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Sign pattern matrices that allow inertia $\mathbb{S}_n$

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Sign pattern matrices of order n that allow inertias in the set $\mathbb{S}_n$ are considered. All sign patterns of order 3 (up to equivalence) that allow $\mathbb{S}_3$ are classified and organized according to their associated directed graphs. Furthermore, a minimal set of such matrices is found. Then, given a pattern of order n that allows $\mathbb{S}_n$, a construction is given that generates families of irreducible sign patterns of order $n+1$ that allow $\mathbb{S}_{n+1}$.

1. Introduction

The *inertia* of a real matrix A of order n is an ordered triple $\mathrm{i}(A) = (n_+, n_-, n_0)$ of nonnegative integers summing to n, where n_+, n_-, n_0 are the number of eigenvalues of A with positive, negative, and zero real parts, respectively.

A *sign pattern matrix* is a matrix $\mathcal{A}$ of order n with entries in $\{+, -, 0\}$. The set $Q(\mathcal{A})$ denotes the set of all real-valued matrices A with corresponding sign pattern $\mathcal{A}$. Alternatively, we say that $A \in Q(\mathcal{A})$ is a *realization* of $\mathcal{A}$. If $\mathcal{A}$ is a sign pattern of order n, then we say that $\mathcal{A}$ has inertia $\mathrm{i}(\mathcal{A}) = \{\mathrm{i}(A) : A \in Q(\mathcal{A})\}$.

In a dynamical system, the presence of a zero eigenvalue of the Jacobian matrix at an equilibrium may signal onset of instability. Varying a parameter may move eigenvalues from all having negative real parts to having a simple zero eigenvalue, which then moves to have a positive real part, while the other eigenvalues maintain negative real parts. Thus, the inertia begins at $(0, n, 0)$, and with parameter variation, changes to $(0, n-1, 1)$ and then finally to $(1, n-1, 0)$.

With this motivation in mind, the inertia set $\mathbb{S}_n$ (for $n \geq 2$) is defined as

$$\mathbb{S}_n = \{(0, n, 0), (0, n-1, 1), (1, n-1, 0)\}.$$

We are particularly interested in studying irreducible sign patterns that *allow* $\mathbb{S}_n$, i.e., $\mathbb{S}_n \subseteq \mathrm{i}(\mathcal{A}_n)$.

Introduced in [Bodine et al. 2012], the *refined inertia* of a matrix $\mathcal{A}$ is the 4-tuple $\mathrm{ri}(A) = (n_+, n_-, n_z, 2n_p)$, where n_+, n_- are defined as before, n_z is the number

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of zero eigenvalues, and $2n_p$ is the number of nonzero pure imaginary eigenvalues. Using the notation of refined inertia, $\mathbb{S}_n = \{(0,n,0,0),(0,n-1,1,0),(1,n-1,0,0)\}$. Several results regarding other sets of refined inertias can be found in [Gao et al. 2016a; 2016b; Garnett et al. 2013; 2014]. Many similar techniques and ideas are used in this paper.

For simplicity, we identify sign patterns up to *equivalence*. Any combination of transposition, permutation similarity, and signature similarity leaves the eigenvalues of a matrix unchanged. For our purposes, it is convenient to organize sign patterns by their *associated digraph*. For $\mathcal{A} = [\alpha_{ij}]$ (or a realization $A = [a_{ij}]$) of order n, its associated digraph $D(\mathcal{A})$ is a directed graph on n vertices where there is an arc from vertex i to vertex j if and only if $\alpha_{ij} \neq 0$. Two digraphs are equivalent if and only if their associated zero-nonzero patterns are equivalent via transposition and/or permutation similarity.

In order for a sign pattern $\mathcal{A}$ to be irreducible, the associated digraph $D(\mathcal{A})$ must be strongly connected. A sign pattern $\mathcal{A}$ is *sign singular* if $n_0 > 0$ for all $A \in \mathcal{Q}(\mathcal{A})$ and is *sign-nonsingular* if $n_0 = 0$ for all $A \in \mathcal{Q}(\mathcal{A})$. Thus, in order for $\mathcal{A}$ to allow $\mathbb{S}_n$, $\mathcal{A}$ can neither be sign singular nor sign-nonsingular. In particular, this means that the determinant expansion of $\mathcal{A}$ must have at least two nonzero terms. A nonzero term in the determinant expansion of $\mathcal{A}$ corresponds to the existence of a *generalized n-cycle* in the associated digraph $D(\mathcal{A})$ (that is, a disjoint collection of cycles that use all n vertices of $D(\mathcal{A})$). Furthermore, any sign pattern $\mathcal{A}$ where $\mathrm{i}(A) = (0,n,0)$ must have at least one negative diagonal entry. Thus, for our purposes, we need only consider strongly connected digraphs that contain at least one loop and at least two generalized n-cycles.

