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of links in real projective space

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Every link diagram can be represented as a signed ribbon graph. However, different link diagrams can be represented by the same ribbon graphs. We determine how checkerboard colourable diagrams of links in real projective space, and virtual link diagrams, that are represented by the same ribbon graphs are related to each other. We also find moves that relate the diagrams of links in real projective space that give rise to (all-A) ribbon graphs with exactly one vertex.

1. Introduction and overview

It is well known that a classical link diagram can be represented by a unique signed plane graph, called its Tait graph (see, for example, the surveys [Bollobás 1998; Ellis-Monaghan and Moffatt 2013; Welsh 1993]). This construction provides a seminal connection between the areas of graph theory and knot theory, and has found impressive applications, such as in proofs of the Tait conjectures [Murasugi 1987; Thistlethwaite 1987]. Tait graphs can also be constructed for checkerboard colourable link diagrams on other surfaces, in which case the resulting graph is embedded on the surface. However, as this construction requires checkerboard colourability, Tait graphs cannot be constructed for arbitrary link diagrams on a surface, or arbitrary virtual link diagrams. Recently, Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus [Dasbach et al. 2008] extended the idea of a Tait graph by associating a set of signed ribbon graphs to a link diagram (see also [Turaev 1987]). Chmutov and Voltz [2008] extended this construction, giving a way to describe an arbitrary virtual link diagram as a signed ribbon graph. These constructions extend to graphs in other surfaces. The ribbon graphs of link diagrams have found numerous applications, and we refer the reader to the surveys [Champanerkar and Kofman 2014; Ellis-Monaghan and Moffatt 2013] for details.

Every signed plane graph represents a unique classical link diagram. In contrast, a single signed ribbon graph can represent several different link diagrams or virtual link diagrams. This observation leads to the fundamental problem of determining how link diagrams that are presented by the same signed ribbon graphs are related to each other. It is this problem that interests us here. It was solved for classical link diagrams in [Moffatt 2012]. Here we solve it for checkerboard colourable diagrams of links in $\mathbb{R}\mathrm{P}^3$ (in Theorem 7), and for virtual link diagrams (in Theorem 22).

We also examine the one-vertex ribbon graphs of diagrams of links in $\mathbb{R}\mathrm{P}^3$. Every classical link diagram can be represented as a ribbon graph with exactly one vertex. Abernathy et al. [2014] gave a set of moves that provide a way to move between all of the diagrams of a classical link that have one-vertex all-A ribbon graphs. We extend their work to the setting of links in $\mathbb{R}\mathrm{P}^3$.

This paper is structured as follows. In Section 2 we give an overview of diagrams of links in $\mathbb{R}\mathrm{P}^3$ and of ribbon graphs. In Section 3 we describe how diagrams of links in $\mathbb{R}\mathrm{P}^3$ can be represented by ribbon graphs, and we determine how checkerboard colourable diagrams that give rise to the same ribbon graphs are related to one another. In Section 4 we study the ribbon graphs of diagrams of links in $\mathbb{R}\mathrm{P}^3$ that have exactly one vertex. Finally, in Section 5 we describe how virtual link diagrams that give rise to the same ribbon graphs are related to one another.

This work arose from Strömberg's undergraduate thesis at Royal Holloway, University of London, which was supervised by Moffatt.

2. Notation and terminology

2.1. *Links in $\mathbb{R}\mathrm{P}^3$ and their diagrams.*   In this section we provide a brief overview of links in $\mathbb{R}\mathrm{P}^3$ and their diagrams. Further results and details can be found in [Drobotukhina 1994; 1990; Huynh and Le 2008; Mroczkowski 2003; Murasugi 1987; Prasolov and Sossinsky 1997].

A *diagram* of a link in $\mathbb{R}\mathrm{P}^3$ is a disc D^2 in the plane together with a collection of immersed arcs (where an arc is a compact connected 1-manifold possibly with boundary). The end points of arcs with boundary lie on the boundary of the disc ∂D^2, are divided into antipodal pairs, and these are the only points of the arcs that intersect ∂D^2. We further assume that the arcs are generically immersed, in that they have finitely many multiple points and each multiple point is a double point in which the arcs meet transversally. Finally, each double point is assigned an over/under-crossing structure, and is called a *crossing*. Figure 1(a) shows a diagram of a link in $\mathbb{R}\mathrm{P}^3$. Here, D will always refer to a diagram of a link.

A *net* is the real projective plane $\mathbb{R}\mathrm{P}^2$ together with a distinguished projective line, called the *line at infinity*, and a collection of generically immersed closed curves where each double point is assigned an over/under-crossing structure. Let
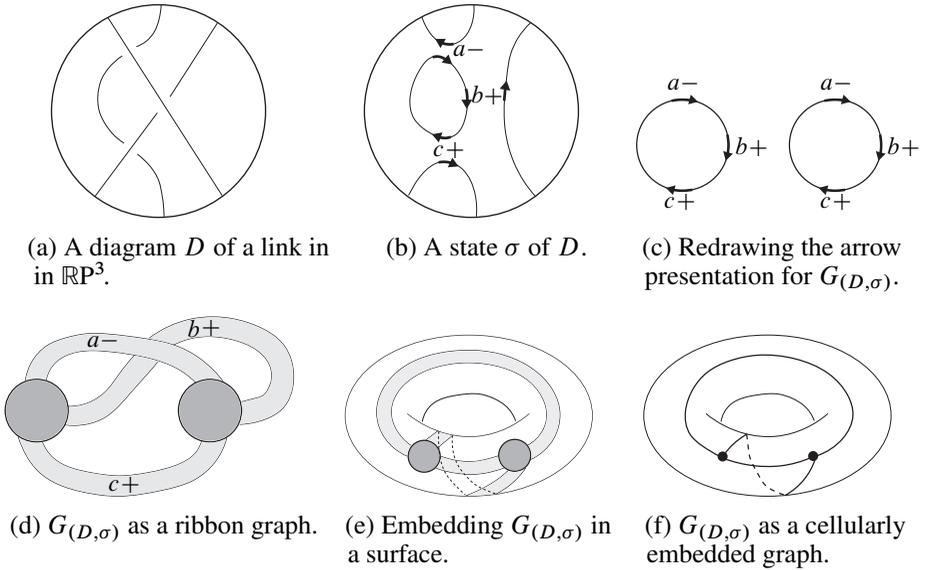
(a) A diagram D of a link in in $\mathbb{R}P^3$.

(b) A state σ of D.

(c) Redrawing the arrow presentation for $G_{(D,\sigma)}$.

(d) $G_{(D,\sigma)}$ as a ribbon graph.

(e) Embedding $G_{(D,\sigma)}$ in a surface.

(f) $G_{(D,\sigma)}$ as a cellularly embedded graph.

Figure 1. A diagram D of a link in $\mathbb{R}P^3$ and one of its ribbon graphs.

D be a diagram of a link in $\mathbb{R}P^3$, then the *net of D*, denoted $\mathcal{N}_D$, is obtained from D by identifying the antipodal points of ∂D^2. The image of ∂D^2 in the net gives the line at infinity.

