involve
a journal of mathematics

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2018 vol. 11, no. 5

INVOLVE 11:5 (2018)
dx.doi.org/10.2140/involve.2018.11.827

Counting eta-quotients of prime level

Allison Arnold-Roksandich, Kevin James and Rodney Keaton

(Communicated by Kenneth S. Berenhaut)

It is known that a modular form on $\mathrm{SL}_2(\mathbb{Z})$ can be expressed as a rational function in $\eta(z)$, $\eta(2z)$ and $\eta(4z)$. By using known theorems and calculating the order of vanishing, we can compute the eta-quotients for a given level. Using this count, knowing how many eta-quotients are linearly independent, and using the dimension formula, we can figure out a subspace spanned by the eta-quotients. In this paper, we primarily focus on the case where the level is $N = p$, a prime. In this case, we will show an explicit count for the number of eta-quotients of level p and show that they are linearly independent.

1. Introduction and statement of results

Modular forms and cusp forms encode important arithmetic information, and are therefore important to study. An easy way to accomplish this is to study the Dedekind eta-function:

$$\eta(z) := q^{1/24} \prod_{n\geq 1}(1-q^n), \quad \text{where} \quad q = e^{2\pi i z}. \tag{1-1}$$

In particular, we focus on functions of the form

$$f(z) = \prod_{d \mid N} \eta^{r_d}(dz), \quad r_d \in \mathbb{Z}, \tag{1-2}$$

which we call eta-quotients, as they provide nice examples of modular forms.

The following theorem is the primary motivation behind this paper.

Theorem 1.1 [Ono 2004, Theorem 1.67]. *Every modular form on* $\mathrm{SL}_2(\mathbb{Z})$ *may be expressed as a rational function in* $\eta(z)$, $\eta(2z)$, *and* $\eta(4z)$.

While the recent work of Rouse and Webb [2015] has shown that Theorem 1.1 does not generalize to all levels, the subspace of eta-quotients for fixed level at least 2 is still interesting. The goal of this paper is to look at the vector space of modular

MSC2010: 11F11, 11F20, 11F37.
Keywords: modular forms, eta-quotients, Dedekind eta-function, number theory.
This study was supported by the NSF sponsored REU grant DMS-1156761.

forms with prime level, $M_k(\Gamma_1(p))$, and count the number of eta-quotients for fixed weight k and level p, and compare the span of these eta-quotients with $M_k(\Gamma_1(p))$. In other words, this paper focuses on explicitly counting the eta-quotients that are modular forms for the congruence subgroups $\Gamma_0(p)$ and $\Gamma_1(p)$, where p is a prime.

Theorem 1.2. *Let $p > 3$ be a prime and k be an integer. Then there exists $f(z) = \eta^{r_1}(z)\eta^{r_p}(pz)$ such that $f(z)$ is a weakly holomorphic modular form with weight k of level p if and only if k is divisible by $h = \frac{1}{2}\gcd(p-1, 24)$.*

This first theorem provides a condition on k that is necessary and sufficient for showing that the space of weakly holomorphic modular forms with weight k and level p contains eta-quotients. With some effort, we can create similar conditions to guarantee when $f(z)$ is in $M_k(\Gamma_1(p))$. The next theorem gives an explicit count of the number of eta-quotients that are cusp forms of weight k and level p.

Theorem 1.3. *Let $p > 3$ be a prime. Let $k = hk'$, where h is the needed divisor of k given by Theorem 1.2. Let $d = (p-1)/(2h)$, and let c be the smallest positive integer representative of $k'h/12$ modulo d:*

(1) *For $c = k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$, the number of eta-quotients in $S_k(\Gamma_1(p))$ is*
$$\frac{k(p+1)}{12d} - 1.$$
(2) *For $c < k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$, the number of eta-quotients in $S_k(\Gamma_1(p))$ is*
$$\left\lceil \frac{k(p+1)}{12d} \right\rceil.$$
(3) *For $c > k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$, the number of eta-quotients in $S_k(\Gamma_1(p))$ is*
$$\left\lfloor \frac{k(p+1)}{12d} \right\rfloor.$$

There are also eta-quotients in $M_k(\Gamma_1(p))$ that are not cusp forms that are given by the following theorem.

Theorem 1.4. *Let $p > 3$ be a prime. Then, $M_k(\Gamma_1(p)) \setminus S_k(\Gamma_1(p))$ contains at least one eta-quotient if and only if $\frac{1}{2}(p-1) \mid k$. Furthermore, for $k > 0$ and $\frac{1}{2}(p-1) \mid k$, there are exactly two eta-quotients in $M_k(\Gamma_1(p)) \setminus S_k(\Gamma_1(p))$, which are of the forms*
$$\frac{\eta^{2pk/(p-1)}(pz)}{\eta^{2k/(p-1)}(z)} \quad \textit{and} \quad \frac{\eta^{2pk/(p-1)}(z)}{\eta^{2k/(p-1)}(pz)}.$$

Finally, the following theorem also tells us the size of the subspace spanned by eta-quotients.

Theorem 1.5. *Let $p > 3$ be a prime. Then, the eta-quotients in $M_k(\Gamma_1(p))$ given by the previous theorems are linearly independent.*

Section 2 of this paper provides the necessary background for the results. The background includes information on modular forms, the dimension formula, and eta-quotients. Section 3 provides the proofs of the results given in this section. Finally, Section 4 details still-open questions and some ideas of how to extend these results further.

2. Background

2A. *Modular forms.* In this section, we present some definitions and basic facts from the theory of modular forms. For further details, the interested reader is referred to [Koblitz 1993, Chapter 3].

Definition 2.1. The *modular group*, denoted by $\mathrm{SL}_2(\mathbb{Z})$, is the group of all matrices of determinant 1 which have integral entries.

