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Nonultrametric triangles
in diametral additive metric spaces

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We prove that a diametral additive metric space is not ultrametric if and only if it contains a diameter attaining nonultrametric triangle.

1. Introduction

Diameter and diametrical pairs of points in ultrametric spaces have been the subject of recent extensive studies, including [Dordovskyi et al. 2011]. In this paper we show that if a diametral additive metric space of diameter Δ is not ultrametric, then it must contain a nonultrametric triangle of diameter Δ .

We begin by recalling some preliminary definitions and background information.

Definition 1.1. A metric space (X, d) is said to be *ultrametric* if for all $x, y, z \in X$, we have

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}.$$

Equivalently, a metric space (X, d) is ultrametric if and only if any given three points in X can be relabeled as x, y, z so that $d(x, y) \leq d(x, z) = d(y, z)$.

Interesting examples of ultrametric spaces include the rings Z_p of p -adic integers, the Baire space B_{\aleph_0} , non-Archimedean normed fields and rings of meromorphic functions on open regions of the complex plane. There is an immense literature surrounding ultrametrics, as they have been intensively studied by topologists, analysts, number theorists and theoretical biologists. For example, [de Groot 1956] characterized ultrametric spaces up to homeomorphism as the strongly zero-dimensional metric spaces. In numerical taxonomy, on the other hand, every finite

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ultrametric space is known to admit a natural hierarchical description called a *dendrogram*. This has significant ramifications in theoretical biology. See, for instance, [Gordon 1987].

In fact, ultrametrics are special instances of a more general class of metrics which are termed additive. As we have noted in Definition 1.1, ultrametrics are defined by a stringent three point criterion. The class of additive metrics satisfy a more relaxed four point criterion. The formal definition is as follows.

Definition 1.2. A metric space (X, d) is said to be *additive* if for all x, y, z, w in X , we have

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\}.$$

Equivalently, a metric space (X, d) is additive if and only if any given four points in X can be relabeled as x, y, z, w so that $d(x, y) + d(z, w) \leq d(x, z) + d(y, w) = d(x, w) + d(y, z)$.

Recall that a *metric tree* is a connected graph (T, E) without cycles or loops in which each edge $e \in E$ is assigned a positive length $|e|$. The distance $d_T(x, y)$ between any two vertices $x, y \in T$ is then defined to be the sum of the lengths of the edges that make up the unique minimal geodesic from x to y . A brief but important paper, [Buneman 1974, Theorem 2], showed that a finite metric space is additive if and only if it is a tree metric in the sense of the following definition.

Definition 1.3. A metric d on a set X is said to be a *tree metric* if there exists a finite metric tree (T, E, d_T) such that

- (1) X is contained in the vertex set T of the tree, and
- (2) $d(x, y) = d_T(x, y)$ for all $x, y \in X$.

In other words, d is a tree metric if (X, d) is isometric to a metric subspace of some metric tree.

Ultrametrics form a very special subclass of the collection of all additive metrics. Indeed, there is a close relationship between ultrametric spaces and the leaf sets or end spaces of certain trees. This type of identification is discussed more formally in [Holly 2001; Fiedler 1998].

The notion of a diametral metric space is recalled in the following definition. It is a well-known result of mathematical analysis that all compact metric spaces are diametral [Kaplansky 1977, Theorem 68].

Definition 1.4. Let (X, d) be a metric space.

- (1) The *diameter* of a metric space (X, d) is defined to be the quantity $\Delta = \sup\{d(x, y) : x, y \in X\}$. If we need to be more explicit about the underlying metric space, we will write $\text{diam } X$ or $\text{diam}(X, d)$ instead of Δ .
- (2) (X, d) is *diametral* if there exist points $x, y \in X$ such that $d(x, y) = \Delta$.

A metric space (X, d) is not ultrametric if it contains a “bad” triangle $\{x, y, z\} \subseteq X$; i.e., x, y, z such that

$$d(x, y) > \max\{d(x, z), d(y, z)\}.$$

In the case of a nonultrametric diametral additive metric space (X, d) , we will see that there is always a bad triangle whose base length equals $\text{diam } X$. Such triangles are the subject of the following definition.

Definition 1.5. Let (X, d) be a metric space of diameter $\Delta < \infty$. We say that a subset $T = \{x, y, z\}$ of three distinct points from X forms a *diameter nonultrametric triangle* if (T, d) is not ultrametric and $\text{diam } T = \text{diam } X$.

2. Nonultrametric triangles in diametral additive metric spaces

In this section we show that every nonultrametric diametral additive metric space (X, d) contains a diameter nonultrametric triangle. We further note that this result is not true in the more general class of diametral metric spaces. Thus the assumption of additivity is necessary.

Henceforth we will assume that $|X| \geq 3$. The following lemma treats the cases $|X| = 3$ or 4.