A *component* of D is a collection of its arcs that give rise to a single closed curve in its net $\mathcal{N}_D$. A component is *null-homologous* if the corresponding curve in $\mathcal{N}_D$ is trivial in $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$ and is *1-homologous* otherwise. We will say that a diagram is *null-homologous* if each of its components is. The *faces* of D (respectively, $\mathcal{N}_D$) are the components of $D\setminus\alpha$ (respectively, $\mathcal{N}_D\setminus\alpha$), where α is the set of immersed curves. A *region* of D is a collection of its faces that correspond to a single face in its net $\mathcal{N}_D$. A diagram D is *checkerboard colourable* if there is an assignment of the colours black and white to its regions such that no two adjacent regions (those meeting a common arc) are assigned the same colour. A diagram may or may not be checkerboard colourable. For example, the diagram in Figure 1(a) is not, but that in Figure 7(d) is.

The *Reidemeister moves* for diagrams of links in $\mathbb{R}P^3$ consist of isotopy of the disc that preserves the antipodal pairing (which we call the *R0-move*), together with the five moves in Figure 2 that change the diagram locally as shown (the diagrams are identical outside of the given region). In the figure, the bold lines represent the boundary of the disc. Two diagrams are *equivalent* if they are related by a sequence of Reidemeister moves.

For brevity we work a little informally in this paragraph, referring the reader to [Drobotukhina 1990] for details. Links in $\mathbb{R}P^3$ give rise to diagrams by representing
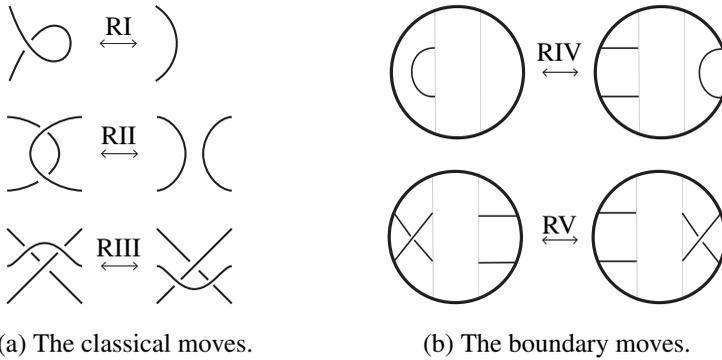
(a) The classical moves.   (b) The boundary moves.

Figure 2. The Reidemeister moves for diagrams of links in $\mathbb{R}P^3$.

$\mathbb{R}P^3$ as a ball D^3 with antipodal points of its boundary identified, lifting the link from $\mathbb{R}P^3$ to D^3 and projecting to the equatorial disc D^2. Conversely, given a diagram, regarding D^2 as the equatorial disc of such a representation of $\mathbb{R}P^3$ and "pulling the over-crossings up a little" gives rise to a link in $\mathbb{R}P^3$. With this, we have from [Drobotukhina 1990] that two links in $\mathbb{R}P^3$ are ambient isotopic if and only if their diagrams are equivalent.

2.2. *Ribbon graphs.*

Definition 1. A *ribbon graph* $G = (V(G), E(G))$ is a (possibly nonorientable) surface with boundary represented as the union of two sets of discs, a set $V(G)$ of *vertices*, and a set of *edges* $E(G)$ such that

(1) the vertices and edges intersect in disjoint line segments;

(2) each such line segment lies on the boundary of precisely one vertex and precisely one edge;

(3) every edge contains exactly two such line segments.

An example of a ribbon graph can be found in Figure 1(d), and additional details about them can be found in, for example, [Ellis-Monaghan and Moffatt 2013; Gross and Tucker 2001].

Two ribbon graphs are *equivalent* if there is a homeomorphism taking one to the other that sends vertices to vertices, edges to edges, and preserves the cyclic ordering of the edges at each vertex. The homeomorphism should be orientation-preserving if the ribbon graphs are orientable. Note that any embedding of a ribbon graph is 3-space is irrelevant.

A ribbon graph is topologically a surface with boundary and the *genus* of a ribbon graph is its genus when it is viewed as a surface. It is *orientable* if it is orientable as a surface. A ribbon graph is said to be *plane* if it is homeomorphic
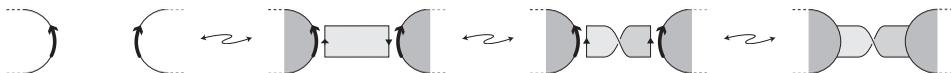
Figure 3. Moving between arrow presentations and ribbon graphs.

to a sphere with holes (or equivalently if it is connected and of genus zero), and is said to be $\mathbb{RP}^2$ if it is homeomorphic to a real projective plane with holes (or equivalently it is connected, nonorientable and of genus one).

Since a ribbon graph is a surface with boundary, each ribbon graph G admits a unique (up to homeomorphism) cellular embedding into a closed surface Σ. (The cellular condition here means that $\Sigma \backslash G$ is a collection of discs). Using this embedding, it is easy to see that ribbon graphs are equivalent to cellularly embedded graphs (in one direction, contract the ribbon graph to obtain a graph drawn on the surface; in the other direction take a neighbourhood of the graph in a surface) and so are the main object of topological graph theory. See Figure 1(d)–(f).

We will make use of the following combinatorial description of ribbon graphs which is due to Chmutov [2009].

Definition 2. An *arrow presentation* consists of a set of closed curves, each with a collection of disjoint, labelled arrows, called *marking arrows*, lying on them. Each label appears on precisely two arrows.

A ribbon graph can be obtained from an arrow presentation as follows. View each closed curve as the boundary of a disc (the disc becomes a vertex of the ribbon graph). Edges are then added to the vertex discs in the following way: take an oriented disc for each label of the marking arrows; choose two nonintersecting arcs on the boundary of each of the edge discs and direct these according to the orientation; identify these two arcs with two marking arrows, both with the same label, aligning the direction of each arc consistently with the orientation of the marking arrow. This process is illustrated in Figure 3.

Conversely, to describe a ribbon graph G as an arrow presentation, start by arbitrarily labelling and orienting the boundary of each edge disc of G. On each arc where an edge disc intersects a vertex disc, place an arrow on the vertex disc, labelling the arrow with the label of the edge it meets and directing it consistently with the orientation of the edge-disc boundary. The boundaries of the vertex set marked with these labelled arrows give the arrow-marked closed curves of an arrow presentation. See Figure 1(c)–(d) for an example, and [Chmutov 2009; Ellis-Monaghan and Moffatt 2013] for further details.

Arrow presentations are *equivalent* if they describe equivalent ribbon graphs.

