

involve

a journal of mathematics

Factorization of Temperley–Lieb diagrams

Dana C. Ernst, Michael G. Hastings and Sarah K. Salmon



Factorization of Temperley–Lieb diagrams

Dana C. Ernst, Michael G. Hastings and Sarah K. Salmon

(Communicated by Scott T. Chapman)

The Temperley–Lieb algebra is a finite-dimensional associative algebra that arose in the context of statistical mechanics and occurs naturally as a quotient of the Hecke algebra arising from a Coxeter group of type A . It is often realized in terms of a certain diagram algebra, where every diagram can be written as a product of “simple diagrams”. These factorizations correspond precisely to factorizations of the so-called fully commutative elements of the Coxeter group that index a particular basis. Given a reduced factorization of a fully commutative element, it is straightforward to construct the corresponding diagram. On the other hand, it is generally difficult to reconstruct the factorization given an arbitrary diagram. We present an efficient algorithm for obtaining a reduced factorization for a given diagram.

1. Introduction

The Temperley–Lieb algebra [1971] is a finite-dimensional associative algebra that arose in the context of statistical mechanics. Penrose [1971] and Kauffman [1990] showed that this algebra can be faithfully represented by a diagram algebra that has a basis given by certain diagrams. Jones [1999] showed that the Temperley–Lieb algebra occurs naturally as a quotient of the Hecke algebra arising from a Coxeter group of type A (whose underlying group is the symmetric group). This realization of the Temperley–Lieb algebra as a Hecke algebra quotient was later generalized to the case of an arbitrary Coxeter group by Graham [1995]. These generalized Temperley–Lieb algebras have a basis indexed by the fully commutative elements (in the sense of Stembridge [1996]) of the underlying Coxeter group. In cases when diagrammatic representations are known to exist, it turns out that every diagram can be written as a product of “simple diagrams”. Each factorization of a diagram corresponds precisely to a factorization of the fully commutative element that indexes the diagram. Given a diagrammatic representation and a reduced factorization of a fully commutative element, it is easy to construct the corresponding diagram. However, given an arbitrary basis diagram, it is generally

MSC2010: 20C08, 20F55, 57M15.

Keywords: diagram algebra, Temperley–Lieb algebra, Coxeter group, heap.

difficult to reconstruct the factorization of the corresponding group element. In the (type A) Temperley–Lieb algebra, we have devised an algorithm for obtaining a reduced factorization for a given diagram.

This paper is organized as follows. In [Section 2](#), we recall the basic terminology of Coxeter groups, fully commutative elements, heaps, and the Temperley–Lieb algebra, as well as establish our notation and review several necessary results. [Section 3](#) describes the construction of the diagram algebra that is a faithful representation of the Temperley–Lieb algebra. This section includes a description of both the so-called simple diagrams that generate the algebra, as well as the basis that is indexed by the fully commutative elements of the Coxeter group of type A . We present our algorithm for factoring a given Temperley–Lieb diagram in terms of the heap associated to the corresponding fully commutative element in [Section 4](#). We conclude with [Section 5](#), which details potential further research.

2. Preliminaries

Coxeter groups. A *Coxeter system* is a pair (W, S) consisting of a finite set S of generating involutions and a group W , called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It follows that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st [[Humphreys 1990](#)]. Coxeter groups are generalizations of reflection groups, where each generator $s \in S$ can be thought of as a reflection. Recall that the composition of two reflections is a rotation by twice the angle between the corresponding hyperplanes. So if $s, t \in S$, we can think of st as a rotation with order $m(s,t)$.

Since elements of S have order 2, the relation $(st)^{m(s,t)} = e$ can be written as

$$\underbrace{sts \cdots}_{m(s,t)} = \underbrace{tst \cdots}_{m(s,t)} \quad (1)$$

with $m(s,t) \geq 2$ factors. If $m(s,t) = 2$, then $st = ts$ is called a *commutation relation* since s and t commute. Otherwise, if $m(s,t) \geq 3$, then the corresponding relation is called a *braid relation*. The replacement

$$\underbrace{sts \cdots}_{m(s,t)} \mapsto \underbrace{tst \cdots}_{m(s,t)}$$

will be referred to as a *commutation* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

We can represent the Coxeter system (W, S) with a unique *Coxeter graph* Γ with

- (1) vertex set $S = \{s_1, \dots, s_n\}$ and
- (2) edges $\{s_i, s_j\}$ for each $m(s_i, s_j) \geq 3$.

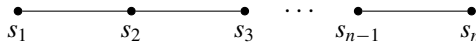


Figure 1. Coxeter graph of type A_n .

Each edge $\{s_i, s_j\}$ is labeled with its corresponding bond strength $m(s_i, s_j)$. Since bond strength 3 is the most common, we typically omit the labels of 3 on those edges.

There is a one-to-one correspondence between Coxeter systems and Coxeter graphs. Given a Coxeter graph Γ , we can construct the corresponding Coxeter system (W, S) . In this case, we say that (W, S) , or just W , is of type Γ . If (W, S) is of type Γ , for emphasis, we may write (W, S) as $(W(\Gamma), S(\Gamma))$. Note that generators s_i and s_j are connected by an edge in the Coxeter graph Γ if and only if s_i and s_j do not commute [Humphreys 1990].

The Coxeter system of type A_n is given by the Coxeter graph in Figure 1. In this case, $W(A_n)$ is generated by $S(A_n) = \{s_1, s_2, \dots, s_n\}$ and has defining relations

- (1) $s_i s_i = e$ for all i ;
- (2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;
- (3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group S_{n+1} under the mapping that sends s_i to the adjacent transposition $(i, i + 1)$. This paper focuses on an associative algebra whose underlying structure is a Coxeter system of type A_n .

Let S^* denote the free monoid over S . If a word $w = s_{x_1} s_{x_2} \dots s_{x_m} \in S^*$ is equal to w when considered as an element of W , we say that w is an *expression* for w . (Expressions will be written in sans serif font for clarity.) Furthermore, if m is minimal among all possible expressions for w , we say that w is a *reduced expression* for w , and we call m the *length* of w , denoted $\ell(w)$. Each element $w \in W$ can have several different reduced expressions that represent it. The following theorem, called Matsumoto’s theorem [Geck and Pfeiffer 2000], indicates how all of the reduced expressions for a given group element are related.

Proposition 2.1. *In a Coxeter group W , any two reduced expressions for the same group element differ by a finite sequence of commutations and braid moves.* □

Let w be a reduced expression for $w \in W$. We define a *subexpression* of w to be any subsequence of w . We will refer to a consecutive subexpression of w as a *subword*.

Example 2.2. Let $w = s_1 s_2 s_4 s_5 s_2 s_6 s_5$ be an expression for $w \in W(A_6)$. Then

$$s_1 s_2 s_4 s_5 s_2 s_6 s_5 = s_1 s_4 s_2 s_5 s_2 s_6 s_5 = s_1 s_4 s_5 s_2 s_2 s_6 s_5 = s_1 s_4 s_5 s_6 s_5,$$

where the blue subword indicates the location where a commutation is applied to obtain the next expression and the green subword indicates the location where

two adjacent occurrences of the same generator are canceled to obtain the last expression. This shows that w is not reduced. It turns out that $s_1s_4s_5s_6s_5$ is a reduced expression for w and hence $\ell(w) = 5$.

