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An exploration of ideal-divisor graphs

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Zero-divisor graphs have given some interesting insights into the behavior of commutative rings. Redmond introduced a generalization of the zero-divisor graph called an ideal-divisor graph. This paper expands on Redmond's findings in an attempt to find additional information about the structure of commutative rings from ideal-divisor graphs.

1. Definitions and introduction

Throughout, we assume that R is a finite commutative ring with identity, though in some instances the proofs given can be extended to more general rings. A *zero-divisor* in R is an element x such that there exists a nonzero $y \in R$ with $xy = 0$. The set of all zero-divisors in R is denoted by $Z(R)$. The set of all nonzero zero-divisors is denoted by $Z(R)^*$.

A graph G is defined by a vertex set $V(G)$ and an edge set

$$E(G) \subseteq \{\{a, b\} \mid a, b \in V(G)\}.$$

Two vertices x and y joined by an edge are said to be *adjacent*, denoted $x - y$. A vertex x is said to be *looped* if $x - x$. A *path* between two elements $a_1, a_n \in V(G)$ is an ordered sequence $\{a_1, a_2, \ldots, a_n\}$ of distinct vertices of G such that $a_{i-1} - a_i$ for all $1 < i \leq n$. If there exists a path between any two distinct vertices, then the graph is said to be *connected*. A graph is said to be *complete* if every vertex is adjacent to every other vertex, and we denote the complete graph on n vertices by K^n. A graph G is a *finite graph* if $V(G)$ is a finite set.

If the vertices of a graph G can be partitioned into two sets with vertices adjacent only if they are in distinct sets, then G is *bipartite*. If vertices in a bipartite graph are adjacent if and only if they are in distinct vertex sets, then the graph is called *complete bipartite*. We will denote the complete bipartite graph with distinct vertex sets of cardinalities m and n by $K^{m,n}$. A *star graph* is a complete bipartite graph

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such that one of its vertex sets has cardinality one. In general, we say a graph G is a *refinement* of a graph H if $V(G) = V(H)$ and $E(H) \subseteq E(G)$. We note that any graph of radius one is a refinement of a star graph.

For any other terms not defined here, see [Chartrand 1985] for a graph theory reference, and see [Herstein 1990] for a ring theory reference. The figures in this paper were generated by Mathematica using programs originally written by Brendan Kelly, Darrin Weber, and Elisabeth Wilson and modified to suit our needs.

Beck [1988] was the first to define the zero-divisor graph of a commutative ring. However, it was in the seminal paper [Anderson and Livingston 1999] that the structure was first used extensively to reveal ring-theoretic properties. In this paper, the *zero-divisor graph* of R, denoted $\Gamma(R)$, is the simple graph with vertex set $V(\Gamma(R)) = Z(R)^*$ and edge set

$$E(\Gamma(R)) = \{\{a, b\} \mid a, b \in V(\Gamma(R)),\ ab = 0 \ \text{ and } \ a \neq b\}.$$

Redmond [2003] introduced *ideal-divisor graphs*, a generalization of zero-divisor graphs. For I an ideal of R, an element $x \in R$ is an *ideal-divisor* if there exists some $y \in R \setminus I$ such that $xy \in I$. The set of ideal-divisors of R with respect to I is denoted $Z_I(R)$. The *ideal-divisor graph* of a R with respect to an ideal I, denoted $\Gamma_I(R)$, is the simple graph with vertex set $V(\Gamma_I(R)) = Z_I(R)^*$ and edge set

$$E(\Gamma_I(R)) = \{\{x, y\} \mid x, y \in V(\Gamma_I(R)),\ x \neq y \ \text{ and } \ xy \in I\}.$$

Redmond [2003] proved that if I is an ideal of R, then $\Gamma_I(R)$ is connected with $\text{diam}(\Gamma_I(R)) \leq 3$. He proved further that if $\Gamma_I(R)$ contains a cycle, then $g(\Gamma_I(R)) \leq 7$, and he developed an algorithm for constructing the graph of $\Gamma_I(R)$ from $\Gamma(R/I)$.