We will need to make use of signed ribbon graphs. A *signed ribbon graph* is a ribbon graph G together with a function from $E(G)$ to $\{+, -\}$. Thus it consists of a ribbon graph with a sign associated to each of its edges. Similarly, a *signed*
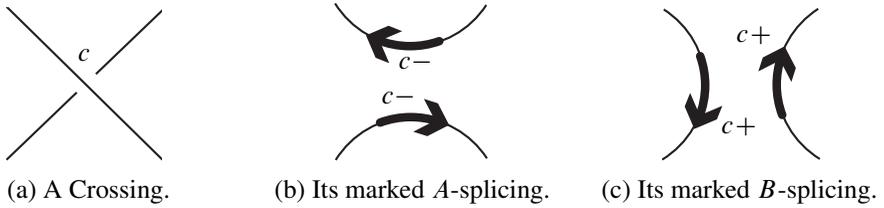
(a) A Crossing.   (b) Its marked A-splicing.   (c) Its marked B-splicing.

Figure 4. Marked splicings of a link diagram.

arrow presentation consists of an arrow presentation together with a function from its set of labels to $\{+, -\}$. Signed ribbon graphs and signed arrow presentations are equivalent in the obvious way.

3. The ribbon graphs of links in $\mathbb{RP}^3$

3.1. *The ribbon graphs of link diagrams.* We now describe how a set of ribbon graphs can be associated to a link diagram. Let D be a diagram of a link in $\mathbb{RP}^3$. Assign a unique label to each crossing of D. A *marked A-splicing* or a *marked B-splicing* of a crossing c is the replacement of the crossing with one of the schemes shown in Figure 4.

Notice that we decorate the two arcs in the splicing with signed labelled arrows that are chosen to be consistent with an arbitrary orientation of the disc. The labels of the arrows are determined by the label of the crossing, and the signs are determined by the choice of splicing.

A *state* σ of D is the result of marked A- or B-splicing each of its crossings. Observe that a state is a signed arrow presentation of a signed ribbon graph. We denote the signed ribbon graph corresponding to the state σ of D by $G_{(D,\sigma)}$. These ribbon graphs are the ribbon graphs of a link diagram:

Definition 3. Let D be a diagram of a link in $\mathbb{RP}^3$. Then the *set of signed ribbon graphs associated with D*, denoted $\mathbb{G}_D$, is defined by

$$\mathbb{G}_D = \{G_{(D,\sigma)} \mid \sigma \text{ is a marked state of } D\}.$$

If $G \in \mathbb{G}_D$ then we say that G is a *signed ribbon graph of D*. We will also say that *G represents D*.

An example of a ribbon graph $G_{(D,\sigma)}$ for a state σ of a link diagram D is given in Figure 1(a)–(d). The construction of $\mathbb{G}_D$ is a direct extension of the construction for classical links from [Dasbach et al. 2008; Turaev 1987].

If D is checkerboard coloured, then we can construct a signed ribbon graph of D by choosing the splicing that follows the black regions at each crossing. The resulting signed ribbon graph is called a *Tait graph* of D. If D is checkerboard

colourable, then it has exactly two Tait graphs, one corresponding to each of the two checkerboard colourings.

Proposition 4. *Let D be a checkerboard colourable diagram of a link in* $\mathbb{R}\mathrm{P}^3$. *Then its Tait graphs are either plane or* $\mathbb{R}\mathrm{P}^2$ *ribbon graphs.*

Proof. Checkerboard colour D and let G be its Tait graph. If D is not null-homologous then all of its regions are discs. Since the marked splicings follow the black regions and the black regions are discs, we can embed G in $\mathbb{R}\mathrm{P}^2$ by taking the black regions bounded by the curves of the splicings as vertices, and embedding the edge disc between the pairs of labelled arrows in the obvious way. Since D is checkerboard coloured, all regions of the embedded ribbon graph are discs, and no two face regions or vertex regions share a boundary. Thus G is cellularly embedded in the net and is therefore $\mathbb{R}\mathrm{P}^2$.

If D is null-homologous replace the face of its net that is a Möbius band with a disc to obtain a diagram on the sphere, and repeat the above argument with this embedding. $\square$

We note that it follows from the proof of Proposition 4 that the Tait graphs defined here coincide with the "usual" Tait graphs obtained by placing vertices in black regions and embedding edges through each crossing.

Remark 5. One of the significant applications of the ribbon graphs of links is that they provide a way to connect graph and knot polynomials. A seminal result of Thistlethwaite [1987] expresses the Jones polynomial of an alternating classical link as an evaluation of the Tutte polynomial of either of its Tait graphs. There have been several recent extensions of this result that express the Jones polynomial and Kauffman bracket of virtual and classical links as evaluations of Bollobás and Riordan's extension of the Tutte polynomial to ribbon graphs; see [Bradford et al. 2012; Chmutov 2009; Chmutov and Pak 2007; Chmutov and Voltz 2008; Dasbach et al. 2008; Moffatt 2010; 2011].

Kauffman brackets and Jones polynomials of links in $\mathbb{R}\mathrm{P}^3$ can similarly be expressed in terms of the (multivariate) Bollobás–Riordan polynomials of ribbon graphs that represent their diagrams. In fact, the statement and proofs of the results for links in $\mathbb{R}\mathrm{P}^3$ follow those for the existing results with almost no change. Accordingly we only remark here that they hold. Following the notation of the exposition [Ellis-Monaghan and Moffatt 2013] gives that for a diagram D of a link in $\mathbb{R}\mathrm{P}^3$,

$$\langle D \rangle = d^{k(\mathbb{A})-1} A^{n(\mathbb{A})-r(\mathbb{A})} R(\mathbb{A}; -A^4, A^{-2}d, d^{-1}, 1),$$

and

$$\langle D \rangle = d^{-1} A^{e_-(G_D)-e_+(G_D)} Z(G_D; 1, \boldsymbol{w}, d, 1),$$

where

$$w_e = \begin{cases} A^{-2} & \text{if } e \text{ is negative,} \\ A^2 & \text{if } e \text{ is positive.} \end{cases}$$
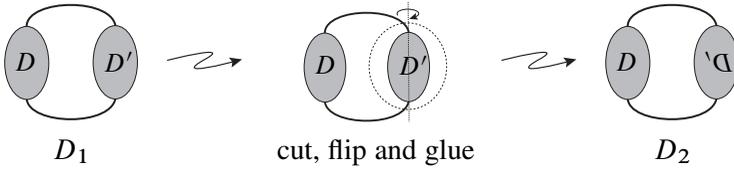
Figure 5. A summand-flip.

In these equations, $\langle D \rangle$ is the Kauffman bracket of [Drobotukhina 1990], $d = -A^2 - A^{-2}$, $\mathbb{A}$ is the ribbon graph of D obtained by choosing the A-splicing at each crossing, R is the Bollobás–Riordan polynomial [2002], and Z is the multivariate Bollobás–Riordan polynomial of [Moffatt 2008]. These identities can be obtained by following Section 5.4.2 of [Ellis-Monaghan and Moffatt 2013].

