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The failed zero forcing number of a graph

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Given a graph G, the *zero forcing number* of G, $\mathrm{Z}(G)$, is the smallest cardinality of any set S of vertices on which repeated applications of the color change rule results in all vertices joining S. The *color change rule* is: if a vertex v is in S, and exactly one neighbor u of v is not in S, then u joins S in the next iteration.

In this paper, we introduce a new graph parameter, the failed zero forcing number of a graph. The *failed zero forcing number* of G, $\mathrm{F}(G)$, is the maximum cardinality of any set of vertices on which repeated applications of the color change rule will never result in all vertices joining the set.

We establish bounds on the failed zero forcing number of a graph, both in general and for connected graphs. We also classify connected graphs that achieve the upper bound, graphs whose failed zero forcing numbers are zero or one, and unusual graphs with smaller failed zero forcing number than zero forcing number. We determine formulas for the failed zero forcing numbers of several families of graphs and provide a lower bound on the failed zero forcing number of the Cartesian product of two graphs.

We conclude by presenting open questions about the failed zero forcing number and zero forcing in general.

1. Introduction

The concept of zero forcing has been explored over the past few years because of its application to minimum rank problems in linear algebra [Barioli et al. 2008; 2010]. For an introduction to minimum rank problems, see [Fallat and Hogben 2007]. While we do not discuss the details of minimum rank problems here, the zero forcing number of a graph provides an upper bound on the maximum nullity of any matrix associated with the graph, which in turn leads to a bound on the minimum rank of these matrices. This has led to active research on zero forcing, particularly on graphs for which the minimum rank is difficult to determine. Programs have been developed to determine the zero forcing number of a graph in Sage [DeLoss et al. 2008].

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Keywords: zero forcing number, vertex labeling, graph coloring.

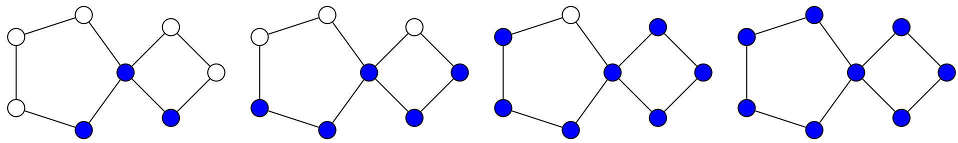

Figure 1. A starting set S and three iterations of the color change rule.

In this paper, we explore the other side of the problem, sets that fail to zero force.

Definitions. Let G be a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. We specify a coloring by choosing a set, usually called S, of vertices. The vertices in the set are filled in, and the others are left blank. Hence, our coloring consists only of two colors: filled, or unfilled. In much of the existing literature, the color black is used to represent filled in, and white is used to represent blank. We simply use *filled* and *unfilled*.

Unlike proper colorings, there are no rules to determine how we choose our initial set or coloring. Instead, we are interested in what happens when we apply the color change rule to our initial set. The standard *color change rule*, as described in [Barioli et al. 2008; 2010] among others, works as follows. Examine each filled vertex, one at a time. If a filled vertex u has exactly one unfilled neighbor, v, then we will fill v at the next iteration. In this case, we say that u *forces* v. Once we have examined all filled vertices, we iterate, and repeat. We repeat this process until no more color changes are possible. In Figure 1 we show a starting set S followed by three iterations of the color change rule.

We use the following term when no more color changes are possible.

Definition 1.1. Let S be a set of vertices in a graph. Suppose that no color changes are possible from S. Then we say that S is *stalled*.

If S is stalled, there are two possible scenarios: either $S = V(G)$ or there are some unfilled vertices that can never be filled. That is, we may be stuck. The two possible conditions under which a set is stalled distinguish a zero forcing set from a failed zero forcing set.

The next two definitions were formalized in [Barioli et al. 2008], although we use slightly different terminology.

Definition 1.2. Let S be a set of vertices in a graph such that repeated applications of the color change rule to S result in all vertices in the graph becoming filled. Then S is a *zero forcing set*.

It is easy to see that $V(G)$ itself is a trivial zero forcing set. The difficult problem is to find the smallest zero forcing set in G. There is considerable work in the literature on this problem, specifically because this parameter provides a bound useful in minimum rank problems [Barioli et al. 2008; 2010].

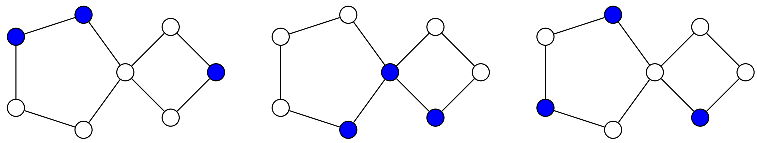

Figure 2. A failed zero forcing set, a zero forcing set, and a stalled failed zero forcing set.

Definition 1.3. The *zero forcing number* of a graph G, denoted $\mathrm{Z}(G)$, is the cardinality of a smallest zero forcing set in the graph.

In this paper, we are interested in subsets of a graph's vertex set that are not zero forcing sets. If a set of vertices is not a zero forcing set, then we will call it *failed*.

Definition 1.4. A *failed zero forcing set* is an initial set S of vertices in a graph such that, no matter how many times we apply the color change rule, some vertices in the graph will never be filled.

In Figure 2 we show a failed zero forcing set that is not stalled, a zero forcing set, and a failed zero forcing set that is stalled.

This new concept of failed zero forcing sets is the main topic of this paper. In particular, we are interested in *maximum failed zero forcing sets*, that is, finding failed zero forcing sets of largest cardinality in a graph. We define this parameter.

Definition 1.5. The *failed zero forcing number* of a graph G, denoted $\mathrm{F}(G)$, is the maximum cardinality of any failed zero forcing set in the graph.

At times, we will be interested in the concept of *maximal* failed zero forcing sets. Note the difference between maximum and maximal failed zero forcing sets. A maximal failed zero forcing set S is a set of vertices such that adding any other vertex in $V(G)$ to S will change S into a zero forcing set. A maximal failed zero forcing set may not be maximum, but a maximum failed zero forcing set is maximal.

We use the concept of a subgraph, as well as an induced subgraph, in this paper. If G is a graph and G' is a subgraph of G, then $V(G') \subseteq V(G)$, and any two vertices $u, v \in V(G')$ may be adjacent in G' if they are adjacent in $V(G)$, but they may not. If H is an induced subgraph of G, however, then if we have two vertices $u, v \in V(H)$ and $uv \in E(G)$, then $uv \in E(H)$ as well. If $S \subseteq V(G)$, then we use the notation $G[S]$ for the induced subgraph of G with vertex set S.

The concept of a module will be important in this paper.