Redmond, like Anderson and Livingston, did not include looped vertices in his definition of the ideal-divisor graph. The following definitions have therefore been modified to include looped vertices. The *zero-divisor graph* of R (denoted $\Gamma(R)$) has vertex set $V(\Gamma(R)) = Z(R)^*$ and edge set

$$E(\Gamma(R)) = \{\{a, b\} \mid a, b \in V(\Gamma(R)) \ \text{ and } \ ab = 0\}.$$

The *ideal-divisor graph* of R with respect to an ideal I, denoted $\Gamma_I(R)$, has vertex set $V(\Gamma_I(R)) = Z_I(R)^*$ and edge set

$$E(\Gamma_I(R)) = \{\{x, y\} \mid x, y \in V(\Gamma_I(R)) \ \text{ and } \ xy \in I\}.$$

These modified definitions allow a vertex b in $\Gamma(R)$ or $\Gamma_I(R)$ to be adjacent to itself if and only if $b^2 = 0$ or $b^2 \in I$ for each graph, respectively.

In Sections 2 and 3, we expand upon Redmond's results by examining the structure of $\Gamma_I(R)$. We also consider the relationships between $\Gamma_I(R)$ and $\Gamma(R/I)$. In particular, we establish conditions for $\Gamma_I(R)$ to be finite, demonstrate several relationships between the cut-sets of $\Gamma(R/I)$ and $\Gamma_I(R)$, and prove a result on the connectivity of $\Gamma_I(R)$. In Section 4, we modify and prove a modification of a proposition presented in [Redmond 2003]. A brief discussion at the end of this paper examines the structure of $\Gamma_I(R)$ when I is a radical, primary, or weakly prime ideal.

The following results are included for reference. Although these results were proven for graphs without loops, it is straightforward to check that they still hold when the graphs are looped.

Theorem 1.1 [Redmond 2003, Theorem 2.5]. *Let I be an ideal of R, and let $x, y \in R \setminus I$. Then*:

(1) *If $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$, then x is adjacent to y in $\Gamma_I(R)$.*

(2) *If x is adjacent to y in $\Gamma_I(R)$ and $x + I \neq y + I$, then $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$.*

(3) *If x is adjacent to y in $\Gamma_I(R)$ and $x + I = y + I$, then $x^2, y^2 \in I$.*

Corollary 1.2 [Redmond 2003, Corollary 2.6]. *If x and y are* (*distinct*) *adjacent vertices in $\Gamma_I(R)$, then all* (*distinct*) *elements of $x + I$ and $y + I$ are adjacent in $\Gamma_I(R)$. If $x^2 \in I$, then all the distinct elements of $x + I$ are adjacent in $\Gamma_I(R)$.*

2. Structure of $\Gamma_I(R)$

In this section, we investigate the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$, and provide some results about the general structure of $\Gamma_I(R)$.

A few definitions are needed for clarification in this section. Elements of the vertex set of $\Gamma_I(R)$ which are elements of the same coset in R/I form a *column* in $\Gamma_I(R)$ [Redmond 2003, Theorem 2.9]. Corollary 1.2 gives that if a vertex $a + I$ is looped (i.e., $(a + I)^2 = 0 + I$) in $\Gamma(R/I)$, then all the vertices in the corresponding column of $\Gamma_I(R)$ are adjacent to one another. Finally, the *ideal annihilator* of an element $a \in R \setminus I$ with respect to some ideal I is the set $(I : a) = \{b \mid ab \in I \text{ and } b \in R \setminus I\}$.

Proposition 2.1. *Let I be an ideal of R. If $(a + I)$ and $(b + I)$ are distinct vertices in $\Gamma(R/I)$ with $(a + I) - (b + I)$, then the columns corresponding to $a + I$ and $b + I$, taken as a pair, form a subgraph that is a refinement of a complete bipartite graph in $\Gamma_I(R)$. Moreover, for any $a + i \in V(\Gamma_I(R))$, $|(I : a + i)|$ is equal to $k|I|$ for some $k \in \mathbb{N}$.*

Proof. This result follows directly from Theorem 1.1. □

Example 2.2. In Figure 1, each adjacent pair of columns of $\Gamma_{(8)}(\mathbb{Z}_{24})$ is a refinement of a complete bipartite graph.