Furthermore, a connection between the Bollobás–Riordan polynomial and the HOMFLY-PT polynomial of links in $\mathbb{RP}^3$ from [Mroczkowski 2004] that is analogous to Jaeger's connection [1988] between the Tutte polynomial of a plane graph and the HOMFLY-PT polynomial of a classical link (see also [Jin and Zhang 2012; Moffatt 2008; Traldi 1989]) can also be found:

$$P(\mathcal{L}(G); x, y) = \left(\frac{1}{xy}\right)^{v(G)-1} \left(\frac{y}{x}\right)^{e(G)} (x^2-1)^{k(G)-1} R\left(G; x^2, \frac{x-x^{-1}}{xy^2}, \frac{y}{x-x^{-1}}\right).$$

Again the notation here is from [Ellis-Monaghan and Moffatt 2013], and the result can be obtained by following Section 5.5.2 of that text.

3.2. *Relating link diagrams with the same ribbon graph.* As mentioned in the introduction, two diagrams can give rise to the same set of signed ribbon graphs. That is, it is possible that $D \neq D'$ but $\mathbb{G}_D = \mathbb{G}_{D'}$. A fundamental question is then if D and D' are diagrams such that $\mathbb{G}_D = \mathbb{G}_{D'}$, how are D and D' related? Here we answer this question in the case when D and D' are both checkerboard colourable. To describe the result, we need to introduce some notation.

Definition 6. Let D and D' be diagrams of links in $\mathbb{RP}^3$. We say that D and D' are related by a *summand-flip* if D' can be obtained from D by the following process: Orient the disc D^2 and choose a disc $\mathfrak{D}$ in D^2 whose boundary intersects D transversally in exactly two points a and b. Cut out $\mathfrak{D}$ and glue it back in such a way that the orientations of $\mathfrak{D}$ and $D^2 \backslash \mathfrak{D}$ disagree, the points a on the boundaries of $\mathfrak{D}$ and $S^2 \backslash \mathfrak{D}$ are identified, and the points b on the boundaries of $\mathfrak{D}$ and $S^2 \backslash \mathfrak{D}$ are identified. See Figure 5. We say that two link diagrams D and D' are *related by summand-flips* if there is a sequence of summand-flips and R0-moves taking D to D'.

Our first main result is the following.

Theorem 7. *Let D and D' be checkerboard colourable diagrams of links in $\mathbb{RP}^3$. Then $\mathbb{G}_D = \mathbb{G}_{D'}$ if and only if D and D' are related by summand-flips.*
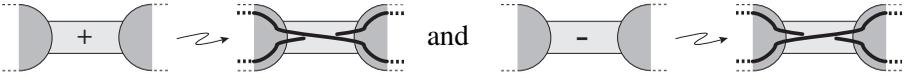
Figure 6. Forming a diagram D_G from a signed ribbon graph G.

Before proving Theorem 7, we note that the requirement that the link diagrams are checkerboard colourable is essential to our approach, and we pose the following.

Open problem. Let D and D' be diagrams of links in $\mathbb{RP}^3$ (that are not necessarily checkerboard colourable). Determine necessary and sufficient conditions for $\mathbb{G}_D$ and $\mathbb{G}_{D'}$ to be equal.

To prove Theorem 7, we need to be able to recover link diagrams from ribbon graphs. Given a signed $\mathbb{RP}^2$ or plane ribbon graph, it is straight-forward to recover a link diagram that it represents. Let G be a signed $\mathbb{RP}^2$ ribbon graph, fill in the holes to obtain a cellular embedding of it in $\mathbb{RP}^2$, as in Section 2.2. Represent $\mathbb{RP}^2$ as a disc D^2 with antipodal points identified, and lift the embedding of G to a drawing on D^2. Finally, draw the configuration of Figure 6 on each of its edges, and connect the configurations by following the boundaries of the vertices of G, to obtain the link diagram. See Figure 7 for an example.

If G is a signed plane ribbon graph, fill in all but one of the holes to obtain a cellular embedding of it in a disc D^2. Drawing the configuration of Figure 6 on each of its edges and connecting the configurations by following the boundaries of the vertices of G gives the required link diagram. In either case, we denote the resulting diagram of a link in $\mathbb{RP}^3$ by D_G.

Proposition 8. *Let G be a signed $\mathbb{RP}^2$ or plane ribbon graph. Then D_G is checkerboard colourable.*

Proof. This follows by colouring the regions of D_G that correspond to the vertices of the ribbon graph black. □

To recover a link diagram from a ribbon graph that is not plane or $\mathbb{RP}^2$ requires more work, and for our application, Chmutov's concept [2009] of a partial dual of a ribbon graph. The idea behind a partial dual is to form the geometric dual of an embedded graph but with respect to only some of its edges. We approach partial duals and geometric duals via arrow presentations as this is particularly convenient for us here. Other descriptions of partial duality can be found in, for example, [Chmutov 2009; Ellis-Monaghan and Moffatt 2013].

Definition 9. Let G be a ribbon graph viewed as an arrow presentation, and let $A \subseteq E(G)$. Then the *partial dual* G^A of G with respect to A is the arrow presentation (or ribbon graph) obtained as follows. For each $e \in A$, suppose α and β are the two arrows labelled e in the arrow presentation of G. Draw a line segment with an

(a) A ribbon graph G.

(b) A partial dual $G^{\{3\}}$ of G.

(c) Drawing $G^{\{3\}}$ in a disc.

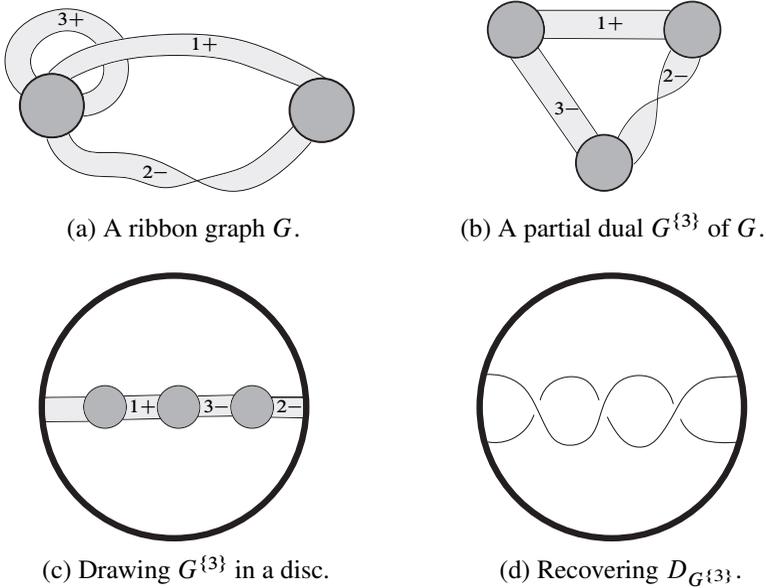(d) Recovering $D_{G^{\{3\}}}$.

Figure 7. Recovering a link diagram from a ribbon graph.

arrow on it directed from the head of α to the tail of β, and a line segment with an arrow on it directed from the head of β to the tail of α. Label both of these arrows e, and delete α and β and the arcs containing them. This process is illustrated locally at a pair of arrows in Figure 8. The ribbon graph $G^{E(G)}$ is the *geometric dual* of G.

