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a journal of mathematics

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msp

2015 vol. 8, no. 4

INVOLVE 8:4 (2015)
dx.doi.org/10.2140/involve.2015.8.629

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Differentiation with respect to parameters of solutions of nonlocal boundary value problems for difference equations

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(Communicated by Kenneth S. Berenhaut)

For the n-th order difference equation, $\Delta^n u = f(t, u, \Delta u, \ldots, \Delta^{n-1}u, \lambda)$, the solution of the boundary value problem satisfying $\Delta^{i-1}u(t_0) = A_i$, $1 \le i \le n-1$, and $u(t_1) - \sum_{j=1}^{m} a_j u(\tau_j) = A_n$, where $t_0, \tau_1, \ldots, \tau_m, t_1 \in \mathbb{Z}$, $t_0 < \cdots < t_0+n-1 < \tau_1 < \cdots < \tau_m < t_1$, and $a_1, \ldots, a_m, A_1, \ldots, A_n \in \mathbb{R}$, is differentiated with respect to the parameter λ.

1. Introduction

With differences defined by $\Delta u(t) = u(t+1) - u(t)$ and $\Delta^i u(t) = \Delta(\Delta^{i-1}u(t))$ for $i > 1$, we will be concerned with solutions of the n-th order difference equation,

$$\Delta^n u = f(t, u, \Delta u, \ldots, \Delta^{n-1}u, \lambda), \tag{1-1}$$

satisfying Dirichlet conditions

$$\Delta^{i-1}u(t_0) = A_i, \quad 1 \le i \le n-1, \tag{1-2}$$

and nonlocal boundary conditions

$$u(t_1) - \sum_{j=1}^{m} a_j u(\tau_j) = A_n, \tag{1-3}$$

where $t_0, \tau_1, \ldots, \tau_m \in \mathbb{Z}$, $t_0 + n - 1 < \tau_1 < \cdots < \tau_m < t_1$, $A_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $a_j \in \mathbb{R}$, $j = 1, \ldots, m$.

Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{N}$ denote, respectively, the integers, the real numbers and the natural numbers. Given $\varnothing \ne S \subseteq \mathbb{R}$, let $S_{\mathbb{Z}} := S \cap \mathbb{Z}$. We assume throughout the paper that for (1-1):

MSC2010: primary 39A10, 34B08; secondary 34B10.
Keywords: difference equation, boundary value problem, nonlocal, differentiation with respect to parameters.

(A) $f(t, s_1, \dots, s_n, \lambda) : \mathbb{Z} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous.

(B) $(\partial f/\partial s_i)(t, s_1, \dots, s_n, \lambda) : \mathbb{Z} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous for $i = 1, \dots, n$.

(C) $(\partial f/\partial \lambda)(t, s_1, \dots, s_n, \lambda) : \mathbb{Z} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous.

Given a solution $u(t)$ of (1-1), two linear equations playing fundamental roles for our results are the *variational equation along* $u(t)$ given by

$$\Delta^n z = \sum_{i=1}^{n} \frac{\partial f}{\partial s_i}(t, u(t), \dots, \Delta^{n-1}u(t), \lambda)\Delta^{i-1}z, \tag{1-4}$$

and the corresponding nonhomogeneous equation along $u(t)$ given by

$$\Delta^n z = \sum_{i=1}^{n} \frac{\partial f}{\partial s_i}(t, u(t), \dots, \Delta^{n-1}u(t), \lambda)\Delta^{i-1}z + \frac{\partial f}{\partial \lambda}(t, u(t), \dots, \Delta^{n-1}u(t), \lambda). \tag{1-5}$$

Our primary motivation arises from results by Henderson, Horn and Howard [Henderson et al. 1994] dealing with differentiation with respect to parameters for solutions of difference equations satisfying multipoint boundary conditions. Study of the relationship between a solution to a differential or difference equation and the associated variational equation can trace its origin to a result that Hartman [1982] attributed to Peano concerning differentiation of solutions of a differential equation with respect to initial conditions. Since then, these results have been extended and refined in various ways including boundary value problems for differential equations and difference equations [Datta 1998; Ehme and Henderson 1992; Henderson and Lee 1991; Spencer 1975]. Datta and Henderson [1992] did research on differentiation of solutions of difference equations with respect to boundary conditions. Benchohra et al. [2007] extended these results to nonlocal boundary value problems for second order difference equations. Also, interest in multipoint and nonlocal boundary value problems has grown significantly [Ashyralyev et al. 2004; Benchohra et al. 2007; Henderson et al. 2008; Lyons 2011]. Hopkins et al. [2009] proved a theorem about boundary data smoothness for solutions of nonlocal boundary value problems for second order difference equations. Then, Lyons [2014] generalized those results to n-th order difference equations.

