + h(t, u(t)) = 0, \quad t \in (0, 1),$$
$$u'(0) = u'(1) = 0.$$

They assumed $f(t, u) = h(t, u) + \omega^2 u$ and considered the equivalent boundary value problem

$$-u''(t) + \omega^2 u(t) = f(t, u(t)), \quad u'(0) = u'(1) = 0.$$

The Krasnosel'skii–Guo fixed point theorem was also used in their analysis.

More recently, Almansour and Eloe [2015] and Al Mosa and Eloe [2016] studied two-point boundary value problems

$$y''(t) = f(t, y(t)), \quad t \in [0, 1],$$
$$y'(0) = 0, \quad y'(1) = 0,$$

and

$$y''(t) = f(t, y(t), y'(t)), \quad t \in [0, 1],$$
$$y(0) = 0, \quad y'(0) = y'(1),$$

using shift arguments and the Krasnosel'skii–Guo fixed point theorem, the Schauder fixed point theorem, the Leray–Schauder nonlinear alternative [Zeidler 1990] in the former, and monotone methods coupled with upper and lower solutions in the latter.

When considering the first boundary value problem, they assumed $g(t, y) = f(t, y) + \beta^2 y$ and studied the equivalent boundary value problem

$$y''(t) + \beta^2 y(t) = g(t, y(t)), \quad y'(0) = y'(1) = 0,$$

and when considering second, they assumed $g(t, x, y) = f(t, x, y) + \beta y$ and studied the equivalent boundary value problem

$$y''(t) + \beta y'(t) = g(t, y(t), y'(t)), \quad y(0) = 0, \ y'(0) = y'(1).$$

Here, we make use of two substitutions, one of which has not been used previously in the literature. In Section 2, we study solutions of (1-1), (1-2) by employing the substitution $g(t, x, y) := f(t, x, y) + \beta y$. The shifted boundary value problem is no longer at resonance, and so a Green's function can be constructed. An appropriate integral operator is defined and fixed point methods are used to show the existence of solutions. In Section 3, we study solutions of (1-1), (1-3). The substitutions mentioned above do not help because in both cases the shifted boundary value problem is still at resonance. Thus, we use the substitution $k(t, x, y) = f(t, x, y) + 2\alpha y + (\alpha^2 + \beta^2)x$. This substitution has not been used in the prior literature. A similar approach to that in Section 2 is then used to show existence of solutions. The construction of the two Green's functions and the shift employed in Section 3 can both lead to more research in this area.

2. Solutions of (1-1), (1-2)

Notice that for $\beta > 0$, $\beta \neq n\pi$, $n \in \mathbb{N}$, the boundary value problem $u'' + \beta^2 u = 0$, (1-2) is at resonance, since $u(t) = \cos\beta t - ((1+\cos\beta)/\sin\beta)\sin\beta t$ is a nontrivial solution. If $\beta = n\pi$, $n \in \mathbb{N}$, then $u(t) = \sin\beta t$ is a nontrivial solution of the boundary value problem. Thus the substitution $g(t, x, y) = f(t, x, y) + \beta^2 x$ cannot be applied.

Let $\beta > 0$ be a constant and assume $g(t, x, y) := f(t, x, y) + \beta y$. We study the shifted differential equation

$$u'' + \beta u' = g(t, u, u'), \quad t \in (0, 1), \tag{2-1}$$

satisfying boundary conditions (1-2). The boundary value problem (2-1), (1-2) is not at resonance, since the unique solution of $u'' + \beta u' = 0$, (1-2), is $u \equiv 0$. Notice if $u(t)$ is a solution of (2-1), (1-2), then

$$u''(t) = g(t, u(t), u'(t)) - \beta u'(t) = f(t, u(t), u'(t)),$$

implying u is a solution of (1-1), (1-2).

We first construct the Green's function associated with $u'' + \beta u' = 0$, (1-2).

Lemma 2.1. *Let $h(t)$ be a continuous function. Then $u(t)$ is the unique solution of the boundary value problem*

$$u'' + \beta u' = h(t), \quad t \in (0, 1), \tag{2-2}$$

satisfying boundary conditions (1-2) if and only if

$$u(t) = \int_0^1 G(t, s)\, h(s)\, ds,$$

where

$$G(t,s) = \frac{1}{2\beta(1-e^{-\beta})} \begin{cases} 2e^{-\beta(1-s)} - 2e^{-\beta} e^{-\beta(t-s)} + e^{-\beta} - 1, & 0 \le t \le s \le 1, \\ 2e^{-\beta(1-s)} - 2e^{-\beta(t-s)} - e^{-\beta} + 1, & 0 \le s \le t \le 1. \end{cases} \tag{2-3}$$