Example 2.3. Let $w = s_1s_2s_3s_4s_2$ be a reduced expression for $w \in W(A_4)$. Then the set of all reduced expressions for w is given by

$$\{s_1s_2s_3s_4s_2, s_1s_2s_3s_2s_4, s_1s_3s_2s_3s_4, s_3s_1s_2s_3s_4\},$$

where the blue subwords indicate the location where a commutation is applied to obtain the next expression in the set and the pink subword indicates the location where a braid move is applied to obtain the third expression from the second expression. Note that $\ell(w) = 5$.

Fully commutative elements. Let (W, S) be a Coxeter system of type Γ and let $w \in W$. Following [Stembridge 1996], we define a relation \sim on the set of reduced expressions for w . Let w and w' be two reduced expressions for w . We define $w \sim w'$ if we can obtain w' from w by applying a single commutation move of the form $st \mapsto ts$, where $m(s, t) = 2$. Now, define the equivalence relation \approx by taking the reflexive transitive closure of \sim . Each equivalence class under \approx is called a *commutation class*. Two reduced expressions are said to be *commutation equivalent* if they are in the same commutation class.

Example 2.4. Let $w = s_1s_2s_3s_4s_5s_2$ and $w' = s_1s_2s_3s_2s_4s_5$ be two different reduced expressions for $w \in W(A_5)$. Then w and w' are commutation equivalent since

$$s_1s_2s_3s_4s_5s_2 = s_1s_2s_3s_4s_2s_5 = s_1s_2s_3s_2s_4s_5,$$

where the blue subwords indicate the location where a commutation is applied to obtain the next expression. By applying a braid relation to w' , we obtain

$$s_1s_2s_3s_2s_4s_5 = s_1s_3s_2s_3s_4s_5,$$

where the location of the braid move has been highlighted in pink. It turns out that the last reduced expression above is neither commutation equivalent to w nor w' , and hence w has more than one commutation class. Specifically, the commutation classes are

$$\{s_1s_2s_3s_4s_5s_2, s_1s_2s_3s_4s_2s_5, s_1s_2s_3s_2s_4s_5\} \quad \text{and} \quad \{s_1s_3s_2s_3s_4s_5, s_3s_1s_2s_3s_4s_5\}.$$

Example 2.5. Let $w = s_2s_1s_3s_4s_2$ be a reduced expression for $w \in W(A_4)$. In this case, w has exactly five reduced expressions, including w . From this, it is easy to verify that all reduced expressions for w are commutation equivalent. This implies that there is a unique commutation class for w :

$$\{s_2s_1s_3s_4s_2, s_2s_3s_1s_4s_2, s_2s_1s_3s_2s_4, s_2s_3s_1s_2s_4, s_2s_3s_4s_1s_2\}.$$

If w has exactly one commutation class, then we say that w is *fully commutative*, or just FC. The set of all fully commutative elements of W is denoted by $\text{FC}(\Gamma)$, where Γ is the corresponding Coxeter graph. For consistency, we say that a reduced expression w is FC if it is a reduced expression for some $w \in \text{FC}(\Gamma)$. Note that the element in [Example 2.4](#) is not FC since there are two commutation classes, while the element in [Example 2.5](#) is FC.

Given some $w \in \text{FC}(\Gamma)$ and a starting reduced expression for w , observe that the definition of fully commutative states that one only needs to perform commutations to obtain all the reduced expression for w , but the following result due to Stembridge [1996] states that when w is FC, performing commutations is the only possible way to obtain another reduced expression for w .

Proposition 2.6. *An element $w \in W$ is FC if and only if no reduced expression for w contains*

$$\underbrace{sts \cdots}_{m(s,t)}$$

as a subword when $m(s, t) \geq 3$. □

In other words, an element is FC if and only if there is no opportunity to apply a braid move. For example, we can conclude that the element in [Example 2.4](#) is not FC without actually computing the commutation classes since there is an opportunity to apply a braid move, which we highlighted in pink.

Stembridge classified the irreducible Coxeter groups that contain only finitely many fully commutative elements, called the *FC-finite Coxeter groups*. This paper is mainly concerned with $W(A_n)$, which is a finite group, so it has finitely many FC elements. However, there exist infinite Coxeter groups that contain only finitely many FC elements. For example, Coxeter groups of type E_n with $n \geq 9$ are infinite, but they have only finitely many FC elements. It is well known that the number of FC elements in $W(A_n)$ is equal to the $(n+1)$ -th Catalan number, where the k -th Catalan number is given by

$$C_k = \frac{1}{k+1} \binom{2k}{k}.$$

Heaps. Each reduced expression is associated with a labeled partially ordered set called a heap. Heaps provide a visual representation of the reduced expression while preserving the relations of the generators. We follow the development in [Ernst 2010; Stembridge 1996].

Let (W, S) be a Coxeter system. Suppose $w = s_{x_1} s_{x_2} \cdots s_{x_k}$ is a reduced expression for $w \in W$, and as in [Stembridge 1996], define a partial ordering $<$ on the indices $\{1, \dots, k\}$ by the transitive closure of the relation $j < i$ if $i < j$ and s_{x_i} and s_{x_j} do not commute. In particular, $j < i$ if $i < j$ and $s_{x_i} = s_{x_j}$, by transitivity and the fact that w is reduced. This partial order with i labeled s_{x_i} is called the *heap* of w . Note



Figure 2. Labeled Hasse diagram and lattice point representation of a heap.

that for simplicity, we are omitting the labels of the underlying poset but retaining the labels of the corresponding generators.

Example 2.7. Let $w = s_2 s_1 s_3 s_2 s_4 s_5$ be a reduced expression for $w \in W(A_5)$. Since $\ell(w) = 6$, the expression w is indexed by $\{1, 2, 3, 4, 5, 6\}$. We see that $4 < 3$ since $3 < 4$ and the third and fourth factors (namely, s_3 and s_2) do not commute. The labeled Hasse diagram for the heap of w is shown in Figure 2 (left).

Let w be a fixed reduced expression for $w \in W(A_n)$. As in [Billey and Jones 2007; Ernst 2010], we represent a heap for w as a set of lattice points embedded in $\{1, \dots, n\} \times \mathbb{N}$. To do so, we assign coordinates $(x, y) \in \{1, \dots, n\} \times \mathbb{N}$ to each entry of the labeled Hasse diagram for the heap of w in such a way that

- (1) an entry labeled s_i in the heap has coordinates (x, y) if and only if $x = i$;
- (2) an entry with coordinates (x, y) is greater than an entry with coordinates (x', y') in the heap if and only if $y > y'$.

It follows from the definition that there is an edge in the Hasse diagram from (x, y) to (x', y') if and only if $x = x' \pm 1, y > y'$, and there are no entries (x'', y'') such that $x'' \in \{x, x'\}$ and $y' < y'' < y$. This implies that we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation.

Note that our heaps are upside-down versions of the heaps that appear in [Billey and Jones 2007] and several other papers. That is, in this paper, entries on top of a heap correspond to generators occurring to the left, as opposed to the right, in the corresponding reduced expression. One can form similar lattice point representations for heaps when Γ is a straight line Coxeter graph.

Let $w = s_{x_1} \cdots s_{x_k}$ be any reduced expression for $w \in W(A_n)$. We let $H(w)$ denote a lattice representation of the heap poset in $\{1, \dots, n\} \times \mathbb{N}$ described in the paragraph above. There are many possible coordinate assignments for the entries of $H(w)$, yet the x -coordinates for each entry will be fixed. If s_{x_i} and s_{x_j} are adjacent generators in the Coxeter graph with $i < j$, then we must place the point labeled by s_{x_i} at a level that is *above* the level of the point labeled by s_{x_j} . In particular, two entries labeled

by the same generator may only differ by the amount of vertical space between them while maintaining their relative vertical position to adjacent entries in the heap.

