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We construct a basis for free Lie algebras via a *left greedy* bracketing algorithm on Lyndon–Shirshov words. We use a new tool — the configuration pairing between Lie brackets and graphs of Sinha and Walter — to show that the left greedy brackets form a basis. Our constructions further equip the left greedy brackets with a dual monomial Lie coalgebra basis of *star* graphs. We end with a brief example using the dual basis of star graphs in a Lie algebra computation.

1. Introduction

Lie algebras are classical objects with applications in differential geometry, theoretical physics, and computer science. A Lie algebra is a vector space which has an extra nonassociative (bilinear) operation called a *Lie bracket*, written $[a, b]$. The Lie bracket operation satisfies anticommutativity and Jacobi relations:

(anticommutativity) $\quad 0 = [a, b] + [b, a],$

(Jacobi) $\quad 0 = [a, [b, c]] + [c, [a, b]] + [b, [c, a]].$

A free Lie algebra is a Lie algebra whose bracket operation satisfies no extra relations — only the two written above and any relations which can be generated by combining them together. For example,

$$[a, [b, c]] - [[a, b], c] = [[c, a], b]$$

is a relation for free Lie algebras since $[c, [a, b]] = -[[a, b], c]$ by anticommutativity, and similarly for $[b, [c, a]]$. Free Lie algebras are fundamental in that every Lie algebra can be written *via generators and relations* as a free Lie algebra with extra relations placed on its bracket operation.

Recall that a set of elements generates an algebra if all other elements in the algebra can be obtained via sums of products of elements from the set. A minimal

generating set is called an *algebra basis*. We are interested in linear bases for algebras — these are minimal sets consisting of an algebra basis along with enough products of these so that all further algebra elements can be reached using only sums.

The current work describes a new linear basis for free Lie algebras, along with a new method for finding, computing with, and proving theorems about general free Lie algebra bases. Our method uses the graph/tree pairing developed in [Sinha 2005; 2006], which yields a new way to describe Lie coalgebras via graphs as applied in [Sinha and Walter 2011] and explained further in [Walter 2010]. Since we introduce a new and different way to perform calculations in free Lie algebras, we give many detailed examples throughout. For readers interested in a history of bases of free Lie algebras, we suggest [Bokut and Chibrikov 2006, §3.2].

2. Notation and classical constructions

In this paper we will say *alphabet* for a collection of abstract letters (or variables). A *word* in an alphabet is a (noncommutative, associative) string (or product) of letters from the alphabet. An ordering on an alphabet (such as the standard alphabetical ordering in English) induces an ordering on words called the *lexicographical* (or dictionary) ordering. The cyclic permutations of a word are given by removing letters from the beginning of the word and moving them to the end.

Example 2.1. The cyclic permutations of the word $abcd$ are $bcda$, $cdab$, $dabc$. The cyclic permutations of the word $aaabb$ are $aabba$, $abbaa$, $bbaaa$ and $baaab$.

A word that is lexicographically strictly less than all its cyclic permutations is called a *Lyndon–Shirshov word* (or just a *Lyndon word*). For instance, $aaabb$ is a Lyndon–Shirshov word, but $aabba$ and $abbaa$ are not. Neither is $abab$, since it equals one of its cyclic permutations.

A linear basis for the free Lie algebra on an alphabet can be built on Lyndon–Shirshov words using *standard bracketing* [Reutenauer 1993]i (see [Melançon and Reutenauer 1989; Chibrikov 2006; Stöhr 2008] for other methods). Given a Lyndon–Shirshov word w, its standard bracketing $[w]$ is recursively defined by $[w] = w$ if w has length 1, and by $[w] = [uv] = [[u], [v]]$ if v is a maximally long Lyndon–Shirshov suffix of $w = uv$, with u nonempty.

Example 2.2. The standard bracketing of $aaabb$ is

$$[aaabb] = \big[[a], [aabb]\big] = \big[a, [\,[a], [abb]]\big]$$
$$= \big[a, [a, [[ab], [b]]]\big] = \big[a, [a, [[a, b], b]]\big].$$

The collection of all standard bracketings of Lyndon–Shirshov words gives a linear basis for the free Lie algebra on the underlying alphabet.