Proof. Using Laplace transforms, one can show the general solution of (2-2) is given by

$$u(t) = c_1 + c_2 e^{-\beta t} + \frac{1}{\beta}\int_0^t (1 - e^{-\beta(t-s)})\, h(s)\, ds.$$

Since $u'(0) - u'(1) = 0$, we have

$$-c_2\beta + c_2\beta e^{-\beta} - \int_0^1 e^{-\beta(1-s)}\, h(s)\, ds = 0.$$

Solving for c_2 gives

$$c_2 = -\frac{1}{\beta(1-e^{-\beta})}\int_0^1 e^{-\beta(1-s)}\, h(s)\, ds.$$

The boundary condition $u(0) + u(1) = 0$ gives

$$c_1 + c_2 + c_1 + c_2 e^{-\beta} + \frac{1}{\beta}\int_0^1 (1 - e^{-\beta(t-s)})\, h(s)\, ds = 0.$$

By substituting c_2 from above, solving for c_1, and simplifying, we have

$$c_1 = \frac{1}{2\beta(1-e^{-\beta})}\int_0^1 (-1 + e^{-\beta} + 2e^{-\beta(1-s)})\, h(s)\, ds.$$

Thus

$$\begin{aligned} u(t) &= \frac{1}{2\beta(1-e^{-\beta})}\int_0^1 (-1 + e^{-\beta} + 2e^{-\beta(1-s)})\, h(s)\, ds \\ &\qquad - \frac{e^{-\beta t}}{\beta(1-e^{-\beta})}\int_0^1 e^{-\beta(1-s)}\, h(s)\, ds + \frac{1}{\beta}\int_0^t (1 - e^{-\beta(t-s)})\, h(s)\, ds \\ &= \int_0^1 G(t,s)\, h(s)\, ds, \end{aligned}$$

where

$$G(t,s) = \begin{cases} \dfrac{-1+e^{-\beta}+2e^{-\beta(1-s)}}{2\beta(1-e^{-\beta})} - \dfrac{e^{-\beta t} e^{-\beta(1-s)}}{\beta(1-e^{-\beta})}, & 0 \le t \le s \le 1, \\ \dfrac{-(1-e^{-\beta})+2e^{-\beta(t-s)}}{2\beta(1-e^{-\beta})} - \dfrac{e^{-\beta t} e^{-\beta(1-s)}}{\beta(1-e^{-\beta})} + \dfrac{1-e^{-\beta(t-s)}}{\beta}, & 0 \le s \le t \le 1. \end{cases}$$

Simplifying $G(t,s)$ gives (2-3).

The reverse direction of the proof can be shown by direct computation. □

Notice that

$$\frac{\partial}{\partial t} G(t,s) = \frac{1}{1-e^{-\beta}} \begin{cases} e^{-\beta} e^{-\beta(t-s)}, & 0 \le t \le s \le 1, \\ e^{-\beta(t-s)}, & 0 \le s \le t \le 1. \end{cases} \tag{2-4}$$

We point out several properties of the Green's function.

Lemma 2.2. *$G(t,s)$ satisfies the following properties*:

(1) $G \in C([0,1]\times[0,1])$.

(2) $G(0,s) = -\dfrac{1}{2\beta} < 0$ *for all* $s \in [0,1]$.

(3) $G(1,s) = \dfrac{1}{2\beta} > 0$ *for all* $s \in [0,1]$.

(4) $\dfrac{\partial}{\partial t}G(t,s) > 0$ *for all* $(t,s) \in [0,1]\times[0,1]$.

(5) $\max_{t\in[0,1]} |G(t,s)| = \dfrac{1}{2\beta}$ *for all* $s \in [0,1]$.

(6) $\max\limits_{t\in[0,1]} \dfrac{\partial}{\partial t}G(t,s) \le \dfrac{1}{1-e^{-\beta}}$ *for all* $s \in [0,1]$.

(7) $\max\limits_{t\in[0,1]} \displaystyle\int_0^1 |G(t,s)|\,ds \le \dfrac{(4+\beta)e^{\beta}+\beta-4}{2\beta^2(e^{\beta}-1)}$.

(8) $\max\limits_{t\in[0,1]} \displaystyle\int_0^1 \dfrac{\partial}{\partial t}G(t,s)\,ds = \dfrac{1}{\beta}$.

All of these properties can be shown directly, so a proof is not given. We point out that property (8) is obtained by making all the terms in $G(t,s)$ positive, integrating, and finding an upper bound when $t \in [0,1]$.