Because generators that are not adjacent in the Coxeter graph commute, points whose x -coordinates differ by more than 1 can slide past each other or land at the same level. To visualize the labeled heap poset of a lattice representation we will enclose each entry of the heap in a block in such a way that if one entry covers another, the blocks overlap halfway.

It follows from [Stembridge 1996, Proposition 2.2] that heaps are well-defined up to commutation class. In particular, there is a one-to-one correspondence between commutation classes and heaps. That is, if w and w' are two reduced expressions for $w \in W$ that are in the same commutation class, then the heaps of w and w' are equal. Conversely, if w and w' belong to different commutation classes, then the corresponding heaps will be different. In particular, if w is FC, then it has a single commutation class, and so there is a unique heap associated to w . In this case, if w is FC, then we may write $H(w)$ to denote the heap of any reduced expression for w . We will not make a distinction between $H(w)$ and its lattice point representation.

There are potentially many different ways to represent a heap, each differing by the vertical placement of blocks. For example, we can place blocks in vertical positions that are as high as possible, as low as possible, or some combination of high and low. When w is FC, we wish to make a canonical choice for the representation of $H(w)$ by giving all blocks at the top of the heap the same vertical position and placing all other blocks as high as possible. Note that our canonical representation of heaps of FC elements corresponds precisely to the unique heap factorization of [Viennot 1986, Lemma 2.9] and to the Cartier–Foata normal form for monomials [Cartier and Foata 1969; Green 2006].

Example 2.8. The canonical lattice point representation of $H(w)$ for the reduced expression given in Example 2.7 is shown in Figure 2 (right).

Example 2.9. Consider $w \in W(A_5)$ from Example 2.4. This element has two commutation classes, and hence two heaps as given in Figure 3, where we have color-coded in pink the blocks of each heap that correspond to the braid relation $s_2 s_3 s_2 = s_3 s_2 s_3$. Figure 3 (left) corresponds to the commutation class $\{s_1 s_2 s_3 s_4 s_5 s_2, s_1 s_2 s_3 s_4 s_2 s_5, s_1 s_2 s_3 s_2 s_4 s_5\}$, while Figure 3 (right) corresponds to the commutation class $\{s_1 s_3 s_2 s_3 s_4 s_5, s_3 s_1 s_2 s_3 s_4 s_5\}$.

Given a heap, we can write a reduced expression for the corresponding group element by reading off the generators, starting at the top, moving left to right and then down. The expression we obtain is commutation equivalent to any expression to which the heap corresponds.

The Temperley–Lieb algebra. Given a Coxeter graph Γ , we can form the associative algebra $\text{TL}(\Gamma)$, which we call the Temperley–Lieb algebra of type Γ [Graham

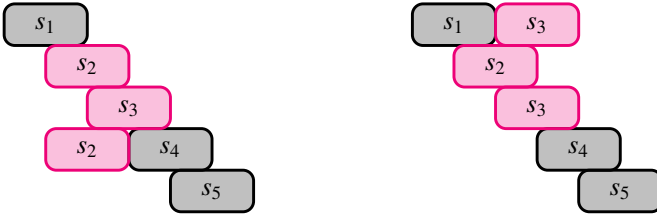


Figure 3. Two different heaps corresponding to the same non-FC element.

1995]. For a complete description of the construction of $\text{TL}(\Gamma)$, see [Ernst 2010; Graham 1995; Green 2006]. For our purposes, it suffices to define $\text{TL}(A_n)$ in terms of generators and relations. We are using [Green 2006, Proposition 2.6] (also see [Graham 1995, Proposition 9.5]) as our definition.

The *Temperley–Lieb algebra of type A_n* , denoted by $\text{TL}(A_n)$, is the unital $\mathbb{Z}[\delta]$ -algebra generated by $\{b_1, b_2, \dots, b_n\}$ with defining relations

- (1) $b_i^2 = \delta b_i$ for all i ;
- (2) $b_i b_j = b_j b_i$ if $|i - j| > 1$;
- (3) $b_i b_j b_i = b_i$ if $|i - j| = 1$.

Suppose w lies in $\text{FC}(A_n)$ and has reduced expression $w = s_{x_1} s_{x_2} \cdots s_{x_k}$. Define the element $b_w \in \text{TL}(A_n)$ via

$$b_w = b_{x_1} b_{x_2} \cdots b_{x_k}.$$

Notice that since w is required to be fully commutative, the definition of b_w is independent of choice of reduced expression for w . It is well known (and follows from [Green 2006, Proposition 2.4]) that the set $\{b_w \mid w \in \text{FC}(A_n)\}$ forms a $\mathbb{Z}[\delta]$ -basis for $\text{TL}(A_n)$, called the *monomial basis*.

3. The Temperley–Lieb diagram algebra

Next, we establish our notation and introduce all of the terminology required to define an associative diagram algebra that is a faithful representation of $\text{TL}(A_n)$.

Let k be a nonnegative integer. The *standard k -box* is a rectangle with $2k$ points, called *nodes*, labeled as in Figure 4. We will refer to the top of the rectangle as the *north face* and the bottom as the *south face*.

A *concrete pseudo k -diagram* is composed of a finite number of disjoint curves (planar), called *edges*, embedded in the standard k -box with two restrictions:

- (1) Every node of the box is the endpoint of exactly one edge, which meets the box transversely.
- (2) All other edges must be closed (isotopic to circles) and disjoint from the box.

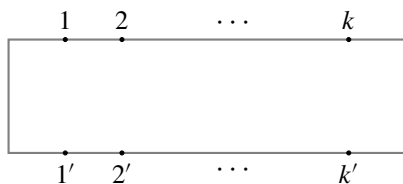


Figure 4. Standard k -box.



Figure 5. Example of a concrete pseudo 6-diagram together with a nonexample.

Example 3.1. The diagram in Figure 5 (left) is an example of a concrete pseudo 6-diagram, whereas the diagram in Figure 5 (right) does not represent a concrete pseudo 6-diagram since the diagram contains edges that are not disjoint (i.e., they intersect), node 4 is the endpoint for more than one edge, nodes 3 and 6' are not endpoints for any edge, and the edge leaving node 6 does not have a node as its second endpoint.

We now define an equivalence relation on the set of concrete pseudo k -diagrams. Two concrete pseudo k -diagrams are (*isotopically*) *equivalent* if one concrete diagram can be obtained from the other by isotopically deforming the edges such that any intermediate diagram is also a concrete pseudo k -diagram. Note that an isotopy of the k -box is a 1-parameter family of homeomorphisms of the k -box to itself that are stationary on the boundary.

A *pseudo k -diagram* is defined to be an equivalence class of equivalent concrete pseudo k -diagrams. We denote the set of pseudo k -diagrams by T_k . Note that we used the word “pseudo” in our definition to emphasize that we allow loops to appear in our diagrams.

Remark 3.2. When representing a pseudo k -diagram with a drawing, we pick an arbitrary concrete representative among a continuum of equivalent choices. When no confusion can arise, we will not make a distinction between a concrete pseudo k -diagram and the equivalence class that it represents. We say that two concrete pseudo k -diagrams are *vertically equivalent* if they are equivalent in the above sense by an isotopy that preserves setwise each vertical cross-section of the k -box.