Lyons [2014] has obtained extensive results for solutions of (1-1)–(1-3) when f is independent of λ. Our main results concern differentiation of solutions of (1-1)–(1-3) with respect to the parameter λ. Section 2 is devoted to results for initial value problems. We state theorems concerning solutions of initial value problems for (1-1) and their continuity and differentiability properties with respect to initial values and parameters. Then, in Section 3, we present two uniqueness assumptions and state theorems concerning continuous dependence with respect to both boundary values and parameters. Finally, in Section 4, we provide our result dealing with solutions of (1-1)–(1-3) and their differentiability properties with respect to the parameter λ.

2. Initial value problems

The n-th order difference equation (1-1) along with the conditions

$$\Delta^{i-1}v(\sigma_0) = c_i, \quad 1 \le i \le n, \tag{2-1}$$

where $\sigma_0 \in \mathbb{Z}$, $c_i \in \mathbb{R}$, $1 \le i \le n$, is called an initial value problem. For notational purposes, we let $v(t) = v(t, \sigma_0, c_1, \ldots, c_n, \lambda)$ denote the solution of the initial value problem (1-1), (2-1) on $[\sigma_0, +\infty)_{\mathbb{Z}}$. Results stated in this section concerning continuous dependence and differentiability of v with respect to initial conditions and parameters can be found in [Datta and Henderson 1992; Henderson and Lee 1991].

Theorem 2.1 (continuous dependence with respect to initial values). *Assume that condition (A) is satisfied. Let $\sigma_0 \in \mathbb{Z}$, $c_1, \ldots, c_n \in \mathbb{R}$, and $\lambda_0 \in \mathbb{R}$ be given. Then, for each $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists a $\delta(\varepsilon, \sigma_0, k, c_1, \ldots, c_n, \lambda_0) > 0$ such that if $|c_i - d_i| < \delta$, $1 \le i \le n$, and $|\lambda_0 - p_0| < \delta$, then*

$$|\Delta^{i-1}v(t, \sigma_0, c_1, \ldots, c_n, \lambda_0) - \Delta^{i-1}v(t, \sigma_0, d_1, \ldots, d_n, p_0)| < \varepsilon$$

on $[\sigma_0, k]_{\mathbb{Z}}$ for $i = 1, \ldots, n$.

Theorem 2.2 (discrete Peano). *Assume that conditions (A), (B) and (C) are satisfied. Let $\sigma_0 \in \mathbb{Z}$, $c_1, \ldots, c_n \in \mathbb{R}$, and let $\lambda \in \mathbb{R}$ be given. Then, for each $1 \le j \le n$, given $r_1, \ldots, r_n \in \mathbb{R}$ and $\lambda_0 \in \mathbb{R}$,*

$$\alpha_j(t) := \frac{\partial v}{\partial c_j}(t, \sigma_0, r_1, \ldots, r_n, \lambda_0), \quad 1 \le i \le n,$$

exists, is the solution of the variational equation (1-4) along $v(t, \sigma_0, r_1, \ldots, r_n, \lambda_0)$ and satisfies the initial conditions

$$\Delta^{i-1}\alpha_j(\sigma_0) = \delta_{ij}, \quad 1 \le i \le n.$$

Moreover,

$$\beta(t) := \frac{\partial v}{\partial \lambda}(t, \sigma_0, r_1, \ldots, r_n, \lambda_0)$$

exists, is the solution of the nonhomogeneous equation (1-5) along $v(t, \sigma_0, r_1, \ldots, r_n, \lambda_0)$, and satisfies the initial conditions

$$\Delta^{i-1}\beta(\sigma_0) = 0, \quad 1 \le i \le n.$$

3. Boundary value problems

In order to establish a relation between the work in the last section and boundary value problems, we need two uniqueness assumptions.

