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We study the dynamical behavior of functions on vertices of a graph that are contractions in the graph metric. We show that the fixed point set of such functions must be convex. If a function has no fixed points and the graph is a tree, we prove that every dynamical cycle must have an even period and the function behaves eventually like a symmetry.

## 1. Introduction

This work was inspired by dynamics of analytic functions on the unit disk. The key property of such functions is the point-invariant Schwarz lemma, i.e., that analytic functions are contractions in the hyperbolic metric of the disk. This property allows the proof of various results about iteration of analytic functions; see, for example, the survey paper [Poggi-Corradini 2011].

Our purpose is to study dynamics of contractions in a discrete setting. In particular, we study dynamics on finite graphs (in most cases, trees). A connected graph can be considered as a discrete metric space of vertices with the *graph metric*. Let  $G = (V, E)$  be a finite, connected, simple graph with the set of vertices  $V$  and the set of edges  $E$ . Then for all vertices  $x, y \in V$ , we say the distance between  $x$  and  $y$ , denoted  $d(x, y)$ , is the number of edges in the shortest path connecting  $x$  to  $y$ . Such path is called a *geodesic*. Note that trees as metric spaces are 0-hyperbolic [Anderson 1999], so we expect them to have some similar properties to the unit disk with hyperbolic metric.

We wish to study contractions (in the graph metric) on the vertices of a graph. Let  $f$  be a function on the vertices of  $G$  to the vertices of  $G$ . We say  $f$  is a *contraction* if, for all vertices  $x, y \in V$ , we have  $d(f(x), f(y)) \leq d(x, y)$ . We will need some terminology from dynamics. Let  $f$  be a function. We denote by  $f^{on}(x) = f \circ f \circ f \circ \cdots \circ f(x)$  ( $n$  terms) the  $n$ -th *iterate* of  $f$ . If for some point  $x$  and some positive integer  $n$ , we have  $f^{on}(x) = x$ , then we say  $x$  is a periodic point,  $x$  lies on a *dynamical cycle* of  $f$  of period  $n$ , or that  $x$  lies on a *dynamical  $n$ -cycle*

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of  $f$ . If  $f(x) = x$ , we say  $x$  is a *fixed point* of  $f$ . We use the term *dynamical cycle* to distinguish these cycles from the graph cycles.

It is easy to show by induction that, given a contraction  $f$ , the map  $f^{\circ n}$  is also a contraction for any positive integer  $n$ . Dynamical cycles and fixed points will be the main focus of our study.

2. Fixed point sets

Our goal is to characterize the set of fixed points of a contraction on graph vertices. Note that in the general case, the fixed point set can be empty:

**Example 2.1.** Let  $G_1$  be a graph with four vertices  $x, y, z, w$ . Let  $f$  be a function on the vertices of  $G_1$  defined by  $f(x) = y, f(y) = z, f(z) = w$ , and  $f(w) = x$ . Then  $\{x, y, z, w\}$  forms a dynamical 4-cycle of  $f$  (see Figure 1). The map  $f$  is clearly a contraction since for all  $a, b \in \{x, y, w, z\}$ , we have  $d(f(a), f(b)) = d(a, b)$ . In this case, the set of fixed points of  $f$  is empty.

**Example 2.2.** Let  $G_2$  be a graph with vertices  $x_0, x_1, x_2, y_0, y_1, z_0$  and  $z_1$  as shown in Figure 2. Let  $f$  be a contraction on the vertices of  $G_2$  such that  $x_0, x_1, x_2$  are fixed by  $f$ , and  $\{y_0, y_1\}$  and  $\{z_0, z_1\}$  are dynamical 2-cycles of  $f$ .

Note that one main difference between the two examples is that for any two vertices in  $G_2$ , the geodesic connecting them is unique, whereas this is not the case with  $G_1$ . Notice also that for any two fixed points in  $G_2$ , the geodesic connecting them contains only fixed points.

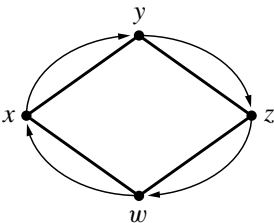


Figure 1. Dynamical 4-cycle.