The modular group acts on the upper half-plane $\mathcal{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Furthermore, if we define $\mathcal{H}^*$ to be the set $\mathcal{H} \cup \mathbb{Q} \cup \{i\infty\}$, then the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ extends to an action on $\mathcal{H}^*$ [Koblitz 1993].

There are only certain specific subgroups of $\mathrm{SL}_2(\mathbb{Z})$ which we will use for our purposes. They are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Each of these subgroups is called a congruence subgroup of level N. Note that if $N = 1$, then $\Gamma_0(N) = \Gamma_1(N) = \mathrm{SL}_2(\mathbb{Z})$. This brings us to our next definition.

Definition 2.2. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and define an equivalence relation on $\mathbb{Q} \cup \{\infty\}$ by $z_1 \sim z_2$ if there is a $\gamma \in \Gamma$ such that $\gamma \cdot z_1 = z_2$. We call each equivalence class under this relation a *cusp of* Γ.

Now, for an integer k and a function $f : \mathcal{H}^* \to \mathbb{C}$ and a $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we define the weight-k slash operator by

$$f|_k\gamma(z) = (cz+d)^{-k} f(\gamma \cdot z).$$

Note that we will often suppress the weight from the notation when it is clear from context or irrelevant for our purposes.

We can now define the objects which will be of primary interest to us.

Definition 2.3. A function $f : \mathcal{H}^* \to \mathbb{C}$ is called a *weakly holomorphic modular form* of weight k and level Γ if

(1) f is holomorphic on $\mathcal{H}$,

(2) f is modular, i.e., for every $\gamma \in \Gamma$ and $z \in \mathcal{H}$ we have $f \mid \gamma(z) = f(z)$, and

(3) f is meromorphic at each cusp of Γ.

Furthermore, if we replace condition (3) by "f is holomorphic at each cusp of Γ", then we call f a modular form. If we further replace condition (3) with "f vanishes at each cusp of Γ", then we call f a cusp form.

Consider a form f of level N. We will clarify what we mean by a function being "holomorphic at a cusp". First, consider the cusp $\{i\infty\}$, which we call "the cusp at ∞". Note that the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is an element of $\Gamma_1(N)$ and hence $\Gamma_0(N)$ for every N. As our function satisfies condition (2), we have $f(Tz) = f(z+1) = f(z)$; i.e., our function is periodic. It is a basic fact from complex analysis that such a function has a Fourier expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n, \quad \text{where} \quad q := e^{2\pi i z}.$$

Using this, we say that f is meromorphic at $\{i\infty\}$ if there is some $c < 0$ such that $a_n = 0$ for all $n < c$. We say that f holomorphic at $\{i\infty\}$ if $a_n = 0$ for all $n < 0$, and we say that f vanishes at $\{i\infty\}$ if $a_n = 0$ for all $n \le 0$. We call the smallest n such that $a_n \neq 0$ the order of vanishing on the cusp at ∞. To cover another cusp α, let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be such that $\gamma \cdot \infty = \alpha$. Then, we need

$$(cz+d)^{-k} f(\gamma \cdot z) = \sum_{n=-\infty}^{\infty} c_n q^n.$$

If this holds, then we say f is meromorphic at α if there is some $c < 0$ such that $c_n = 0$ for all $n < c$. We also similarly say that the smallest n such that $c_n \neq 0$ is the order of vanishing at α.

Now, we set some notation which we will use throughout. For $\Gamma \le \mathrm{SL}_2(\mathbb{Z})$ we denote the spaces of weakly holomorphic modular forms, modular forms, and cusp forms of level Γ and weight k by $M_k^!(\Gamma)$, $M_k(\Gamma)$, and $S_k(\Gamma)$, respectively. Note that the spaces $S_k(\Gamma) \le M_k(\Gamma)$ are finite-dimensional complex vector spaces.

Throughout, we will also need the notion of a modular form with an associated character. We define a Dirichlet character of modulus N as a map $\chi : \mathbb{Z} \to \mathbb{C}$ such that:

(1) $\chi(m) = \chi(m+N)$ for all $m \in \mathbb{Z}$.

(2) If $\gcd(m, N) > 1$ then $\chi(m) = 0$. If $\gcd(m, N) = 1$, then $\chi(m) \neq 0$.

(3) $\chi(mn) = \chi(m)\chi(n)$ for all integers m, n.

Furthermore, if we let c be the minimal integer such that χ factors through $(\mathbb{Z}/c\mathbb{Z})^\times$, then we say χ has conductor c.

Let $f \in M_k(\Gamma_1(N))$ and suppose further that f satisfies

$$f \mid \gamma(z) = \chi(d) f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Then we say that f is a modular form of level N and character χ, and we denote the space of such functions by $M_k(N, \chi)$. Note that this is defined similarly for weakly holomorphic modular forms and cusp forms.

It is well known that we have the decomposition

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi \bmod N} M_k(N, \chi),$$

where the direct sum is over all Dirichlet characters modulo N. We can further decompose $M_k(N, \chi)$ into

$$M_k(N, \chi) = S_k(N, \chi) \oplus E_k(N, \chi),$$

where $S_k(N, \chi)$ is the space of cusp forms and $E_k(N, \chi)$, called the Eisenstein subspace, is orthogonal complement of $M_k(N, \chi)$ with respect to the Petersson inner product.

2B. *Dimension formulas.* In this section we present formulas for the dimensions of spaces of cusp and modular forms. For more details regarding dimension formulas, the interested reader is referred to [Stein 2007].

2B1. *The dimension formula for level $\Gamma_0(p)$ with trivial character.* We present a formula for the dimension of $E_k(\Gamma_0(p))$ and $S_k(\Gamma_0(p))$ for a rational prime $p \geq 5$.