The standard bracketing of a Lyndon–Shirshov word can also be described recursively from the innermost brackets to the outermost roughly as follows: Read the letters of a Lyndon–Shirshov word from right to left looking for the first occurrence of consecutive letters, $\ldots a_i a_{i+1} \ldots$, where $a_i < a_{i+1}$ (called the *rightmost inversion*). Replace $a_i a_{i+1}$ by the bracket $[a_i, a_{i+1}]$, which we will consider to be a new *letter* placed in the ordered alphabet in the lexicographical position of the word $a_i a_{i+1}$. Repeat. (For a more detailed description see [Melançon and Reutenauer 1989, §2].)

Our construction of left greedy brackets will also proceed from the innermost bracket the to outermost bracket, similar to the *rewriting system* presented above. However, just as with the standard bracketing, left greedy brackets can also be described from the outermost to the innermost brackets.

3. Left greedy brackets and star graphs

Simple words.

Definition 3.1. Given a fixed letter a in an alphabet, an a-simple word is a word of the form $w = aa \cdots ax$ (written $w = a^n x$ for short), where x is any single letter not equal to a. The single-letter word $w = x$ (i.e., $w = a^0 x$) is also an a-simple word (for $x \neq a$).

The collection of all words in an alphabet is itself an (infinite) ordered alphabet (with the lexicographical ordering). A word in the alphabet whose letters are words in another alphabet will be casually referred to as a *word of words*. Note that such an expression of a word as a product of subwords is equivalent to a partition of the word. Considering words as ordered sets of letters, partitions are order-preserving surjections of sets; hence our notation for partitioning a word will be a double-headed arrow, $\twoheadrightarrow$, as defined below.

Remark 3.2. A partition of a word is equivalent to a rewriting which combines multiple subwords in parallel (compare [Mélançon 1997]). For our construction and proofs we will critically make use of the levels of nesting of partitions. We use the term *partition* so that our notation and our terminology reflect this emphasis.

Definition 3.3. A simple partition of a word w is an expression of w as subwords $w = \alpha_1 \alpha_2 \cdots \alpha_k$ where each α_i is an a-simple word and a is the first letter of w. We will write $w \twoheadrightarrow \alpha_1 \alpha_2 \cdots \alpha_k$.

Note that words have at most one simple partition. The subword α_1 must consist of the initial string of a's as well as the first letter of w other than a. If the letter in w following α_1 is a, then α_2 must consist of the next string of a's as well as the next letter other than a. If the letter following α_1 is not a, then α_2 will consist of only that one letter. (See the first line of Example 3.6.)

Remark 3.4. A simple partition of a word is equivalent to performing Lazard elimination [Lothaire 1997, Chapter 5] on the word, eliminating the first letter a via the bisection $(a^*(A \backslash a), a)$. It appears likely that the constructions of nested partitions and fully partitioned words which will follow may also be performed via a recursive series of Lazard elimination steps along the lines of: Order all words lexicographically. Eliminate them, one at time, beginning with the least ordered word a. (Possibly it will be best to restrict to words of length $\leq n$ at first.)

This would give an alternate proof that the left greedy brackets form a basis. However, making the previous statement precise and showing that it gives a well-defined recursive operation which will terminate is complicated. Also, following such a path would not yield the dual basis of star graphs, which we wish to exploit in later work.

Given a simple partition $w \twoheadrightarrow \alpha_1 \alpha_2 \cdots \alpha_k$, we may recurse: The a-simple subwords α_1, α_2, etc. are letters in the alphabet of words. They may have a further simple partition (now as α_1-simple words). This process constructs a unique nested partition of a word such that each nested level is a simple partition.

Definition 3.5. A word *fully partitions* if it has a series of simple partitions,

$$w \twoheadrightarrow \omega_1 \twoheadrightarrow \cdots \twoheadrightarrow \omega_\ell,$$

where ω_ℓ is the trivial coarse partition.

Colloquially, a word fully partitions if it is a simple word of simple words of simple words etc.

Example 3.6. Words fully partition as follows (for clarity we will use distinct delimiters (, [and { to indicate different nested levels of partition):

- $aaaab \twoheadrightarrow (aaaab)$.
- $ababb \twoheadrightarrow (ab)\,(ab)\,(b) \twoheadrightarrow [(ab)(ab)(b)]$.
- $aabcb \twoheadrightarrow (aab)\,(c)\,(b) \twoheadrightarrow [(aab)(c)]\,[(b)] \twoheadrightarrow \{[(aab)(c)]\,[(b)]\}$.
- $ababbabaab \twoheadrightarrow (ab)\,(ab)\,(b)\,(ab)\,(aab) \twoheadrightarrow [(ab)(ab)(b)]\,[(ab)(aab)]$ $\twoheadrightarrow \{[(aab)(b)]\,[(ab)(aab)]\}$.