Definition 1.6. A set X of vertices in $V(G)$ is a *module* if all vertices in X have the same set of neighbors among vertices not in X.

For example, in Figure 3 the set $\{a, b, c\}$ is a module of order 3; $\{b, c\}$ is a module of order 2.

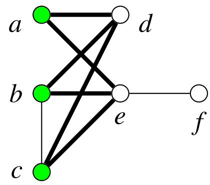

Figure 3. $S = \{a, b, c\}$ is a module.

Throughout the paper, we assume that G is a simple finite graph and use $n = |V(G)|$, unless n is otherwise defined. We now move on to exploring the basic properties of failed zero forcing sets.

Basic properties of failed zero forcing sets. We establish some fundamental observations about failed zero forcing sets. We compare these to known properties of zero forcing sets.

Note that any subset of $V(G)$ is either a zero forcing set or a failed zero forcing set. If S is a zero forcing set of a graph G, then note that any superset of S is also a zero forcing set. We can make a similar statement about failed zero forcing sets.

Observation 1.7. Suppose that G is a graph with failed zero forcing set $S \subseteq V(G)$. Then any subset of S is also a failed zero forcing set.

Next we consider how the color change rule may or may not act on a set of vertices. Let G be a graph, and suppose that S is a proper subset of $V(G)$. If S is a zero forcing set, then a color change must be possible from S. For failed zero forcing sets in general, we cannot make such a statement. For maximum failed zero forcing sets, however, we can.

Observation 1.8. If S is a maximum failed zero forcing set in a graph G, then S is stalled.

To see this, note that since S is a failed zero forcing set, it will not force all vertices in G. Therefore, at some iteration, no more color changes are possible. However, since S is maximum, it must also be maximal. Put simply, if a color change is possible from S, and S is a failed zero forcing set, then clearly, S is not maximum. Hence, any maximum failed zero forcing set S is stalled.

We also note two observations about subgraphs.

Observation 1.9. Let G be a graph with failed zero forcing set S, and let H be a subgraph of G. Then S restricted to H may not be a failed zero forcing set of H.

For example, let $G = P_4$, and let $S = \{v\}$, where v is an internal vertex. If we construct H by deleting the leaf adjacent to v, then S restricted to H is a zero forcing set. However, in a special case, the property is hereditary.

Observation 1.10. Let G be a graph with failed zero forcing set S, and let H be an induced subgraph of G where all vertices in $V(G)\backslash V(H)$ are in S. Then $S \cap V(H)$ is a failed zero forcing set in H.

Goals of this paper. While the zero forcing numbers of many graphs have been determined, the introduction of this relatively new topic has brought with it a large collection of open questions. Note that we can consider zero forcing to be a graph labeling problem with only two labels: *filled* or *unfilled*.

One major difference between zero forcing and other graph labeling problems is that the question of which labelings do *not* work is interesting. In proper coloring, for example, we can construct a failed proper coloring simply by coloring two adjacent vertices the same color. Thus, any graph with an edge has a trivial failed proper coloring. For zero forcing, however, there is no rule to determine how the vertices are labeled. We can choose any starting labeling; whether the labeling is successful or not depends on whether the color change rule leads to all vertices eventually being filled in. Therefore, in general it is not trivial to construct a failed zero forcing set.

Zero forcing opens up a wealth of new problems in graph theory. In this paper, we focus on the failed zero forcing number of different graph families and how these numbers relate to zero forcing numbers. In Section 2, we provide bounds on the failed zero forcing number of a graph, classify graphs with extreme failed zero forcing numbers, such as $\mathrm{F}(G) = 0$, 1, $n-2$ or $n-1$, and classify the unusual set of graphs for which $\mathrm{F}(G) < \mathrm{Z}(G)$. In Section 3, we establish this parameter for several classes of graphs. In Section 4, we explore the failed zero forcing number of the Cartesian product of graphs, including a lower bound in general and determination of the explicit value of the parameter for certain graph families.

We end with a set of open questions about zero forcing in general. While zero forcing numbers have been well studied for their applications to linear algebra, they have also opened up a new area of problems. We list some of these open questions in Section 5.

2. Bounds on failed zero forcing numbers

Whether G is connected or not, there are some fairly immediate bounds on the maximum failed zero forcing number.

Observation 2.1. For any graph G, we have $\mathrm{Z}(G) - 1 \leq \mathrm{F}(G) \leq n - 1$.

We explain both sides of the inequality here. If $\mathrm{Z}(G) - 1 > \mathrm{F}(G)$, then any set of order $\mathrm{Z}(G) - 1$ forces the graph, contradicting the definition of $\mathrm{Z}(G)$ as the minimum order of any zero forcing set. This gives us the lower bound. The upper bound is trivial: if a set S has order $n = V(G)$, then the set is not failed by definition.

It is fairly straightforward to see that $\mathrm{F}(G) = n - 1$ if and only if G has an isolated vertex. For the reverse direction, note that if G has an isolated vertex v_0, letting $S = V(G)\backslash\{v_0\}$ makes S a failed zero forcing set. For the forward direction, assume that $\mathrm{F}(G) = n-1$. Then there is some set S of $n-1$ vertices that does not force the lone vertex $v \in V(G)\backslash S$. If any vertex $u \in S$ is adjacent to v, however, u would force v. Hence, no vertex in S is adjacent to v; that is, v is an isolated vertex.

Hence, if G is connected, we can improve our bound from Observation 2.1.

Lemma 2.2. *Let G be a connected graph on n vertices where $n \geq 2$. Then*

$$\mathrm{Z}(G) - 1 \leq \mathrm{F}(G) \leq n-2.$$

Extreme values. We will show that the upper bound is sharp, that is, that there is a graph G that achieves $\mathrm{F}(G) = n-2$. In fact, we will classify such graphs. First, we prove a related lemma that will help us in classifying graphs with the upper bound.

Lemma 2.3. *Let G be a graph with module X of order $k > 1$. Then* $\mathrm{F}(G) \geq n-k$.

Proof. Let $S = V(G)\backslash X$. No vertices in X can be forced by vertices in S since if w is a vertex in S that is adjacent to some vertex $v \in X$, then w is adjacent to all vertices in X, of which there are $k > 1$. Hence, we have found a failed zero forcing set of order $n-k$. □

Note that if $G[X]$ is connected, we can improve this by letting $S = (V(G)\backslash X) \cup X'$, where X' is a failed zero forcing set of $G[X]$.

We now use Lemma 2.3 to classify connected graphs with failed zero forcing number $n-2$.

Theorem 2.4. *Let G be connected. Then* $\mathrm{F}(G) = n-2$ *if and only if G has a module of order* 2.