For $n = 2$, there are two nonequivalent sign patterns that allow $\mathbb{S}_2$, namely $\left[\begin{smallmatrix} - & - \\ - & - \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} - & - \\ + & + \end{smallmatrix}\right]$ (see [Olesky et al. 2013]). The first sign pattern requires $\mathbb{S}_2$. The second pattern attains every possible spectrum allowed by a real matrix, and such a pattern is called *spectrally arbitrary*. A sign pattern $\widehat{\mathcal{A}}$ is a *superpattern* of $\mathcal{A}$ if $\mathcal{A}$ can be obtained from $\widehat{\mathcal{A}}$ by changing any number of nonzero entries to 0. In [Berliner et al. 2018], sufficient conditions for a sign pattern and all of its superpatterns to allow $\mathbb{S}_n$ are given. Suppose $A = [a_{ij}]$ is a real matrix of order n having $m \geq n$ nonzero entries and $\mathrm{i}(A) = (0, n-1, 1)$. If the m nonzero entries $a_{i_1,j_1}, \ldots, a_{i_m,j_m}$ are replaced by variables $x_1, \ldots, x_m$ to obtain the matrix X, the characteristic polynomial of X is

$$c_X(z) = z^n + p_1 z^{n-1} + \cdots + p_{n-1} z + p_n,$$

with coefficients $p_1, \ldots, p_n$ depending on $x_1, \ldots, x_m$. The $n \times m$ *Jacobian matrix* J of A has (i, j)-entry equal to $\partial p_i(x_1, \ldots, x_m)/\partial x_j$ evaluated at $(x_1, \ldots, x_m) = (a_{i_1,j_1}, \ldots, a_{i_m,j_m})$. If J has rank n, then A *allows a Jacobian matrix of full rank*. This definition, which uses a rectangular Jacobian matrix as in [Garnett and Shader

2013], is equivalent to the determinantal property that A "allows a nonzero Jacobian" as defined in [Cavers and Vander Meulen 2005]. The following theorem is proved in [Berliner et al. 2018, Theorem 2.2].

Theorem 1.1. *Let $\mathcal{A}$ be an $n \times n$ sign pattern that allows inertia $(0, n-1, 1)$ and let $A \in \mathcal{Q}(\mathcal{A})$ with $\mathrm{i}(A) = (0, n-1, 1)$. If A allows a Jacobian matrix of full rank, then every superpattern $\widehat{\mathcal{A}}$ of $\mathcal{A}$ (including $\mathcal{A}$ itself) allows $\mathbb{S}_n$.*

In Section 2, we classify all nonequivalent sign patterns of order 3 that allow $\mathbb{S}_3$. In Section 3, we give a construction that, using a sign pattern of order m that allows $\mathbb{S}_m$, creates sign patterns of any order $n > m$ that allow $\mathbb{S}_n$. This construction allows us to use the sign patterns of order 3 that allow $\mathbb{S}_3$ to create sign patterns of order $n > 3$ that allow $\mathbb{S}_n$.

2. Sign patterns allowing $\mathbb{S}_3$

In this section, we classify all sign patterns of order 3 that allow $\mathbb{S}_3$. First, we may restrict our attention to sign patterns $\mathcal{A}$ whose associated digraph $D(\mathcal{A})$ is strongly connected, has at least one loop, and contains two or more generalized 3-cycles. Without loops included, there are only five nonequivalent strongly connected digraphs of order 3, as shown in Figure 1. For these, we use the same digraph labeling as in [Berliner et al. 2017] (up to equivalence). Adding loops in and enforcing the generalized 3-cycle requirement, we then focus solely on sign patterns associated with the looped digraphs described in Table 1 (again up to equivalence).

If $\mathcal{A}$ is a sign pattern of order 3 having a realization with inertia $(0, 2, 1)$ that allows a Jacobian of full rank, then by Theorem 1.1 any superpattern of $\mathcal{A}$ will

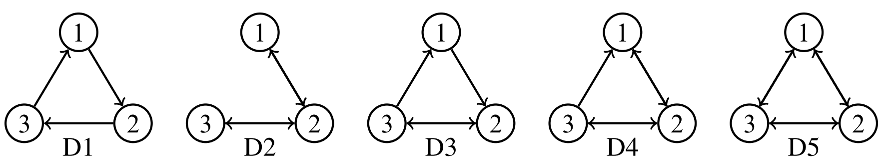

Figure 1. Strongly connected digraphs of order 3.

strongly connected digraph	nonequivalent loop locations
D1	123
D2	13, 123
D3	1, 13, 123
D4	1, 12, 13, 123
D5	1, 13, 123

Table 1. Nonequivalent strongly connected digraphs with two or more generalized 3-cycles.

$$\begin{bmatrix} - & + & 0 \\ 0 & - & + \\ + & 0 & - \end{bmatrix} \begin{bmatrix} + & - & 0 \\ 0 & - & - \\ - & 0 & - \end{bmatrix} \begin{bmatrix} - & - & 0 \\ - & 0 & - \\ 0 & + & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix} \begin{bmatrix} - & - & 0 \\ - & - & - \\ 0 & - & - \end{bmatrix} \begin{bmatrix} - & - & 0 \\ - & - & - \\ 0 & + & + \end{bmatrix}$$

$$\begin{bmatrix} - & - & 0 \\ + & + & - \\ 0 & + & - \end{bmatrix} \begin{bmatrix} + & - & 0 \\ + & - & - \\ 0 & - & + \end{bmatrix} \begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ + & - & 0 \end{bmatrix} \begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ - & + & - \end{bmatrix} \begin{bmatrix} + & + & 0 \\ 0 & 0 & - \\ + & + & - \end{bmatrix} \begin{bmatrix} - & - & 0 \\ + & 0 & + \\ + & + & 0 \end{bmatrix} \begin{bmatrix} + & + & 0 \\ - & - & + \\ + & + & 0 \end{bmatrix}$$

Figure 2. $\mathbb{S}_3$-minimal sign patterns.

automatically allow $\mathbb{S}_3$. Thus, we will focus on *$\mathbb{S}_3$-minimal sign patterns*, i.e., patterns having a realization with inertia $(0,2,1)$ that allows a Jacobian of full rank that are not superpatterns of a pattern having a realization with inertia $(0,2,1)$ that allows a Jacobian of full rank.