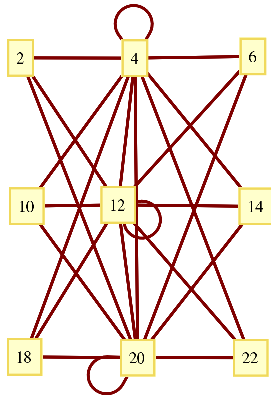

Figure 1. $\Gamma_{(8)}(\mathbb{Z}_{24})$.

Theorem 2.3. *Let S be a commutative ring. Then $\Gamma_I(S)$ is finite if and only if either S is a finite ring or I is a prime ideal. In particular, if $1 \leq |\Gamma_I(S)| < \infty$, then S is a finite ring and I is not a prime ideal.*

Proof. ($\Rightarrow$) If I is prime, then $\Gamma_I(S) = \varnothing$. So, assume I is not prime.

(1) If I is infinite, then by [Redmond 2003, Corollary 2.7], $\Gamma_I(S)$ is infinite.

(2) If I is finite and S is infinite, then S/I is infinite and not an integral domain, so $\Gamma(S/I)$ is infinite (see [Ganesan 1964]). By [Redmond 2003, Theorem 2.5], since $\Gamma(S/I)$ is isomorphic to a subgraph of $\Gamma_I(S)$, $\Gamma_I(S)$ is also infinite.

($\Leftarrow$) Clear. □

3. Cut-sets and connectivity

In a connected graph, a *cut-vertex* is a vertex that, when it and any edges incident to it are removed, separates the graph into two or more connected components. Cut-vertices were introduced into the analysis of zero-divisor graphs in [Axtell et al. 2009] and were further studied in [Axtell et al. 2011]. In [Redmond 2003, Theorem 3.2], Redmond proved that $\Gamma_I(R)$ contains no cut-vertices whenever I is a nonzero proper ideal of R. Cut-sets, a generalization of the cut-vertex, were also introduced into the analysis of zero-divisor graphs in [Coté et al. 2011]. For a connected graph G, a subset $A \subset V(G)$ is a *cut-set* if there exist $c, d \in V(G) \setminus A$ such that every path from c to d contains at least one vertex from A, and no proper subset of A satisfies the same condition. It is easy to show that for a given nonempty set of vertices A, the existence of such c and d is equivalent to the existence of two subgraphs X and Y of G whose (vertexwise and edgewise) union equals G, and whose vertex sets satisfy $V(X) \cap V(Y) = A$, $V(X) \setminus A \neq \varnothing$, and $V(Y) \setminus A \neq \varnothing$. When this happens we say that A *separates* X and Y.

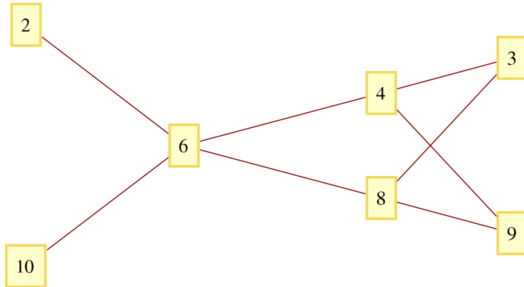

Figure 2. $\Gamma(\mathbb{Z}_{12})$, using Anderson and Livingston's definition.

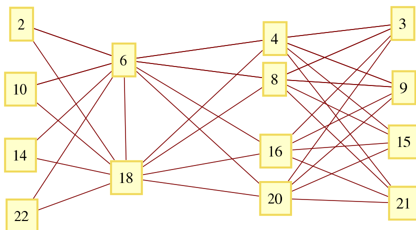

Figure 3. $\Gamma_{(12)}(\mathbb{Z}_{24})$.