(D) Given $\lambda \in \mathbb{R}$, $t_0, \tau_1, \dots, \tau_n, t_1 \in \mathbb{Z}$, $t_0+n-1 < \tau_1 < \cdots < \tau_n < t_1$, and $A_i \in \mathbb{R}$, $1 \le i \le n$, if $u_1(t)$ and $u_2(t)$ are solutions of (1-1)–(1-3), then $u_1(t) \equiv u_2(t)$ on $[t_0, +\infty)_{\mathbb{Z}}$.

(E) For each $\lambda \in \mathbb{R}$ and $t_0, \tau_1, \dots, \tau_n, t_1 \in \mathbb{Z}$, and for each solution $u(t)$ of (1-1), the only solution $\rho(t)$ of the boundary value problem for the variational equation (1-4) along $u(t)$ and satisfying

$$\Delta^{(i-1)}\rho(t_0) = 0, \quad 1 \le i \le n-1,$$

and

$$\rho(t_1) - \sum_{j=1}^{m} a_j \rho(\tau_j) = 0,$$

where $t_0 + n - 1 < \tau_1 < \cdots < \tau_m < t_1$, is

$$\rho(t) \equiv 0 \text{ on } [t_0, +\infty)_{\mathbb{Z}}.$$

Theorem 3.1 (continuous dependence with respect to boundary values and parameters). *Assume conditions (A) and (D) are satisfied. Let $y(t)$ be a solution of (1-1) for some $\lambda \in \mathbb{R}$ on $[a, +\infty)_{\mathbb{Z}}$. Let $t_0 < \cdots < t_0 + n - 1 < \tau_1 < \cdots < \tau_m < t_1$ in $[a, +\infty)_{\mathbb{Z}}$ be given. Then, there exists $\varepsilon > 0$ such that if $|\Delta^{i-1}y(t_0) - A_i| < \varepsilon$, $1 \le i \le n-1$, and $|y(t_1) - \sum_{j=1}^{m} a_j y(\tau_j) - A_n| < \varepsilon$, and if $|\lambda - \mu| < \varepsilon$, then the boundary value problem for (1-1) with respect to the parameter μ satisfying*

$$\Delta^{i-1}h(t_0) = A_i, \quad 1 \le i \le n-1,$$

and

$$h(t_1) - \sum_{j=1}^{m} a_j h(\tau_j) = A_n$$

has a unique solution, $h(t, t_0, t_1, \tau_1, \dots, \tau_m, A_1, \dots, A_n, \mu)$, on $[t_0, +\infty)_{\mathbb{Z}}$, and moreover,

$$h(t, t_0, t_1, \tau_1, \dots, \tau_m, A_1, \dots, A_n, \mu) \to y(t),$$

as $\varepsilon \to 0$, on $[t_0, +\infty)_{\mathbb{Z}}$.

4. Main result

Now, we provide our main result concerning differentiation of solutions of (1-1)–(1-3) with respect to the parameter λ.

Theorem 4.1. *Assume conditions (A)–(E) are satisfied. For $t_0 < \cdots < t_0 + n - 1 < \tau_1 < \cdots < \tau_m < t_1$ in $\mathbb{Z}$, let $u(t, t_0, t_1, \tau_1, \dots, \tau_m, A_1, \dots, A_n, \lambda)$ denote the solution of (1-1)–(1-3) on $[t_0, +\infty)_{\mathbb{Z}}$. Then, $\partial u/\partial\lambda$ exists on $[t_0, +\infty)_{\mathbb{Z}}$, and*

$w(t) := (\partial u/\partial\lambda)(t)$ is the solution of the nonhomogeneous linear equation (1-5) along $u(t)$ and satisfies

$$\Delta^{i-1} w(t_0) = 0, \quad 1 \le i \le n-1,$$

and

$$w(t_1) - \sum_{j=1}^{m} a_j w(\tau_j) = 0.$$

Proof. Let $\varepsilon > 0$ be given. For $0 < |h| < \varepsilon$, we consider the difference quotient

$$w_h(t) := \frac{1}{h}\big(u(t, t_0, t_1, \tau_1, \ldots, \tau_n, A_1, \ldots, A_n, \lambda + h) \\ - u(t, t_0, t_1, \tau_1, \ldots, \tau_n, A_1, \ldots, A_n, \lambda)\big).$$