For visual clarity, we have found that indicating nested partitions via underlining is often more understandable than using nested parentheses:

$$aaaab \twoheadrightarrow \underline{aaaab}, \qquad ababb \twoheadrightarrow \underline{ab}\ \underline{ab}\ \underline{b},$$

$$aabcb \twoheadrightarrow \underline{\underline{ab}\ \underline{c}\ \underline{b}}, \qquad ababbabaab \twoheadrightarrow \underline{ab}\ \underline{ab}\ \underline{b}\ \underline{ab}\ \underline{aab},$$

$$abcabcabbabcaab \twoheadrightarrow \underline{\underline{ab}\ \underline{c}\ \underline{ab}\ \underline{c}\ \underline{ab}\ \underline{b}\ \underline{ab}\ \underline{c}\ \underline{aab}}.$$

Example 3.7. Some words do not fully partition:

- *aaaa* contains repetitions of only one letter.
- *aaba* has the same initial and final letter.
- *abab* ⟶ (*ab*)(*ab*), which is a repetition of a single subword (*ab*).
- *abaabab* ⟶ (*ab*)(*aab*)(*ab*), which has the same initial and final subword (*ab*).
- *ababbcababb* ⟶ [(*ab*)(*ab*)(*b*)][(*c*)][(*ab*)(*ab*)(*b*)], which has the same initial and final subword [(*ab*)(*ab*)(*b*)].

The following simple lemma follows immediately from standard facts about Lyndon–Shirshov words. We give a proof below for completeness.

Lemma 3.8. *Every Lyndon–Shirshov word fully partitions.*

Proof. Fix a word w with initial letter a. The only obstacle to the partition of w into a-simple words is whether the final letter and the initial letter match. More generally, each step of the recursive partitioning can be completed as long as the initial and final subword do not match. This fails only if the word has the form $w = \alpha \chi \alpha$, where α and χ are subwords (the subword χ may be empty and is likely not simple).

However no Lyndon–Shirshov word has this form. If χ is empty then $w = \alpha\alpha$, which is not Lyndon–Shirshov. If χ is nonempty, then one of the cyclic reorderings of w is lexicographically lower: either $\alpha\alpha\chi < \alpha\chi\alpha$ (if $\alpha < \chi$) or else $\chi\alpha\alpha < \alpha\chi\alpha$ (if $\chi < \alpha$). $\qquad\square$

Remark 3.9. Many non-Lyndon–Shirshov words also fully partition. The requirement that $w \neq \alpha \chi \alpha$ for any subwords α and χ is much weaker than the Lyndon–Shirshov requirement. Via some experimentation, we have found that it is possible to use methods similar to those presented in the current work to find new bases for Lie algebras which are constructed from sets of words other than the Lyndon–Shirshov words. It is unclear if these sets of words are also bases for the shuffle algebra.

Left greedy brackets.

Definition 3.10. The left greedy bracketing of the a-simple word $w = a^n x$, which we denote by $\llbracket w \rrbracket$, is $\llbracket a^n x \rrbracket = [a, [a, \cdots [a, [a, x]] \cdots]]$, the standard right-normed bracketing. The left greedy bracketing of a simple word of simple words (and, more generally, any fully partitioned word) is defined recursively:

$$\llbracket \alpha^n \chi \rrbracket = \big[\llbracket \alpha \rrbracket, [\llbracket \alpha \rrbracket, \ldots [\llbracket \alpha \rrbracket, [\llbracket \alpha \rrbracket, \llbracket \chi \rrbracket]] \cdots]\big].$$

Example 3.11. Following are some examples of left greedy bracketings of fully partitioned words. To aid understanding in the examples below, we underline to indicate their full partition into simple words. (Note that we do not require words to be Lyndon–Shirshov in order to define their left greedy bracketing.)

- $\llbracket \underline{aaab} \rrbracket = \big[a, [a, [a, b]]\big]$.
- $\llbracket \underline{ab}\,\underline{ab}\,\underline{b} \rrbracket = \big[[a, b], [[a, b], b]\big]$.
- $\llbracket \underline{\underline{aab}\,\underline{c}\,\underline{b}} \rrbracket = \big[[[a, [a, b]], c], b]\big]$.
- $\llbracket \underline{\underline{ab}\,\underline{ab}\,\underline{b}\,\underline{ab}\,\underline{aab}} \rrbracket = \big[[[a, b], [[a, b], b]], \ [[a, b], [a, [a, b]]]\big]$.