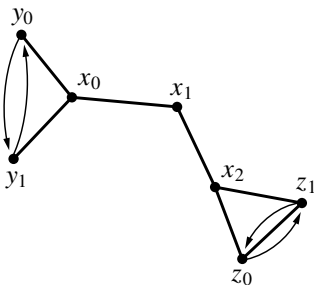
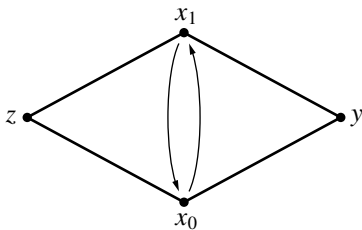


Figure 2. Cycles and fixed points.



**Figure 3.** The graph  $G_3$  with nonunique geodesics.

**Definition 2.3.** Let  $G = (V, E)$  be a graph and let  $H \subset V$ . We say  $H$  is *convex* if for any two vertices in  $H$ , the geodesic connecting them contains only vertices in  $H$ . (See, for example, [Gross and Yellen 2006].)

Thus in [Example 2.2](#), the set of fixed points of  $f$  is convex. In fact, this is true in general:

**Theorem 2.4.** Let  $G = (V, E)$  be a graph such that the geodesic between any two vertices is unique. Let  $f$  be a contraction on the vertices of  $G$ . Then the set of fixed points of  $f$  is convex.

*Proof.* Let  $x, y \in V$  be fixed by  $f$ . Let  $L$  be the unique geodesic connecting them. Let  $z \in L$ . We need to show that  $f(z) = z$ . We will first show that  $f(z) \in L$  and it will follow that  $f(z) = z$ .

By way of contradiction, suppose  $f(z) \notin L$ . Then there exist unique geodesics connecting  $x$  to  $f(z)$  and  $y$  to  $f(z)$ , respectively. We can concatenate these geodesics to construct a walk  $K$  connecting  $x$  to  $y$ , so the length of  $K$  is  $d(x, f(z)) + d(f(z), y)$  and the length of  $L$  is  $d(x, z) + d(z, y)$ . Since  $f$  is a contraction and  $x$  and  $y$  are fixed points, we have  $d(f(z), x) \leq d(z, x)$  and  $d(f(z), y) \leq d(z, y)$ . Then it follows that  $d(x, f(z)) + d(f(z), y) \leq d(x, z) + d(z, y)$ . If  $d(x, f(z)) + d(f(z), y) = d(x, z) + d(z, y)$ , then  $L$  is not a unique geodesic between  $x$  and  $y$ , a contradiction. If  $d(x, f(z)) + d(f(z), y) < d(x, z) + d(z, y)$ , then  $K$  is shorter than  $L$ , which is also a contradiction. Thus it must be that  $z \in L$ .

Now we will show  $f(z) = z$ . Suppose  $f(z) \neq z$ . Since  $f(z)$  lies on the geodesic  $L$  connecting  $x$  to  $y$ , we have  $d(x, z) + d(z, y) = d(x, f(z)) + d(f(z), y) = d(x, y)$ . We can assume without loss of generality that  $d(x, f(z)) < d(x, z)$ , in which case we obtain  $d(y, f(z)) = d(x, y) - d(x, f(z)) > d(x, y) - d(x, z) = d(y, z)$ , contradicting the fact that  $f$  is a contraction. Thus we conclude that  $f(z) = z$ .  $\square$

Note that if for any two points in  $G$  the geodesic connecting them is not unique then the conclusion of [Theorem 2.4](#) does not necessarily hold, as can be seen in the following counterexample.

**Example 2.5.** Let  $G_3$  be a graph with vertices  $x_0, x_1, y$  and  $z$  as shown in [Figure 3](#).

Let  $f$  be a contraction such that the vertices  $z$  and  $y$  are fixed and the points  $x_0$  and  $x_1$  form a dynamical 2-cycle. Note that the geodesic connecting  $z$  to  $y$  is

not unique, since the path from  $z$  to  $y$  through  $x_0$  is the same length as the path through  $x_1$ . Despite the fact that  $z$  and  $y$  are fixed and that  $x_0, x_1$  lie on the geodesics connecting them,  $x_0$  and  $x_1$  are clearly not fixed. Thus the conclusion of [Theorem 2.4](#) does not hold in this case.