We show that $\lim_{h\to 0} w_h(t)$ exists on $[t_0, +\infty)_{\mathbb{Z}}$. For $h \ne 0$, we first observe that, for $1 \le i \le n-1$,

$$\begin{aligned} \Delta^{i-1} w_h(t_0) &= \frac{1}{h}\big(\Delta^{i-1} u(t_0, t_0, t_1, \tau_1, \ldots, \tau_m, A_1, \ldots, A_n, \lambda + h) \\ &\qquad - \Delta^{i-1} u(t_0, t_0, t_1, \tau_1, \ldots, \tau_m, A_1, \ldots, A_n, \lambda)\big) \\ &= \frac{1}{h}(A_i - A_i) = 0, \end{aligned}$$

and

$$\begin{aligned} w_h(t_1) - \sum_{j=1}^{m} \alpha_j w_h(\tau_j) &= \frac{1}{h}\big(u(t_1, t_0, t_1, \tau_1, \ldots, \tau_m, A_1, \ldots, A_n, \lambda + h) \\ &\qquad - \sum_{j=1}^{m} a_j u(\tau_j, t_0, t_1, \tau_1, \ldots, \tau_m, A_1, \ldots, A_n, \lambda + h) \\ &\qquad - u(t_1, t_0, t_1, \tau_1, \ldots, \tau_m, A_1, \ldots, A_n, \lambda) \\ &\qquad + \sum_{j=1}^{m} a_j u(\tau_j, t_0, t_1, \tau_1, \ldots, \tau_m, A_1, \ldots, A_n, \lambda)\big) \\ &= \frac{1}{h}(A_n - A_n) = 0. \end{aligned}$$

Next, we set

$$D := \Delta^{n-1} u(t_0, t_0, t_1, \tau_1, \ldots, \tau_m, A_1, \ldots, A_n, \lambda)$$

and

$$\varepsilon_h := \varepsilon_0(h) = \Delta^{n-1} u(t_0, t_0, t_1, \tau_1, \ldots, \tau_m, A_1, \ldots, A_n, \lambda + h) - D.$$

By Theorem 3.1, $\varepsilon_h \to 0$ as $h \to 0$. With $v(t, t_0, c_1, \ldots, c_n, \lambda)$ being our notation for solutions of initial value problems (1-1), (2-1) corresponding to λ in (1-1), we have, by using a telescoping sum,

$$\begin{aligned} w_h(t) &= \frac{1}{h}\big(v(t, t_0, A_1, \dots, A_{n-1}, D+\varepsilon, \lambda+h) - v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda)\big) \\ &= \frac{1}{h}\big(v(t, t_0, A_1, \dots, A_{n-1}, D+\varepsilon, \lambda+h) - v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda+h) \\ &\qquad + v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda+h) - v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda)\big). \end{aligned}$$

By Theorem 2.2, $\alpha_n = \partial v/\partial c_n$ and $\beta = \partial v/\partial\lambda$ both exist. So, by the mean value theorem,

$$\begin{aligned} w_h(t) &= \frac{1}{h}\big(\alpha_n(t, v(t, t_0, A_1, \dots, A_{n-1}, D+\bar{\varepsilon}, \lambda+h))(D+\varepsilon-D) \\ &\qquad + \beta(t, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda+\bar{h})(\lambda+h-\lambda))\big) \\ &= \frac{\varepsilon}{h}\alpha_n(t, v(t, t_0, A_1, \dots, A_{n-1}, D+\bar{\varepsilon}, \lambda+h)) \\ &\qquad + \beta(t, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda+\bar{h})), \end{aligned}$$

where

$$\alpha_n(t, v(t, t_0, A_1, \dots, A_{n-1}, D+\bar{\varepsilon}, \lambda+h)) = \frac{\partial v}{\partial c_n}(t, t_0, A_1, \dots, A_{n-1}, D+\bar{\varepsilon}, \lambda+h),$$

$$\beta(t, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda+\bar{h})) = \frac{\partial v}{\partial\lambda}(t, t_0, A_1, A_{n-1}, D, \lambda+\bar{h}),$$

$\bar{\varepsilon}$ is between 0 and ε, and $\bar{h}$ is between 0 and h.