Remark 3.12. The name *left greedy* is due to the fact that the bracketing of the word *aaabcd* begins with innermost bracket $[a, b]$ and then brackets leftwards i.e., $[a, [a, [a, b]]]$ before bracketing to the right. An alternative *right greedy* bracketing, would go to the right i.e., $[[[a, b], c], d]$ before bracketing leftwards. Both of these yield free Lie algebra bases, but the left greedy bracketing has a cleaner basis proof and appears to have better properties. We leave the discussion of the beneficial properties of the left greedy bracketing to a later paper.

Remark 3.13. Left greedy bracketing of Lyndon–Shirshov words is different than other bracketing methods considered in the literature. We give a few examples for comparison with other methods. Consider the Lyndon–Shirshov word $w = aababb$:

- $\llbracket aababb \rrbracket = \big[[[a, [a, b]], [a, b]], b\big]$, the left greedy bracketing.
- $[aababb] = \big[a, [[a, b], [[a, b], b]]\big]$, the standard Lyndon–Shirshov bracketing [Reutenauer 1993].
- $[aababb] = \big[[a, [a, b]], [[a, b], b]\big]$, the bracketing of [Chibrikov 2006, §4].

Star graphs. By a *graph* we mean a finite directed graph whose vertices are labeled by letters.

Definition 3.14. The star graph of the a-simple word $w = a^n x$, denoted $\star(w)$, is the graph with n vertices: labeled a, one vertex labeled x, and an edge from each a vertex to the vertex x:

$$\star(a^n x) = \underset{a \quad\quad a}{\overset{a}{\underset{x}{\longrightarrow}}}.$$
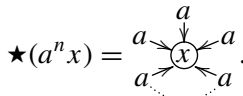
The vertex x is called the *anchor vertex*. The star graph of a simple word of simple words (and, more generally, any fully partitioned word) is defined recursively. The graph $\star(\alpha^n \chi)$ consists of n disjoint subgraphs $\star(\alpha)$ and one disjoint subgraph $\star(\chi)$ with edges connecting the anchor vertices of the $\star(\alpha)$ to the anchor vertex of $\star(\chi)$:

$$\star(\alpha^n \chi) = \quad\quad.$$
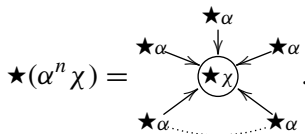
The anchor vertex of the subgraph $\star(\chi)$ serves as the anchor vertex of the star graph $\star(\alpha^n \chi)$.

Remark 3.15. The star graph of an a-simple word consisting of one letter $w = x \neq a$ is a single anchor vertex:

$$\star(x) = \circledcirc.$$

Example 3.16. Following are some examples of star graphs. In the examples below, the anchor vertex of the subgraphs are indicated with dotted circles and the anchor vertex of the entire graph is indicated with a solid circle.

$$\star(\underline{aaab}) =$$

$$\star(\underline{ab\,ab\,b}) =$$

$$\star(\underline{aab\,c\,b}) =$$

$$\star(\underline{ab\,ab\,b\,ab\,aab}) =$$
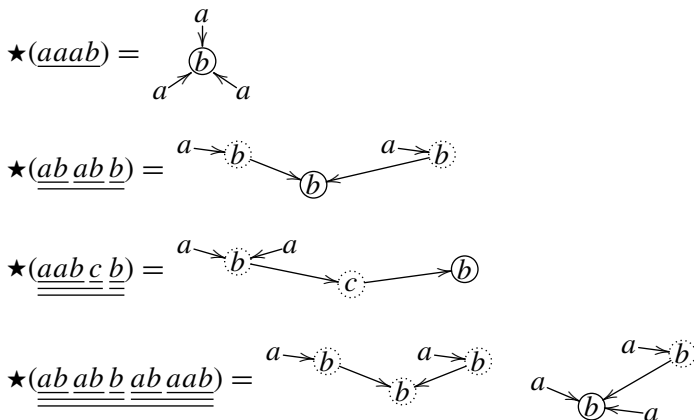
Remark 3.17. The name *star graph* comes from imagining the graph $\star(a^n b)$ as a sun (b) with planets (a) orbiting around it. The recursive construction of star graphs then composes suns and their planetary systems into orbiting star clusters, into galaxies, etc.