We employ Schauder's fixed point theorem in our analysis. Because of the fact that $G(t,s)$ changes sign, many fixed point theorems using cones cannot be used.

Theorem 2.3 (Schauder fixed point theorem [Hale and Verduyn Lunel 1993]). *If $\mathcal{M}$ is a closed*, *bounded*, *convex subset of a Banach space $\mathcal{B}$ and $T:\mathcal{M}\to\mathcal{M}$ is completely continuous*, *then T has a fixed point in $\mathcal{M}$.*

Let $\mathcal{B} = C^{(1)}[0,1]$ be the Banach space of functions whose first derivatives are continuous endowed with the norm

$$\|u\| = \max\{|u|_0, |u'|_0\},$$

where $|u|_0 = \max_{t\in[0,1]}|u(t)|$. Let $M>0$. Define $\mathcal{M} = \{u \in \mathcal{B} : \|u\| \le M\}$. Notice that $\mathcal{M}$ is a closed, bounded, convex subset of $\mathcal{B}$.

Define the operator $T:\mathcal{B}\to\mathcal{B}$ by

$$Tu(t) = \int_0^1 G(t,s)\,g(s,u(s),u'(s))\,ds.$$

Thus if u is a fixed point of T, then u is a solution of (2-1), (1-2). A standard application of the Arzelà–Ascoli theorem gives us that T is completely continuous.

Define

$$\max_{t\in[0,1]}\int_0^1 |G(t,s)|\,ds := \overline{G} \quad\text{and}\quad \max_{t\in[0,1]}\int_0^1 \frac{\partial}{\partial t}G(t,s)\,ds := \overline{G}'.$$

Theorem 2.4. *Assume $f(t,x,y)$ is continuous in $[0,1]\times\mathbb{R}\times\mathbb{R}$ with*

$$|f(t,x,y)+\beta y| \le \min\left\{\frac{M}{\overline{G}}, \frac{M}{\overline{G'}}\right\}$$

for all $(t,x,y)\in[0,1]\times[-M,M]\times[-M,M]$. Then (1-1), (1-2) has a solution $u^*\in\mathcal{M}$.

Proof. Since $g(t,x,y)=f(t,x,y)+\beta y$,

$$|g(t,x,y)| \le \min\left\{\frac{M}{\overline{G}}, \frac{M}{\overline{G'}}\right\}$$

for all $(t,x,y)\in[0,1]\times[-M,M]\times[-M,M]$.

Now, for $u\in\mathcal{M}$,

$$|Tu(t)| \le \int_0^1 |G(t,s)|\,|g(s,u(s),u'(s))|\,ds \le \frac{M}{\overline{G}}\int_0^1 |G(t,s)|\,ds = M,$$

$$|(Tu)'(t)| \le \int_0^1 \frac{\partial}{\partial t}G(t,s)|g(s,u(s),u'(s))|\,ds \le \beta M\int_0^1 \frac{\partial}{\partial t}G(t,s)\,ds = M.$$

So $\|Tu\|\le M$, and $T:\mathcal{M}\to\mathcal{M}$. Thus T has a fixed point $u^*\in\mathcal{M}$ which is a solution of (2-1), (1-2). Therefore, u^* is a solution of (1-1), (1-2). □

Example 2.5. Define

$$f(t,x,y)=\frac{5x^2t^2}{y^2+2}-5y.$$

Let $\beta=5$. Then from Lemma 2.2

$$\min\left\{\frac{M}{\overline{G}}, \frac{M}{\overline{G'}}\right\} \le \min\left\{\frac{2\beta^2(e^\beta-1)}{(4+\beta)e^\beta+\beta-4}M, \beta M\right\} = 5M.$$

So

$$|f(t,x,y)+5y| = \frac{5x^2t^2}{y^2+2} \le 5M^2 \le 5M$$

if $M\le 1$. So the boundary value problem

$$u''=\frac{5u^2t^2}{(u')^2+2}-5u', \quad t\in(0,1),$$
$$u(0)+u(1)=0, \quad u'(0)-u'(1)=0,$$

has a solution u^* with $\|u^*\|\le 1$.