Of the 200 nonequivalent sign patterns of order 3, 111 allow $\mathbb{S}_3$ and 13 of these are $\mathbb{S}_3$-minimal sign patterns (see Figure 2). Of the $\mathbb{S}_3$-minimal sign patterns, two have associated digraph D1, six are associated to D2, three are associated to D3, and two are associated to D4. All other nonequivalent sign patterns of order 3 that allow $\mathbb{S}_3$ are equivalent to a superpattern of one of these 13, and thus automatically allow $\mathbb{S}_3$. These superpatterns can be found in Appendix A.

Below, we illustrate the method for one of the 13 $\mathbb{S}_3$-minimal sign patterns.

Example 2.1. The sign pattern $\mathcal{A}$ and a realization $A \in Q(\mathcal{A})$ (with associated digraph D2) are given by

$$\mathcal{A} = \begin{bmatrix} - & - & 0 \\ - & 0 & - \\ 0 & + & - \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -1 & -1 & 0 \\ -a & 0 & -1 \\ 0 & b & -c \end{bmatrix},$$

with $a, b, c > 0$. Without loss of generality, one diagonal entry and two other entries in the digraph associated with $\mathcal{A}$ (corresponding to a spanning tree) can be set equal to ± 1. The characteristic polynomial of A is

$$c_A(z) = z^3 + (1+c)z^2 + (b+c-a)z + (b-ac).$$

In order to realize inertia $(0,2,1)$, we must have $b = ac$. If $a = 1$ and $b = c = 2$, then $\mathrm{i}(A) = (0,2,1)$, as desired. We now check if A allows a nonzero Jacobian. We replace the nonzero entries of A with variables to get

$$X_A = \begin{bmatrix} x_1 & x_2 & 0 \\ x_3 & 0 & x_4 \\ 0 & x_5 & x_6 \end{bmatrix},$$

which has characteristic polynomial

$$c_{X_A}(z) = z^3 - (x_1 + x_6)z^2 + (x_1x_6 - x_2x_3 - x_4x_5)z + (x_1x_4x_5 + x_2x_3x_6).$$

Calculating the Jacobian matrix of X_A, we get

$$J_{X_A} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ x_6 & -x_3 & -x_2 & -x_5 & -x_4 & x_1 \\ x_4x_5 & x_3x_6 & x_2x_6 & x_1x_5 & x_1x_4 & x_2x_3 \end{bmatrix},$$

which when evaluated at $x_1 = x_2 = x_3 = x_4 = -1,\ x_5 = 2,\ x_6 = -2$ has full rank. By Theorem 1.1, $\mathcal{A}$ and all of its superpatterns allow $\mathbb{S}_3$.

The other 89 nonequivalent sign patterns do not allow $\mathbb{S}_3$. Several (57) of these sign patterns do not allow $\mathbb{S}_3$ because they are sign-nonsingular and cannot possibly allow the inertia $(0, n-1, 1)$. These patterns, for $n = 3$, can be found in Appendix B.1. The 32 remaining patterns (see Appendix B.2) are not sign-nonsingular, but nonetheless do not allow inertia $(0, 2, 1)$ for other algebraic reasons. Here, we illustrate the method for one of the sign patterns that is not sign-nonsingular yet does not allow inertia $(0, 2, 1)$.

Example 2.2. The sign pattern $\mathcal{A}$ and a realization $A \in Q(\mathcal{A})$ (with associated digraph D3) are given by

$$\mathcal{A} = \begin{bmatrix} - & - & 0 \\ 0 & 0 & + \\ + & + & + \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix},$$

with $a, b, c > 0$. Without loss of generality, one diagonal entry and two entries on the 3-cycle in the digraph associated with $\mathcal{A}$ can be set equal to ± 1. The characteristic polynomial of A is

$$c_A(z) = z^3 + (1-c)z^2 + (-b-c)z + (a-b).$$

Since $b, c > 0$, it must be the case that $-b-c < 0$. However, in order to allow inertia $(0, 2, 1)$, the quadratic and linear coefficients of $c_A(z)$ must be able to be simultaneously positive. Thus $\mathcal{A}$ does not allow $\mathbb{S}_3$.