Theorem 3.1. *Let I be an ideal of R. If A is a cut-set in $\Gamma_I(R)$, then A is a column or a union of columns.*

Proof. Assume A is a cut-set of $\Gamma_I(R)$. Let $x, y \in V(\Gamma_I(R)) \setminus A$. Let $x - \cdots - a + i - \cdots - y$ be a path from x to y, where $a + i \in A$. Since $x - \cdots - a + \bar{i} - \cdots - y$ is also a path from x to y for all $\bar{i} \in I$, we must have $a + I \subseteq A$. □

As an example, let $R = \mathbb{Z}_{24}$ and let $I = (12)$. Since $R/I \cong \mathbb{Z}_{12}$, we can identify $\Gamma(R/I)$ with Figure 2. We notice that the vertices 4 and 8 form a cut-set. Likewise, looking at Figure 3, in $\Gamma_{(12)}(\mathbb{Z}_{24})$ the set {4, 8, 16, 20} is a cut-set. We note that in this figure 4 and 16 form the column associated with $4 + (12)$, while 8 and 20 form the column associated with $8 + (12)$.

Theorem 3.2. *If A is a cut-set in $\Gamma(R/I)$, then $B = \{a + i \mid a + I \in A, i \in I\}$ is a cut-set in $\Gamma_I(R)$.*

Proof. Let X and Y be subgraphs of $\Gamma(R/I)$ separated by the cut-set A. Let $x, y \in V(\Gamma_I(R))$ such that $x + I$ and $y + I$ are vertices of X and Y, respectively. Let $x + I - \cdots - y + I$ be a path from $x + I$ to $y + I$. Then since A is a cut-set, this path must contain at least one element from A.

Suppose there exists a path $x - z_1 - \cdots - z_n - y$ from x to y that does not contain at least one element from B. From Corollary 1.2, it can be assumed without loss of generality that each z_j is in a distinct column of $\Gamma_I(R)$, where $1 \leq j \leq n$. Thus, by

Figure 4. $\Gamma((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)/(\{0\} \times \{0\} \times \mathbb{Z}_2))$.

Theorem 1.1, $(x+I)-(z_1+I)-\cdots-(z_n+I)-(y+I)$ is a path in $\Gamma(R/I)$. This path does not contain at least one element from A, contradicting the fact that A is a cut-set. Therefore, every path between x and y contains at least one element of B.

Suppose B is not the minimal such set. Then there exists some $b \in B$ such that every path from x to y contains at least one vertex from $B \setminus \{b\}$. Then $b+I \in A$, and every path from x to y contains at least one element from $A \setminus \{b+I\}$. This contradicts that A is a cut-set of $\Gamma(R/I)$. $\square$

The converse is not always true. Consider the graph of

$$\Gamma((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)/(\{0\} \times \{0\} \times \mathbb{Z}_2)),$$

shown in Figure 4, which is isomorphic to K^2.

There are no cut-vertices or cut-sets in the above graph. However, the sets $\{(0,1,0),(0,1,1)\}$ and $\{(1,0,0),(1,0,1)\}$ are cut-sets in $\Gamma_{(\{0\}\times\{0\}\times\mathbb{Z}_2)}(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2)$.

Lemma 3.3. *If $x, y \in \Gamma_I(R)$ are distinct and every path connecting x to y contains a vertex $z \in A \subseteq V(\Gamma_I(R))$, then every path connecting $x+I$ to $y+I$ in $\Gamma(R/I)$ contains an element of $B = \{a+I \mid a \in A\}$.*

Proof. Let $(x+I)-(w_1+I)-\cdots-(w_k+I)-(y+I)$ be a path from $x+I$ to $y+I$ in $\Gamma(R/I)$. If $w_n \notin A$ for all $1 \le n \le k$, then there exists a path $x-w_1-\cdots-w_k-y$ in $\Gamma_I(R)$ that does not contain an element of A, a contradiction. $\square$