3. Solutions of (1-1), (1-3)

For $\beta > 0$, the boundary value problem $u'' + \beta^2 u = 0$, (1-3) is at resonance, since

$$u(t) = \cos\beta t - \left(\frac{1-\cos\beta}{\sin\beta}\right)\sin\beta t$$

gives a nontrivial solution. If $\beta = n\pi$, $n \in \mathbb{N}$, then $u(t) = \cos\beta t$ is a nontrivial solution of the boundary value problem. Thus the substitution $k(t, x, y) = f(t, x, y) + \beta^2 x$ cannot be applied. Also, the boundary value problem $u'' + \beta u' = 0$, (1-3) is at resonance, since $u(t) \equiv 1$ gives a nontrivial solution. This implies the substitution $k(t, x, y) = f(t, x, y) + \beta^2 y$ cannot be used. Thus, neither substitution used in previous literature can be employed.

Let $\alpha > 0$, $\beta \in \left(0, \frac{\pi}{2}\right)$ and define

$$k(t, x, y) = f(t, x, y) + 2\alpha y + (\alpha^2 + \beta^2)x.$$

Here we consider the equivalent boundary value problem

$$u'' + 2\alpha u' + (\alpha^2 + \beta^2)u = k(t, u, u'), \quad t \in (0, 1), \tag{3-1}$$

satisfying boundary conditions (1-3), which is not at resonance, since the unique solution of $u'' + 2\alpha u' + (\alpha^2 + \beta^2)u = 0$, (1-3) is $u \equiv 0$. If u is a solution of (3-1), (1-3), then u is a solution of (1-1), (1-3).

Again, we construct a corresponding Green's function.

Lemma 3.1. *The unique solution of*

$$u'' + 2\alpha u' + (\alpha^2 + \beta^2)u = h(t), \quad t \in (0, 1), \tag{3-2}$$

satisfying the boundary conditions (1-3) *is given by*

$$u(t) = \int_0^1 H(t, s)\, h(s)\, ds,$$

where

$$H(t, s) = \frac{1}{2\beta(\beta\sinh\alpha - \alpha\sin\beta)}\Psi(t, s), \tag{3-3}$$

with

$$\Psi(t,s) = \begin{cases} e^{-\alpha(t-s)}\big[-\beta e^{-\alpha}\sin(\beta(s-t)) + 2\alpha\sin(\beta(1-s))\sin(\beta t) \\ \qquad -\beta\sin(\beta t)\cos(\beta(1-s)) + \beta\cos(\beta t)\sin(\beta(1-s))\big], & 0 \le t \le s \le 1, \\ e^{-\alpha(t-s)}\big[\beta e^{\alpha}\sin(\beta(t-s)) + 2\alpha\sin(\beta s)\sin(\beta(1-t)) \\ \qquad -\beta\sin(\beta s)\cos(\beta(1-t)) + \beta\cos(\beta s)\sin(\beta(1-t))\big], & 0 \le s \le t \le 1. \end{cases}$$

Proof. If u satisfies (3-2), then, using Laplace transforms,

$$u(t) = e^{-\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t)) + \frac{1}{\beta}\int_0^t (e^{-\alpha(t-s)}\sin(\beta(t-s)))\, h(s)\, ds.$$

Solving the system $u(0)-u(1)=0,\ u'(0)+u'(1)=0$ gives

$$c_1=-\frac{1}{2\alpha e^{-\alpha}\sin(\beta)+\beta e^{-2\alpha}-\beta}\times\left[\int_0^1[e^{-\alpha(1-s)}\sin(\beta(1-s))-e^{-\alpha(2-s)}\sin(\beta s)]h(s)\,ds\right],$$

$$\begin{aligned}c_2=-&\frac{1}{\beta e^{-\alpha}\sin(\beta)(2\alpha e^{-\alpha}\sin(\beta)+\beta e^{-2\alpha}-\beta)}\\ &\times\left[\int_0^1\Big[\beta e^{-\alpha(3-s)}[\cos(\beta)\sin(\beta s)+\sin(\beta(1-s))]\right.\\ &\qquad -e^{-\alpha(2-s)}\big[\beta\cos(\beta)\sin(\beta(1-s))\\ &\qquad\qquad \left.+\beta\sin(\beta s)-2\alpha\sin(\beta)\sin(\beta(1-s))\big]\Big]h(s)\,ds\right]\end{aligned}$$