Example 3.3. The concrete pseudo 5-diagrams in Figure 6 are equivalent since each diagram can be obtained from the other by isotopically deforming the edges.

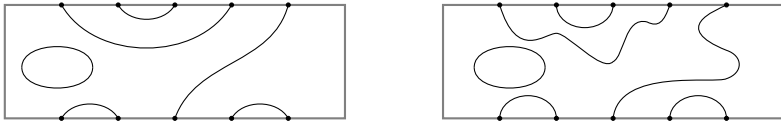


Figure 6. Isotopically equivalent diagrams.

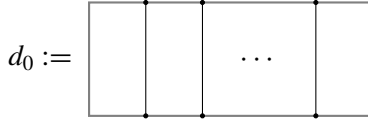


Figure 7. Unique loop-free diagram having only propagating edges.

Let d be a diagram and let e be an edge of d . If e is a closed curve occurring in d , then we call e a *loop*. For example, the diagram in Figure 5 (left) has a single loop. If e joins node i in the north face to node j' in the south face, then e is called a *propagating edge from i to j'* . If e is not propagating, loop or otherwise, it will be called *nonpropagating*. It is clear that there is a unique loop-free diagram consisting only of propagating edges. This diagram, denoted by d_0 , is depicted in Figure 7.

We wish to define an associative algebra that has the pseudo k -diagrams as a basis. Let R be a commutative ring with 1. The associative algebra \mathcal{P}_k over R is the free R -module having T_k as a basis. We define multiplication (referred to as diagram concatenation) in \mathcal{P}_k by defining multiplication in the case where d and d' are basis elements, and then extending bilinearly. If $d, d' \in T_k$, the product $d'd$ is the element of T_k obtained by placing d' on top of d , so that node i' of d' coincides with node i of d , and then removing the identified boundary to recover a standard k -box. If desired, one can then vertically rescale the resulting rectangle.

Example 3.4. Figure 8 depicts the product of two pseudo 5-diagrams in \mathcal{P}_5 .

We now restrict our attention to the base ring $\mathbb{Z}[\delta]$, which is the ring of polynomials in δ with integer coefficients. We define the *Temperley–Lieb diagram algebra* $\mathbb{D}TL(A_n)$ to be the associative $\mathbb{Z}[\delta]$ -algebra equal to the quotient of \mathcal{P}_{n+1} determined by the relation depicted in Figure 9.

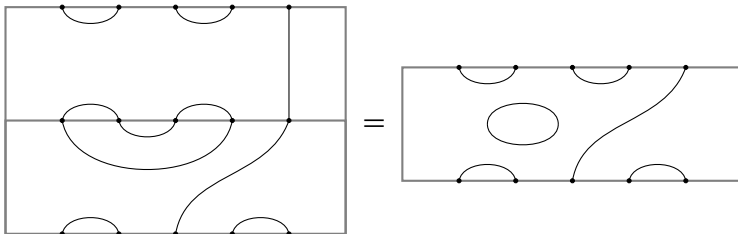


Figure 8. Example of multiplication in \mathcal{P}_5 .



Figure 9. Defining relation of $\mathbb{D}\text{TL}(A_n)$.

It is well known [Kauffman 1987; 1990] that $\mathbb{D}\text{TL}(A_n)$ is the free $\mathbb{Z}[\delta]$ -module with basis given by the elements of T_{n+1} having no loops. The multiplication is inherited from the multiplication on \mathcal{P}_{n+1} except we multiply by a factor of δ for each resulting loop and then discard the loop. It is easy to see that the identity in $\mathbb{D}\text{TL}(A_n)$ is the diagram d_0 given in Figure 7. Technically, the identity diagram is the image of d_0 in the quotient algebra, but there is no danger of identifying the two diagrams.

Example 3.5. Figure 10 depicts the product of three basis diagrams from $\mathbb{D}\text{TL}(A_4)$.

Define the *simple diagrams* d_1, d_2, \dots, d_n as in Figure 11. Note that the simple diagrams are elements of the basis for $\mathbb{D}\text{TL}(A_n)$. It turns out [Kauffman 1987; 1990] (and follows from Proposition 3.6 below) that the set of loop-free diagrams of $\mathbb{D}\text{TL}(A_n)$ is generated as a unital algebra by the set of simple diagrams $\{d_1, d_2, \dots, d_n\}$. In fact, $\mathbb{D}\text{TL}(A_n)$ is often defined to be the unital $\mathbb{Z}[\delta]$ -algebra generated by the simple diagrams subject to the relation given in Figure 9.

It is easy to verify that the simple diagrams of $\mathbb{D}\text{TL}(A_n)$ satisfy the defining relations of $\text{TL}(A_n)$. That is, we have

- (1) $d_i d_i = \delta d_i$ for all i ;
- (2) $d_i d_j = d_j d_i$ when $|i - j| > 1$;
- (3) $d_i d_j d_i = d_i$ when $|i - j| = 1$.

For example, Figure 12 illustrates the third relation above for the case $j = i + 1$. Indeed, $\text{TL}(A_n)$ and $\mathbb{D}\text{TL}(A_n)$ are isomorphic as $\mathbb{Z}[\delta]$ -algebras (for instance, see [Kauffman 1990, §3]).

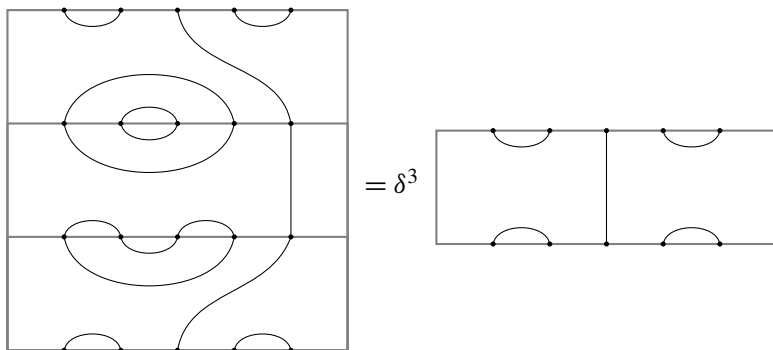


Figure 10. Example of multiplication in $\mathbb{D}\text{TL}(A_4)$.

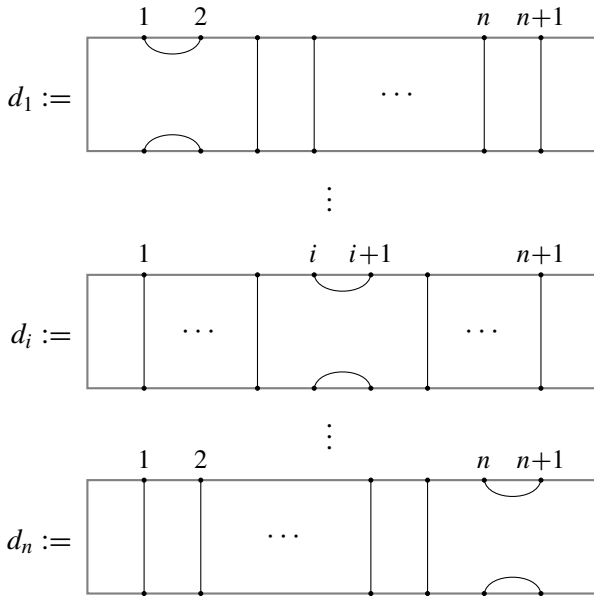


Figure 11. Simple diagrams.