4. Configuration pairing

Throughout, assume that all graphs and Lie bracket expressions have labels and letters from the same alphabet.

Definition 4.1. Given a graph G and Lie bracket expression L, a bijection $\sigma : G \leftrightarrow L$ is a bijection between the vertices of G and the positions in L compatible with labels and letters (vertices of G are sent to positions in L labeled with the identical letter).

Example 4.2. Following are some basic examples investigating bijections between graphs and Lie bracket expressions:

- There are no bijections $\nearrow^{b}\searrow_{a}$ $\leftrightarrow [a, b]$ because there are three vertices in the graph but only two positions in the Lie bracket expression. Similarly, there are no bijections $\nearrow^{b}$ $\leftrightarrow [[a, b], a]$.

- There are no bijections $\nearrow^{b}\searrow_{c}$ $\leftrightarrow [[a, b], a]$ because there is no letter c in the Lie bracket expression.

- There is only one bijection $a\nearrow^{b}\searrow_{c} \leftrightarrow [[b, c], a]$ given by identifying each vertex with the correspondingly labeled position in the bracket expression.

- There are two bijections $a\nearrow^{b}\searrow_{a} \leftrightarrow [[a, b], a]$ since there are two ways to choose an identification between the two vertices a and the two bracket positions a.

- More generally, there are $n!$ bijections $\star(a^n b) \leftrightarrow \llbracket a^n b \rrbracket$.

Given a graph G and a subset V of the vertices of G, write $|V|$ for the full subgraph of G with vertices from V — i.e., two vertices are connected by an edge in $|V|$ if and only if they are connected by an edge in G. Recall that a graph is connected if every two vertices can be connected by a path of edges. We will say that directed graphs are connected if they are connected, ignoring edge directions.

The configuration pairing defined in [Sinha 2006] between directed graphs and rooted trees gives a pairing between graphs and Lie bracket expressions which can be defined as follows [Walter 2010].

Definition 4.3. Given a graph G and a Lie bracket expression L as well as a bijection $\sigma : G \leftrightarrow L$, the σ-configuration pairing of G and L is

$$\langle G, L \rangle_\sigma = \begin{cases} 0 & \text{if } L \text{ contains a subbracket expression } [H, K] \text{ so} \\ & \text{that the corresponding subgraphs } |\sigma^{-1}H| \text{ and} \\ & |\sigma^{-1}K| \text{ are not connected graphs with exactly one} \\ & \text{edge between them in } G, \\ (-1)^n & \text{otherwise (where } n \text{ is the number of edges of } G \\ & \text{whose orientation corresponds under } \sigma \text{ to the right-} \\ & \text{to-left orientation of positions in } L). \end{cases}$$

The configuration pairing of G and L is the sum over all bijections σ,

$$\langle G, L \rangle = \sum_{\sigma : G \leftrightarrow L} \langle G, L \rangle_\sigma.$$

Casually, we will say that an edge $a\nearrow^{b}$ in G whose orientation corresponds under σ to the right-to-left orientation of L (i.e., $\sigma(a)$ is to the right of $\sigma(b)$ in L) moves leftwards in L under σ.

Example 4.4. Following are some example computations of configuration pairings.

- $\langle a\nearrow^{b}\searrow_{c}, [[b, c], a] \rangle = -1$. There is only one bijection. In this bijection only the edge $a\nearrow^{b}$ moves leftwards in $[[b, c], a]$.

- $\langle a\nearrow^{a}\searrow_{b}, [[a, b], a] \rangle = -1 - 1 = -2$.

- $\langle a\nearrow^{b}\searrow_{a}, [[a, a], b] \rangle = 0$. For each of the two bijections, $|\sigma^{-1}([a, a])|$ (the subgraph) is disconnected in G.

- $\left\langle \vphantom{x}_a\nearrow^b\searrow_c\nearrow^a, [[a, b], [a, c]] \right\rangle = -1 + 1 = 0$. There are two bijections. One bijection makes $_c\nearrow^a$ go leftwards. The other bijection makes $_a\nearrow^b$ and also $_c\nearrow^a$ go leftwards.

- The pairing of a linear (or *long*) graph $_{a_1}\nearrow^{a_2}\searrow..\nearrow^{a_n}$ with a bracket expression L is equal to the coefficient of the term $(a_1 a_2 \cdots a_n)$ in the associative polynomial for L [Walter 2010].