First, we set

$$\mu_{0,2}(p) = \begin{cases} 0 & \text{if } p \equiv 3 \pmod 4, \\ 2 & \text{if } p \equiv 1 \pmod 4, \end{cases} \qquad \mu_{0,3}(p) = \begin{cases} 0 & \text{if } p \equiv 2 \pmod 3, \\ 2 & \text{if } p \equiv 1 \pmod 3. \end{cases}$$

Then define

$$g_0(p) = \tfrac{1}{12}(p+1) - \tfrac{1}{4}\mu_{0,2}(p) - \tfrac{1}{3}\mu_{0,3}(p).$$

$\dim S_k(\Gamma_0(p))$				
$k(12)\downarrow$ $p(12)\rightarrow$	1	5	7	11
0	$\frac{1}{12}(u+2)$	$\frac{1}{12}(u-6)$	$\frac{1}{12}(u-4)$	$\frac{1}{12}(u-12)$
1	0	0	0	0
2	$\frac{1}{12}(u-26)$	$\frac{1}{12}(u-18)$	$\frac{1}{12}(u-20)$	$\frac{1}{12}(u-12)$
3	0	0	0	0
4	$\frac{1}{12}(u-6)$	$\frac{1}{12}(u-6)$	$\frac{1}{12}(u-12)$	$\frac{1}{12}(u-12)$
5	0	0	0	0
6	$\frac{1}{12}(u-10)$	$\frac{1}{12}(u-18)$	$\frac{1}{12}(u-4)$	$\frac{1}{12}(u-12)$
7	0	0	0	0
8	$\frac{1}{12}(u-14)$	$\frac{1}{12}(u-6)$	$\frac{1}{12}(u-20)$	$\frac{1}{12}(u-12)$
9	0	0	0	0
10	$\frac{1}{12}(u-18)$	$\frac{1}{12}(u-18)$	$\frac{1}{12}(u-12)$	$\frac{1}{12}(u-12)$
11	0	0	0	0

Table 1. The dimension of $S_k(\Gamma_0(p))$ with trivial character and $k>2$. Note that $u=(p+1)(k-1)$.

Using this we have $\dim S_2(\Gamma_0(p)) = g_0(p)$ and $\dim E_2(\Gamma_0(p)) = 1$, and for even $k \geq 4$ we have $\dim E_k(\Gamma_0(p)) = 2$ and

$$\dim S_k(\Gamma_0(p)) = (k-1)(g_0(p)-1) + (k-2) + \mu_{0,2}(p)\left\lfloor \tfrac{1}{4}k \right\rfloor + \mu_{0,3}(p)\left\lfloor \tfrac{1}{3}k \right\rfloor.$$

From this, we see that our formula depends on the congruence class which k and p lie in modulo 12, so compiling these different congruences together we have Table 1.

2B2. *The dimension formula for* $\Gamma_0(p)$ *with quadratic character.* We will consider the case that our level is $\Gamma_0(p)$ for some rational prime p and that our associated character is quadratic. Note that at the end of the section we compile all of our computations together in a table for convenience.

In Section 2B1 we considered the trivial character case, so we now set $\chi(\cdot) = \left(\frac{\cdot}{p}\right)$. We must compute the summations

$$\sum_{x\in A_4(p)} \chi(x) \quad \text{and} \quad \sum_{x\in A_3(p)} \chi(x),$$

where $A_4(N) = \{x \in \mathbb{Z}/N\mathbb{Z} : x^2+1=0\}$ and $A_3(p) = \{x \in \mathbb{Z}/p\mathbb{Z} : x^2+x+1=0\}$.

First, we will consider $\sum_{x\in A_4(p)} \chi(x)$. This is clearly zero if $A_4(p)$ is empty, which occurs precisely when $p \equiv 3 \pmod 4$. Also, it is immediate that our summation equals 1 when $p=2$. Now suppose $p \equiv 1 \pmod 4$. Then $\#A_4(p) = 2$.

Note that if $r \in A_4(p)$ then $-r \in A_4(p)$ and $\chi(r) = \chi(-r)$ since $\chi(-1) = 1$. Furthermore, it is not hard to see that $\chi(r) = 1$ if and only if there is an element of order 8 in $(\mathbb{Z}/p\mathbb{Z})^\times$, i.e., $p \equiv 1 \pmod 8$. Thus, we have

$$\sum_{x \in A_4(p)} \chi(x) = \begin{cases} 1 & \text{if } p = 2, \\ 0 & \text{if } p \equiv 3 \pmod 4, \\ 2 & \text{if } p \equiv 1 \pmod 8, \\ -2 & \text{if } p \equiv 5 \pmod 8. \end{cases}$$

Now we consider the summation $\sum_{x \in A_3(p)} \chi(x)$. Similar to the above, we have $A_3(p)$ is empty if $p \equiv 2 \pmod 3$, in which case our summation is zero. Also, if $p = 3$ then our summation is 1. Now, suppose that $p \equiv 1 \pmod 3$. Note: it is immediate that if $r \in A_3(p)$ then so is r^2. Similar to the previous situation, we have $\chi(r) = 1$ if and only if there is an element of order 6 in $(\mathbb{Z}/p\mathbb{Z})^\times$, i.e., $p \equiv 1 \pmod 6$. Note that as p is prime, it follows that $p \equiv 1 \pmod 6$ is equivalent to $p \equiv 1 \pmod 3$. Thus, we have

$$\sum_{x \in A_3(p)} \chi(x) = \begin{cases} 1 & \text{if } p = 3, \\ 0 & \text{if } p \equiv 2 \pmod 3, \\ 2 & \text{if } p \equiv 1 \pmod 3. \end{cases}$$

We summarize our calculations in Table 2.

2C. *Eta-quotients.* We introduce the eta-function and present some results relating this to modular forms. For further details regarding the eta-function, the interested reader is referred to [Köhler 2011].