If G is a signed ribbon graph then G^A is also a signed ribbon graph with the signs of G^A given by the rule that if an edge e of G has sign $\varepsilon \in \{+, -\}$, then the corresponding edge in G^A has sign $-\varepsilon$ if $e \in A$, and ε if $e \notin A$. (Thus taking the dual of an edge toggles its sign.)

Figure 7(a)–(b) gives an example of a partial dual.

We will need the following properties of partial duals from [Chmutov 2009].

Proposition 10. *Let G be a (signed) ribbon graph and $A, B \subseteq E(G)$. Then the following hold.*

(1) $G^\varnothing = G$.

(2) $G^{E(G)} = G^*$, *where G^* is the geometric dual of G.*

(3) $(G^A)^B = G^{(A \triangle B)}$, *where $A \triangle B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference of A and B.*

(4) G *is orientable if and only if G^A is orientable.*

We emphasise that the construction of the geometric dual G^* of G agrees with the usual graph theoretic construction of the geometric dual of a cellularly embedded
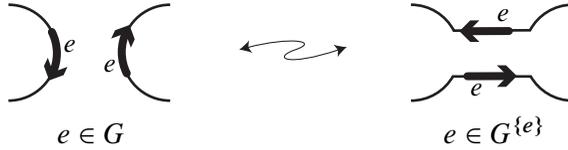
Figure 8. Taking the partial dual of an edge in an arrow presentations.

graph in which a cellularly embedded graph G^* is obtained from a cellularly embedded graph G by placing one vertex in each of its faces, and embedding an edge of G^* between two of these vertices whenever the faces of G they lie in are adjacent, and the edges of G^* are embedded so that they cross the corresponding face boundary (or edge of G) transversally.

Proposition 11. *Let G be a signed $\mathbb{R}P^2$ or plane ribbon graph. Then $D_G = D_{G^*}$.*

Proof. Upon remembering that taking the dual of a signed ribbon graph changes the sign of each edge, the result is readily seen by comparing Figures 6 and 8. $\square$

Lemma 12. *Let D be a diagram of a link in $\mathbb{R}P^3$. Then all of the signed ribbon graphs in $\mathbb{G}_D$ are partial duals of each other.*

Proof. Let $G, H \in \mathbb{G}_D$. Then $G = G_{(D,\sigma)}$ and $H = H_{(D,\sigma')}$. It can be seen from Figure 8 that taking partial duals corresponds exactly to choosing another state of D as in Figure 4. $\square$

Lemma 13. *Let D be a checkerboard colourable diagram of a link in $\mathbb{R}P^3$. Then G represents D if and only if $D = D_{G^A}$, where G^A is a signed plane or $\mathbb{R}P^2$ ribbon graph.*

Proof. We begin by assuming that $D = D_{G^A}$, where G^A is a signed plane or $\mathbb{R}P^2$ ribbon graph. Then $G^A = G_{(D,\sigma)}$ for some state σ of D. By Lemma 12, it follows that the partial dual $(G^A)^A = G$ also represents D.

Conversely, assume that G represents D. Since D is checkerboard colourable, it can be represented by a Tait graph T. Clearly $D = D_T$. Then from Lemma 12, it follows that $T = G^A$ for some $A \subseteq E(G)$. Since T is a plane or $\mathbb{R}P^2$ ribbon graph (by Proposition 4), T is the ribbon graph required by the lemma. $\square$

Lemma 13 provides a way to construct all of the checkerboard colourable link diagrams represented by a given signed ribbon graph: find all of its plane or $\mathbb{R}P^2$ partial duals and construct the links associated with them. This process is illustrated in Figure 7. The checkerboard colorability requirement here cannot be dropped. For example, if D is the diagram from Figure 1(a), then $\mathbb{G}_D$ contains no plane or $\mathbb{R}P^2$ ribbon graphs. This leads to the following problem.

Open problem. Let G be a signed ribbon graph. Find an efficient way to construct all of the diagrams of links in $\mathbb{R}P^3$ that have G as a representative.

We continue with some corollaries of Lemma 13.

Corollary 14. *Let D and D' be checkerboard colourable diagrams of links in $\mathbb{RP}^3$ such that $\mathbb{G}_D = \mathbb{G}_{D'}$. Then D is null-homologous if and only if D' is.*

Proof. D is null-homologous if and only if it has a plane Tait graph. The result then follows since partial duality preserves orientability. □

Corollary 15. *Let D and D' be checkerboard colourable diagrams of links in $\mathbb{RP}^3$ such that $\mathbb{G}_D = \mathbb{G}_{D'}$. Then there exists a plane or, respectively, $\mathbb{RP}^2$ ribbon graph G, and $A \subseteq E(G)$ such that G^A is plane or, respectively, $\mathbb{RP}^2$ and such that $D = D_G$ and $D' = D_{G^A}$.*

Proof. We have that D and D' give rise to the same set of ribbon graphs. Since D is checkerboard colourable, it gives rise to a plane or $\mathbb{RP}^2$ ribbon graph G (namely one of its Tait graphs, by Proposition 4). Moreover, since D' is also checkerboard colourable, it also gives rise to a plane or $\mathbb{RP}^2$ ribbon graph H. We also have that $H \in \mathbb{G}_D$, so $H = G^A$ for some $A \subseteq E(G)$ by Lemma 12. □

Corollary 15 is of key importance here: it tells us that if two checkerboard colourable diagrams of links in $\mathbb{RP}^3$, D and D', are represented by the same ribbon graphs, then they are both diagrams associated with partially dual plane or $\mathbb{RP}^2$ ribbon graphs G and G'. Thus if we understand how G and G' are related to each other, we can deduce how D and D' are related to each other. This is our strategy for proving Theorem 7.

In [Moffatt 2012; 2013], rough structure theorems for the partial duals of plane ribbon graphs and $\mathbb{RP}^2$ ribbon graphs were given. These papers also contained local moves that allow us to move between all partially dual plane or $\mathbb{RP}^2$ ribbon graphs. To describe this move, we need a little additional terminology.

Let G be a ribbon graph, $v \in V(G)$, and P and Q be nontrivial ribbon subgraphs of G. Then G is said to be the *join* of P and Q, written $P \vee Q$, if $G = P \cup Q$ and $P \cap Q = \{v\}$ and if there exists an arc on v with the property that all edges of P meet it there, and none of the edges of Q do. See the left-hand side of Figure 9, which illustrates a ribbon graph of the form $P \vee Q$. We do not require the ribbon graphs G, P or Q to be connected. Note that since genus is additive under joins, if G is plane then both P and Q are plane, and if G is $\mathbb{RP}^2$ then exactly one of P or Q is $\mathbb{RP}^2$ and the other is plane.

Let $G = P \vee Q$ be a ribbon graph. We say that the ribbon graph $G^{E(Q)} = P \vee Q^{E(Q)} = P \vee Q^*$ is obtained from G by a *dual-of-a-join-summand move*. We say that two ribbon graphs are related by *dualling join-summands* if there is a sequence of dual-of-a-join-summand moves taking one to the other, or if they are geometric duals. See Figure 9.