The configuration pairing encodes a duality between free Lie algebras and graphs modulo the Arnold and arrow-reversing identities [Sinha and Walter 2011]. In the current work we will use only that the configuration pairing is well-defined on Lie algebras — i.e., the configuration pairing vanishes on Jacobi and anticommutativity Lie bracket expressions. Thus the configuration pairing with graphs can be used to distinguish Lie bracket expressions, and in particular can be used to establish linear independence.

The main theorem will be proven essentially via recursive application of the following proposition, whose proof is trivial.

Proposition 4.5. *Let w_1 and w_2 be Lyndon–Shirshov words. If $w_1 = a^n b$ is a-simple then*

$$\left\langle \star(w_1), \llbracket w_2 \rrbracket \right\rangle = \begin{cases} n! & \text{if } w_2 = w_1, \\ 0 & \text{otherwise.} \end{cases}$$

A similar result holds if w_2 is a-simple.

Proof. Suppose that w_1 and w_2 are Lyndon–Shirshov words with $\left\langle \star(w_1), \llbracket w_2 \rrbracket \right\rangle \neq 0$. Note that w_1 and w_2 must be written with the same letters for any bijections $\sigma : \star(w_1) \leftrightarrow \llbracket w_2 \rrbracket$ to exist. Furthermore w_1 and w_2 must share the same initial letter, since Lyndon–Shirshov words always begin with their lowest-ordered letter. Thus $w_1 = w_2$.

If $w_1 = w_2$, then there are $n!$ possible bijections $\sigma : \star(a^n b) \leftrightarrow \llbracket a^n b \rrbracket$. For each of these $\left\langle \star(a^n b), \llbracket a^n b \rrbracket \right\rangle_\sigma = 1$. $\square$

5. The basis theorem

Theorem 5.1. *If w_1 and w_2 are Lyndon–Shirshov words, then $\left\langle \star(w_1), \llbracket w_2 \rrbracket \right\rangle \neq 0$ if and only if $w_1 = w_2$ (in this case it is a product of factorials).*

Our desired result follows as a simple corollary.

Corollary 5.2. *The left greedy bracketing of Lyndon–Shirshov words gives a basis for free Lie algebras.*

Proof. A perfect pairing of graphs with left greedy brackets of Lyndon–Shirshov words implies that the left greedy brackets of Lyndon–Shirshov words are linearly

independent. Since the number of Lyndon–Shirshov words of length n equals the dimension of the vector space of length-n Lie bracket expressions, this is enough to show that left greedy brackets of Lyndon–Shirshov words form a basis for the free Lie algebra. □

Proof of Theorem 5.1. Suppose that w_1 and w_2 are Lyndon–Shirshov words with nonzero pairing:

$$\big\langle \star(w_1), \lfloor w_2 \rfloor \big\rangle \neq 0.$$

Fix a bijection $\sigma : \star(w_1) \leftrightarrow \lfloor w_2 \rfloor$. We will show that $w_1 = w_2$ by inducting on the depth of the nested partition resulting from fully partitioning the Lyndon–Shirshov words w_1 and w_2.

First note, as in the proof of Proposition 4.5, that w_1 and w_2 must be written with the same letters and must share the same initial letter, call it a. Thus w_1 and w_2 both fully partition, where the innermost partitions are a-simple words.

Write $w_2 \twoheadrightarrow (a^{n_1} b_1)(a^{n_2} b_2) \cdots (a^{n_k} b_k)$ for the innermost partition of w_2. According to its recursive definition, the bracket expression $\lfloor w_2 \rfloor$ will have subbracket expressions $\lfloor a^{n_i} b_i \rfloor$. From the definition of the configuration pairing, these must correspond under σ to connected, disjoint subgraphs of $\star(w_1)$. However, the only possible connected subgraph of a star graph (with initial letter a) using the letters $a^{n_i} b_i$ is $\star(a^{n_i} b_i)$. Note that this implies w_1 is composed of the subwords $(a^{n_i} b_i)$ (though possibly written in a different order). Furthermore, the first subword of w_1 must be $(a^{n_1} b_1)$ (just as in w_2), because Lyndon–Shirshov words must begin with their lexicographically least subword.

The induction step is equivalent to the previous case, treating subwords as letters. At the end of the previous case, for each simple subword u of w_2 the subbracket expressions $\lfloor u \rfloor$ of $\lfloor w_2 \rfloor$ correspond to disjoint connected subgraphs $\star(u)$ of $\star(w_1)$. Furthermore, the initial subword of w_2 coincides with the initial subword of w_1.