Theorem 3.4. *If a cut-set A in $\Gamma_I(R)$ is a union of n columns and $|Z(R/I)^*|-n \ge 2$, then $B = \{a+I \mid a \in A\}$ is a cut-set in $\Gamma(R/I)$.*

Proof. Suppose A is a cut-set in $\Gamma_I(R)$. Since $|Z(R/I)^*|-n \ge 2$, there are $b, c \in \Gamma_I(R) \setminus A$ such that b and c are in different columns, and any path connecting b and c contains an element $a \in A$. To see this, note that if two vertices are in the same column and are separated by A, then these vertices must be isolated when A is removed, because vertices in the same column are adjacent to the same set of vertices by Corollary 1.2. Since there are at least two columns left after the removal of A, we can now choose b and c in different columns that meet the desired conditions.

By Lemma 3.3, any path from $b+I$ to $c+I$ in $\Gamma(R/I)$ must contain $a+I$ for some $a \in A$. The set of all such points is B; thus, B is a cut-set or contains a cut-set.

Suppose B is not minimal. Then there exists some $a_i+I \in B$ such that $C \subseteq B \setminus \{a_i+I\} = \{a+I \mid a \in A \setminus \{a_i\}\}$ is a cut-set in $\Gamma(R/I)$. Then by Theorem 3.2, $D = \{a+i \mid a+I \in C, i \in I\} \subset A$ is a cut-set in $\Gamma_I(R)$, a contradiction. $\square$

Figure 5. $\Gamma(\mathbb{Z}_{27}/(9))$.

If $|Z(R/I)^*|-n<2$, then B would certainly not be a cut-set in $\Gamma(R/I)$. If there was only one column remaining after the removal of A from $\Gamma_I(R)$, then there would only be one coset representative remaining after the removal of B from $\Gamma(R/I)$.

It is proved in [Coté et al. 2011] that if R is not local and if B is a cut-set of $\Gamma(R)$, then $B\cup\{0\}$ is an ideal. A similar theorem for cut-sets in $\Gamma_I(R)$ is provided.

Theorem 3.5. *Let I be an ideal of R such that R/I is nonlocal, let A be a cut-set in $\Gamma(R/I)$, and let $B=\{a+i \mid a+I\in A, i\in I\}$. Then $B\cup I$ is an ideal of R.*

Proof. Let A be a cut-set in $\Gamma(R/I)$. Then $A\cup\{0+I\}$ is an ideal of R/I by [Coté et al. 2011]. Then $B\cup I=\phi^{-1}(A\cup\{0+I\})$, where $\phi: R\to R/I$ is the canonical homomorphism, is an ideal of R. □

The *connectivity* of a connected graph G, denoted $\kappa(G)$, is the minimum number of vertices that must be removed from G to produce a disconnected graph. It is customary to define the connectivity of the complete graph K^n to be $\kappa(K^n)=n-1$. In other words, $\kappa(G)$ is the order of the smallest cut-set of G, when G is not isomorphic to K^n. The following result on the connectivity of $\Gamma_I(R)$ is Theorem 3.3 of [Redmond 2003].

Theorem 3.6. *Let I be a nonzero proper ideal of R.*

(1) *If $\Gamma(R/I)$ is the graph on one vertex, then $\kappa(\Gamma_I(R))=|I|-1$.*

(2) *If $\Gamma(R/I)$ has at least two vertices, then $2\le\kappa(\Gamma_I(R))\le|I|\cdot\kappa(\Gamma(R/I))$.*

(3) $|I|-1\le\kappa(\Gamma_I(R))$.

In light of this theorem, consider $\Gamma(\mathbb{Z}_{27}/(9))$, shown in Figure 5. The connectivity of $\kappa(\Gamma(\mathbb{Z}_{27}/(9)))$ is 1. So, by the above theorem, $\kappa(\Gamma_{(9)}(\mathbb{Z}_{27}))$ should be 2 or 3. However, since $\Gamma_{(9)}(\mathbb{Z}_{27})$ (shown in Figure 6) is complete, $\kappa(\Gamma_{(9)}(\mathbb{Z}_{27}))=|\Gamma_{(9)}(\mathbb{Z}_{27})|-1=5$. A reading of the proof of this theorem in [Coté et al. 2011] shows this problem arises only when $\Gamma(R/I)$ is complete. We provide the following modification of this theorem to take into account complete graphs.