The following result is an amalgamation of Theorem 7.3 of [Moffatt 2012] and Theorem 5.8 of [Moffatt 2013].
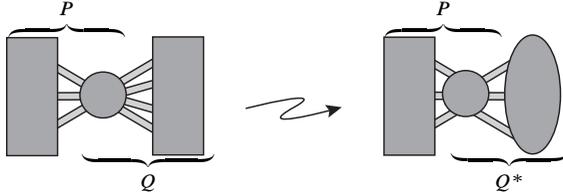
Figure 9. The dual of a join-summand move.

Theorem 16. *Let G and H be connected plane or $\mathbb{R}P^2$ ribbon graphs. Then G and H are partial duals if and only if they are related by dualling join-summands.*

Theorem 16 allows us to prove the following key result.

Lemma 17. *If two $\mathbb{R}P^2$ ribbon graphs G and G' are related by dualling join-summands, then the link diagrams D_G and $D_{G'}$ they represent are related by summand-flips.*

Proof. It suffices to show that if G and G' are related by a single dual-of-a-join-summand move then D_G and $D_{G'}$ are related by a summand-flip. Suppose that $G = A \vee B$, that $A \cap B = \{v\}$, and that $G' = A^* \vee B$ or $G' = A \vee B^*$. Since we know that genus is additive under joins, we have that one of A or B is $\mathbb{R}P^2$ and the other is plane. Without loss of generality, suppose that A is the $\mathbb{R}P^2$ summand.

First suppose that $G' = A^* \vee B$. We start by determining how the cellular embeddings of G and G' are related. From this, we will deduce how the corresponding link diagrams are related. Start by taking the cellular embedding of G in $\mathbb{R}P^2$. This is illustrated in Figure 10(a). For each edge of B that meets v, place a labelled arrow on the intersection of the edge with v. We can then "detach" B from G, as indicated in Figure 10(b), so that G is recovered from A and B by identifying the corresponding arrows in A and in B with its copy of v removed. After detaching B, we obtain a cellular embedding of A in $\mathbb{R}P^2$. From this, form the cellular embedding of A^* by interchanging the vertices and faces. (In detail, $A^* \subset \mathbb{R}P^2$ is obtained from $A \subset \mathbb{R}P^2$ by reassigning the face (respectively, vertex) discs of $A \subset \mathbb{R}P^2$ as vertex (respectively, face) discs of $A^* \subset \mathbb{R}P^2$. Edge discs are unchanged.) This is indicated in Figure 10(c). Finally, obtain an embedding of $G' = A^* \vee B$ by reattaching B according to the labelled arrows, as is indicated in arrows as in Figure 10(d), and notice that B has been "flipped over". Finally consider the diagrams D_G and $D_{G'}$ drawn using these embeddings. Since A and A^* have the same edges and vertex/face boundaries, and by Proposition 11, $D_A = D_{A^*}$, we see that D_G and $D_{G'}$ are related by a summand-flip, as in Figure 10(e)–(f).

Next suppose that $G' = A \vee B^*$. Then, using Proposition 10 and that duality preserves joins, we have $G' = (A \vee B^*) = (A \vee B^*)^{**} = (A^* \vee B^{**})^* = (A^* \vee B)^*$. Then since $D_{(A^* \vee B)^*} = D_{(A^* \vee B)}$, by Proposition 11, this case reduces to the first, completing the proof. $\qquad\square$

(a) $G \subset \mathbb{RP}^2$.   (b) $A \subset \mathbb{RP}^2$.   (c) $A^* \subset \mathbb{RP}^2$.

(d) G'.   (e) D_G.   (f) $D_{G'}$.
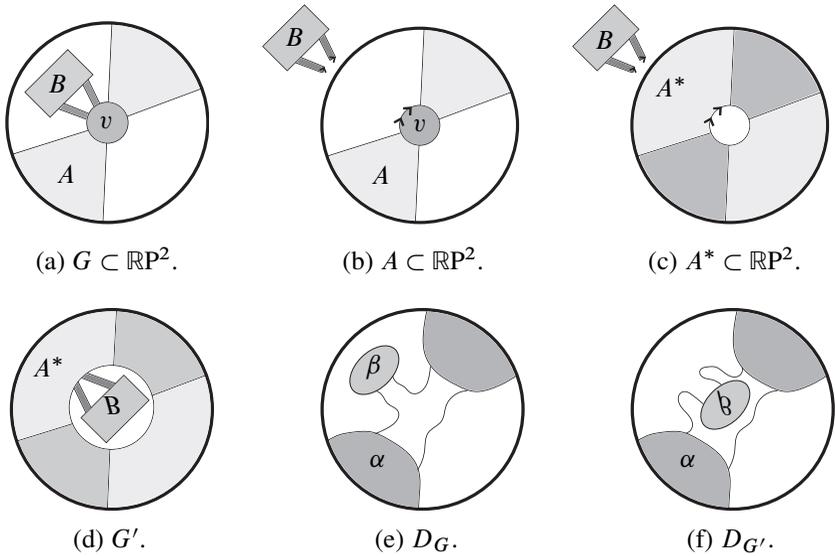
Figure 10. A figure used in the proof of Lemma 17.

Proof of Theorem 7. It is readily seen that if D and D' are related by summand-flips then $\mathbb{G}_D = \mathbb{G}_{D'}$.

For the converse, assume that D and D' are checkerboard colourable link diagrams on $\mathbb{RP}^2$ such that $\mathbb{G}_D = \mathbb{G}_{D'}$. If D and D' are not null-homologous then, by Corollary 15, for some G, we have $D = D_G$ and $D' = D_{G^A}$, where G and G^A are both $\mathbb{RP}^2$. We know by Theorem 16 that G and G^A are related by dualling join-summands. Thus either $G^A = G^*$, in which case the result follows from Proposition 11, or G^A is obtained from G by a sequence of dual-of-a-join-summand moves, in which case the result follows from Lemma 17.   □

4. One vertex ribbon graphs

We let $\mathbb{A}_D$ denote the *all-A ribbon graph* of D, which is the ribbon graph obtained from D by choosing the marked A-splicing at each crossing. The all-A ribbon graph is of particular interest since all of the signs are the same, and so a link diagram can be represented by an unsigned ribbon graph (see also Remark 5). It was shown in [Abernathy et al. 2014] that every classical link (i.e., in S^3) can be represented as a ribbon graph with exactly one vertex. Furthermore, the authors of that paper gave a set of moves, analogous to the Reidemeister moves, that provide a way to move between all of the diagrams of a classical link that have one-vertex all-A ribbon graphs. In this section we extend their result to links in $\mathbb{RP}^3$.