To finish the proof, we must note that $\big\langle \star(w), \lfloor w \rfloor \big\rangle \neq 0$ when w is a Lyndon–Shirshov word, since all bijections $\sigma : \star(w) \leftrightarrow \lfloor w \rfloor$ have $\big\langle \star(w), \lfloor w \rfloor \big\rangle_\sigma > 0$. In fact, a few short computations show that

$$\big\langle \star(a^n b), \lfloor a^n b \rfloor \big\rangle = n!,$$
$$\big\langle \star\big((a^{n_1} b_1)^m (a^{n_2} b_2)\big), \lfloor (a^{n_1} b_1)^m (a^{n_2} b_2) \rfloor \big\rangle = m!(n_1!)^m n_2!,$$
$$\vdots \qquad\qquad \square$$

6. Projection onto the left greedy basis

Theorem 5.1 is of independent interest because it gives a direct, computational method for writing Lie bracket elements in terms of the left greedy Lyndon–Shirshov basis via projection.

Given a Lie bracket expression L, write $\{w_k\}$ for the set of Lyndon–Shirshov words written using the letters in L (with multiplicity). Left greedy brackets of Lyndon–Shirshov words form a linear basis, so it is possible to write L as a linear combination of the $\llbracket w_k \rrbracket$:

$$L = c_1 \llbracket w_1 \rrbracket + \cdots + c_n \llbracket w_n \rrbracket.$$

We may compute the constants c_k by pairing with $\star(w_k)$ since $\langle \star(w_k), \llbracket w_j \rrbracket \rangle = 0$ for $j \neq k$ by Theorem 5.1. This proves the following.

Corollary 6.1. *Given a Lie bracket expression L,*

$$L = \sum_{\substack{\text{Lyndon–Shirshov} \\ \text{words } w}} \frac{\langle \star(w), L \rangle}{\langle \star(w), \llbracket w \rrbracket \rangle} \llbracket w \rrbracket.$$

Recall that the denominators $\langle \star(w), \llbracket w \rrbracket \rangle$ are products of factorials. Interestingly, each coefficient in the expression above must be an integer (despite their large denominators).

Pairing computations are aided by the bracket/cobracket compatibility property of the configuration pairing. Bracket/cobracket compatibility states that pairings of a graph G with a bracket expression $[L, K]$ may be computed by calculating pairings of L and K with the subgraphs obtained by cutting G into two pieces by removing an edge. The following is Proposition 3.14 of [Sinha and Walter 2011].

Proposition 6.2. *Bracketing Lie expressions is dual to cutting graphs:*

$$\langle G, [H, K] \rangle = \sum_e \langle G_1^{\hat{e}}, H \rangle \cdot \langle G_2^{\hat{e}}, K \rangle \; - \; \langle G_1^{\hat{e}}, K \rangle \cdot \langle G_2^{\hat{e}}, H \rangle,$$

Where $G_1^{\hat{e}}$ and $G_2^{\hat{e}}$ are the graphs obtained by removing edge e from G, ordered so that e pointed from $G_1^{\hat{e}}$ to $G_2^{\hat{e}}$ in G.

Remark 6.3. Applying bracket/cobracket duality and the definition of the configuration pairing yields a recursive method for computation of $\langle G, L \rangle$. Consider the outermost bracketing $L = [H, K]$. Look for edges of G which can be removed to separate G into subgraphs $G_1^{\hat{e}}$ and $G_2^{\hat{e}}$ whose sizes matches that of H and K, and check that the subgraphs are written using the same letters as H and K. If this is not possible, then the bracketing is 0. Otherwise the bracketing is given by summing $\langle G_1^{\hat{e}}, H \rangle \cdot \langle G_2^{\hat{e}}, K \rangle$ (or the negative $- \langle G_1^{\hat{e}}, K \rangle \cdot \langle G_2^{\hat{e}}, H \rangle$ if e pointed so that $G_1^{\hat{e}}$ corresponded to K instead of H) over all such edges. Recurse. Note that removing an edge from a star graph will always result in subgraphs which are themselves star graphs (though possibly not star graphs of Lyndon–Shirshov words).