Recall Dedekind's eta-function given in (1-1). The eta-function satisfies the following transformation properties with respect to our matrices S, T defined in Section 2A:

$$\eta(Sz) = \eta(-z^{-1}) = \sqrt{-iz}\,\eta(z), \qquad \eta(Tz) = \eta(z+1) = e^{2\pi i/24}\eta(z).$$

More generally, we have the following general transformation formula for the eta-function:

$$\eta(\gamma z) = \epsilon(\gamma)(cz+d)^{1/2}\eta(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

where

$$\epsilon(\gamma) = \begin{cases} \left(\dfrac{d}{|c|}\right) e^{(2\pi i/24)((a+d)c - bd(c^2-1) - 3c)} & \text{if } c \text{ is odd}, \\ (-1)^{(1/4)(\mathrm{sgn}(c)-1)(\mathrm{sgn}(d)-1)} \left(\dfrac{d}{|c|}\right) e^{(2\pi i/24)((a+d)c - bd(c^2-1) + 3d - 3 - 3cd)} & \text{if } c \text{ is even}, \end{cases}$$

and $\mathrm{sgn}(x) = x/|x|$. For a proof of this transformation formula, the reader is referred to [Knopp 1970, Theorem 10, Chapter 3].

$\dim S_k\big(p, \big(\frac{\cdot}{p}\big)\big)$				
$k(12) \downarrow \quad p(24) \rightarrow$	1	5	7	11
0	$\frac{1}{12}(u+8)$	$\frac{1}{12}(u-12)$	0	0
1	0	0	$\frac{1}{12}u$	$\frac{1}{12}(u-6)$
2	$\frac{1}{12}(u-20)$	$\frac{1}{12}u$	0	0
3	0	0	$\frac{1}{12}(u+2)$	$\frac{1}{12}(u-6)$
4	$\frac{1}{12}u$	$\frac{1}{12}(u-12)$	0	0
5	0	0	$\frac{1}{12}(u-14)$	$\frac{1}{12}(u-6)$
6	$\frac{1}{12}(u-4)$	$\frac{1}{12}u$	0	0
7	0	0	$\frac{1}{12}u$	$\frac{1}{12}(u-6)$
8	$\frac{1}{12}(u-4)$	$\frac{1}{12}(u-12)$	0	0
9	0	0	$\frac{1}{12}(u+2)$	$\frac{1}{12}(u-6)$
10	$\frac{1}{12}(u-12)$	$\frac{1}{12}u$	0	0
11	0	0	$\frac{1}{12}(u-14)$	$\frac{1}{12}(u-6)$
$k(12) \downarrow \quad p(24) \rightarrow$	13	17	19	23
0	$\frac{1}{12}(u-4)$	$\frac{1}{12}u$	0	0
1	0	0	$\frac{1}{12}u$	$\frac{1}{12}(u-6)$
2	$\frac{1}{12}(u-8)$	$\frac{1}{12}(u-12)$	0	0
3	0	0	$\frac{1}{12}(u+2)$	$\frac{1}{12}(u-6)$
4	$\frac{1}{12}(u-12)$	$\frac{1}{12}u$	0	0
5	0	0	$\frac{1}{12}(u-14)$	$\frac{1}{12}(u-6)$
6	$\frac{1}{12}(u+8)$	$\frac{1}{12}(u-12)$	0	0
7	0	0	$\frac{1}{12}u$	$\frac{1}{12}(u-6)$
8	$\frac{1}{12}(u-20)$	$\frac{1}{12}u$	0	0
9	0	0	$\frac{1}{12}(u+2)$	$\frac{1}{12}(u-6)$
10	$\frac{1}{12}u$	$\frac{1}{12}(u-12)$	0	0
11	0	0	$\frac{1}{12}(u-14)$	$\frac{1}{12}(u-6)$

Table 2. Dimension of $S_k\big(p, \big(\frac{\cdot}{p}\big)\big)$. Note that $u = (p+1)(k-1)$.

In addition to the eta-function, we will also need to consider the related function $\eta(\delta z)$ for a positive integer δ. If we set $f(z) = \eta(\delta z)$ then $f(z)$ satisfies

$$f(\gamma z) = \epsilon\left(\begin{pmatrix} a & \delta b \\ c/\delta & d \end{pmatrix}\right)(cz+d)^{1/2} f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\delta).$$

Finally, we will need the transformation

$$f(Tz) = e^{2\pi i\delta/24} f(z).$$

Notice that this function is "almost" a modular form. With this in mind, we consider certain products of these functions with the goal of eliminating the "almost". This brings us to eta-quotients, which we defined in (1-2). We are interested in when these eta-quotients are modular forms. We have the following theorem which partially answers this question.

Theorem 2.4 [Ono 2004, Theorem 1.64]. *Define the eta-quotient*

$$f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z),$$

and set

$$k = \frac{1}{2}\sum_{\delta \mid N} r_\delta \in \mathbb{Z}.$$

Suppose our exponents $r_1, \dots, r_N$ satisfy

$$\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24} \quad \textit{and} \quad \sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Then,

$$f|_k\gamma(z) = \chi(d) f(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, where

$$\chi(n) = \left(\frac{(-1)^k s}{n}\right)$$

with $s = \prod_{\delta \mid N} \delta^{r_\delta}$.

This theorem provides conditions on when an eta-quotient is a weakly holomorphic modular form. However, to answer the question of when an eta-quotient is a modular form we need the following theorem, which provides information concerning the order of vanishing at the cusps of $\Gamma_0(N)$.