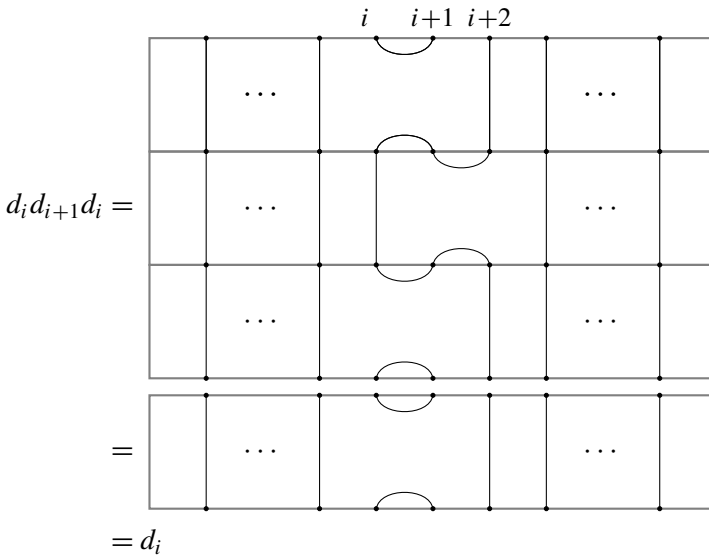


Figure 12. Special case of one of the relations in $\mathbb{D}\text{TL}(A_n)$.

Proposition 3.6. *Let $\theta : \text{TL}(A_n) \rightarrow \mathbb{D}\text{TL}(A_n)$ be the function determined by $b_i \mapsto d_i$. Then θ is a well-defined $\mathbb{Z}[\delta]$ -algebra isomorphism that maps the monomial basis of $\text{TL}(A_n)$ to the set of loop-free diagrams in $\mathbb{D}\text{TL}(A_n)$. \square*

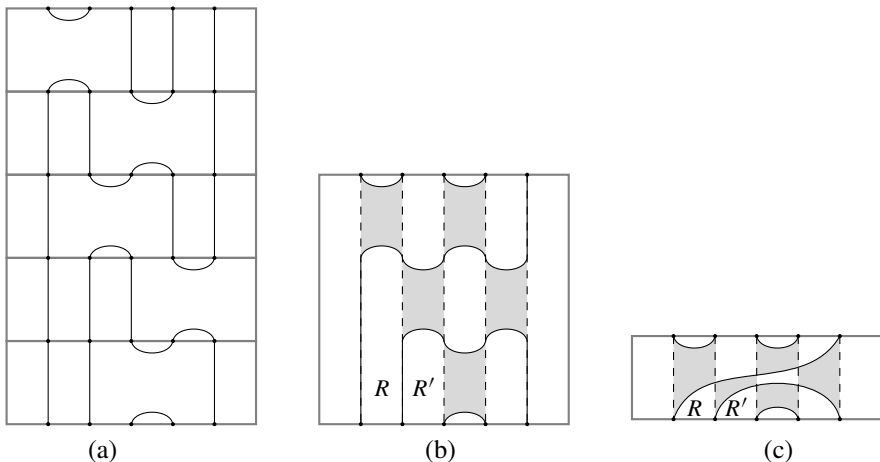


Figure 13. Multiplication of simple diagrams together with the corresponding simple representation and resulting product. The 1-regions of the simple representation and product have been shaded.

Let w be an element of $FC(A_n)$ and define d_w to be the image of the monomial b_w . It follows that if w has $s_{x_1} \cdots s_{x_k}$ as reduced expression, then $d_w = d_{x_1} \cdots d_{x_k}$. That is, given a reduced factorization for $w \in FC(A_n)$, we can easily obtain a reduced factorization of d_w in terms of simple diagrams. However, given a loop-free diagram d , it is more difficult to obtain a factorization. Resolving this difficulty is the content of Section 4.

Let $w = s_{x_1} \cdots s_{x_k}$ be a reduced expression for $w \in FC(A_n)$. For each simple diagram d_{x_i} , fix a concrete representation such that the propagating edges are straight and the pair of nonpropagating edges never double-back on themselves (i.e., the nonpropagating edges never intersect any vertical line more than once). Now, consider the concrete diagram that results from concatenating the concrete simple diagrams d_{x_1}, \dots, d_{x_k} , rescaling vertically to recover the standard $(n+1)$ -box, but not deforming any of the nonpropagating edges. Since w is FC and vertical equivalence respects commutation, given any two reduced expressions for w , the corresponding concrete diagrams constructed as above will be vertically equivalent (see Remark 3.2). We define the corresponding vertical equivalence class to be the *simple representation* of d_w . The simple representation of d_w is designed to replicate the structure of the corresponding heap.

Example 3.7. Let $w = s_1 s_3 s_2 s_4 s_3$ be a reduced expression for $w \in FC(A_4)$. The factorization for d_w determined by w together with its simple representation is shown in Figure 13(a) and (b), respectively. The resulting product is d_w , which is shown in Figure 13(c). The shaded regions in Figure 13(b) and (c) indicate that the pair of edges bounding the top and bottom of the region arise from the same factor.

In light of [Proposition 3.6](#), it follows that if d is a loop-free diagram from $\mathbb{D}\text{TL}(A_n)$, then there exists a unique $w \in \text{FC}(A_n)$ such that $d_w = d$. The upshot is that it makes sense to refer to the simple representation of d .

4. Main results

In this section, we assume that all diagrams are loop-free and that no edge ever double-backs on itself (i.e., other than vertical propagating edges, edges never intersect any vertical line more than once). If d is a k -diagram, we section the corresponding k -box into *columns* by connecting node i in the north face to node i' in the south face. The i -th column C_i lies between nodes i and $i + 1$. The connected components of the complement of the edges in each column are called *regions*. For example, the columns and regions for the diagram given in [Figure 14\(a\)](#) are depicted in [Figure 14\(b\)](#).

Lemma 4.1. *The number of edges within a single column of a diagram is even.*

Proof. This is clear for the simple representation of a diagram as each simple diagram d_i contributes precisely two edges to the column C_i . Isotopically deforming

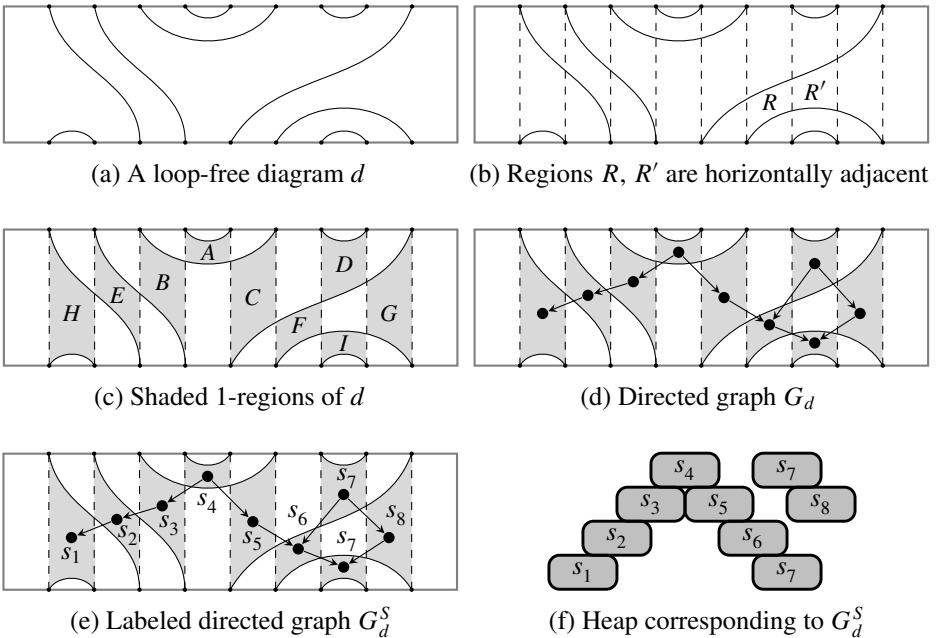


Figure 14. Shaded 1-regions, directed graph G_d , and labeled directed graph G_d^S for a diagram d together with the lattice point representation of the corresponding heap.