Proof. Suppose $\mathrm{F}(G) = n-2$. Let S be a maximum failed zero forcing set. Then $V(G)\backslash S = \{u, v\}$ for some vertices u and v. Since neither u nor v can be forced, every neighbor of u in S must also be a neighbor of v, and vice versa. Thus, $\{u, v\}$ is a module of order 2.

The converse follows from Lemmas 2.2 and 2.3. □

For trees, we can be even more specific.

Corollary 2.5. *Let T be a tree. Then* $\mathrm{F}(T) = n-2$ *if and only if either T has two leaves adjacent to a single vertex or $T = K_2$.*

Proof. We know by Theorem 2.4 that $\mathrm{F}(T) = n-2$ if and only if T has a module $X = \{u, v\}$ of order 2. If u and v each have two neighbors x and y, then $uxvyu$ forms a cycle; therefore u and v have at most one neighbor. It follows that T has a module $X = \{u, v\}$ if and only if u and v have one or less neighbors. That is, u and v are adjacent to a single common vertex or $T = K_2$. □

We now examine the lower bound from Lemma 2.2. This is of particular interest because a graph G that achieves this bound has $\mathrm{F}(G) < \mathrm{Z}(G)$, while we intuitively expect that the failed zero forcing number should be at least as large as the zero forcing number. This property is indeed unusual. Before providing our classification of graphs with $\mathrm{F}(G) = \mathrm{Z}(G) - 1$, we state two results that will be of use.

Observation 2.6 [Row 2011]. $\mathrm{Z}(G) = 1$ if and only if $G = P_n$ for some $n \geq 1$.

Theorem 2.7 [Row 2011]. *Let G be a connected graph with $n = |V(G)| \geq 2$. Then $\mathrm{Z}(G) = n - 1$ if and only if $G = K_n$.*

It turns out that complete graphs and their complements are the only graphs with $\mathrm{F}(G) < \mathrm{Z}(G)$, as we now show.

Theorem 2.8. *For any graph G, $\mathrm{F}(G) < \mathrm{Z}(G)$ if and only if $G = K_n$ or $G = \overline{K}_n$.*

Proof. We start with the reverse direction. By Theorem 2.7, the zero forcing number of a complete graph is $n - 1$. We also see from Theorem 2.4 that $\mathrm{F}(K_n) = n - 2$ since any pair of vertices forms a module. Hence, $\mathrm{F}(K_n) = \mathrm{Z}(K_n) - 1$. For the null graph (the complement of the complete graph), note that any zero forcing set must consist of the entire vertex set. To fail, we must remove one vertex from this set. Hence, $\mathrm{F}(\overline{K}_n) = \mathrm{Z}(\overline{K}_n) - 1$.

We now prove the forward direction. Let G be a graph with $\mathrm{F}(G) < \mathrm{Z}(G)$. Then we know that $\mathrm{F}(G) = \mathrm{Z}(G) - 1$ by Observation 2.1.

It follows that any set of cardinality $\mathrm{Z}(G)$ must be a zero forcing set. Otherwise, we would have a failed zero forcing set of cardinality $\mathrm{Z}(G)$, which would contradict our assumption that $\mathrm{F}(G) < \mathrm{Z}(G)$. Similarly, any set of cardinality $\mathrm{F}(G) = \mathrm{Z}(G) - 1$ is a failed zero forcing set. Otherwise, we would have a zero forcing set of cardinality $\mathrm{Z}(G) - 1$, which contradicts the definition of $\mathrm{Z}(G)$.

Let $S \subseteq V(G)$ with $|S| = \mathrm{Z}(G)$. If $|S| = 1$, then G is a path P_n by Observation 2.6. By our assumption, any vertex in G is a zero forcing set. But no internal vertex of P_n can force P_n, which means that G has no internal vertices. That is, $n = 2$. Since $P_2 = K_2$, in this case, the proof is complete.

Hence, we assume that $|S| \geq 2$. Now, S is a zero forcing set, which means that either some color change is possible from S or $S = V(G)$. If $S = V(G)$, then by assumption, any set of cardinality $n - 1$ or less fails to force the graph. That is, G must have no edges and is therefore $\overline{K}_n$, which completes the proof. Otherwise, some color change is possible from S. This means that there exists at least one vertex in S that is adjacent to exactly one vertex in $V(G) \backslash S$. Let S' be this nonempty set of vertices,

$$S' = \{v \in S \mid uv \in E(G) \text{ for exactly one } u \in V(G) \backslash S\}.$$

Let $w \in S$. Note that $S \backslash \{w\}$ is stalled since $|S \backslash \{w\}| = \mathrm{Z}(G) - 1 = F(G)$, and we saw above that any set of cardinality $\mathrm{F}(G)$ is a maximum failed zero forcing set.

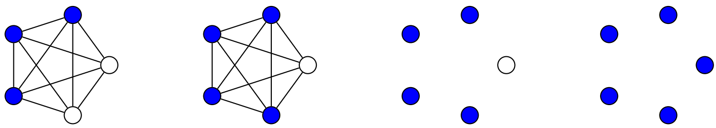

Figure 4. $\mathrm{F}(K_5) = 3 < \mathrm{Z}(K_5) = 4$; $\mathrm{F}(\overline{K}_5) = 4 < \mathrm{Z}(\overline{K}_5) = 5$.

Therefore, by Observation 1.8, $S\backslash\{w\}$ is stalled. It follows that w is adjacent to every vertex in S' (except w itself, if $w \in S'$). Hence, every vertex in S is adjacent to every vertex in S'. Additionally, by assumption, any set of cardinality $\mathrm{Z}(G) = |S|$ is a zero forcing set. Therefore, we can swap any vertex $u \in V(G)\backslash S$ with any vertex $w \in S$. Therefore, every vertex $u \in V(G)\backslash S$ is adjacent to every vertex in S'.

However, by the definition of S', each vertex in S' is adjacent to exactly one vertex in $V(G)\backslash S$. Hence, we must have that $|V(G)\backslash S| = 1$. That is, $\mathrm{Z}(G) = n-1$. By Theorem 2.7, $G = K_n$, completing the proof. □

In Figure 4 we illustrate that the failed zero forcing number of the complete graph on five vertices is less than its zero forcing number and that the failed zero forcing number of the null graph on five vertices is less than its zero forcing number.

Corollary 2.9. *A graph has $F(G) < Z(G)$ if and only if the automorphism group of G is doubly transitive.*

This is a result of the fact that only the complete graph and its complement have doubly transitive automorphism groups [Babai 1995].

Very small values. We have determined which graphs have large failed zero forcing numbers, such as $\mathrm{F}(G) = n-2$ or $n-1$. We now look at which graphs have very small failed zero forcing numbers.