Lemma 18. *Every link in $\mathbb{RP}^3$ has a diagram D for which $\mathbb{A}_D$ has exactly one vertex.*
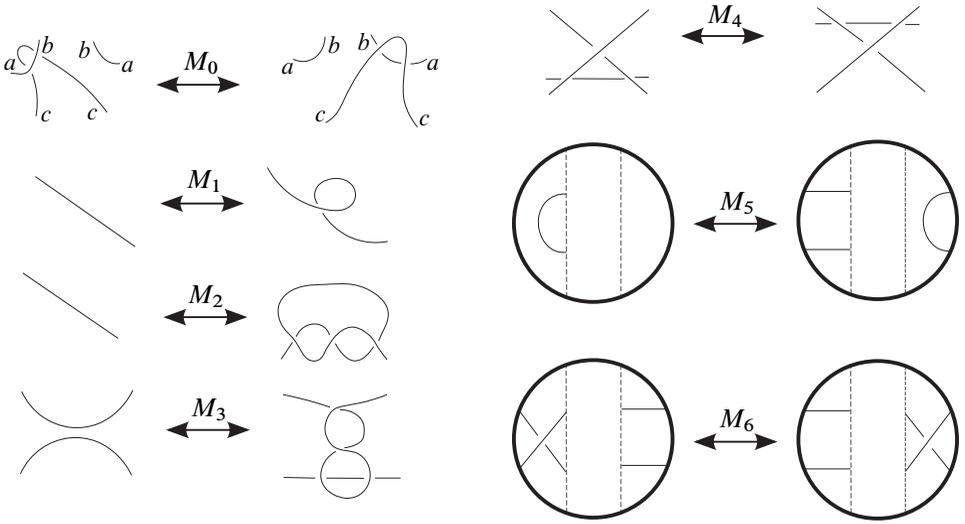
Figure 11. The M-moves.

Proof. Let D be a diagram of a link in $\mathbb{R}P^3$. Let σ_A denote the *all-A state* of D obtained by choosing the marked A-splicing at each crossing. If σ_A has exactly one component then $\mathbb{A}_D$ has exactly one vertex. Otherwise, consider the all-A state $\bar{\sigma}_A$ of the net $\mathcal{N}_D$ of D. There must be two closed curves of $\bar{\sigma}_A$ that can be joined by an embedded arc $\bar{\alpha}$ in $\mathbb{R}P^2 \setminus \bar{\sigma}_A$. Performing an RII-move (possibly with some RIV-moves) along the image of this arc in D gives a new diagram D'. Then $\mathbb{A}_{D'}$ has one less vertex than $\mathbb{A}_D$. Repeat this process until only one curve remains. $\square$

The *M-moves* for diagrams of links in $\mathbb{R}P^3$ consist of isotopy of the disc that preserves the antipodal pairing, together with the moves shown in Figure 11 that change the diagram locally as shown (the diagrams are identical outside of the shown region). For the M_0-move, we require the diagram to be connected in a specific way, as indicated by the labels.

Lemma 19. *Let D be a diagram of a given link in $\mathbb{R}P^3$. Then the M-moves do not change the number of vertices in $\mathbb{A}_D$.*

Proof. For moves M_0–M_4, we refer the reader to [Abernathy et al. 2014]. It is easy to see that the M_5 move does not affect the number of components of the all-A state σ_A of D, since it does not affect the number of, or type of, crossings. It is also easy to see that M_6 does not change the number of vertices of the all-A ribbon graph. $\square$

Let $\mathcal{D}$ denote the set of all diagrams of links in $\mathbb{R}P^3$, $\tilde{\mathcal{D}}$ denote $\mathcal{D}$ modulo the Reidemeister moves, $\mathcal{D}_1 \subset \mathcal{D}$ denote the subset of diagrams such that their all-A ribbon graphs have exactly one vertex, and $\tilde{\mathcal{D}}_1$ denote $\mathcal{D}_1$ modulo the M-moves. Now consider the two natural projections $\phi : \mathcal{D} \to \tilde{\mathcal{D}}$ and $\phi_1 : \mathcal{D}_1 \to \tilde{\mathcal{D}}_1$.

Theorem 20. *Given $D, D' \in \mathcal{D}_1$, we have $\phi(D) = \phi(D')$ if and only if $\phi_1(D) = \phi_1(D')$.*

Proof. First assume that $\phi_1(D) = \phi_1(D')$. Then the link diagrams are related by M-moves. It is easy to see that the link diagrams are then related by Reidemeister moves, so we have that $\phi(D) = \phi(D')$.

Conversely, suppose that $\phi(D) = \phi(D')$. Hence the diagrams are related by Reidemeister moves. We need to show that each Reidemeister move can be described as a sequence of M-moves. For RI–RIII, we refer the reader to [Abernathy et al. 2014]. RIV and RIV are exactly M_5 and M_6 moves, so we have that all the Reidemeister moves can be described as a sequence of M-moves. Hence $\phi_1(D) = \phi_1(D')$, as required.   □

5. Virtual link diagrams with same the signed ribbon graphs.

A *virtual link diagram* consists of n closed piecewise-linear plane curves in which there are finitely many multiple points and such that at each multiple point exactly two arcs meet and they meet transversally. Moreover, each double point is assigned either a *classical crossing* structure or is marked as a *virtual crossing*. See the left-hand side of Figure 12, where the virtual crossings are marked by circles. A virtual link is *oriented* if each of its plane curves is. Further details on virtual knots can be found in, for example, the surveys [Kauffman 1999; 2000; 2012; Kaufman and Manturov 2006; Manturov 2004].

Virtual links are considered up to the *generalised Reidemeister moves*. These consist of orientation-preserving homeomorphisms of the plane (which we include in any subset of the moves), the classical Reidemeister moves of Figure 2(a), and the virtual Reidemeister moves of Figure 13. Two virtual link diagrams are *equivalent* if there is a sequence of generalised Reidemeister moves taking one diagram to the other.

Virtual knots are the knotted objects that can be represented by Gauss diagrams. Here a *Gauss diagram* consists of a set of oriented circles together with a set of oriented signed chords whose end points lie on the circles (see the right-hand
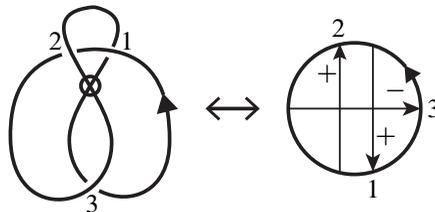
Figure 12. A virtual link (on the left) and its Gauss diagram (on the right). The crossings and chords are numbered for clarity.
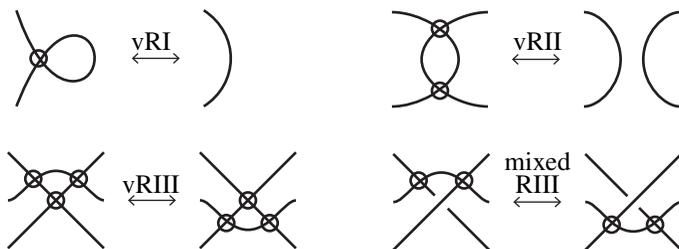
Figure 13. The virtual Reidemeister moves.

side of Figure 12). A Gauss diagram is obtained from an oriented n component virtual link diagram D as follows. Start by numbering each classical crossing. For each component, choose a base point and travel around the component from the base point following the orientation and reading off the numbers of the classical crossings as they are met. Whenever a crossing is met as an over-crossing, label the corresponding number with the letter O. Place each number, in the order met, on an oriented circle corresponding to the component. Connect the points on the circles that have the same number by a chord that is directed away from the O-labelled number. Finally, label each chord with the *oriented sign* of the corresponding crossing, shown in Figure 14, and delete the numbers. The resulting Gauss diagram describes D. See Figure 12 for an example.