Theorem 3.7. *Let I be a nonzero proper ideal of R.*

(1) *If $\Gamma(R/I)$ is complete on more than two vertices, then*

$$\kappa(\Gamma_I(R))=|I|\cdot|V(\Gamma(R/I))|-1.$$

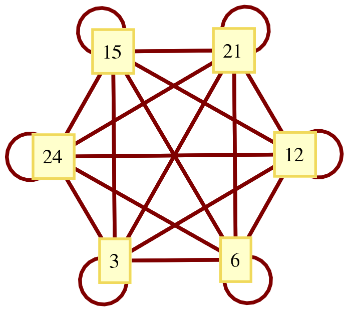

Figure 6. $\Gamma_{(9)}(\mathbb{Z}_{27})$.

(2) *If* $\Gamma(R/I)$ *is the graph on two vertices*, *then*

$$\kappa(\Gamma_I(R)) = |I| \quad \textit{or} \quad |I| \cdot |V(\Gamma(R/I))| - 1.$$

(3) *If* $\Gamma(R/I)$ *is not complete and has at least three vertices*, *then*

$$2 \le \kappa(\Gamma_I(R)) \le |I| \cdot \kappa(\Gamma(R/I)).$$

(4) $|I| - 1 \le \kappa(\Gamma_I(R))$.

Proof. Parts 3 and 4 are proved in [Coté et al. 2011].

(1) Suppose $\Gamma(R/I)$ is complete. Then for all $a + I, b + I \in \Gamma(R/I)$, we have $(a+I)-(b+I)$. By [Anderson and Livingston 1999, Theorem 2.8], $Z(R/I)^2 = \{0\}$. Thus, by Theorem 1.1, $\Gamma_I(R)$ is complete. Hence $\kappa(\Gamma_I(R)) = |\Gamma_I(R)| - 1 = |I| \cdot |\Gamma(R/I)| - 1$. (See [Coté et al. 2011, Remark 28].)

(2) Suppose $\Gamma(R/I)$ is the graph on two vertices, $x+I$ and $y+I$. Then by [Anderson and Livingston 1999, Theorem 2.8], either $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Z(R/I)^2 = \{0\}$. Thus, there are two cases:

(a) Suppose $x^2+I = 0+I = y^2+I$. Then $\Gamma_I(R)$ is complete. Thus, $\kappa(\Gamma_I(R)) = |I| \cdot |\Gamma(R/I)| - 1$.

(b) Suppose $x^2+I \neq 0+I \neq y^2+I$. Then $\Gamma_I(R)$ is isomorphic to $K_{|I|,|I|}$. Without loss of generality, let $x \in x+I \subset V(\Gamma_I(R))$. Then x is adjacent to every vertex in $y+I$. Thus, to create a disconnected graph from $\Gamma_I(R)$, every vertex in $y+I$ is removed, i.e., we remove $|I|$ vertices. □

4. Classifying ideals via ideal-divisor graphs

Let I be an ideal of R. The *radical* of I is the set $\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}$. For any ideal I, $\sqrt{I}$ is an ideal of R, and if $\sqrt{I} = I$, then I is called a *radical ideal*. Note that for a radical ideal I of R, if $|I| \ge 2$, then there are no connected columns.

Lemma 4.1. *Let I be an ideal of R and let $a \in Z_I(R)^*$. If no vertex of $\Gamma_I(R)$ is looped, then $a^n \notin I$ for all $n \in \mathbb{N}$.*

Proof. Suppose $a^n \in I$ for some least $n \in \mathbb{N}$. Then, $a^{n-1} \in Z_I(R)^*$ and $(a^{n-1})^2 \in I$. Thus, a^{n-1} is looped, a contradiction. □

Theorem 4.2. *Let I be an ideal of R. Then I is a radical ideal if and only if no vertex in $\Gamma_I(R)$ is looped (equivalently, $\Gamma_I(R)$ has no connected columns).*

Proof. (⇒) Consider $a \in V(\Gamma_I(R))$. Since $a \notin I$, $a^n \notin I$ for all $n \in \mathbb{N}$. Thus, $a^2 \notin I$.