The Green's function given in (3-3) can then be obtained. □

Notice

$$\frac{\partial}{\partial t}H(t,s)=\frac{1}{2\beta(\beta\sinh\alpha-\alpha\sin\beta)}\Phi(t,s), \tag{3-4}$$

where

$$\Phi(t,s)=\begin{cases} e^{-\alpha(t-s)}\big[e^{-\alpha}\beta^2\cos(\beta(s-t))+2\alpha\beta\sin(\beta(1-s))\cos(\beta t)\\ \qquad -\beta^2\sin(\beta(1-s))\sin(\beta t)-\beta^2\cos(\beta(1-s))\cos(\beta t)\big]\\ -\alpha e^{-\alpha(t-s)}\big[2\alpha\sin(\beta(1-s))\sin(\beta t)-e^{-\alpha}\beta\sin(\beta(s-t))\\ \qquad +\beta\sin(\beta(1-s))\cos(\beta t)-\beta\cos(\beta(1-s))\sin(\beta t)\big], & 0\le t\le s\le 1,\\ e^{-\alpha(t-s)}\big[e^{\alpha}\beta^2\cos(\beta(t-s))-2\alpha\beta\sin(\beta s)\cos(\beta(1-t))\\ \qquad -\beta^2\sin(\beta s)\sin(\beta(1-t))-\beta^2\cos(\beta s)\cos(\beta(1-t))\big]\\ -\alpha e^{-\alpha(t-s)}\big[2\alpha\sin(\beta s)\sin(\beta(1-t))+e^{\alpha}\beta\sin(\beta(t-s))\\ \qquad -\beta\sin(\beta s)\cos(\beta(1-t))+\beta\cos(\beta s)\sin(\beta(1-t))\big], & 0\le s\le t\le 1.\end{cases}$$

We point out several properties of the Green's function.

Lemma 3.2. *$H(t,s)$ satisfies the following properties*:

(1) $H\in C([0,1]\times[0,1])$.

(2) $H(0,s)=H(1,s)=\dfrac{e^{\alpha s}(\beta\sin(\beta(1-s))-e^{-\alpha}\beta\sin(\beta s))}{2\beta(\beta\sinh\alpha-\alpha\sin\beta)}$ *for all* $s\in[0,1]$.

(3) $\max\limits_{t\in[0,1]}|H(t,s)|\le\dfrac{\beta e^{\alpha}+2\alpha+2\beta}{2\beta(\beta\sinh\alpha-\alpha\sin\beta)}$ *for all* $s\in[0,1]$.

(4) $\max\limits_{t\in[0,1]}\left|\dfrac{\partial}{\partial t}H(t,s)\right|\le\dfrac{\alpha\beta e^{\alpha}+2\alpha^2+2\beta^2+2\alpha\beta+\beta^2e^{\alpha}}{2\beta(\beta\sinh\alpha-\alpha\sin\beta)}$ *for all* $s\in[0,1]$.

(5) $\max\limits_{t\in[0,1]}\displaystyle\int_0^1|H(t,s)|\,ds\le\dfrac{\beta+\beta\sinh\alpha}{(\alpha^2+\beta^2)(\beta\sinh\alpha-\alpha\sin\beta)}$.

(6) $$\max_{t\in[0,1]}\int_0^1\left|\frac{\partial}{\partial t}H(t,s)\right|ds \le \frac{\alpha^2e^{\alpha}+\alpha^2+\beta^2+\beta^2e^{\alpha}+\alpha e^{\alpha}+\alpha\beta+3\beta}{(\alpha^2+\beta^2)(\beta\sinh\alpha-\alpha\sin\beta)}.$$

Again, a proof is not given, since all these properties can be verified directly. Properties (5) and (6) are obtained by making all the terms in $H(t,s)$ and $(\partial/\partial t)H(t,s)$, respectfully, positive, integrating, and finding an upper bound when $t\in[0,1]$.

Define the operator $T:\mathcal{B}\to\mathcal{B}$ by

$$Tu(t)=\int_0^1 H(t,s)\,k(s,u(s),u'(s))\,ds.$$

Thus if u is a fixed point of T, then u is a solution of (3-1), (1-3). A standard application of the Arzelà–Ascoli theorem gives us that T is completely continuous.

Define

$$\max_{t\in[0,1]}\int_0^1|H(t,s)|\,ds:=\overline{H}\quad\text{and}\quad\max_{t\in[0,1]}\int_0^1\left|\frac{\partial}{\partial t}H(t,s)\right|ds:=\overline{H}'.$$

Theorem 3.3. *Assume $f(t,x,y)$ is continuous in $[0,1]\times\mathbb{R}\times\mathbb{R}$ with*

$$\left|f(t,x,y)+2\alpha y+(\alpha^2+\beta^2)x\right|\le\min\left\{\frac{M}{\overline{H}},\frac{M}{\overline{H}'}\right\}$$

for all $(t,x,y)\in[0,1]\times[-M,M]\times[-M,M]$. Then (1-1), (1-3) has a solution $u^*\in\mathcal{M}$.

The proof is similar to the proof of Theorem 2.4 and is therefore omitted.

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