To show that $\lim_{h\to 0} w_h(t)$ exists, it suffices to show that $\lim_{h\to 0} \varepsilon/h$ exists. We have the $n-1$ conditions, $\Delta^{i-1} w_h(t_0) = 0$, $i = 1, \dots, n-1$, and the condition $w_h(t_1) - \sum_{j=1}^m a_j w_h(\tau_j) = 0$. So, from the last condition,

$$\begin{aligned} &\frac{\varepsilon}{h}\frac{\partial v}{\partial c_n}(t_1, t_0, A_1, \dots, A_{n+1}, D+\bar{\varepsilon}, \lambda+h) + \frac{\partial v}{\partial\lambda}(t_1, t_0, A_1, \dots, A_{n-1}, D, \lambda+\bar{h}) \\ &\quad - \frac{\varepsilon}{h}\sum_{j=1}^m a_j \frac{\partial v}{\partial c_n}(t_1, t_0, A_1, \dots, A_{n+1}, D+\bar{\varepsilon}, \lambda+h) \\ &\qquad - \sum_{j=1}^m a_j \frac{\partial v}{\partial\lambda}(t_1, t_0, A_1, \dots, A_{n-1}, D, \lambda+\bar{h}) = 0. \end{aligned}$$

Hence, we have

$$\begin{aligned} \frac{\varepsilon}{h} = \frac{1}{M_{h,\bar{\varepsilon}}}\Big(&-\beta(t_1, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda+\bar{h})) \\ &+ \sum_{j=1}^m a_j \beta(\tau_j, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda+\bar{h}))\Big), \end{aligned}$$

where

$$M_{h,\bar{\varepsilon}} := \alpha_n(t_1, v(t, t_0, A_1, \dots, A_{n-1}, D+\bar{\varepsilon}, \lambda+h)) - \sum_{j=1}^{m} a_j \alpha_n(\tau_j, v(t, t_0, A_1, \dots, A_{n-1}, D+\bar{\varepsilon}, \lambda+h)).$$

Now, $\Delta^{n-1}\alpha_n(t_0, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda)) = 1$, so

$$\alpha_n(t, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda) \not\equiv 0.$$

By uniqueness assumption (E),

$$\alpha_n(t_1, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda)) - \sum_{j=1}^{m} a_j \alpha_n(\tau_j, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda)) \neq 0.$$

By Theorem 3.1, for h sufficiently small, $M_{h,\bar{\varepsilon}} \neq 0$. So, $\lim_{h\to 0} \varepsilon/h$ exists, and

$$\lim_{h\to 0} \frac{\varepsilon}{h} = \lim_{h\to 0} \frac{-1}{M_{h,\bar{\varepsilon}}} \Big(\beta(t_1, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda+\bar{h})) - \sum_{j=1}^{m} a_j \beta(\tau_j, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda+\bar{h}))\Big) := J.$$

Hence, $\lim_{h\to 0} w_h(t)$ exists, or in particular, $(\partial u/\partial\lambda)(t) = \lim_{h\to 0} w_h(t)$ exists on $[t_0, +\infty)_{\mathbb{Z}}$, and

$$\begin{aligned} w(t) :=& \lim_{h\to 0} w_h(t) \\ =& \frac{\partial u}{\partial \lambda}(t) \\ =& J \cdot \alpha_n(t, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda)) + \beta(t, v(t, t_0, A_1, \dots, A_{n-1}, D, \lambda)) \\ =& J \cdot \alpha_n(t, u(t, t_0, t_1, \tau_1, \dots, \tau_m, A_1, \dots, A_n, \lambda)) \\ &\quad + \beta(t, u(t, t_0, t_1, \tau_1, \dots, \tau_m, A_1, \dots, A_n, \lambda)), \end{aligned}$$

which is a solution of (1-5) along $u(t)$, and from above satisfies the boundary conditions,

$$\Delta^{i-1} w(t_0) = \lim_{h\to 0} \Delta^{i-1} w_h(t_0) = 0, \quad 1 \le i \le n-1,$$

and

$$w(t_1) - \sum_{j=1}^{m} a_j w(\tau_j) = \lim_{h\to 0} \Big(w_h(t_1) - \sum_{j=1}^{m} a_j w_h(\tau_j)\Big) = 0. \qquad \square$$

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Received: 2014-05-12 Revised: 2014-05-21 Accepted: 2014-05-31

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Cover: Alex Scorpan

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

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nonprofit scientific publishing

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involve

2015 vol. 8 no. 4