Theorem 2.5 [Ono 2004, Theorem 1.65]. *Let $f(z)$ be an eta-quotient satisfying the conditions of Theorem 2.4. Let $c, d \in \mathbb{N}$ with $d \mid N$ and $(c, d) = 1$. Then, the order of vanishing of $f(z)$ at the cusp c/d is*

$$v_d = \frac{N}{24}\sum_{\delta \mid N} \frac{(d,\delta)^2 r_\delta}{(d, N/d) d\delta}.$$

3. Proofs of results

We will provide the proofs for the results given in Section 1. We will assume that the eta-quotients being discussed always have $N = p > 3$, which is a prime, unless otherwise stated. From Theorems 2.4 and 2.5, we have conditions that tell us when an eta-quotient is a holomorphic modular form. Thus, we will use the equations

$$\tfrac{1}{2}(r_1 + r_p) = k, \tag{3-1}$$

$$r_1 + pr_p \equiv 0 \pmod{24}, \tag{3-2}$$

$$pr_1 + r_p \equiv 0 \pmod{24}, \tag{3-3}$$

$$v_1 = \tfrac{1}{24}(pr_1 + r_p), \tag{3-4}$$

$$v_p = \tfrac{1}{24}(r_1 + pr_p), \tag{3-5}$$

where v_1 and v_p are the orders of vanishing at the two cusps of $\Gamma_1(p)$, $i\infty$ and $1/p$, respectively.

For a fixed prime p and a fixed weight k, we see that it is possible to express r_p in terms of r_1 by (3-1). It is convenient to rewrite (3-4) and (3-5) as

$$24v_1 = 2k + (p-1)r_1, \tag{3-6}$$

$$24v_p = 2kp + (1-p)r_1. \tag{3-7}$$

It is now clear that we can relate the orders of vanishing to the weight of an eta-quotient by

$$24(v_1 + v_p) = 2k(p+1). \tag{3-8}$$

We begin the discussion for counting eta-quotients of level $\Gamma_0(p)$ by looking at possible conditions on k. These conditions were stated in Theorem 1.2, which we restate here for convenience.

Theorem 1.2. *Let $p > 3$ be a prime and k be an integer. Then there exists $f(z) = \eta^{r_1}(z)\eta^{r_p}(pz)$ such that $f(z)$ is a weakly holomorphic modular form with weight k of level p if and only if k is divisible by $h = \frac{1}{2}\gcd(p-1, 24)$.*

Proof. ($\rightarrow$) Suppose that $f(z) \in M^!(\Gamma_1(p))$. We note that it suffices to show that we can satisfy (3-7) and (3-8) since (3-6) can be gained from these two.

From (3-8), we see that we want $\frac{1}{12}k(p+1)$ to be an integer, as the orders of vanishing, v_1 and v_p, are integers. From here we can find a divisor d of k that would make this possible. Then by (3-7), we know that we need $24 \mid (2kp - (p-1)r_1)$. This gives us

$$2pdn \equiv (p-1)r_1 \pmod{24}, \tag{3-9}$$

where $dn = k$. Let $\delta = \gcd(24, 2dp, p-1)$. Then, we get that $2dp/\delta$, $(p-1)/\delta \in (\mathbb{Z}/(24/\delta)\mathbb{Z})^\times$ and obtain our desired conclusion, where $d = h = \frac{1}{2}\gcd(p-1, 24)$; except for when p is congruent to 1 or 17 modulo 24.

Suppose that $p \equiv 1 \pmod{24}$. Then we can rewrite (3-9) as

$$12n \equiv 0 \pmod{24}.$$

This tells us that n must be even. Thus, we have $k \equiv 0 \pmod{12}$. We further note that $12 = \frac{1}{2}\gcd(24\ell, 24)$, therefore showing our result for this case.

Suppose that $p \equiv 17 \pmod{24}$. Rewriting (3-9), we get

$$68n \equiv 16r_1 \pmod{24}.$$

This tells us that

$$5n \equiv 4r_1 \pmod{6}.$$

Since $5 \in \mathbb{Z}/6\mathbb{Z}^\times$, we have

$$n \equiv 2r_1 \pmod{6}.$$

Therefore, n must be even, and we have $4 \mid k$. As $4 = \frac{1}{2}\gcd(24\ell + 16, 24)$, we reach our desired conclusion.

($\leftarrow$) Suppose that $h = \frac{1}{2}\gcd(p-1, 24)$ divides k. We want to show that there exists $f(z) = \eta^{r_1}(z)\eta^{r_p}(pz)$ in $M_k^!(\Gamma_1(p))$. It is sufficient to show that there exists r_1 such that $r + p(2k - r_1) \equiv 0 \pmod{24}$. We can interpret this as

$$r_1(1-p) + 2pk = 24N,$$

where $N \in \mathbb{Z}$. As $2h$ divides every term, we can get

$$-r_1 d + p\frac{2k}{2h} = \frac{24}{2h}N.$$

Therefore, we have

$$dr_1 \equiv p\frac{k}{h} \left(\operatorname{mod} \frac{24}{2h}\right).$$

Since d and $24/(2h)$ are relatively prime, d has an inverse in $\mathbb{Z}/(24/(2h))\mathbb{Z}$. Thus, there exists a unique $r_1 \in \mathbb{Z}/(24/(2h))\mathbb{Z}$ such that

$$r_1 \equiv p\frac{k}{h}(d)^{-1} \left(\operatorname{mod} \frac{24}{2h}\right). \qquad \square$$

As mentioned in Section 1, we can extend Theorem 1.2 to show when there exists $f(z) = \eta^{r_1}(z)\eta^{r_p}(pz) \in M_k(\Gamma_1(p))$. Before we do so, we need a lemma.

Lemma 3.1. *Let N be an integer such that $\gcd(N, 6) = 1$. Let $f(z)$ be given by*

$$f(z) = \prod_{d \mid N} \eta^{r_d}(dz).$$

If $f \in M_k(\Gamma_0(N), \chi)$, then it must be the case that

$$\sum_{d \mid N} dr_d \equiv 0 \pmod{24} \quad \textit{and} \quad \sum_{d \mid N} \frac{N}{d} r_d \equiv 0 \pmod{24}.$$

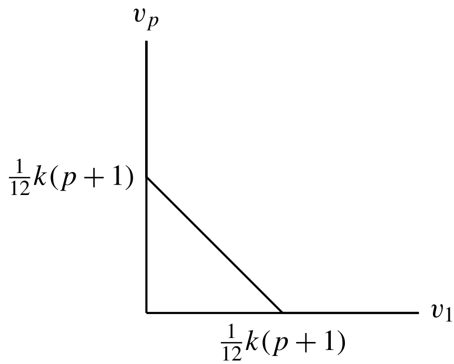

Figure 1. The line $v_1 + v_p = \frac{1}{12}k(p+1)$.