(⇐) Let $a \in V(\Gamma_I(R))$. By Lemma 4.1 and the definition of an ideal divisor, $a^n \notin I$ for all $n \in \mathbb{N}$. Hence, if $b^n \in I$ for some $n \in \mathbb{N}$, we must have $b \in I$. Thus, I is a radical ideal. □

We now move to a classification of primary and weakly prime ideals. Let Q be an ideal of R. We say Q is a *primary ideal* if whenever $ab \in Q$, either $a \in Q$ or $b^n \in Q$ for $n \in \mathbb{N}$. Let P be a proper ideal of R. Then, P is *weakly prime* if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$ (see [Anderson and Smith 2003]).

Lemma 4.3. *Let I be an ideal of R. Let $K = \{k_1, k_2, \dots, k_n\} \subseteq R \setminus I$ such that for each $k_i \in K$, there exists a minimal $m_i \in \mathbb{N}$ such that $k_i^{m_i} \in I$. Then there exists $a \in R \setminus I$ such that $ak_i \in I$ for all $k_i \in K$.*

Proof. There exists a minimal $m_1 \geq 2$ such that $k_1^{m_1} \in I$. Let $a_1 = k_1^{m_1-1}$. Clearly, $a_1 k_1 \in I$ and $a_1 \notin I$. Now, there exists a minimal n_2 with $1 \leq n_2 \leq m_2$ such that $a_1 k_2^{n_2} \in I$. Now let $a_2 = a_1 k_2^{n_2-1}$. Again, $a_2 k_2 \in I$ and $a_2 \notin I$. Continuing in this fashion, there exists an $n_j - 1$ (possibly zero, in which case $a_j = a_{j-1}$) such that $a_j = a_{j-1} k_j^{n_j-1} \notin I$ but $a_j k_j \in I$. Let $a = a_n$. By construction, a is connected to every $k_i \in K$. □

Theorem 4.4. *Let I be a nonzero ideal of R that is not prime. Then I is a primary ideal if and only if $\Gamma_I(R)$ is a refinement of a star graph.*

Proof. (⇒) Let $a, b \in V(\Gamma_I(R))$ with $ab \in I$. By definition of $V(\Gamma_I(R))$ and the fact that I is primary, we have $a^r, b^s \in I$ for some $r, s \geq 2$. Therefore, we have $V(\Gamma_I(R)) \subseteq R \setminus I$, and for each $x \in V(\Gamma_I(R))$ there is some $n \in \mathbb{N}$ such that $x^n \in I$. By Lemma 4.3, we have at least one $y \in V(\Gamma_I(R))$ with $xy \in I$ for all $x \in V(\Gamma_I(R))$. That is, the vertex y connects to every other vertex in $\Gamma_I(R)$. Thus, $\Gamma_I(R)$ is a refinement of a star graph.

(⇐) If $\Gamma_I(R)$ is a refinement of a star graph, then the diameter of $\Gamma_I(R)$ is 2. According to Corollary 2.7 in [Redmond 2003], since I is an ideal of R, $\Gamma_I(R)$ contains a subgraph that is isomorphic to $\Gamma(R/I)$. Using Theorem 1.1 and Corollary 1.2, there exists an element $x + I$ that is connected to every other element, including itself, in $\Gamma(R/I)$. Then, applying Lemma 3.1 in [Axtell et al. 2009] gives us that $Z(R/I)$ is an ideal. If the zero-divisors of a finite ring form an ideal, then that ideal