edges (while avoiding edges doubling-back on themselves) does not change the parity of the number of edges within the column. \square

Lemma 4.2. *The number of regions in each column is odd.*

Proof. Since the number of edges within any column is even, there must be an odd number of regions within each column. \square

Note that isotopically deforming the edges of a concrete diagram preserves the relative adjacency of the regions in each column. We say that regions R and R' of column C are *vertically adjacent* if they are adjacent across a common edge. Within a single column, we will label the first region just below the north face with a 0. Moving south, the next region will be labeled with a 1 and we continue this way, alternating labels 0 and 1. We will refer to the labeled regions as 0-regions and 1-regions, respectively. By Lemma 4.2, it is clear that the southernmost region in each column is a 0-region. Figure 14(c) depicts a diagram and its 0-regions and 1-regions, where we have shaded the 1-regions.

Observe that if d is a diagram from $\mathbb{DTL}(A_n)$, then each 1-region in column C_i of the simple representation for d corresponds precisely to the regions bounded above and below by the pair of edges corresponding to a unique factor d_i .

Suppose R and R' are regions of adjacent columns C and C' , respectively, of some diagram d . We say that R and R' are *horizontally adjacent* if there exist points p and p' in R and R' , respectively, such that the line segment joining p and p' does not cross any edge of d . Loosely speaking, R and R' are horizontally adjacent if they are adjacent across the common vertical boundary of C and C' .

Since we forbid edges from doubling-back on themselves, horizontal adjacency of regions is preserved when isotopically deforming the edges of d . This implies that horizontal adjacency is well-defined.

Figure 14(b) depicts two horizontally adjacent regions, R and R' . However, the regions labeled R and R' in the simple representation depicted in Figure 13(b) may appear at first glance to be horizontally adjacent, but they are not. This is evident by looking at the corresponding regions R and R' in Figure 13(c).

If R is a region of column C in diagram d , then the *depth* of R , $\text{depth}(R)$, is defined to be the number of regions in C strictly between the north face of d and R . For example, we have $\text{depth}(R) = 1$ and $\text{depth}(R') = 2$ for the regions R and R' in Figure 14(b). Note that for any diagram, the northernmost region in each column has depth 0. Moreover, every 1-region has an odd depth while every 0-region has an even depth.

Lemma 4.3. *If R and R' are horizontally adjacent regions of a diagram d , then*

- (1) $|\text{depth}(R) - \text{depth}(R')| = 1$, and
- (2) R is a 1-region if and only if R' is a 0-region.

Proof. Induction on depth quickly yields (1), while (2) is an immediate consequence of (1). \square

We say that regions R and R' of a diagram d are *diagonally adjacent* if there exists a region S that is vertically adjacent to R and horizontally adjacent to R' . In particular, if S lies below R , then we write $R \rightarrow R'$.

Lemma 4.4. *If $R \rightarrow R'$ in a diagram d , then R is a 1-region if and only if R' is a 1-region.*

Proof. The result follows immediately from the construction of 0-regions and 1-regions together with Lemma 4.3. \square

Example 4.5. In Figure 14(c), we see that

$$A \rightarrow B \rightarrow E \rightarrow H, \quad A \rightarrow C \rightarrow F \rightarrow I, \quad D \rightarrow F \rightarrow I, \quad \text{and} \quad D \rightarrow G \rightarrow I.$$

Remark 4.6. If C_i is not the leftmost or rightmost column, edges bounding 1-regions in column C_i must pass into its adjacent columns unless an edge connects directly to a node at the top or bottom of C_i . In the leftmost column, C_1 , no edge will pass through to the left. Similarly, in the rightmost column, C_n , no edge will pass through to the right. This implies that if R and R' are both 1-regions in the same column C_i with $\text{depth}(R') = \text{depth}(R) + 2$ (i.e., R and R' are consecutive 1-regions in C_i with R' below R), then there exist 1-regions T and T' in C_{i-1} and C_{i+1} , respectively, such that $R \rightarrow T \rightarrow R'$ and $R \rightarrow T' \rightarrow R'$. Loosely speaking, this determines a local checkerboard pattern of 1-regions, as seen in Figure 14(c).

The checkerboard pattern of 0-regions and 1-regions motivates the following definition. Let d be a diagram having 1-regions R_1, \dots, R_n . Define G_d to be the directed graph having

- (1) vertex set $V(G_d) := \{R_1, \dots, R_n\}$ and
- (2) directed edges (R_k, R_l) whenever $R_k \rightarrow R_l$.

Since we require the edges of d to not double-back on themselves, it is clear that G_d is independent of choice of concrete representation for d ; indeed, isotopically deforming the edges and rescaling the rectangle preserves horizontal and vertical adjacency and so diagonal adjacency is also preserved. In particular, if w indexes d , then we can construct G_d using the simple representation of d_w .

Figure 14(d) shows the directed graph G_d for the diagram d given in Figure 14(a). Observe that directed paths correspond to chains of diagonally adjacent regions.

Next, we will append labels from the generating set of the Coxeter group to the vertices of G_d . Define the vertex labeling function $\nu : V(G_d) \rightarrow S$ as follows. If R is a 1-region that lies in column i , then $\nu(R) = s_i$. That is, each region is labeled with the generator of the corresponding column. Now, define G_d^S to be the directed

graph G_d together with the labels on the vertices assigned by ν . Figure 14(e) shows the labeled directed graph G_d^S for the diagram d given in Figure 14(a).

Each labeled directed graph G_d^S naturally corresponds to a unique labeled Hasse diagram of a heap for some element in $W(A_n)$. It follows from Remark 4.6 and Proposition 2.6 that this element is FC. Figure 14(f) shows the heap that corresponds to the diagram d given in Figure 14(a). It remains to show that the heap determined by G_d^S corresponds to the group element that indexes the diagram d .