**Corollary 2.6.** *Let  $G = (V, E)$  be a graph such that for any two vertices in  $G$  the geodesic connecting them is unique. Let  $f$  be a contraction on  $V$ . Suppose  $f$  has a dynamical cycle  $J$  of period  $k$ . Let  $z$  be a point which lies on the geodesic connecting two consecutive points in  $J$ . Then  $z$  lies on a dynamical cycle whose period divides  $k$ .*

*Proof.* Let  $x, y \in J$ . Let  $z \in V$  such that  $z$  lies on the geodesic between  $x$  and  $y$ . Since  $J$  is a dynamical cycle of period  $k$ , we know  $f^{ok}(x) = x$  and  $f^{ok}(y) = y$ . Thus  $x$  and  $y$  are fixed by the  $k$ -th iterate of  $f$ . Since  $f$  is a contraction, any iterate of  $f$  is also a contraction. Thus [Theorem 2.4](#) applied to  $f^{ok}$  implies  $f^{ok}(z) = z$ . So  $z$  must lie on a dynamical cycle whose period divides  $k$ .  $\square$

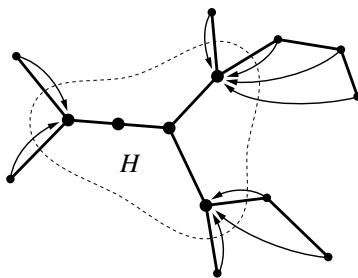
Now we will consider a particular case when the graph is a tree. For any tree, a path connecting any two points is unique, hence geodesics are unique, so [Theorem 2.4](#) holds. But the converse is also true for trees: any convex set of vertices will be a fixed point set for some contraction.

We will need the following property of a tree structure: in a tree, a concatenation of two geodesics from  $x$  to  $y$  and from  $y$  to  $z$  is either a geodesic from  $x$  to  $z$  or a walk that follows the geodesic connecting  $x$  to  $y$  until the first common point of the two geodesics,  $y'$ , then follows the geodesic from  $y'$  to  $y$ , then goes back to  $y'$  along the same geodesic and finally follows the geodesic from  $y'$  to  $z$ . Note that concatenation of geodesics from  $x$  to  $y'$  and from  $y'$  to  $z$  will form a geodesic that connects  $x$  to  $z$ .

**Proposition 2.7.** *Let  $T = (V, E)$  be a tree and  $H \subset V$  be convex set. Then there exists a contraction  $f$  such that  $H$  is the fixed point set of  $f$ .*

*Proof.* Given  $H$ , we define the desired contraction  $f$  as follows: for all  $x \in V$ ,  $f(x) = y$ , where  $y \in H$  is the closest vertex to  $x$  in  $H$ ; see [Figure 4](#). Note that such a  $y$  is unique. Indeed, suppose  $y_1, y_2 \in H$  are at the same shortest distance from  $x \notin H$ . Apply the property mentioned above the proposition to the concatenation of geodesics connecting  $y_1$  to  $x$  and  $x$  to  $y_2$ . If it is a geodesic, then  $x \in H$ , which is a contradiction. If instead there is a common point  $y'$ , then  $y' \in H$  and it is closer to  $x$  than  $y_1$  and  $y_2$  are, again a contradiction. Thus the point  $y$  is unique and the function  $f$  is well-defined. Also,  $H$  is clearly fixed point set of  $f$ .

Now we need to show that  $f$  is a contraction. Let  $f(x_1) = y_1$  and  $f(x_2) = y_2$ . Consider a walk following the geodesic from  $x_1$  to  $y_1$ , then from  $y_1$  to  $y_2$ . If there is a common point of these geodesics other than  $y_1$ , then this point is in  $H$  and within a shorter distance to  $x_1$  than  $y_1$ , which contradicts the construction of  $y_1$ . So the



**Figure 4.** Constructing the contraction  $f$  for a given convex subset of vertices  $H$ .

concatenation of these two geodesics is the geodesic from  $x_1$  to  $y_2$ . Similarly, the geodesic from  $y_1$  to  $x_2$  passes through  $y_2$ , and finally, the geodesic from  $x_1$  to  $x_2$  is just a concatenation of those from  $x_1$  to  $y_1$ ,  $y_1$  to  $y_2$  and  $y_2$  to  $x_1$ . So we have

$$d(x_1, x_2) = d(x_1, y_1) + d(y_1, y_2) + d(y_2, x_2) \geq d(y_1, y_2),$$

and  $f$  is a contraction.  $\square$

### 3. Contractions with no fixed points

In the previous section, we characterized the set of fixed points of a contraction on the vertices of a graph with unique geodesics, in particular a tree. Next we want to consider the case when a contraction has no fixed points. Then there must exist a dynamical cycle. We will use the following property of periodic points:

**Lemma 3.1.** *Let  $G$  be a finite graph, and  $f$  be a contraction on vertices of  $G$ . If  $x$  and  $y$  are two periodic points of  $f$  (not necessarily from the same dynamical cycle), then  $d(f(x), f(y)) = d(x, y)$ .*

*Proof.* Assume  $x$  belongs to a dynamical  $m$ -cycle and  $y$  belongs to a dynamical  $n$ -cycle. Let  $K$  be a common multiple of  $m$  and  $n$ . Then we have

$$d(x, y) \geq d(f(x), f(y)) \geq \cdots \geq d(f^{\circ K}(x), f^{\circ K}(y)) = d(x, y).$$

So all inequalities must be, in fact, equalities and in particular,  $d(f(x), f(y)) = d(x, y)$ .  $\square$

Now let us introduce some notation. Let  $G = (V, E)$  be a graph and  $f$  a contraction on  $V$ . Let  $J \subset V$  be a dynamical cycle of  $f$ . Then we denote by  $J'$  the set of all vertices which lie on geodesics connecting consecutive points in  $J$ , together with the vertices in  $J$ .

**Theorem 3.2.** *Let  $T$  be a finite tree. Let  $f$  be a contraction on the vertices of  $T$ . If  $f$  has no fixed points, then  $f$  has a dynamical 2-cycle such that the points in the cycle are connected by an edge. Moreover, such a cycle is unique.*

*Proof.* Suppose  $f$  has no fixed points. Since the number of vertices of  $T$  is finite, every vertex of  $T$  either lies on a dynamical cycle of period greater than 1 or is eventually mapped into one. Let  $k$  be the smallest period of all dynamical cycles of  $f$ . Let  $J$  be a dynamical cycle of period  $k$  such that the distance between consecutive points in  $J$  is least among all dynamical cycles of  $f$  of period  $k$ . We want to show that  $k = 2$ .

We claim that for  $k > 2$  there must exist two geodesics connecting consecutive points in  $J$  that intersect at a point other than their endpoints. If not, the points in  $J$  would form a graph cycle, which is a contradiction since  $T$  is a tree. Thus there must exist two geodesics which intersect at a point which is not one of their endpoints.

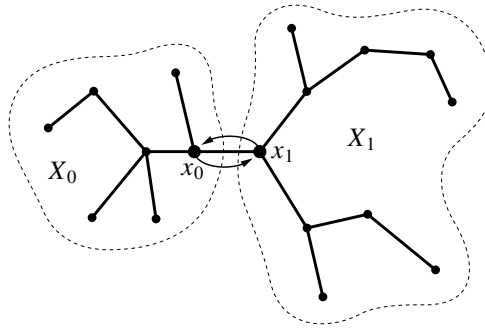
Suppose two nonconsecutive geodesics intersect at some point  $y$ . Then we claim that there must exist two consecutive geodesics which intersect at point  $z$  which is not one of their endpoints. Indeed, if we start from the point  $y$  of intersection of two nonconsecutive geodesics and follow one of the geodesics to the point  $x_j$  on the cycle  $J$ , then follow the next geodesic to the point  $x_{j+1} = f(x_j)$ , and so on, we will eventually return to the point  $y$ . Since the graph is a tree, the walk constructed this way must go over each edge in this walk at least twice. In particular, there must exist a vertex  $w$  which is farthest away from  $y$  on this walk and an edge  $\{w, z\}$  such that our walk will follow the edge from  $z$  to  $w$  and then immediately return to  $z$  through the same edge. Note that  $w$  must be an endpoint of two consecutive geodesics, because one geodesic cannot follow the same edge twice. Then  $z$  lies on the intersection of two consecutive geodesics.

Without loss of generality, let  $x_0, x_1, x_2$  be the endpoints of the two consecutive geodesics constructed above. By [Corollary 2.6](#),  $z$  must lie on a dynamical cycle whose period divides  $k$ , but since  $k$  is the smallest possible cycle length,  $z$  must lie on a dynamical  $k$ -cycle.