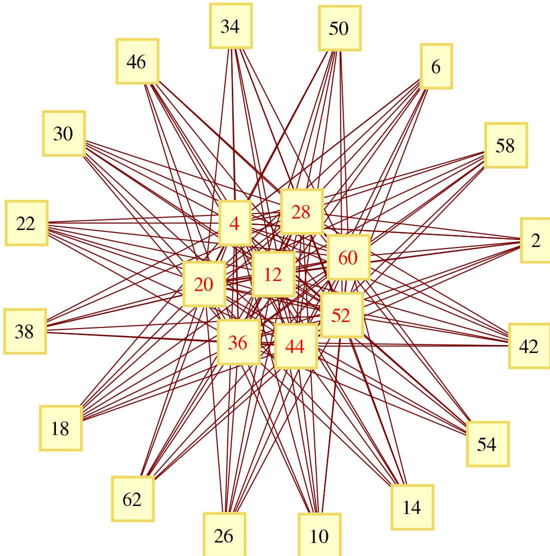

Figure 7. $\Gamma_{(8)}(\mathbb{Z}_{64})$.

is the maximal ideal of the ring, and the ring is local. It is well known that if a ring is local and finite, every zero-divisor is nilpotent. Every zero-divisor in R/I is nilpotent, so I is a primary ideal. □

Example 4.5. Let $R = \mathbb{Z}_{64}$ and $I = (8)$. We see I is a primary ideal of R, and in the figure below, we see that we have a refinement of a star graph. Note that any of {4, 12, 20, 28, 36, 44, 52, 60} could work as our central vertex (see Figure 7).

Note that the condition that I is a nonzero ideal of R in Theorem 4.4 is necessary for the "if" portion on the proof, for if $R = \mathbb{Z}_2 \times F$, where F is a field, and $I = \{(0, 0)\}$, then $\Gamma(R/I)$ is a star graph, but I is not a primary ideal. The issue that arises in this case is that $(1, 0)$ is connected to every other vertex in $\Gamma(R/I)$, but it is not looped.

Lemma 4.6. *Let I be a weakly prime ideal and let $a \in R \setminus I$. If $a^k \in I$ for some $k \in \mathbb{N}$, then $a^k = 0$.*

Proof. Let $a \in R \backslash I$ and assume $a^k \in I^*$. Then $0 \neq a \cdot a^{k-1} \in I$. Since $a \notin I$, we have $a^{k-1} \in I$ because I is weakly prime. Continuing, we obtain $0 \neq a \cdot a \in I$, but $a \notin I$, a contradiction. □

Theorem 4.7. *Let I be a nonzero ideal of R that is not prime. Then I is weakly prime if and only if $\Gamma_I(R)$ is the induced subgraph of $\Gamma(R)$ on $Z(R) \setminus I$.*

Proof. (⇒) According to Theorem 7 in [Anderson and Smith 2003], R is not decomposable, so R is either local or a field. Supposing R is local, it is well known that every zero-divisor is nilpotent. Let $a \in Z(R) \setminus I$. Since a is nilpotent, there exists a minimal $n \in \mathbb{N}$ such that $a^n = 0 \in I$. So, by Lemma 4.6, $a \cdot a^{n-1} \in I$ and $a^{n-1} \notin I$. Hence, $a \in V(\Gamma_I(R))$. Now let $a, b \in V(\Gamma_I(R))$ with $ab \in I$. Since I is weakly prime, $ab = 0$. Hence, $Z(R) \setminus I = V(\Gamma_I(R))$, and $a - b \in \Gamma_I(R)$ if and only if $ab = 0$. Thus, $\Gamma_I(R)$ is the induced subgraph of $\Gamma(R)$ on $Z(R) \setminus I$.

($\Leftarrow$) Assume the ideal-divisor graph is the induced subgraph of $\Gamma(R)$ on $Z(R) \setminus I$. Let $a, b \notin I$ and $ab \in I$. Since $\Gamma_I(R)$ is the induced subgraph of $\Gamma(R)$ on $Z(R) \setminus I$, $ab = 0$. Thus, I is a weakly prime ideal. □

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