Since diagonal adjacency is preserved when isotopically deforming the edges of d , as in the simple representation, a 1-region R in column C_i is bounded above and below by a pair of edges corresponding to the simple diagram d_i . This region

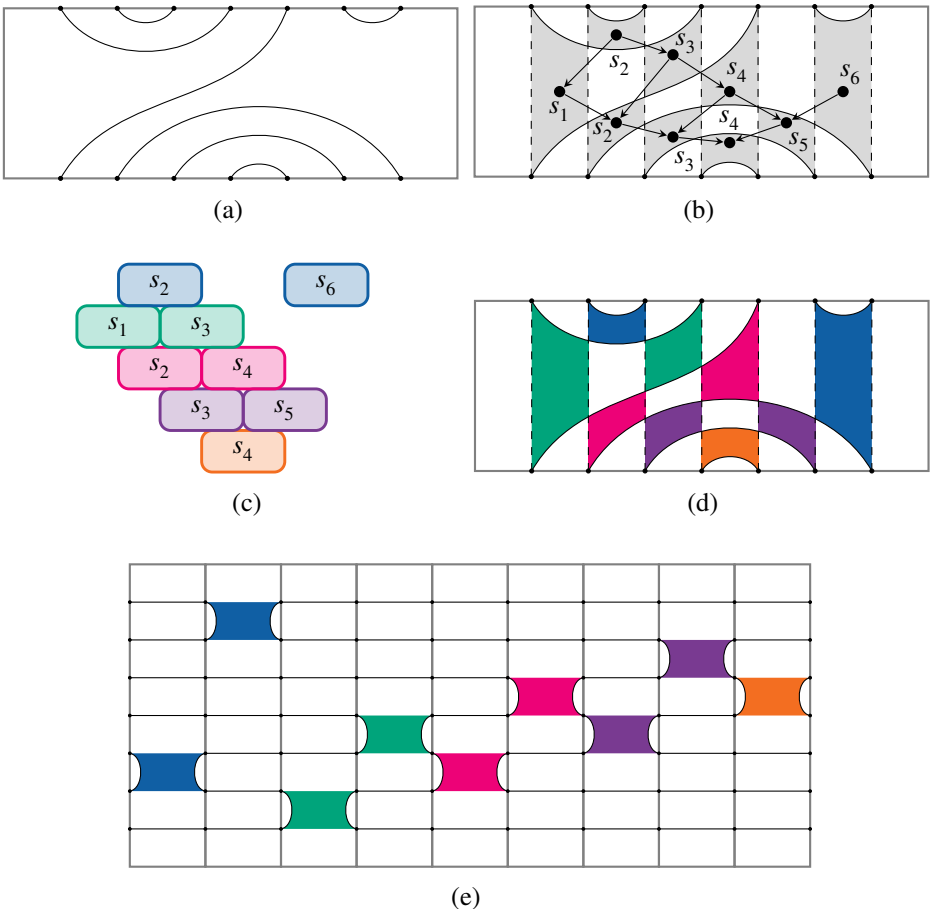


Figure 15. Given a diagram d , we can obtain a reduced factorization by constructing the corresponding labeled directed graph G_d^S , which yields the canonical representation of the heap that indexes d . We have color-coded the corresponding 1-regions and entries of the heap.

is labeled s_i in G_d^S . The structure of G_d^S determines $w \in \text{FC}(A_n)$ satisfying $d = d_w$, and it follows that G_d^S corresponds to $H(w)$. Note that we find w by writing the elements of S corresponding to the labels of each row of the heap left to right starting from the top row and working toward the bottom of the heap.

The above discussion together with the preceding lemmas justifies the following theorem.

Theorem 4.7. *If d is a loop-free diagram in $\text{TL}(A_n)$, then d is indexed by the heap determined by G_d^S . \square*

An immediate consequence of the above theorem is that we nonrecursively obtain a factorization of a diagram d by reading off from top to bottom and left to right the entries of the heap determined by G_d^S . If we choose the canonical representation of the heap, then the factorization of d corresponds to the Cartier–Foata normal form of [Cartier and Foata 1969; Green 2006]. Our construction also yields the following corollary, which appeared independently as Lemma 3.3 in [Green 1998].

Corollary 4.8. *If d is a loop-free diagram in $\text{TL}(A_n)$, then the number of occurrences of the simple diagram d_i in any factorization for d is equal to half the number of edges passing through the column C_i .*

Example 4.9. Consider the diagram d given in Figure 15(a). After forming columns, we obtain a checkerboard of 0-regions and 1-regions, which yields the labeled directed graph G_d^S depicted in Figure 15(b). Then G_d^S determines the canonical representation of the heap given in Figure 15(c), where each row of the heap has a unique color. In Figure 15(d), we have color-coded the 1-regions of d to match the corresponding entries in the heap. By reading off the entries of the heap, we see that $d = d_w$, where

$$w = s_2 s_6 s_1 s_3 s_2 s_4 s_3 s_5 s_4.$$

Equivalently, we obtain the factorization

$$d = d_2 d_6 d_1 d_3 d_2 d_4 d_3 d_5 d_4,$$

which is shown (rotated counterclockwise by a quarter turn in the interest of space) in Figure 15(e).

5. Closing remarks

If (W, S) is a Coxeter system of type Γ , the associated Hecke algebra $\mathcal{H}(\Gamma)$ is an algebra with a basis given by $\{T_w \mid w \in W\}$ and relations that deform the relations of W by a parameter q . As mentioned in Section 1, the ordinary Temperley–Lieb algebra $\text{TL}(A_n)$ is a quotient of the corresponding Hecke algebra $\mathcal{H}(A_n)$. This realization of the Temperley–Lieb algebra as a Hecke algebra quotient was generalized by Graham [1995] to the case of an arbitrary Coxeter system. In

general, the Temperley–Lieb algebra $TL(\Gamma)$ is a quotient of $\mathcal{H}(\Gamma)$ having several bases indexed by the FC elements of W [Graham 1995, Theorem 6.2].

When a faithful diagrammatic representation of $TL(\Gamma)$ is known to exist, multiplication in the diagram algebra is given by applying local combinatorial rules to the diagrams. In each case, one can choose a basis for the diagram algebra so that each basis diagram is indexed by an FC element, where the diagrams indexed by the distinguished generators of the Coxeter group form a set of “simple diagrams” that generate the algebra. Every factorization of a basis diagram in terms of simple diagrams corresponds precisely to a factorization of the FC element that indexes the diagram.

Given a reduced expression for an FC element, obtaining the corresponding diagram is straightforward. All one needs to do is concatenate the sequence of simple diagrams determined by the reduced expression and then apply the appropriate local combinatorial rules in the diagram algebra. However, it is another matter to reverse this process. That is, given a basis diagram, can one obtain a factorization in terms of simple diagrams, or equivalently obtain a reduced expression for the FC element that indexes the diagram? [Theorem 4.7](#) answers this question in the affirmative in the case of type A_n . What happens with the other types where faithful diagrammatic representations are known to exist? For example, can we find factorization algorithms for the Temperley–Lieb diagram algebras of types B_n , D_n , E_n , \tilde{A}_n , and \tilde{C}_n ?

References

- [Billey and Jones 2007] S. C. Billey and B. C. Jones, “Embedded factor patterns for Deodhar elements in Kazhdan–Lusztig theory”, *Ann. Comb.* **11**:3–4 (2007), 285–333. [MR](#) [Zbl](#)
- [Cartier and Foata 1969] P. Cartier and D. Foata, *Problèmes combinatoires de commutation et réarrangements*, Lecture Notes in Mathematics **85**, Springer, Berlin, 1969. [MR](#) [Zbl](#)
- [Ernst 2010] D. C. Ernst, “Non-cancellable elements in type affine C Coxeter groups”, *Int. Electron. J. Algebra* **8** (2010), 191–218. [MR](#) [Zbl](#)
- [Geck and Pfeiffer 2000] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori–Hecke algebras*, London Mathematical Society Monographs (N. S.) **21**, Oxford University Press, New York, NY, 2000. [MR](#) [Zbl](#)
- [Graham 1995] J. J. Graham, *Modular representations of Hecke algebras and related algebras*, thesis, University of Sydney, 1995.
- [Green 1998] R. M. Green, “Generalized Temperley–Lieb algebras and decorated tangles”, *J. Knot Theory Ramifications* **7**:2 (1998), 155–171. [MR](#) [Zbl](#)
- [Green 2006] R. M. Green, “Star reducible Coxeter groups”, *Glasg. Math. J.* **48**:3 (2006), 583–609. [MR](#) [Zbl](#)
- [Humphreys 1990] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, 1990. [MR](#) [Zbl](#)
- [Jones 1999] V. F. R. Jones, “Planar algebras, I”, preprint, 1999. [arXiv](#)
- [Kauffman 1987] L. H. Kauffman, “State models and the Jones polynomial”, *Topology* **26**:3 (1987), 395–407. [MR](#) [Zbl](#)