Theorem 2.10. *Let G be a nonempty graph. Then* $\mathrm{F}(G) = 0$ *if and only if G is either a single vertex or K_2.*

Proof. The reverse direction is clear: For the case that G is a single vertex v, if we allow v to be in S, then the graph is forced; therefore $\mathrm{F}(G) = 0$. For the case that $G = K_2$, allowing either of the vertices to be in S will force the other vertex in the next iteration; therefore $\mathrm{F}(G) = 0$.

For the forward case, assume G is a graph with $\mathrm{F}(G) = 0$. Then any set $S \subseteq V(G)$ with $|S| = 1$ forces the graph. This means that G consists of a single vertex, or every vertex in G has degree one, and G is connected. But the only connected graph with every vertex of degree one is K_2. Hence the theorem. □

Theorem 2.11. $\mathrm{F}(G) = 1$ *if and only if G is one of the following graphs: a pair of isolated vertices, K_3, P_3 or P_4.*

Proof. The reverse direction is clear: if G is a pair of isolated vertices, then we can pick at most one of them to be in the set, otherwise the graph is trivially forced. If $G = K_3$, then any pair of vertices is a module of order 2, and if $G = P_3$, the end vertices form a module of order 2. Hence, in both cases, $\mathrm{F}(G) = n - 2 = 1$. For $G = P_4$, note that a single internal vertex is the largest subset of $V(P_4)$ that is not a zero forcing set.

For the forward direction, assume $\mathrm{F}(G) = 1$. If we allow G to be disconnected, it follows that G has at most two maximal connected components because if there are three nonempty components, we can take one vertex each from two of them, and this set will fail to force the third component. Since any pair of vertices in G can force G, each component has at most one vertex because otherwise we could take two vertices in a single component, leaving the other component unforced. Hence, if G is not connected, G is a pair of isolated vertices, and the proof is complete.

We now assume that G is connected. Since $\mathrm{F}(G) = 1$, any pair of vertices in G can force. We know that $\mathrm{F}(K_n) = n - 2$ for any n; thus, if $G = K_n$, then $n - 2 = 1$ implies that $G = K_3$, completing the proof. Assume that $G \neq K_n$. Then there is some pair of vertices, u and v, that are not adjacent. Let P be the shortest path from u to v. Since the set $S = \{u, v\}$ forces the graph by assumption, either u or v must force a vertex w in G. Assume without loss of generality that u forces w. Then u is adjacent only to w. Hence, w is the vertex along P that is adjacent to u. The vertex w can force the next vertex along the path (and continue this process) until we reach a vertex w' possibly with $w' = w$, where either w' is adjacent to an unforced vertex not on P in addition to the next vertex on P or the next vertex along P is already forced.

Assume the former. That is, assume that w' is adjacent to the next vertex on P as well as a vertex not on P. Since we assume that S is a zero forcing set, and must therefore eventually force the graph, it follows that one of these two vertices will be forced by some other vertex than w. But so far, the $u - w'$ path is forced, with no vertex except w' adjacent to any other vertex; we also have v forced. Hence, we must have that from v, a sequence of vertices is forced, resulting in one of the two vertices adjacent to w' being forced. Hence, v is only adjacent to a single vertex, and since we assume P is a path from u to v, it must be the vertex along P. By a similar argument, we have that no other vertices along P are adjacent to vertices not on P, except w'. Thus, G consists of P and a set of vertices connected to P only through w', as in Figure 5, where zigzag lines indicate paths. But then, we could take $S = \{u, w\}$, which will fail to force G, contradicting our assumption.

Similarly, if we assume that u forces a sequence of vertices until reaching w', which is adjacent to some vertex already forced, we have either $G = P$ or the situation from Figure 5 again, which leads to a contradiction. If $G = P$, then G is a path on n vertices. If $G = P_3$ or $G = P_4$, we're done. Otherwise, we can take any

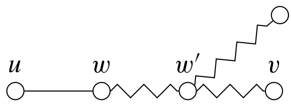

Figure 5. $\{u, v\}$ is a zero forcing set, but $\{u, w\}$ is not.

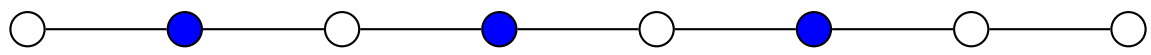

Figure 6. A maximum failed zero forcing set of P_8.

pair of nonadjacent vertices of degree 2 in G to be S, contradicting our assumption. Hence, $G = P_3$ or P_4. □

There are many examples of graphs with $\mathrm{F}(G) = 2$. For example, three isolated vertices, two copies of K_2, or an isolated vertex and K_2 all have failed zero forcing numbers of 2. Also, any connected graph G on four vertices, except for P_4, has $\mathrm{F}(G) = 2$, as does any connected graph on five vertices that does not have a module of order two, such as P_5 or the house graph. However, there are many such graphs. We stop at $\mathrm{F}(G) = 1$ and move on to determining failed zero forcing numbers of different families of graphs.

3. Failed zero forcing numbers of various families of graphs

We have already seen the failed zero forcing numbers of several graphs, including that of complete graphs, $\mathrm{F}(K_n) = n-2$. We now consider several families of graphs, including paths, cycles, complete bipartite graphs, binary trees, wheels, and the Petersen graph. We also give a formula for the failed zero forcing number of graphs with multiple connected components.

Theorem 3.1. *The failed zero forcing number of a path P_n on n vertices is*

$$\mathrm{F}(P_n) = \left\lceil \frac{n-2}{2} \right\rceil.$$

Proof. If S is a failed zero forcing set in P_n, then neither end vertex is in S because either end vertex is a zero forcing set. Further, S contains no pairs of adjacent vertices because any pair of adjacent vertices is a zero forcing set. Therefore, S can have at most $\lceil (n-2)/2 \rceil$ vertices in it. We construct such a set by starting with the vertex adjacent to either end vertex in P_n and adding it to S. From there, we take every other vertex until we reach the other end vertex, which we do not add to S. Thus, $|S| = \lceil (n-2)/2 \rceil$, and it does not force the graph because every vertex in S has exactly two neighbors not in S. □

In Figure 6, the construction of a maximum failed zero forcing set in P_8 is shown.

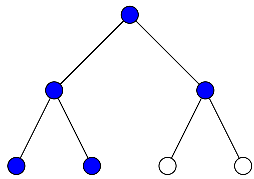

Figure 7. The failed zero forcing number of a binary tree with n vertices is $n-2$.