3. The Jacobian and patterns of higher order

We begin with a sign pattern of order n that allows $\mathbb{S}_n$ and give a construction that yields a sign pattern that allows $\mathbb{S}_{n+1}$. If $\mathcal{A}$ is a sign pattern of order n, we consider the $(n+1) \times (n+1)$ sign pattern

$$\mathcal{A}^- = \left[\begin{array}{c|c} \mathcal{A} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \ \cdots \ 0 & - \end{array}\right].$$

Then, $(n_+, n_-, n_0) \in \mathrm{i}(\mathcal{A})$ if and only if $(n_+, n_- + 1, n_0) \in \mathrm{i}(\mathcal{A}^-)$. It follows that $\mathcal{A}^-$ allows $\mathbb{S}_{n+1}$ if and only if $\mathcal{A}$ allows $\mathbb{S}_n$. An analogous result holds for n_+ if we create the $(n+1)\times(n+1)$ sign pattern $\mathcal{A}^+$ by replacing the lower-right corner entry of $\mathcal{A}^-$ by $+$.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{A}^-$ be defined as above, where $\mathcal{A}$ has at least n nonzero entries. If A is a realization of $\mathcal{A}$ that allows a Jacobian of full rank for which* $\mathrm{i}(A)=(n_+,n_-,n_0)$, *then there exists a realization B of $\mathcal{A}^-$ that allows a Jacobian of full rank and* $\mathrm{i}(B)=(n_+,1+n_-,n_0)$.

Proof. Let A be a realization of $\mathcal{A}$ that has inertia (n_+,n_-,n_0) and allows a Jacobian of full rank. Then, replacing the $m\ge n$ nonzero entries $a_{i_1,j_1},\dots,a_{i_m,j_m}$ of A with variables $x_1,\dots,x_m$, the characteristic polynomial is

$$p_A(z)=z^n+p_1z^{n-1}+\cdots+p_{n-1}z+p_n,$$

where $p_1,\dots,p_n$ are functions of $x_1,\dots,x_m$. Furthermore, we know the matrix $J_{X_A}=[\partial p_i/\partial x_j]$ has full rank when evaluated at $(x_1,\dots,x_m)=(a_{i_1,j_1},\dots,a_{i_m,j_m})$.

In order to obtain the Jacobian matrix for B, we replace the lower-right corner entry with variable $\hat{x}$ and the other m entries with the same variables $x_1,\dots,x_m$ as with A. Thus, we obtain the characteristic polynomial

$$\begin{aligned} p_B(z)&=p_A(z)(z-\hat{x})=(z^n+p_1z^{n-1}+\cdots+p_{n-1}z+p_n)(z-\hat{x})\\ &=z^{n+1}+(p_1-\hat{x})z^n+(p_2-\hat{x}p_1)z^{n-1}+\cdots+(p_n-\hat{x}p_{n-1})z-\hat{x}p_n \end{aligned}$$

and we have

$$J_{X_B}=\begin{bmatrix} \frac{\partial p_1}{\partial x_1} & \cdots & \frac{\partial p_1}{\partial x_m} & -1\\ \frac{\partial p_2}{\partial x_1}-\hat{x}\frac{\partial p_1}{\partial x_1} & \cdots & \frac{\partial p_2}{\partial x_m}-\hat{x}\frac{\partial p_1}{\partial x_m} & -p_1\\ \vdots & & \vdots & \vdots\\ \frac{\partial p_n}{\partial x_1}-\hat{x}\frac{\partial p_{n-1}}{\partial x_1} & \cdots & \frac{\partial p_n}{\partial x_m}-\hat{x}\frac{\partial p_{n-1}}{\partial x_m} & -p_{n-1}\\ -\hat{x}\frac{\partial p_n}{\partial x_1} & \cdots & -\hat{x}\frac{\partial p_n}{\partial x_m} & -p_n \end{bmatrix}.$$

We sequentially perform $n-1$ row operations on J_{X_B}, where the i-th row operation adds $\hat{x}$ times row i to row $i+1$. The resulting matrix is

$$J=\begin{bmatrix} \frac{\partial p_1}{\partial x_1} & \cdots & \frac{\partial p_1}{\partial x_m} & -1\\ \frac{\partial p_2}{\partial x_1} & \cdots & \frac{\partial p_2}{\partial x_m} & (-p_1-\hat{x})\\ \vdots & & \vdots & \vdots\\ \frac{\partial p_n}{\partial x_1} & \cdots & \frac{\partial p_n}{\partial x_m} & (-p_{n-1}-\hat{x}p_{n-2}-\cdots-\hat{x}^{n-2}p_1-\hat{x}^{n-1})\\ 0 & \cdots & 0 & (-p_n-\hat{x}p_{n-1}-\cdots-\hat{x}^{n-1}p_1-\hat{x}^n) \end{bmatrix},$$

which has the same rank as J_{X_B}. The leading principal $n \times n$ submatrix of J has full rank. Furthermore, if we substitute the original values corresponding to the entries of A, the $(n+1, n+1)$-entry is a degree-n real polynomial in $\hat{x}$. Thus, there must exist $b > 0$ such that, if $\hat{x} = -b$, this entry is nonzero. Therefore, J has rank $n+1$ after this evaluation. Since $\mathrm{i}(A) = (n_+, n_-, n_0)$, it follows that

$$B = \left[\begin{array}{ccc|c} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & -b \end{array}\right]$$

is a realization of $\mathcal{A}^-$ that allows a Jacobian of full rank and $\mathrm{i}(B) = (n_+, 1+n_-, n_0)$. □

Combining this result with Theorem 1.1, we obtain the following:

Corollary 3.2. *Let $\mathcal{A}$ and $\mathcal{A}^-$ be defined as above. If $\mathcal{A}$ allows $\mathbb{S}_n$ and has a realization A that allows a Jacobian of full rank and* $\mathrm{i}(A) = (0, n-1, 1)$, *then every superpattern of $\mathcal{A}^-$ (including $\mathcal{A}^-$ itself) allows $\mathbb{S}_{n+1}$.*

Although $\mathcal{A}^-$ is a reducible matrix, adding at least one additional nonzero entry in the last row and last column of $\mathcal{A}^-$ will yield irreducible patterns of order $n+1$ that allow $\mathbb{S}_{n+1}$. We may repeatedly apply this construction to any matrix that allows $\mathbb{S}_m$ to create large families of irreducible sign patterns that allow $\mathbb{S}_n$ for $n > m$. Below is an example of a family created in such a way. In particular, all $\mathbb{S}_3$-minimal sign patterns were found in Section 2, and all such patterns have a realization A that allows a Jacobian of full rank for which $\mathrm{i}(A) = (0, 2, 1)$. Thus, many families may be created using these patterns as the starting point.

Example 3.3. Using the notation and results of [Berliner et al. 2018], the zero-nonzero pattern

$$\begin{bmatrix} * & * & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & 0 & 0 \end{bmatrix},$$

which corresponds to the digraph G16 with a loop at vertex 1, allows $\mathbb{S}_4$. In fact, there is a corresponding sign pattern $\mathcal{A}$ that allows $\mathbb{S}_4$. The sign pattern $\mathcal{A}$ and a realization $A \in Q(\mathcal{A})$ are given by

$$\mathcal{A} = \begin{bmatrix} - & + & 0 & - \\ 0 & 0 & + & 0 \\ 0 & - & 0 & + \\ + & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -1 & 1 & 0 & -a \\ 0 & 0 & 1 & 0 \\ 0 & -b & 0 & 1 \\ c & 0 & 0 & 0 \end{bmatrix},$$

with $a, b, c > 0$. Without loss of generality, one diagonal entry and two other entries in the digraph associated with $\mathcal{A}$ (corresponding to a spanning tree) can be set equal to ± 1. The characteristic polynomial of A is

$$c_A(z) = z^4 + z^3 + (b + ac)z^2 + bz + (abc - c).$$

In order to realize inertia $(0, 3, 1)$, we must have $c = abc$. If $a = b = c = 1$, then $\mathrm{i}(A) = (0, 3, 1)$, as desired. We now check if $\mathcal{A}$ allows a nonzero Jacobian. We replace the nonzero entries of $\mathcal{A}$ with variables to get

$$X_{\mathcal{A}} = \begin{bmatrix} x_1 & x_2 & 0 & x_3 \\ 0 & 0 & x_4 & 0 \\ 0 & x_5 & 0 & x_6 \\ x_7 & 0 & 0 & 0 \end{bmatrix},$$

which has characteristic polynomial

$$c_{X_{\mathcal{A}}}(z) = z^4 - x_1 z^3 - (x_3x_7 + x_4x_5)z^2 + (x_1x_4x_5)z - (x_2x_4x_6x_7 + x_3x_4x_5x_7).$$

Calculating the Jacobian matrix of $X_{\mathcal{A}}$, we get

$$J_{X_{\mathcal{A}}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_7 & -x_5 & -x_4 & 0 & -x_3 \\ x_4x_5 & 0 & 0 & x_1x_5 & x_1x_4 & 0 & 0 \\ 0 & -x_4x_6x_7 & x_4x_5x_7 & x_3x_5x_7 - x_2x_6x_7 & x_3x_4x_7 & -x_2x_4x_7 & x_3x_4x_5 - x_2x_4x_6 \end{bmatrix},$$

which when evaluated at $x_1 = x_3 = x_5 = -1$, $x_2 = x_4 = x_6 = x_7 = 1$ has full rank. By Corollary 3.2, any $n \times n$ sign pattern ($n \geq 4$) of the form

$$\begin{bmatrix} - & + & 0 & - & \oplus & \cdots & \cdots & \oplus \\ 0 & 0 & + & 0 & \oplus & & & \vdots \\ 0 & - & 0 & + & \oplus & & & \vdots \\ + & 0 & 0 & 0 & \oplus & & & \vdots \\ \oplus & \oplus & \oplus & \oplus & - & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & \oplus \\ \oplus & \cdots & \cdots & \cdots & \cdots & \cdots & \oplus & - \end{bmatrix}$$

allows $\mathbb{S}_n$ (where the $\oplus$ entries may be any of $+$, $-$, or 0).

Appendix A: Nonequivalent superpatterns of $\mathbb{S}_3$-minimal sign patterns

The following sign patterns are the nonequivalent superpatterns of the $\mathbb{S}_3$-minimal sign patterns in Figure 2. All sign patterns with associated digraph D1 that allow

$\mathbb{S}_3$ are $\mathbb{S}_3$-minimal. Thus, the patterns here are organized into four groups corresponding to their associated (loopless) digraph D2–D5. The use of the symbol $\pm$ indicates that a particular entry could be either $-$ or $+$.