Since  $f$  is a contraction and  $x_0, x_1, x_2$  are points on a dynamical cycle,  $f$  must map the geodesic from  $x_0$  to  $x_1$  bijectively to the geodesic from  $x_1$  to  $x_2$ . Since  $z$  lies on the geodesic from  $x_0$  to  $x_1$ , the point  $f(z)$  must lie on the geodesic from  $x_1$  to  $x_2$ . Thus both  $z$  and  $f(z)$  lie on the geodesic from  $x_1$  to  $x_2$  and we have  $d(z, f(z)) < d(x_1, x_2) = d(x_0, x_1)$ . So we have found a dynamical  $k$ -cycle  $\{z, f(z), \dots, f^{\circ(k-1)}(z)\}$  such that the distance between two consecutive points in this cycle is less than  $d(x_0, x_1)$ .

This contradicts the way we selected  $J$ , so  $k$  must be equal to 2 and the geodesic from  $x_0$  to  $x_1$ , which is the same as the geodesic from  $x_1$  to  $x_0$ , must contain no other points. This means there is a dynamical 2-cycle  $\{x_0, x_1\}$  and  $x_0$  and  $x_1$  are connected by an edge.

Now we need prove that such a dynamical 2-cycle is unique. Let  $\{y_0, y_1\}$  be another such cycle. Without loss of generality assume that the distance  $a$  between  $x_0$  and  $y_0$  is the shortest among all distances from a point in  $\{x_0, x_1\}$  to a point in  $\{y_0, y_1\}$ . Now consider  $x_1$ ; it is connected to  $x_0$  by an edge. If  $x_1$  lies on the geodesic from  $x_0$



**Figure 5.** Unique 2-cycle  $\{x_0, x_1\}$  and sets  $X_0$  and  $X_1$ .

to  $y_0$ , then  $d(x_1, y_0) < d(x_0, y_0)$ , which contradicts the choice of  $x_0, y_0$ . Otherwise, the geodesic from  $y_0$  to  $x_1$  follows the geodesic from  $y_0$  to  $x_0$  and then the edge connecting  $x_0$  to  $x_1$ , so  $d(y_0, x_1) = a + 1$ . Similarly,  $d(x_0, y_1) = a + 1$ , and finally,  $d(x_1, y_1) = a + 2$ . But then  $d(x_1, y_1) = d(f(x_0), f(y_0)) > d(x_0, y_0)$ , which contradicts the assumption that  $f$  is a contraction.  $\square$

It will in fact turn out that every dynamical cycle of a contraction with no fixed points has even period. To prove this, we will need the following corollary to [Theorem 3.2](#). Let us introduce the following notation. Let  $\{x_0, x_1\}$  be the points in the 2-cycle constructed in [Theorem 3.2](#). We let  $X_0$  denote the set of all points which are within shorter distance to  $x_0$  than to  $x_1$ . Similarly we let  $X_1$  denote the set of all points which are within shorter distance to  $x_1$  than to  $x_0$ ; see [Figure 5](#).

**Corollary 3.3.** *Let  $T$  be a finite tree and  $f$  a contraction on the vertices of  $T$  such that  $f$  has no fixed points. Let  $\{x_0, x_1\}$  be the unique dynamical 2-cycle, where  $x_0$  and  $x_1$  are connected by an edge. Then for all vertices  $z$  that lie on any dynamical cycle, if  $z \in X_0$  (respectively  $X_1$ ), then  $f(z) \in X_1$  (respectively  $X_0$ ).*

*Proof.* Let  $z$  lie on a dynamical cycle and  $z \in X_0$ . By way of contradiction, suppose  $f(z) \in X_0$ . Let  $a = d(z, x_0)$ ; then  $d(z, x_1) = a + 1$ . By [Lemma 3.1](#),  $d(f(z), x_1) = a$ , and since  $f(z) \in X_0$ , we must have  $d(f(z), x_0) < d(f(z), x_1) = a$ . But by [Lemma 3.1](#) again,  $d(f(z), x_0) = d(f(z), f(x_1)) = d(z, x_1) = a + 1$ , which is a contradiction. So  $f(z) \in X_1$ .  $\square$

Note that if  $z$  is not a periodic point, then the above claim does not hold.