Theorem 3.2. *The failed zero forcing number of a cycle C_n on n vertices is*

$$\mathrm{F}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. Suppose S is a failed zero forcing set. Then there are no adjacent vertices in S since any pair of adjacent vertices forces C_n. Hence, $|S| \leq \lfloor n/2 \rfloor$. We can construct such a set by starting with any vertex in C_n and adding every other vertex to S. Since every vertex in S has two neighbors in $V(G)\backslash S$, the set S will not force the graph. Therefore, $\mathrm{F}(C_n) = \lfloor n/2 \rfloor$. □

We use $K_{m,n}$ to denote the complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = m \geq 1$ and $|V_2| = n \geq 1$.

Theorem 3.3. *If* $m+n \geq 3$, *then* $\mathrm{F}(K_{m,n}) = m+n-2$.

Proof. Since $m+n \geq 3$, it follows that $m \geq 2$ or $n \geq 2$. Without loss of generality, assume that $n \geq 2$. Then any pair of vertices in V_2 is a module of order 2 since both vertices have the same sets of neighbors, V_1. Hence, by Theorem 2.4, $\mathrm{F}(K_{m,n}) = m+n-2$. □

A full m-ary tree T is a rooted tree whose vertices have m or 0 children, where m is a positive integer of at least 2. Note that if $m = 2$, then T is a full binary tree.

Theorem 3.4. *The failed zero forcing number of a full m-ary tree T with $n > 1$ is* $\mathrm{F}(T) = n-2$.

Proof. Take any two vertices u and v of degree one that have the same parent, w. We know that u and v exist because T is finite and $m \geq 2$. Then, u and v form a module of order two because they each have exactly the same neighbor, w. Hence, by Theorem 2.4, $\mathrm{F}(T) = n-2$. □

In Figure 7, a binary tree with a maximum failed zero forcing set is shown.

The *join* of graphs G_1 and G_2, denoted $G_1 \vee G_2$, consists of a copy of G_1, a copy of G_2, and an edge between every pair of vertices u and v such that $u \in V(G_1)$ and $v \in V(G_2)$.

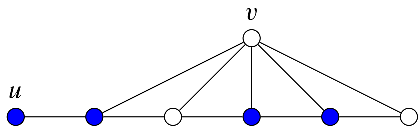

Figure 8. The graph G if $k = 1$.

Lemma 3.5. *Let G be a connected graph, and let $H = G \vee \{v_0\}$. That is, H consists of G and a single vertex v_0 that is adjacent to all vertices in G. Then* $\mathrm{F}(H) \geq \mathrm{F}(G) + 1$.

Proof. Let $S \subsetneq V(G)$ be stalled. Let $S' \subseteq V(H)$ be defined $S' = S \cup \{v_0\}$.

Since S is a failed zero forcing set in G, there are at least two vertices $u, v \in V(G)$ that are not forced by S. Any vertex in $S' \backslash \{v_0\}$ that is adjacent to v in H must also be adjacent to some other unforced vertex, otherwise it would force v in G. Also, v_0 is adjacent to both v and u, so it will not force v. Hence, S' is a failed zero forcing set of H. Since $|S'| = |S| + 1$, we have that $\mathrm{F}(H) \geq \mathrm{F}(G) + 1$. ☐

For any positive integer k, we can construct a graph G such that $\mathrm{F}(G \vee \{v_0\}) \geq \mathrm{F}(G) + k$. Let G consist of a path P_l, where $l = 3(k+1)$, and a vertex v that is adjacent to all vertices in P_l except for one end vertex, u. An example of G for $k = 1$ is shown in Figure 8. We claim that $\mathrm{F}(G) \leq \lfloor (2/3)l \rfloor$. First, suppose that S is a maximum failed zero forcing set. If $v \in S$, then no adjacent vertices from the path can be in S and neither end vertex can be in the path. Hence, if $v \in S$, this implies that $\mathrm{F}(G) \leq \lfloor l/2 \rfloor + 1$. If $v \notin S$, then no more than two consecutive vertices on the path can be in S because if three are in S, then the middle vertex will force v. Hence, $\mathrm{F}(G) \leq \lfloor (2/3)l \rfloor$. Since $l = 3(k+1) \geq 6$, it follows that $\lfloor l/2 \rfloor + 1 \leq \lfloor (2/3)l \rfloor$. Hence, $\mathrm{F}(G) \leq \lfloor (2/3)l \rfloor$.

Letting $H = G \vee \{v_0\}$ for a single vertex v_0, however, we find that $\mathrm{F}(H) \geq l - 1$. Let $S = P_l \backslash \{u\}$. That is, $S = H \backslash \{u, v, v_0\}$. Note that S is stalled because every vertex in S is adjacent to both v and v_0, which are not in S. Hence, S is a failed zero forcing set. Thus, we have that $\mathrm{F}(H) \geq l - 1 = 3(k+1) - 1 = 3k + 2$, and $\mathrm{F}(G) \leq \lfloor (2/3)l \rfloor = 2k + 2$. Thus, $\mathrm{F}(H) - \mathrm{F}(G) \geq k$.

Therefore, joining an additional vertex to a graph will certainly increase the failed zero forcing number of the graph, and the increase may be large.

We use Lemma 3.5 to examine another graph family. Let W_n be a wheel on $n+1$ vertices consisting of C_n and an additional vertex v_0 adjacent to all vertices in C_n.

Theorem 3.6. *Let $n \geq 3$. Then* $\mathrm{F}(W_n) = \lfloor 2n/3 \rfloor$ *if $n \neq 4$, and $F(W_4) = 3$.*

Proof. We know by Theorem 3.2 and Lemma 3.5 that $\mathrm{F}(W_n) \geq \lfloor n/2 \rfloor + 1$. We construct a failed zero forcing set S on W_n as follows. Starting with any vertex along the cycle, add the vertex and one of its neighbors to S. Continuing around the cycle, leave out the third vertex, add the next two to S, and leave out the next,

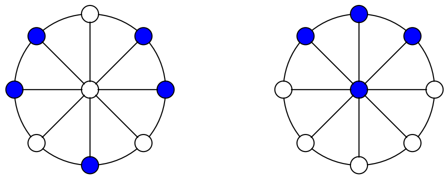

Figure 9. Left: A failed zero forcing set in W_8. Right: A zero forcing set in W_8.

as long as vertices are remaining, making sure at the end not to add any three consecutive vertices to S. Also, $v_0 \notin S$. Figure 9 shows this construction for W_8. Since there are no three consecutive vertices along the cycle in our set and v_0 is not in this set, it follows that every vertex in the set is adjacent to at least two vertices not in the set: v_0 and one vertex along the cycle. Hence, $\mathrm{F}(W_n) \geq \lfloor 2n/3 \rfloor$.