Sign patterns with associated digraph D2:

$$\begin{bmatrix}- & - & 0\\ - & - & -\\ 0 & + & -\end{bmatrix} \begin{bmatrix}- & + & 0\\ - & + & +\\ 0 & + & -\end{bmatrix} \begin{bmatrix}- & + & 0\\ - & - & +\\ 0 & - & +\end{bmatrix} \begin{bmatrix}+ & - & 0\\ + & + & -\\ 0 & + & -\end{bmatrix}$$

Sign patterns with associated digraph D3:

$$\begin{bmatrix}- & + & 0\\ 0 & 0 & +\\ + & - & \pm\end{bmatrix} \begin{bmatrix}- & + & 0\\ 0 & - & +\\ \pm & + & -\end{bmatrix} \begin{bmatrix}- & + & 0\\ 0 & \pm & +\\ + & - & -\end{bmatrix} \begin{bmatrix}- & + & 0\\ 0 & + & +\\ + & - & +\end{bmatrix} \begin{bmatrix}+ & + & 0\\ 0 & - & -\\ + & + & \pm\end{bmatrix} \begin{bmatrix}- & - & 0\\ 0 & + & +\\ + & + & -\end{bmatrix} \begin{bmatrix}- & + & 0\\ 0 & + & -\\ + & + & -\end{bmatrix} \begin{bmatrix}+ & - & 0\\ 0 & - & +\\ + & + & -\end{bmatrix}$$

Sign patterns with associated digraph D4:

$$\begin{bmatrix}- & + & 0\\ + & 0 & +\\ + & - & 0\end{bmatrix} \begin{bmatrix}- & + & 0\\ - & 0 & -\\ - & + & 0\end{bmatrix} \begin{bmatrix}- & - & 0\\ + & 0 & +\\ + & + & \pm\end{bmatrix} \begin{bmatrix}- & + & 0\\ - & 0 & +\\ + & + & -\end{bmatrix} \begin{bmatrix}- & + & 0\\ + & 0 & +\\ - & + & -\end{bmatrix} \begin{bmatrix}- & \pm & 0\\ - & 0 & +\\ - & - & +\end{bmatrix} \begin{bmatrix}- & + & 0\\ - & 0 & -\\ - & + & \pm\end{bmatrix} \begin{bmatrix}- & + & 0\\ + & \pm & +\\ + & - & 0\end{bmatrix}$$

$$\begin{bmatrix}- & + & 0\\ + & - & +\\ - & + & 0\end{bmatrix} \begin{bmatrix}- & + & 0\\ - & \pm & -\\ - & + & 0\end{bmatrix} \begin{bmatrix}- & - & 0\\ + & \pm & +\\ + & + & 0\end{bmatrix} \begin{bmatrix}+ & + & 0\\ + & - & -\\ + & + & 0\end{bmatrix} \begin{bmatrix}+ & + & 0\\ - & - & +\\ - & - & 0\end{bmatrix} \begin{bmatrix}- & \pm & 0\\ + & - & +\\ + & + & -\end{bmatrix} \begin{bmatrix}- & + & 0\\ - & \pm & +\\ + & + & -\end{bmatrix} \begin{bmatrix}- & + & 0\\ + & \pm & +\\ - & + & -\end{bmatrix}$$

$$\begin{bmatrix}- & + & 0\\ - & - & -\\ - & + & -\end{bmatrix} \begin{bmatrix}+ & - & 0\\ + & - & +\\ + & \pm & +\end{bmatrix} \begin{bmatrix}+ & + & 0\\ - & - & +\\ + & + & +\end{bmatrix} \begin{bmatrix}+ & + & 0\\ + & + & -\\ + & + & -\end{bmatrix} \begin{bmatrix}+ & + & 0\\ - & + & +\\ + & + & -\end{bmatrix} \begin{bmatrix}\pm & - & 0\\ + & + & -\\ + & + & -\end{bmatrix} \begin{bmatrix}+ & - & 0\\ + & + & +\\ + & - & -\end{bmatrix} \begin{bmatrix}- & - & 0\\ + & + & +\\ + & \pm & -\end{bmatrix}$$

$$\begin{bmatrix}- & - & 0\\ + & - & \pm\\ + & + & +\end{bmatrix} \begin{bmatrix}- & + & 0\\ + & - & -\\ + & + & +\end{bmatrix} \begin{bmatrix}- & \pm & 0\\ + & - & +\\ + & - & +\end{bmatrix} \begin{bmatrix}- & + & 0\\ + & - & +\\ - & + & +\end{bmatrix}$$

Sign patterns with associated digraph D5:

$$\begin{bmatrix}- & - & \pm\\ + & 0 & +\\ + & + & 0\end{bmatrix} \begin{bmatrix}- & \pm & +\\ + & 0 & -\\ + & + & 0\end{bmatrix} \begin{bmatrix}- & - & -\\ + & 0 & -\\ + & + & 0\end{bmatrix} \begin{bmatrix}- & - & +\\ + & 0 & \pm\\ + & + & -\end{bmatrix} \begin{bmatrix}- & \pm & -\\ + & 0 & +\\ + & + & -\end{bmatrix} \begin{bmatrix}- & - & +\\ - & 0 & +\\ + & + & -\end{bmatrix}$$