**Example 3.4.** Let  $T$  be a tree with vertices  $x_0, x_1$  and  $z$  such that there are edges between  $x_0$  and  $x_1$  and between  $x_0$  and  $z$ , and  $f$  be a contraction such that  $\{x_0, x_1\}$  forms a dynamical 2-cycle and  $f(z) = x_0$  (see [Figure 6](#)). Then  $f$  has no fixed points, and  $x_0$  and  $x_1$  form the unique 2-cycle connected by an edge. Since  $f(z) = x_0$ , we have  $z \in X_0$  and also  $f(z) \in X_0$ . Thus we see that if a point  $z$  is in  $X_0$  but does not lie on a dynamical cycle, it is not necessarily true that  $f(z) \in X_1$ .



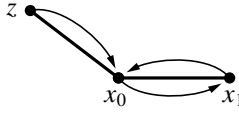


Figure 6

Now we are ready to prove the following:

**Theorem 3.5.** *Let  $T$  be a finite tree and  $f$  a contraction on the vertices of  $T$  such that  $f$  has no fixed points. Then every dynamical cycle of  $f$  has even period.*

*Proof.* Since  $f$  has no fixed points,  $f$  has a dynamical 2-cycle  $\{x_0, x_1\}$  whose points are connected by an edge and sets of vertices  $X_0$  and  $X_1$  as defined above. Let  $\{y_0, y_1, \dots, y_{n-1}\}$  be a dynamical  $n$ -cycle of  $f$ . Without loss of generality, suppose  $y_0 \in X_0$ . Then by [Corollary 3.3](#), we have  $y_1 \in X_1$ , and in general,  $y_{2k} \in X_0$  and  $y_{2k+1} \in X_1$ . If  $n$  is odd, then  $y_0 = f(y_{n-1}) \in X_1$ , which is a contradiction to  $y_0 \in X_0$ . Hence every dynamical cycle of  $f$  has even period.  $\square$

If a contraction  $f$  on the vertices of a tree  $T$  has no fixed points, then  $f$  eventually behaves like a symmetry. More precisely:

**Theorem 3.6.** *Let  $T = (V, E)$  be a finite tree and  $f$  a contraction on  $V$  without fixed points. Then there exists a subset  $H$  of  $V$  and a nonnegative integer  $N$  such that  $f^{\circ N}(V) = H$  and  $f$  is a symmetry on the connected subgraph induced by  $H$ . In particular, there is an edge in the subgraph such that two connected components obtained by removing this edge are isomorphic graphs and  $f$  is an isomorphism.*

*Proof.* Since  $T$  is finite and has no fixed points, each vertex of  $T$  will be mapped eventually to a point on a dynamical cycle. Thus there exists  $N$  such that  $f^{\circ N}(V) = H$  contains only periodic points of  $f$ . Note that by [Corollary 2.6](#), the subgraph induced by  $H$  is connected. Let  $\{x_0, x_1\}$  be the unique dynamical 2-cycle whose points are connected by an edge. Then by [Corollary 3.3](#), for all  $z \in H \cap X_0$ , we have  $f(z) \in H \cap X_1$  and for all  $z \in H \cap X_1$ , we have  $f(z) \in H \cap X_0$ . Moreover, since all points in  $H$  are periodic,  $f$  bijectively maps  $H \cap X_0$  to  $H \cap X_1$ . Now we need to show that any two vertices  $y, z$  in  $H \cap X_0$  are connected by an edge if and only if  $f(y)$  and  $f(z)$  are connected by an edge. But being connected by an edge is equivalent to  $d(y, z) = 1$ , and since by [Lemma 3.1](#),  $d(y, z) = d(f(y), f(z))$ , the required conclusion follows.  $\square$

## 4. Conclusion

Note that in the classical case of the unit disk in the complex plane, any analytic self-map of the disk always has a fixed point in the closed disk. This is the consequence of the classical Denjoy–Wolff theorem (see, for example, [\[Abate 1989\]](#)

and references therein). In our study, a contraction without fixed points must behave like a symmetry. Symmetries are contractions in the unit disk, but they are not analytic (in fact, they are anticonformal, i.e., they preserve the value of angles, but change their orientation). So we can say that our result agrees with the classical case.

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