- [Kauffman 1990] L. H. Kauffman, “An invariant of regular isotopy”, *Trans. Amer. Math. Soc.* **318**:2 (1990), 417–471. [MR](#) [Zbl](#)
- [Penrose 1971] R. Penrose, “Angular momentum: an approach to combinatorial space-time”, pp. 151–180 in *Quantum theory and beyond*, edited by T. Bastin, Cambridge University Press, 1971.
- [Stembridge 1996] J. R. Stembridge, “On the fully commutative elements of Coxeter groups”, *J. Algebraic Combin.* **5**:4 (1996), 353–385. [MR](#) [Zbl](#)
- [Temperley and Lieb 1971] H. N. V. Temperley and E. H. Lieb, “Relations between the ‘percolation’ and ‘colouring’ problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the ‘percolation’ problem”, *Proc. Roy. Soc. London Ser. A* **322**:1549 (1971), 251–280. [MR](#) [Zbl](#)
- [Viennot 1986] G. X. Viennot, “Heaps of pieces, I: Basic definitions and combinatorial lemmas”, pp. 321–350 in *Combinatoire énumérative* (Montreal, QC, 1985), edited by G. Labelle and P. Leroux, Lecture Notes in Mathematics **1234**, Springer, Berlin, 1986. [MR](#) [Zbl](#)

Received: 2015-09-05

Revised: 2016-01-10

Accepted: 2016-01-14

dana.ernst@nau.edu

Department of Mathematics and Statistics, Northern Arizona University, Flagstaff, AZ 86011, United States

mgh64@nau.edu

Department of Mathematics and Statistics, Northern Arizona University, Flagstaff, AZ 86011, United States

sarah.salmon@colorado.edu

Department of Mathematics, University of Colorado Boulder, Boulder, CO 80309, United States

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Suzanne Lenhart	University of Tennessee, USA
John V. Baxley	Wake Forest University, NC, USA	Chi-Kwong Li	College of William and Mary, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Robert B. Lund	Clemson University, USA
Martin Bohner	Missouri U of Science and Technology, USA	Gaven J. Martin	Massey University, New Zealand
Nigel Boston	University of Wisconsin, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA	Emil Minchev	Ruse, Bulgaria
Pietro Cerone	La Trobe University, Australia	Frank Morgan	Williams College, USA
Scott Chapman	Sam Houston State University, USA	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Joshua N. Cooper	University of South Carolina, USA	Zuhair Nashed	University of Central Florida, USA
Jem N. Corcoran	University of Colorado, USA	Ken Ono	Emory University, USA
Toka Diagana	Howard University, USA	Timothy E. O'Brien	Loyola University Chicago, USA
Michael Dorff	Brigham Young University, USA	Joseph O'Rourke	Smith College, USA
Sever S. Dragomir	Victoria University, Australia	Yuval Peres	Microsoft Research, USA
Behrouz Emamizadeh	The Petroleum Institute, UAE	Y.-F. S. Pétermann	Université de Genève, Switzerland
Joel Foisy	SUNY Potsdam, USA	Robert J. Plemmons	Wake Forest University, USA
Erin W. Fulp	Wake Forest University, USA	Carl B. Pomerance	Dartmouth College, USA
Joseph Gallian	University of Minnesota Duluth, USA	Vadim Ponomarenko	San Diego State University, USA
Stephan R. Garcia	Pomona College, USA	Bjorn Poonen	UC Berkeley, USA
Anant Godbole	East Tennessee State University, USA	James Propp	U Mass Lowell, USA
Ron Gould	Emory University, USA	József H. Przytycki	George Washington University, USA
Andrew Granville	Université Montréal, Canada	Richard Rebarber	University of Nebraska, USA
Jerrold Griggs	University of South Carolina, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Jim Haglund	University of Pennsylvania, USA	James A. Sellers	Penn State University, USA
Johnny Henderson	Baylor University, USA	Andrew J. Sterge	Honorary Editor
Jim Hoste	Pitzer College, USA	Ann Trenk	Wellesley College, USA
Natalia Hritonenko	Prairie View A&M University, USA	Ravi Vakil	Stanford University, USA
Glenn H. Hurlbert	Arizona State University, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
Charles R. Johnson	College of William and Mary, USA	Ram U. Verma	University of Toledo, USA
K. B. Kulasekera	Clemson University, USA	John C. Wierman	Johns Hopkins University, USA
Gerry Ladas	University of Rhode Island, USA	Michael E. Zieve	University of Michigan, USA

PRODUCTION

Silvio Levy, Scientific Editor


Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2017 is US \$175/year for the electronic version, and \$235/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2017 Mathematical Sciences Publishers

involve

2017

vol. 10

no. 1

Intrinsically triple-linked graphs in $\mathbb{R}P^3$	1
JARED FEDERMAN, JOEL FOISY, KRISTIN MCNAMARA AND EMILY STARK	
A modified wavelet method for identifying transient features in time signals with applications to bean beetle maturation	21
DAVID MCMORRIS, PAUL PEARSON AND BRIAN YURK	
A generalization of the matrix transpose map and its relationship to the twist of the polynomial ring by an automorphism	43
ANDREW MCGINNIS AND MICHAELA VANCLIFF	
Mixing times for the rook's walk via path coupling	51
CAM MCLEMAN, PETER T. OTTO, JOHN RAHMANI AND MATTHEW SUTTER	
The lifting of graphs to 3-uniform hypergraphs and some applications to hypergraph Ramsey theory	65
MARK BUDDEN, JOSH HILLER, JOSHUA LAMBERT AND CHRIS SANFORD	
The multiplicity of solutions for a system of second-order differential equations	77
OLIVIA BENNETT, DANIEL BRUMLEY, BRITNEY HOPKINS, KRISTI KARBER AND THOMAS MILLIGAN	
Factorization of Temperley–Lieb diagrams	89
DANA C. ERNST, MICHAEL G. HASTINGS AND SARAH K. SALMON	
Prime labelings of generalized Petersen graphs	109
STEVEN A. SCHLUCHTER, JUSTIN Z. SCHROEDER, KATHRYN COKUS, RYAN ELLINGSON, HAYLEY HARRIS, ETHAN RARITY AND THOMAS WILSON	
A generalization of Zeckendorf's theorem via circumscribed m -gons	125
ROBERT DORWARD, PARI L. FORD, EVA FOURAKIS, PAMELA E. HARRIS, STEVEN J. MILLER, EYVINDUR PALSSON AND HANNAH PAUGH	
Loewner deformations driven by the Weierstrass function	151
JOAN LIND AND JESSICA ROBINS	
Rank disequilibrium in multiple-criteria evaluation schemes	165
JONATHAN K. HODGE, FAYE SPRAGUE-WILLIAMS AND JAMIE WOELK	