$$\begin{bmatrix}- & - & +\\ + & 0 & +\\ - & + & -\end{bmatrix} \begin{bmatrix}- & - & -\\ + & 0 & -\\ + & + & \pm\end{bmatrix} \begin{bmatrix}- & - & \pm\\ + & 0 & +\\ + & + & +\end{bmatrix} \begin{bmatrix}- & + & -\\ + & 0 & +\\ + & \pm & +\end{bmatrix} \begin{bmatrix}- & \pm & +\\ + & 0 & -\\ + & + & +\end{bmatrix} \begin{bmatrix}- & - & +\\ + & 0 & +\\ + & - & +\end{bmatrix}$$

$$\begin{bmatrix}- & \pm & +\\ + & - & +\\ + & + & -\end{bmatrix} \begin{bmatrix}- & - & -\\ + & - & \pm\\ + & + & -\end{bmatrix} \begin{bmatrix}\pm & - & +\\ - & - & +\\ + & + & -\end{bmatrix} \begin{bmatrix}- & - & +\\ + & - & -\\ + & + & -\end{bmatrix} \begin{bmatrix}- & - & \pm\\ + & + & +\\ + & + & +\end{bmatrix} \begin{bmatrix}- & \pm & +\\ + & + & -\\ + & + & +\end{bmatrix}$$

$$\begin{bmatrix}- & - & +\\ + & + & +\\ + & - & +\end{bmatrix} \begin{bmatrix}- & - & -\\ + & + & -\\ + & + & +\end{bmatrix} \begin{bmatrix}+ & - & \pm\\ + & - & +\\ + & + & -\end{bmatrix} \begin{bmatrix}+ & \pm & +\\ + & - & -\\ + & + & -\end{bmatrix} \begin{bmatrix}+ & - & +\\ + & - & +\\ - & + & -\end{bmatrix} \begin{bmatrix}+ & - & +\\ + & - & +\\ + & - & -\end{bmatrix} \begin{bmatrix}+ & - & -\\ + & - & -\\ + & + & -\end{bmatrix}$$

Appendix B: Nonequivalent sign-nonsingular sign patterns

Here, we list all nonequivalent sign patterns of order 3 that do not allow $\mathbb{S}_3$. All of these patterns do not allow inertia $(0, 2, 1)$.

B.1. ***Sign-nonsingular sign patterns.*** The following are the nonequivalent sign-nonsingular sign patterns of order 3. Since all realizations of these patterns must be invertible, these patterns do not allow $\mathbb{S}_3$ as their inertias cannot include $(0, 2, 1)$.

The patterns here are organized into five groups corresponding to their associated (loopless) digraph D1–D5.

Sign patterns with associated digraph D1:

$$\begin{bmatrix} - & - & 0 \\ 0 & - & + \\ + & 0 & - \end{bmatrix} \begin{bmatrix} - & - & 0 \\ 0 & + & - \\ - & 0 & + \end{bmatrix} \begin{bmatrix} + & + & 0 \\ 0 & - & + \\ + & 0 & - \end{bmatrix}$$

Sign patterns with associated digraph D2:

$$\begin{bmatrix} - & - & 0 \\ - & 0 & - \\ 0 & - & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & 0 & - \\ 0 & - & + \end{bmatrix} \begin{bmatrix} - & - & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix} \begin{bmatrix} + & + & 0 \\ + & - & + \\ 0 & + & + \end{bmatrix}$$

$$\begin{bmatrix} - & - & 0 \\ - & + & - \\ 0 & - & - \end{bmatrix} \begin{bmatrix} + & + & 0 \\ - & + & + \\ 0 & + & - \end{bmatrix} \begin{bmatrix} - & - & 0 \\ + & - & - \\ 0 & - & + \end{bmatrix} \begin{bmatrix} - & - & 0 \\ + & - & - \\ 0 & + & - \end{bmatrix}$$

Sign patterns with associated digraph D3:

$$\begin{bmatrix} - & + & 0 \\ 0 & 0 & \pm \\ + & + & 0 \end{bmatrix} \begin{bmatrix} - & + & 0 \\ 0 & 0 & \pm \\ + & + & - \end{bmatrix} \begin{bmatrix} + & - & 0 \\ 0 & 0 & + \\ + & + & - \end{bmatrix} \begin{bmatrix} + & + & 0 \\ 0 & 0 & + \\ + & - & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ 0 & 0 & \pm \\ + & + & + \end{bmatrix}$$

$$\begin{bmatrix} - & + & 0 \\ 0 & - & - \\ + & + & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ 0 & + & - \\ + & + & + \end{bmatrix} \begin{bmatrix} + & + & 0 \\ 0 & - & + \\ - & + & + \end{bmatrix} \begin{bmatrix} - & + & 0 \\ 0 & + & + \\ + & + & - \end{bmatrix} \begin{bmatrix} + & + & 0 \\ 0 & - & + \\ + & - & - \end{bmatrix}$$