Example 6.4. Consider the Lie bracket expression $L = \big[[[a, b], b], [[a, b], a]\big]$. There are three Lyndon–Shirshov words with the letters $aaabbb$. These words, along with their partition, left greedy bracketings, and values of $\langle \star(w), \llbracket w \rrbracket \rangle$ are:

- *aaabbb* which, partitions as $\underline{\underline{aaab}}\,\underline{\underline{b}}\,\underline{\underline{b}}$ with

$\lfloor aaabbb \rfloor = \big[[a, [a, [a, b]]], b\big], b\big]$ and $\langle \star(aaabbb), \lfloor aaabbb \rfloor \rangle = 3!.$

- *aababb* which, partitions as $\underline{\underline{aab}}\,\underline{\underline{ab}}\,\underline{\underline{b}}$ with

$\lfloor aababb \rfloor = \big[[[a, [a, b]], [a, b]], b\big]$ and $\langle \star(aababb), \lfloor aababb \rfloor \rangle = 2!.$

- *aabbab* which, partitions as $\underline{\underline{aab}}\,\underline{\underline{b}}\,\underline{\underline{ab}}$ with

$\lfloor aabbab \rfloor = \big[[[a, [a, b]], b], [a, b]\big]$ and $\langle \star(aabbab), \lfloor aabbab \rfloor \rangle = 2!.$

The configuration pairings with L are as follows:

- $\langle \star(aaabbb), \big[[[a, b], b], [[a, b], a]\big] \rangle = 0$, because no edge of $\star(aaabbb)$ can be removed to separate it into subgraphs one of which has a single a and two b's (corresponding to the subbracket $\big[[a, b], b\big]$).

- $\langle \star(aababb), \big[[[a, b], b], [[a, b], a]\big] \rangle = 2$, because only the edge connecting $\star(aab)$ to the remainder of the graph cuts $\star(aababb)$ appropriately. This reduces the computation to

$$-\langle \star(aab), \big[[a, b], a\big] \rangle \cdot \langle \star(abb), \big[[a, b], b\big] \rangle = -(-2) \cdot 1 = 2.$$

- $\langle \star(aabbab), \big[[[a, b], b], [[a, b], a]\big] \rangle = -2$, because only the edge connecting $\star(aab)$ to the remainder of the graph cuts $\star(aabbab)$ appropriately. The computation reduces similarly.

Thus $L = \lfloor aababb \rfloor - \lfloor aabbab \rfloor$.

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References

[Bokut and Chibrikov 2006] L. Bokut and E. S. Chibrikov, "Lyndon-Shirshov words, Gröbner–Shirshov bases, and free Lie algebras", pp. 17–39 in *Non-associative algebra and its applications*, edited by L. Sabinin et al., Lect. Notes Pure Appl. Math. **246**, Chapman & Hall/CRC, Boca Raton, FL, 2006. MR 2203694 Zbl 1330.17001

[Chibrikov 2006] E. S. Chibrikov, "A right normed basis for free Lie algebras and Lyndon–Shirshov words", *J. Algebra* **302**:2 (2006), 593–612. MR 2293773 Zbl 1204.17006

[Lothaire 1997] M. Lothaire, *Combinatorics on words*, 2nd ed., Cambridge University Press, 1997. MR 1475463 Zbl 0874.20040

[Mélançon 1997] G. Mélançon, "Combinatorics of bases of free Lie algebras", 1997.

[Melançon and Reutenauer 1989] G. Melançon and C. Reutenauer, "Lyndon words, free algebras and shuffles", *Canad. J. Math.* **41**:4 (1989), 577–591. MR 1012617 Zbl 0694.17003

[Reutenauer 1993] C. Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs. New Series **7**, The Clarendon Press, Oxford University Press, New York, 1993. MR 1231799 Zbl 0798.17001

[Sinha 2005] D. Sinha, "A pairing between graphs and trees", preprint, 2005. arXiv 0502547

[Sinha 2006] D. P. Sinha, "Operads and knot spaces", *J. Amer. Math. Soc.* **19**:2 (2006), 461–486. MR 2188133 Zbl 1112.57004

[Sinha and Walter 2011] D. Sinha and B. Walter, "Lie coalgebras and rational homotopy theory, I: Graph coalgebras", *Homology Homotopy Appl.* **13**:2 (2011), 263–292. MR 2861231 Zbl 1231.55009

[Stöhr 2008] R. Stöhr, "Bases, filtrations and module decompositions of free Lie algebras", *J. Pure Appl. Algebra* **212**:5 (2008), 1187–1206. MR 2387595 Zbl 1192.17001

[Walter 2010] B. Walter, "Lie algebra configuration pairing", preprint, 2010. arXiv 1010.4732

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