First, consider the special case that $n = 4$. Since $\mathrm{F}(W_n) \geq \lfloor n/2 \rfloor + 1$, we know that $\mathrm{F}(W_4) \geq 3$, but since $|V(W_4)| = 5$, by Lemma 2.2, we know that $\mathrm{F}(W_4) \leq 3$. Hence, $\mathrm{F}(W_4) = 3$.

We continue with the remaining cases, assuming for the remainder of the proof that $n \neq 4$. If $n \geq 6$, then $\lfloor 2n/3 \rfloor \geq \lfloor n/2 \rfloor + 1$ because

$$\lfloor 2n/3 \rfloor = \lfloor n/2 + n/6 \rfloor \geq \lfloor n/2 \rfloor + 1.$$

Also, for the special cases $n = 3$ and $n = 5$, we see that $\lfloor 2n/3 \rfloor = 2$ and 3 respectively. Similarly, for the same cases of $n = 3$ and $n = 5$, we have $\lfloor n/2 \rfloor + 1 = 2$ and 3 respectively. Hence, if $n \neq 4$, we know that $\mathrm{F}(W_n) \geq \lfloor 2n/3 \rfloor \geq \lfloor n/2 \rfloor + 1$.

Before proceeding, note that if at least three consecutive vertices along the cycle are in S and v_0 is in S, then S is a zero forcing set, as shown in Figure 9.

Finally, we show that $\mathrm{F}(W_n) \leq \lfloor 2n/3 \rfloor$. Let S be a set of vertices in W_n with $|S| > \lfloor 2n/3 \rfloor$. Then, either $v_0 \in S$ or there is some set of at least three consecutive vertices along the cycle that are in S. Assume that $v_0 \in S$. Since $|S| > \lfloor 2n/3 \rfloor$, there exists at least one pair of adjacent vertices along the cycle. Let u be in one such pair. If both neighbors of u along the cycle are in S, then we know that S is a zero forcing set and we're done. Otherwise, u has exactly one neighbor, w, not in S. Then, u will force w in the next iteration, and S is not a maximum zero forcing set.

The last possibility is that there is some set of three consecutive vertices, v_1, v_2, and v_3, along the cycle that are in S. Then v_0 is the only neighbor of v_2 that is not in S. Hence, v_2 forces v_0 in the next iteration, and S is a zero forcing set.

Hence, if $n \neq 4$, we have that $\mathrm{F}(W_n) = \lfloor 2n/3 \rfloor$. □

We have found the failed zero forcing numbers of several families of graphs. We now describe how the failed zero forcing number of a disconnected graph can be determined by the failed zero forcing numbers and orders of its components.

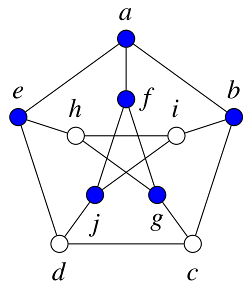

Figure 10. The Petersen graph with a maximum failed zero forcing set.

Theorem 3.7. *Let G be a disconnected graph, that is, a graph with at least two disjoint maximal connected components. Let $G_1, G_2, \ldots, G_k$ be the k maximal connected components of the graph. Then*

$$\mathrm{F}(G) = \max_k \left(\mathrm{F}(G_k) + \sum_{l \neq k} |V(G_l)| \right).$$

Proof. Since the graph is disconnected, if we allow S to consist of all vertices in the graph except those vertices in the component G_k, then clearly, G_k is not forced. We can add any failed zero forcing set of G_k to S, and still not force all vertices in G_k. This will work for any component G_k. Hence, we can pick the component that maximizes the cardinality of the set S. If S' is a set with

$$|S'| > \max_k \left(\mathrm{F}(G_k) + \sum_{l \neq k} |V(G_l)| \right),$$

then every component G_l must have $|S' \cap V(G_l)| > F(G_l)$, forcing all components. Hence, S' is a zero forcing set. □

Theorem 3.8. *Let G be the Petersen graph. Then* $\mathrm{F}(G) = 6$.

Proof. We can find a failed zero forcing set of cardinality six: for example, let $S = \{a, b, e, f, g, j\}$, as in Figure 10. This is clearly a failed zero forcing set, since a and f have all three neighbors in S, while all other vertices have exactly two neighbors in $V(G) \backslash S$. Hence, $\mathrm{F}(G) \geq 6$.

To prove that $\mathrm{F}(G) \leq 6$, suppose S is a maximum failed zero forcing set, and $|S| \geq 7$. By the pigeonhole principle, there are at least four vertices in S that are in the cycle $\{a, b, c, d, e\}$ or in $\{f, g, h, i, j\}$. Since there is an automorphism between these sets, assume without loss of generality that there are at least four from the set $\{a, b, c, d, e\}$ in S. Note that all five vertices cannot be in S because this would force the entire graph. Because of the symmetry, we can assume that $\{a, b, c, d\} \subseteq S$. Since S is a maximum failed zero forcing set, S is stalled. Thus, we must have

$i, g \in S$. Otherwise, b and c would force them. Now, i and g each have exactly two neighbors remaining that we have not assigned to S. These neighbors are f, h and j. If any one of f, h or j is in S, the others will be forced, which will force the graph. Hence, $\{f, h, j\} \subseteq V(G)\backslash S$. But we already know that $e \in V(G)\backslash S$ for the same reason, leaving us with $|S| = 6$. Hence, $\mathrm{F}(G) = 6$. □

4. Cartesian products

We first give a bound on the failed zero forcing number of a Cartesian product graph in terms of the failed zero forcing numbers of the graphs in the product.

Theorem 4.1. *For any graphs G and H,*

$$\mathrm{F}(G \,\Box\, H) \geq \max\big\{\mathrm{F}(G)|V(H)|, \mathrm{F}(H)|V(G)|\big\}.$$

Proof. Consider the Cartesian product $G \,\Box\, H$, where $n = |V(G)|$ and $k = |V(H)|$. Label the vertices of G, u_1 through u_n and the vertices of H, w_1 through w_k. We refer to each vertex in $G \,\Box\, H$ as $v_{i,j}$ where i denotes in which copy of G and j denotes in which copy of H the vertex lies.

Let S be a stalled failed zero forcing set in G. We construct a stalled failed zero forcing set S' in $G \,\Box\, H$ as follows. Suppose $u_\alpha \in S$. Then for all $i \in \{1, 2, \ldots k\}$, let $v_{i,\alpha} \in S'$. Then $|S'| = |S||V(H)|$. We show that S' is a failed zero forcing set of $G \,\Box\, H$.