Sign patterns with associated digraph D4:

$$\begin{bmatrix} - & + & 0 \\ + & 0 & \pm \\ + & + & 0 \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ + & + & 0 \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ - & - & 0 \end{bmatrix} \begin{bmatrix} - & + & 0 \\ + & 0 & + \\ + & + & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ - & - & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ + & + & + \end{bmatrix} \begin{bmatrix} - & - & 0 \\ - & 0 & - \\ - & + & + \end{bmatrix} \begin{bmatrix} - & + & 0 \\ + & - & \pm \\ + & + & 0 \end{bmatrix}$$

$$\begin{bmatrix} - & + & 0 \\ - & \pm & + \\ + & + & 0 \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & \pm & + \\ - & - & 0 \end{bmatrix} \begin{bmatrix} - & + & 0 \\ + & + & \pm \\ + & + & 0 \end{bmatrix} \begin{bmatrix} + & - & 0 \\ + & - & + \\ + & + & 0 \end{bmatrix} \begin{bmatrix} + & + & 0 \\ + & - & + \\ + & - & 0 \end{bmatrix} \begin{bmatrix} + & + & 0 \\ + & - & + \\ - & + & 0 \end{bmatrix} \begin{bmatrix} + & + & 0 \\ - & - & - \\ - & + & 0 \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & - & + \\ - & - & - \end{bmatrix}$$

$$\begin{bmatrix} + & + & 0 \\ + & - & + \\ - & + & + \end{bmatrix} \begin{bmatrix} + & - & 0 \\ + & + & + \\ + & + & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ + & + & + \\ + & + & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & - & + \\ + & + & + \end{bmatrix}$$

Sign patterns with associated digraph D5:

$$\begin{bmatrix} - & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix} \begin{bmatrix} - & - & + \\ + & 0 & + \\ + & - & 0 \end{bmatrix} \begin{bmatrix} - & - & + \\ + & 0 & + \\ - & + & 0 \end{bmatrix} \begin{bmatrix} - & + & + \\ + & 0 & + \\ + & + & - \end{bmatrix} \begin{bmatrix} - & - & + \\ + & 0 & + \\ + & - & - \end{bmatrix} \begin{bmatrix} - & - & + \\ + & 0 & + \\ - & + & + \end{bmatrix} \begin{bmatrix} - & + & - \\ + & 0 & - \\ + & + & + \end{bmatrix}$$

B.2. ***Other sign patterns that do not allow* $(0,2,1)$.** The following sign patterns are not sign-nonsingular, but nevertheless do not allow inertia $(0,2,1)$. In the characteristic polynomial of a realization, it can be shown that it is impossible for the constant term to equal 0 when all of the other coefficients are positive. The patterns are organized into five groups corresponding to their associated (loopless) digraph D1–D5.

Sign pattern with associated digraph D1:

$$\begin{bmatrix} - & + & 0 \\ 0 & + & + \\ + & 0 & + \end{bmatrix}$$

Sign patterns with associated digraph D2:

$$\begin{bmatrix} - & - & 0 \\ - & 0 & - \\ 0 & - & + \end{bmatrix} \begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & \pm & - \end{bmatrix} \begin{bmatrix} - & - & 0 \\ - & - & - \\ 0 & - & + \end{bmatrix} \begin{bmatrix} + & + & 0 \\ - & - & + \\ 0 & - & + \end{bmatrix}$$

Sign patterns with associated digraph D3:

$$\begin{bmatrix} - & - & 0 \\ 0 & 0 & + \\ + & + & 0 \end{bmatrix} \begin{bmatrix} + & + & 0 \\ 0 & 0 & + \\ + & + & - \end{bmatrix} \begin{bmatrix} - & - & 0 \\ 0 & 0 & + \\ + & + & + \end{bmatrix} \begin{bmatrix} - & + & 0 \\ 0 & + & + \\ \pm & + & + \end{bmatrix} \begin{bmatrix} + & + & 0 \\ 0 & - & + \\ + & \pm & + \end{bmatrix} \begin{bmatrix} + & + & 0 \\ 0 & - & + \\ + & + & - \end{bmatrix}$$

Sign patterns with associated digraph D4:

$$\begin{bmatrix} - & + & 0 \\ + & 0 & + \\ - & + & 0 \end{bmatrix} \begin{bmatrix} - & + & 0 \\ + & 0 & + \\ \pm & + & + \end{bmatrix} \begin{bmatrix} - & + & 0 \\ + & + & + \\ - & + & 0 \end{bmatrix} \begin{bmatrix} + & + & 0 \\ + & - & + \\ + & + & 0 \end{bmatrix} \begin{bmatrix} \pm & + & 0 \\ + & - & + \\ + & + & + \end{bmatrix} \begin{bmatrix} + & - & 0 \\ + & - & - \\ + & + & + \end{bmatrix} \begin{bmatrix} + & + & 0 \\ + & + & + \\ + & \pm & - \end{bmatrix} \begin{bmatrix} + & + & 0 \\ + & + & + \\ - & + & - \end{bmatrix}$$

Sign patterns with associated digraph D5:

$$\begin{bmatrix} - & - & + \\ - & 0 & + \\ + & + & 0 \end{bmatrix} \begin{bmatrix} - & + & + \\ + & 0 & + \\ + & + & + \end{bmatrix} \begin{bmatrix} - & - & + \\ - & 0 & + \\ + & + & + \end{bmatrix} \begin{bmatrix} - & + & + \\ + & + & + \\ + & + & + \end{bmatrix} \begin{bmatrix} - & - & + \\ - & + & + \\ \pm & + & + \end{bmatrix} \begin{bmatrix} + & + & + \\ + & - & + \\ + & + & - \end{bmatrix}$$

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