Proof. Since $f \in M_k(\Gamma_0(p), \chi)$, the q-series expansion of f about the cusp at infinity must look like

$$f(z) = \sum_{n \geq 0} c_n q^n.$$

Recall that $\eta(z) = q^{1/24} \prod_{n \geq 1}(1-q^n)$. Thus, we would have

$$\prod_{d \mid N} \eta^{r_d}(dz) = q^{(\sum_{d \mid N} r_d d)/24} \prod_{n \geq 1} \biggl(\prod_{d \mid N} (1-q^{dn})^{r_d} \biggr).$$

Therefore, we need 24 to divide

$$\sum_{d \mid N} d r_d.$$

We also note that for all primes $p \geq 5$, we have $p^2 \equiv 1 \pmod{24}$. Therefore, $Nd \equiv N/d \pmod{24}$. Thus, we have

$$0 \equiv N \sum_{d \mid N} d r_d \equiv \sum_{d \mid N} \frac{N}{d} r_d \pmod{24}. \qquad \square$$

As we wish to focus on holomorphic modular forms, we now want nonnegative orders of vanishing, i.e., $v_1, v_p \geq 0$. Using this condition and (3-8), we also have $v_1, v_p \leq k(p+1)/12$. We use Figure 1 to show the line that relates v_1 to v_p given a fixed k and p. We note that given (3-6), we can define v_1 in terms of r_1, and vice versa. Thus, to count the number of eta-quotients of our desired form, it suffices to count the number of possible orders of vanishing. As orders of vanishing are integer values, we only consider integer points on the line given in Figure 1.

Furthermore, from (3-6), we have

$$(p-1)r_1 = 24v_1 - 2k.$$

This implies that $24v_1 - 2k \equiv 0 \pmod{p-1}$. In other words, we can write $24v_1 - 2k = (p-1)\ell$, where $\ell \in \mathbb{Z}$. Recall how we defined h in Theorem 1.2. Since $2h \mid 2k$ and $2h \mid 24$, we can write $24/(2h)v_1 - k' = d\ell$. We also know that $2h \mid (p-1)$. Therefore, we have

$$\frac{24}{2h}v_1 \equiv k' \pmod{d}.$$

Since we have $2h = \gcd(p-1, 24)$, we get $1 = \gcd(d, 24/(2h))$. This implies that we have a multiplicative inverse of $24/(2h)$ in $\mathbb{Z}/d\mathbb{Z}$. Thus, we have

$$v_1 \equiv \left(\frac{24}{2h}\right)^{-1} k' \pmod{d}. \tag{3-10}$$

From Theorem 2.4 and Lemma 3.1, we get that (3-10) becomes a necessary and sufficient condition for an eta-quotient with order of vanishing v_1 to be in $M_k^!(\Gamma_1(p))$. Now, we have the following corollary which follows from this explanation as well as Theorem 1.2 and Lemma 3.1.

Corollary 3.2. *Let $p \geq 5$ be a prime. There exists $f(z) = \eta^{r_1}(z)\eta^{r_p}(pz)$ in $M_k(\Gamma_1(p))$ if and only if $h = \frac{1}{2}\gcd(p-1, 24)$ divides k and $d \leq \frac{1}{12}k(p+1)$.*

We note that by definition, cusp forms occur on the interior of our line, and noncuspidal modular forms occur at the end points. For this reason it is useful to perform the counts of cusp forms and noncuspidal modular forms separately. First, we prove the count of cusp forms given in Theorem 1.3, which we restate here for convenience.

Theorem 1.3. *Let $p > 3$ be a prime. Let $k = hk'$, where h is the needed divisor of k given by Theorem 1.2. Let $d = (p-1)/(2h)$, and let c be the smallest positive integer representative of $k'h/12$ modulo d.*

(1) *For $c = k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$, the number of eta-quotients in $S_k(\Gamma_1(p))$ is*
$$\frac{k(p+1)}{12d} - 1.$$

(2) *For $c < k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$, the number of eta-quotients in $S_k(\Gamma_1(p))$ is*
$$\left\lceil \frac{k(p+1)}{12d} \right\rceil.$$

(3) *For $c > k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$, the number of eta-quotients in $S_k(\Gamma_1(p))$ is*
$$\left\lfloor \frac{k(p+1)}{12d} \right\rfloor.$$

Proof. Since we are only considering cusp forms, we can assume that $v_1, v_p > 0$. The number of points on our line from Figure 1 which satisfy this inequality and the congruence from (3-10) is the number of eta-quotients. We now consider three cases.

Case 1: Suppose $c = 0 = k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$. Then, we have $v_1 \equiv 0 \pmod{d}$. Furthermore, we note that $d \mid k(p+1)/12$. Thus, we have the number of points which match our congruence is $k(p+1)/(12d)$. However, we note that one of these points gives us $v_p = 0$, which is not desired. Therefore, the number of eta-quotients that are in $S_k(\Gamma_1(p))$ is

$$\frac{k(p+1)}{12d} - 1.$$

Case 2: Suppose $c < k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$. Note that $\lfloor k(p+1)/(12d) \rfloor d$ is less than $k(p+1)/12$. However, since $c < k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$, we have another point to count that is between $\lfloor k(p+1)/(12d) \rfloor d$ and $k(p+1)/12$. Therefore, the number of eta-quotients that are in $S_k(\Gamma_1(p))$ is

$$\left\lceil \frac{k(p+1)}{12d} \right\rceil.$$

Case 3: Suppose $c > k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$. Note that $\lfloor k(p+1)/(12d) \rfloor d$ is less than $k(p+1)/12$. Since $c > k(p+1)/12 - \lfloor k(p+1)/(12d) \rfloor d$, we have no more points to count between $\lfloor k(p+1)/(12d) \rfloor d$ and $k(p+1)/12$. Therefore, the number of eta-quotients that are in $S_k(\Gamma_1(p))$ is

$$\left\lfloor \frac{k(p+1)}{12d} \right\rfloor. \qquad \square$$

Second, we prove the count of noncusp forms given in Theorem 1.4, which we restate here for convenience.