Conversely, an oriented virtual link diagram can be obtained from a Gauss diagram by immersing the circles in the plane so that the ends of chords are identified (there is no unique way to do this), and using the direction and signs to obtain a crossing structure. In general, immersing the circles will create double points that do not arise from chords. Mark these as virtual crossings.

The following theorem of Goussarov, Polyak and Viro [Goussarov et al. 2000] provides an important and fundamental relation between Gauss diagrams and virtual links.

Theorem 21. *Let L and L' be two virtual link diagrams that are described by the same Gauss diagram. Then L and L' are equivalent. Moreover, L and L' are related by the virtual Reidemeister moves.*

(a) A positive crossing.   (b) A negative crossing.
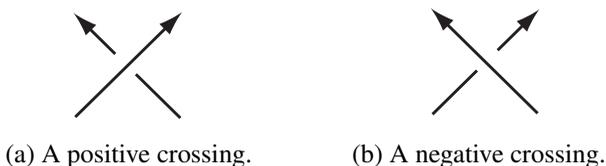
Figure 14. The oriented signs of a link diagram.
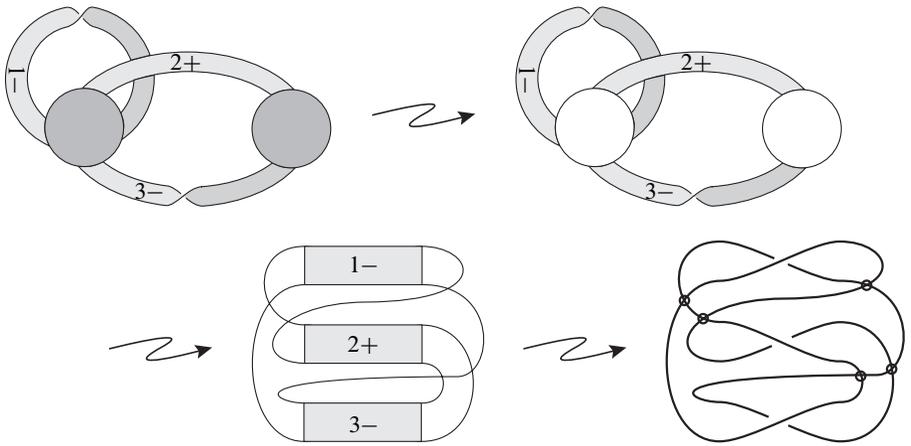
Figure 15. Recovering a virtual link diagram from a signed ribbon graph.

Chmutov and Voltz [2008] observed that the construction of a ribbon graph from a link diagram can be extended to include virtual links. That is, if D is a virtual link diagram and σ is a state of D, then $G_{(D,\sigma)}$ and the set $\mathbb{G}_D$ can be associated with D just as in Section 3.1 (virtual crossings are not smoothed, and the curves of the arrow presentation follow the component of the virtual link through the virtual crossings).

In Theorem 7 we determined how diagrams of links in $\mathbb{RP}^3$ that are represented by the same set of ribbon graphs are related. We will now consider the corresponding problem for virtual links. We start by determining which ribbon graphs represent virtual link diagrams.

If G is a signed ribbon graph, then we can recover a virtual link diagram D with $G = G_D$ as follows: Delete the interiors of the vertices of G (so that we obtain a set of ribbons that are attached to circles). Immerse the resulting object in the plane in such a way that the ribbons are embedded. (Note that as the circles are immersed, they may cross each other and themselves.) Replace each embedded ribbon with a classical crossing with the crossing structure determined by the sign, as in Figure 6. Make all of the intersection points of the immersed circles into virtual crossings. See Figure 15. The resulting virtual link diagram D has the desired property that $G = G_D$ (as G can be obtained for D by reversing the above construction). Moreover, every virtual link diagram that is represented by G can be obtained in this way. This follows since if $G = G_D$, then we can go through the above process drawing the circles and crossings in such a way that they follow D.

Thus we have that every signed ribbon graph is the signed ribbon graph of some virtual link diagram.

We now determine how virtual link diagrams that are represented by the same ribbon graphs are related. For this we need the concept of virtualisation. The

Figure 16. Virtualising a crossing.

virtualisation of a crossing of a virtual link diagram is the flanking of the crossing with virtual crossings as indicated in Figure 16. The crossing in the figure can also be of the opposite type.

Theorem 22. *Let D and D' be two virtual link diagrams. Then D and D' are presented by the same set of signed ribbon graphs if and only if they are related by virtualisation and the virtual Reidemeister moves.*

Proof. Let G be a signed ribbon graph. Label and arbitrarily orient each edge of G. As described above, every virtual link diagram represented by G can be obtained by (1) deleting the interiors of the vertices of G, (2) embedding the edges of G in the plane, (3) immersing the arcs connecting the edges (note that arcs in an immersion may cross each other), and (4) adding the crossing structure as described above.

Suppose D and D' are two virtual link diagrams obtained from G by this procedure. If the edges of G are oriented, in step (2) each embedding of an edge either agrees or disagrees with the orientation of the plane. If in step (2) of the constructions of D and D' the corresponding edges either both agree or both disagree with the orientation of the plane, it is easily seen that for some orientation of their components (in each diagram choose orientations that agree at each pair of crossings that correspond to the same edge of the ribbon graph), D and D' must then be described by the same Gauss diagram. In this case, by Theorem 21, they are related by the purely virtual moves and the semivirtual move.

Now suppose that in step (2) of the construction of D and D', there is an edge e of G such that the orientations of the two plane embeddings disagree with each other, and otherwise the embeddings of the edges and immersions of the arcs in step (3) are identical. Then, by Figure 17, the resulting virtual link diagrams are related by virtualisation.
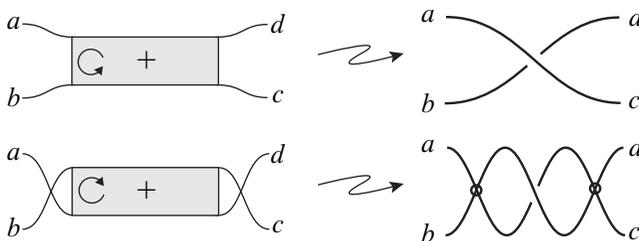
Figure 17. Forming virtual link diagrams from a signed ribbon graph.

It then follows that if *G* is a signed ribbon graph then the link diagrams it represents are related by virtualisation and the virtual moves. The converse of the theorem is easily seen to hold.   □

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