Suppose $v_{\hat{i},\alpha}$ is in S'. Then $u_\alpha \in S$ by construction. Since S is a failed zero forcing set in G, then u_α is either adjacent to no vertices in $V(G)\backslash S$ or u_α is adjacent to two or more vertices in $V(G)\backslash S$, u_β and u_γ. In this latter case, it follows that $v_{\hat{i},\alpha}$ is adjacent to $v_{\hat{i},\beta}$ and $v_{\hat{i},\gamma}$ as well. In the former case, if u_α is adjacent to no vertices in $V(G)\backslash S$, then any neighbors of $v_{\hat{i},\alpha}$ of the form $v_{\hat{i},j}$ for some j are in S'. Since $v_{i,\alpha} \in S'$ for all i by construction, it follows that $v_{\hat{i},\alpha}$ has no neighbors in $V(G \,\Box\, H)\backslash S'$. Thus, S' is a stalled failed zero forcing set.

Since this construction works for any stalled failed zero forcing set in G, and similarly in H, it follows that we can construct in $G \,\Box\, H$ a failed zero forcing set of cardinality $\mathrm{F}(G)|V(H)|$ and similarly a failed zero forcing set of cardinality $\mathrm{F}(H)|V(G)|$. Hence the result. □

Note that the above bound is sharp if $G = P_2$ and $H = K_n$ for $n \geq 4$. Recall that $\mathrm{F}(K_n) = n - 2$ and $\mathrm{F}(P_2) = 0$. Thus, $\max\{\mathrm{F}(G)|V(H)|, \mathrm{F}(H)|V(G)|\} = 2(n-2)$. If we try to construct a failed zero forcing set S of $G \,\Box\, H$ with more vertices than $2(n-2)$, by the pigeonhole principle, one copy of K_n must have at least $n-1$ vertices in S. If one copy has n vertices, then $G \,\Box\, H$ is forced. Therefore, one copy, H_1, must have $n-1$ vertices, and the other, H_2, then must have at least $n-2$ in S. So every vertex in $S \cap V(H_1)$ is adjacent to the single vertex v in $V(H_1)\backslash S$. That means that every vertex in $S \cap V(H_1)$ must have at least one neighbor in $V(H_2)$ that is not

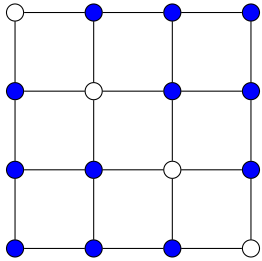

Figure 11. A maximum failed zero forcing set of the square grid $P_4 \square P_4$.

in S. But there are only at most two such vertices, and each has one distinct neighbor in H_1. Since $n \geq 4$, we know that $n-1 \geq 3$, which means that there is at least one vertex in $S \cap V(H_1)$ that has only one neighbor v in $V(G \square H) \backslash S$. Thus, v will be forced, which means that $V(H_1)$ will be completely forced, which will in turn force all vertices in the graph. Hence, $\mathrm{F}(P_2 \square K_n) = \max\{\mathrm{F}(G)|V(H)|, \mathrm{F}(H)|V(G)|\}$, showing that our bound from Theorem 4.1 is sharp.

For most cases, the failed zero forcing number of a Cartesian product of graphs is much greater than our bound. The following theorem establishes an exact value for the square grid graph.

Theorem 4.2. *Let $n \geq 2$. The failed zero forcing number of a square grid, $P_n \square P_n$, is $\mathrm{F}(P_n \square P_n) = n^2 - n$.*

Proof. We can construct such a failed zero forcing set by putting in the set every vertex in the graph, except those vertices along a single main diagonal. That is, if we label every vertex in the graph $v_{i,j}$, where i denotes the row and j denotes the column of the vertex, we let $v_{i,j}$ be in S if and only if $i \neq j$. See Figure 11 for an example.

We will show that S is indeed a failed zero forcing set. The only vertices that can be forced — because they are not in S — are $v_{i,i}$ for $i = 1, 2, \ldots n$. Take any such $v_{i,i}$. Then $v_{i,i}$ is adjacent to four vertices: $v_{i,i+1}$, $v_{i+1,i}$, $v_{i,i-1}$ and $v_{i-1,i}$. Note that if $i = 1$ or $i = n$, only the first two or the last two vertices (respectively) will be adjacent to $v_{i,i}$.

Now, $v_{i,i+1}$ is also adjacent to $v_{i+1,i+1}$, as is $v_{i+1,i}$. Therefore, neither $v_{i,i+1}$ nor $v_{i+1,i}$ will force $v_{i,i}$. Similarly, $v_{i,i-1}$ and $v_{i-1,i}$ are both also adjacent to $v_{i-1,i-1}$, and therefore do not force $v_{i,i}$. Hence, the set is a failed zero forcing set.

It remains to show that S is a maximum zero forcing set. We will show that any set S' with cardinality $|S'| > n^2 - n$ is not a failed zero forcing set.

By the pigeonhole principle, if $|S'| > n^2 - n$, there must be a column in G, say column $\hat{j}$, such that $v_{i,\hat{j}} \in S'$ for all $i = 1, 2, \ldots n$. Note that $1 < \hat{j} < n$ because any end column alone would force G. If every vertex in the first row to the right of column $\hat{j}$ is in S', then the entire graph will be forced; similarly, if all vertices in

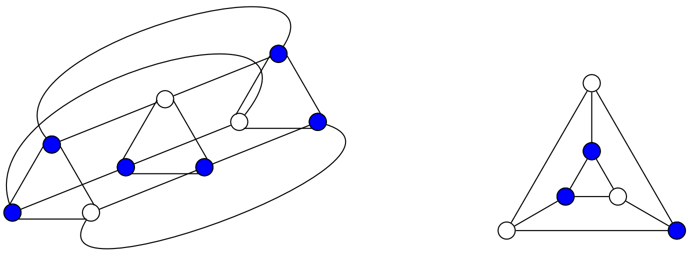

Figure 12. Failed zero forcing sets of $K_3 \square K_3$ and of $K_3 \square K_2$.

any two adjacent rows to the right of column $\hat{j}$ are in S', then the entire graph will be forced. The same holds for the left of column $\hat{j}$. Therefore, there are at least $\lceil (n+1)/2 \rceil$ vertices not in S' on either side of column $\hat{j}$, or at least $n+1$ in G, contradicting the assumption that $|S'| > n^2 - n$.

Therefore, $\mathrm{F}(P_n \square P_n) = n^2 - n$. □

The same construction works for $P_n \square P_m$ if $m = n + (n-1)k$ for a positive integer k. However, the construction does not work for rectangular grids in general.

For Cartesian products of complete graphs, $K_n \square K_m$, we have determined $\mathrm{F}(K_n \square K_m)$ for all cases. Before providing the general result, we must look at two special cases, $K_3 \square K_2$ and $K_3 \square K_3$. Figure 12 shows failed zero forcing sets for each graph. To see that these are optimal, note that neither graph has a module of order two; therefore $F(K_3 \square K_2) \leq 3$ and $F(K_3 \square K_3) \leq 6$, coinciding with the construction in Figure 12. We now move on to determining $\mathrm{F}(K_n \square K_m)$ in general.