Theorem 1.4. *Let $p > 3$ be a prime. Then, $M_k(\Gamma_0(p)) \setminus S_k(\Gamma_1(p))$ contains at least one eta-quotient if and only if $\frac{1}{2}(p-1) \mid k$. Furthermore, for $k > 0$ and $\frac{1}{2}(p-1) \mid k$, there are exactly two eta-quotients in $M_k(\Gamma_1(p)) \setminus S_k(\Gamma_1(p))$, which are of the form*

$$\frac{\eta^{2pk/(p-1)}(pz)}{\eta^{2k/(p-1)}(z)} \quad \textit{and} \quad \frac{\eta^{2pk/(p-1)}(z)}{\eta^{2k/(p-1)}(pz)}.$$

Proof. ($\rightarrow$) Suppose $f(x) \in M_k(\Gamma_1(p)) \setminus S_k(\Gamma_1(p))$ is an eta-quotient that satisfies Theorem 2.4. Then, we know that at least one of the orders of vanishing must be zero. Thus, we have two cases.

Case 1: Suppose $v_1 = 0$. Then, $pr_1 + r_p = 0$, which can be rewritten to get $(p-1)r_1 = -2k$. Therefore, we have $\frac{1}{2}(p-1) \mid k$. Furthermore, we can get that $r_1 = 2k/(p-1)$, and thus $r_p = 2pk/(p-1)$. When plugging these values into v_p we get

$$v_p = \frac{1}{24}\left(\frac{-2k}{p-1} + \frac{2pk}{p-1}\right) = \frac{2k}{24} > 0.$$

Case 2: Suppose $v_p = 0$. Then, $r_1 + pr_p = 0$, which can be rewritten to get $(1-p)r_1 = -2pk$. Therefore, $\frac{1}{2}(p-1) \mid k$ since $p \nmid (p-1)$ and therefore $p \mid r_1$.

Furthermore, we get that $r_1 = 2pk/(p-1)$, and thus $r_p = -2k/(p-1)$. When plugging these values into v_1 we get

$$v_1 = \frac{1}{24}\left(\frac{2pk}{p-1} + \frac{-2k}{p-1}\right) = \frac{2k}{24} > 0.$$

In both cases, the number needed to divide k is the same. Furthermore, both create a single eta-quotient for a fixed k. Therefore, we have $\frac{1}{2}(p-1) \mid k$. Furthermore, there are exactly two eta-quotients which result from looking at either of the orders of vanishing being zero, and they are

$$\frac{\eta^{2pk/(p-1)}(pz)}{\eta^{2k/(p-1)}(z)} \quad \text{and} \quad \frac{\eta^{2pk/(p-1)}(z)}{\eta^{2k/(p-1)}(pz)}.$$

($\leftarrow$) Suppose that $k = \frac{1}{2}(p-1)m > 0$ for some integer m. Also, suppose we have the two eta-quotients

$$\frac{\eta^{2pk/(p-1)}(pz)}{\eta^{2k/(p-1)}(z)} \quad \text{and} \quad \frac{\eta^{2pk/(p-1)}(z)}{\eta^{2k/(p-1)}(pz)}.$$

We consider each eta-quotient as its own case.

Case 1: Consider $\eta^{2pk/(p-1)}(pz)/\eta^{2k/(p-1)}(z)$. Note that $r_1 + r_p = -2k/(p-1) + 2pk/(p-1) = 2k$. Furthermore,

$$r_1 + pr_p = \frac{-2k}{p-1} + \frac{2pk}{p-1}p = (p^2-1)m \equiv 0 \pmod{24}$$

since p is relatively prime to 24, and

$$pr_1 + r_p = p\frac{-2k}{p-1} + \frac{2pk}{p-1} = 0 \equiv 0 \pmod{24}.$$

When looking at the orders of vanishing, we get

$$v_1 = \frac{1}{24}(pr_1 + r_p) = \frac{1}{24}\left(p\frac{-2k}{p-1} + \frac{2pk}{p-1}\right) = 0 \geq 0 \quad \text{and}$$

$$v_p = \frac{1}{24}(r_1 + pr_p) = \frac{1}{24}\left(\frac{-2k}{p-1} + \frac{2pk}{p-1}p\right) = \frac{1}{24}(p^2-1)m \geq 0.$$

Since our orders of vanishing are both ≥ 0 and one of them is equal to 0, we have $\eta^{2pk/(p-1)}(pz)/\eta^{2k/(p-1)}(z) \in M_k(\Gamma_1(p)) \setminus S_k(\Gamma_1(p))$.