Theorem 4.3. *The failed zero forcing number of the rook's graph,* $K_n \square K_m$, *is* $\mathrm{F}(K_n \square K_m) = nm - 4$, *where* $n \geq 4$ *and* $m \geq 2$.

Proof. First, we construct a failed zero forcing set S in $G = K_n \square K_m$ with cardinality $nm - 4$. Let each vertex in the graph be labeled $v_{i,j}$, where i denotes in which copy of K_n and j denotes in which copy of K_m the vertex lies. Let all vertices be in S except $v_{1,1}$, $v_{1,2}$, $v_{2,1}$ and $v_{2,2}$.

We show that S is a failed zero forcing set. The only vertices in S that are adjacent to the vertices in $V(G) \backslash S$ are vertices $v_{1,k}$, $v_{2,k}$, $v_{l,1}$ and $v_{l,2}$, where $3 \leq k \leq n$ and $3 \leq l \leq m$. However, $v_{1,k}$ is adjacent to both $v_{1,1}$ and $v_{1,2}$; $v_{2,k}$ is adjacent to both $v_{2,1}$ and $v_{2,2}$; $v_{l,1}$ is adjacent to both $v_{1,1}$ and $v_{2,1}$; finally, $v_{l,2}$ is adjacent to both $v_{1,2}$ and $v_{2,2}$. Therefore, S is a failed zero forcing set.

We now show that there is no failed zero forcing set larger than S. First, we show that there is no module of order 2 in G. Any two vertices in the same copy of K_n share all the same neighbors in K_n but lie in different copies of K_m and therefore have some distinct neighbors. Similarly, any two vertices in different copies of K_n have different neighbors in their respective copies of K_n. Hence, there is no module of order 2 in G, giving us, by Theorem 2.4, that $nm - 4 \leq \mathrm{F}(G) \leq nm - 3$.

Suppose that S' is a set of vertices in G of cardinality $nm-3$. Let $V(G)\backslash S' = \{x, y, z\}$. If $\{x, y, z\}$ is contained in a single copy of K_n, then take any other copy of K_n. There exist vertices x', y' and z' in this copy such that x' is adjacent to x but not y or z, and similarly for y' and z'. Thus, x' will force x, y' will force y, and z' will force z.

If there exists a copy of K_n, call it H, such that $V(H)\backslash S'$ consists of exactly one vertex, z, then z will be forced by another vertex in H because at most two of the vertices in H can be adjacent to an unforced vertex in any other copy of K_n. Since $\{x, y\}$ is not a module, it will be forced as well. Hence, $\mathrm{F}(G) = nm-4$. □

5. Conclusion and open questions

In this paper, we have defined a new graph parameter, the failed zero forcing number $\mathrm{F}(G)$, and established some properties of this parameter as well as the value of this parameter for several families of graphs. There are many questions about this parameter that remain. More generally, there are many questions that remain about the concept of zero forcing in general. We outline some of these questions here.

As we touched on in the introduction of this paper, the motivation for study of the zero forcing number is minimum rank problems. The maximum nullity of a set of a matrices associated with a graph is bounded above by the zero forcing number of the graph. We would like to know if the failed zero forcing number has any such connection to linear algebra.

There are many graph families whose failed zero forcing numbers are unknown. For example, while we found a value of $\mathrm{F}(P_n \,\square\, P_n)$, we have no formula for $\mathrm{F}(P_n \,\square\, P_m)$ in general. Also, we have not determined the failed zero forcing number of $C_n \,\square\, C_m$ or any Cartesian products of pairs of graphs from different graph families, such as paths and cycles. We know for certain trees — those who have two leaves adjacent to the same vertex — we have $\mathrm{F}(T) = n-2$. Trees in general, however, are open.

We can also look at graphs with failed zero forcing number of 2. We characterized graphs with $\mathrm{F}(G) = 0$ or 1, but many more graphs have $\mathrm{F}(G) = 2$. While we listed some of these, it would be nice to have a full characterization of all graphs with this property. More generally, given a positive integer k, is there an integer l such that any graph G with $|V(G)| > l$ has $\mathrm{F}(G) > k$?

Many graph labeling problems that search for the minimum number of labels required for a given graph are accompanied by a second question: what is the maximum cardinality of any *minimal* labeling? In proper coloring, this is known as the achromatic number [Harary and Hedetniemi 1970]. Failed zero forcing has an analogous problem: the *afailed zero forcing number*. The question is: what is the minimum cardinality of any maximal failed zero forcing set?

Finally, while we were able to classify graphs for which $\mathrm{F}(G) < \mathrm{Z}(G)$, it would be interesting to classify graphs for which $\mathrm{F}(G) = \mathrm{Z}(G)$, since these graphs seem to be unusual.

References

[Babai 1995] L. Babai, "Automorphism groups, isomorphism, reconstruction", Chapter 27, pp. 1447–1540 in *Handbook of combinatorics*, vol. 1, edited by R. L. Graham et al., Elsevier, Amsterdam, 1995. MR 97j:05029 Zbl 0846.05042

[Barioli et al. 2008] F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness, "Zero forcing sets and the minimum rank of graphs", *Linear Algebra Appl.* **428**:7 (2008), 1628–1648. MR 2008m:05166 Zbl 1135.05035

[Barioli et al. 2010] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst, "Zero forcing parameters and minimum rank problems", *Linear Algebra Appl.* **433**:2 (2010), 401–411. MR 2011g:15002 Zbl 1209.05139

[DeLoss et al. 2008] L. DeLoss, J. Grout, T. McKay, J. Smith, and G. Tims, "Program for calculating bounds on the minimum rank of a graph using Sage", preprint, 2008. arXiv 0812.1616

[Fallat and Hogben 2007] S. M. Fallat and L. Hogben, "The minimum rank of symmetric matrices described by a graph: a survey", *Linear Algebra Appl.* **426**:2-3 (2007), 558–582. MR 2008f:05114 Zbl 1122.05057

[Harary and Hedetniemi 1970] F. Harary and S. Hedetniemi, "The achromatic number of a graph", *J. Combinatorial Theory* **8** (1970), 154–161. MR 40 #7143 Zbl 0195.25702

[Row 2011] D. D. Row, *Zero forcing number: results for computation and comparison with other graph parameters*, Ph.D. thesis, Iowa State University, Ames, IA, 2011, available at http://lib.dr.iastate.edu/etd/12003. MR 2941885

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