Case 2: Consider $\eta^{2pk/(p-1)}(z)/\eta^{2k/(p-1)}(pz)$. Note that $r_1 + r_p = 2pk/(p-1) + -2k/(p-1) = 2k$. Furthermore,

$$r_1 + pr_p = \frac{2pk}{p-1} + p\frac{-2k}{p-1} = 0 \equiv 0 \pmod{24},$$

and since p is relatively prime to 24,

$$pr_1 + r_p = p\frac{2pk}{p-1} + \frac{-2k}{p-1} = (p^2-1)m \equiv 0 \pmod{24}.$$

When looking at the orders of vanishing, we get

$$v_1 = \frac{1}{24}(pr_1 + r_p) = \frac{1}{24}\left(p\frac{2pk}{p-1} + \frac{-2k}{p-1}\right) = \frac{1}{24}(p^2-1)m \geq 0 \quad \text{and}$$

$$v_p = \frac{1}{24}(r_1 + pr_p) = \frac{1}{24}\left(\frac{2pk}{p-1} + \frac{-2k}{p-1}p\right) = 0 \geq 0.$$

Since our orders of vanishing are both ≥ 0 and one of them is equal to 0, we have

$$\frac{\eta^{2pk/(p-1)}(z)}{\eta^{2k/(p-1)}(pz)} \in M_k(\Gamma_1(p)) \setminus S_k(\Gamma_1(p)).$$

Thus, $M_k(\Gamma_1(p)) \setminus S_k(\Gamma_1(p))$ contains exactly two eta-quotients. □

From the eta-quotients given in the theorem, let $k = \frac{1}{2}(p-1)m$ where m is a positive integer. Then the eta-quotients have characters

$$\chi_1(n) = \left(\frac{(-1)^{(p-1)/2m}p^{pm}}{n}\right) \quad \text{and} \quad \chi_2(n) = \left(\frac{(-1)^{(p-1)/2m}p^{m}}{n}\right),$$

respectively. In the case where m is even, both of the characters are guaranteed to be the trivial character. When m is odd, we are guaranteed to have a quadratic character. In fact, both quadratic characters are the same.

Now that we know how many eta-quotients there are and can write down what they are if needed, we would like to know the dimension of the space spanned by these eta-quotients. This is provided by Theorem 1.5, which we restate here for convenience.

Theorem 1.5. *Let $p > 3$ be a prime. Then, the eta-quotients in $M_k(\Gamma_1(p))$ given by the previous theorems are linearly independent.*

Proof. Suppose that we are looking at eta-quotients in $M_k(\Gamma_1(p))$ for a prime $p > 3$. Without loss of generality, we look at the Fourier series about the cusp at ∞. By using the Sturm bound [Ono 2004], we get that we need to compare the first $\left\lfloor \frac{1}{12}pk \right\rfloor + 1$ terms of each Fourier series. We can pick a cusp and order the eta-quotients increasingly by looking at the order of vanishing. We can then create a matrix A where the i, j-th entry represents $a(j)$ in the i-th eta-quotient's Fourier series. Since all of the eta-quotients have different orders of vanishing and are in increasing order, we get that A is in echelon form. This tells us that all the rows are linearly independent. Thus all of the eta-quotients are linearly independent. □

The following corollaries can all be obtained by comparing dimension formulas with our counts and applying the previous theorem.

Corollary 3.3. *Let $p \geq 5$ be a prime. Recall that $h = \frac{1}{2}\gcd(p-1, 24)$ from Theorem 1.2. Denote the space of level p, weight k eta-quotients by $\eta_k(p)$.*

(1) *If $p \equiv 3 \pmod 4$, then taking the limit over odd k in the appropriate congruence class from Theorem 1.2 gives*

$$\lim_{k\to\infty} \frac{\dim \eta_k(p)}{\dim S_k\big(p, \big(\frac{\cdot}{p}\big)\big)} = \frac{2h}{p-1}.$$

(2) *If $p \equiv 3 \pmod 4$, then taking the limit over even k in the appropriate congruence class from Theorem 1.2 gives*

$$\lim_{k\to\infty} \frac{\dim \eta_k(p)}{\dim S_k(\Gamma_0(p))} = \frac{2h}{p-1}.$$

(3) *If $p \equiv 1 \pmod 4$, then taking the limit over all k in the appropriate congruence class from Theorem 1.2 gives*

$$\lim_{k\to\infty} \frac{\dim \eta_k(p)}{\dim S_k(\Gamma_0(p)) + \dim S_k\big(p, \big(\frac{\cdot}{p}\big)\big)} = \frac{h}{p-1}.$$

Finally, we would like to consider the case that our v_1 and v_p are integral but do not correspond to integral r_1, r_p. To gain some intuition concerning the properties of the "eta-quotients" formed from these r_1, r_p, we consider the following example.

Example 3.4. Let $p = 11$ and $k = 6$. Note that in this situation we have that in order to have eta-quotients we must have $v_1 \equiv 3 \pmod 5$. So, we will investigate the properties of the function obtained by choosing $v_1 \not\equiv 3 \pmod 5$.

Consider $v_1 = 1$. This implies $v_p = 5$. Then,

$$\begin{pmatrix} r_1 \\ r_p \end{pmatrix} = \begin{pmatrix} 1 & 11 \\ 11 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 24 \\ 120 \end{pmatrix} = \begin{pmatrix} 54/5 \\ 6/5 \end{pmatrix}.$$

Now, we can use these to form the "eta-quotient"

$$f(z) = \eta^{54/5}(z)\eta^{6/5}(11z).$$

Using the transformation properties from Section 2C we have

$$f(Tz) = e^{27\pi i/30}\eta^{54/5}(z)e^{11\pi i/10}\eta^{6/5}(11z) = f(z).$$

Note that if we raise $f(z)$ to the fifth power to cancel denominators of r_1 and r_p then we can use Theorem 2.4 to verify that we obtain $f(z)^5 \in S_{30}(\Gamma_0(11))$, i.e., our lattice point corresponds to a "root" of an η quotient of higher weight.

Note that the remaining choices for v_1 give us similar results.

4. Conclusion and further questions

We detailed the number of eta-quotients in $M_k(\Gamma_0(p))$ of the form $\eta^{r_1}(z)\eta^{r_p}(pz)$. Further work in this project would involve generalizing these results for all levels as well as figuring out linear combinations of eta-quotients that would be possible on a given level.

Acknowledgments

Arnold-Roksandich would like to thank Holly Swisher for aid with editing and organization, and Dania Zantout for help with understanding of background material.

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Received: 2016-12-18 Revised: 2017-07-30 Accepted: 2017-08-24

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2018 vol. 11 no. 5