Lemma 2.1. *Let (X, d) be a three or four point additive metric space. If X is not ultrametric, then X contains a diameter nonultrametric triangle.*

Proof. The lemma is true by inspection if $|X| = 3$, so we will assume that $|X| = 4$. Let $X = \{x, y, z, a\}$ and suppose that $d(a, z) = \Delta$, where Δ is the diameter of X . If X is not ultrametric, then there exist three distinct points in X that do not satisfy the ultrametric inequality. That is, there exists a three point subset of X that is not ultrametric. Consider the three point subsets of X : $\{x, y, z\}$, $\{x, y, a\}$, $\{y, z, a\}$, $\{x, z, a\}$.

Case 1: $\{y, z, a\}$ is not ultrametric. Since $a, z \in \{y, z, a\}$ and $d(a, z) = \Delta$, we see that $\text{diam}\{y, z, a\} = \Delta$. Then $\{y, z, a\}$ forms a diameter nonultrametric triangle by definition.

Case 2: $\{x, z, a\}$ is not ultrametric. The argument proceeds analogously to Case 1 and is omitted.

Case 3: $\{x, y, z\}$ is not ultrametric. If $\max\{d(a, x), d(x, z)\} < \Delta$, then $\{x, z, a\}$ is not ultrametric, and if $\max\{d(a, y), d(y, z)\} < \Delta$, then $\{y, z, a\}$ is not ultrametric. Then we are reduced to Cases 1 and 2. Suppose that $\max\{d(a, x), d(x, z)\} = \max\{d(a, y), d(y, z)\} = \Delta$. If $d(x, z) = \Delta$, then $\text{diam}\{x, y, z\} = \Delta$ and so $\{x, y, z\}$ forms a diameter nonultrametric triangle. The same occurs if $d(y, z) = \Delta$. Now let $d(a, x) = d(a, y) = \Delta$. As we are assuming that the metric space (X, d) is additive and that $d(a, z) = \Delta$, it follows from [Buneman 1974, Theorem 2] that x, y and z are equidistant from a in some finite metric tree. In particular, no three point subset

of $\{x, y, z, a\}$ that includes a can lie on a common geodesic in this tree. Thus x, y and z must be leaves in the minimal subtree generated by the vertices $\{x, y, z, a\}$. The vertex a may or may not be a leaf in this subtree. However, if a is a leaf in this subtree, we may replace it with the vertex a' in the subtree that minimizes $d(x, a')$ subject to the constraint $d(x, a') = d(y, a') = d(z, a')$. So, by proceeding in this way (if necessary) and by ignoring all irrelevant internal vertices in the subtree, it follows that $\{x, y, z\}$ forms the leaf set of a centered metric tree that has at most five vertices. Thus $\{x, y, z\}$ is ultrametric by [Fiedler 1998, Theorem 2.2].

Case 4: $\{x, y, a\}$ is not ultrametric. The argument proceeds analogously to Case 3 and is omitted. \square

Theorem 2.2. *A diametral additive metric space (X, d) is not ultrametric if and only if X contains a diameter nonultrametric triangle.*

Proof. (\Rightarrow) We prove the contrapositive of the forward implication. Let (X, d) be a diametral metric space with diameter Δ . Suppose X contains no diameter nonultrametric triangles. We may choose $a, b \in X$ with $d(a, b) = \Delta$. Let $x, y, z \in X$ be given. We show that the ultrametric inequality holds for x, y, z . Without loss of generality, we may assume that $x \neq a, b$. Consider the set $X' = \{a, b, x\}$. Clearly $\text{diam}(X', d) = \Delta$. If X' is not ultrametric, then X' forms a diameter nonultrametric triangle in X . So X' must be ultrametric. Thus $d(a, x) = \Delta$ or $d(b, x) = \Delta$. Without loss of generality, we may assume that $d(a, x) = \Delta$. Now consider $X'' = \{a, x, y, z\}$. By construction, $\text{diam}(X'', d) = \Delta$. It follows that any diameter nonultrametric triangle of X'' is also a diameter nonultrametric triangle of X . However, X contains no diameter nonultrametric triangles. So X'' contains no diameter nonultrametric triangles. By Lemma 2.1, X'' is ultrametric. Hence $d(x, y) \leq \max\{d(x, z), d(y, z)\}$, and so X is ultrametric.

(\Leftarrow) Any metric space that contains a diameter nonultrametric triangle is not ultrametric. \square

The following example shows that the forward implication of Theorem 2.2 may fail if the metric space is not assumed to be additive. Consider any nonultrametric metric triangle $(\{x, y, z\}, d)$. Let Δ denote the diameter of this triangle. We may assume that $\Delta = d(x, y) > \max\{d(x, z), d(y, z)\}$. Now adjoin a fourth point a at distance $\Delta + \varepsilon$ from x, y and z where $\varepsilon > 0$. The resulting four point diametral metric space is not additive and contains no diameter nonultrametric triangles.

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tef36@drexel.edu

*Department of Mathematics, Drexel University,
Philadelphia, PA 19104, United States*

kdk7rn@virginia.edu

*Department of Mathematics, University of Virginia,
Charlottesville, VA 22904, United States*

mkm233@cornell.edu

*Center for Applied Mathematics, Cornell University,
Ithaca, NY 14853, United States*

heidiv@sas.upenn.edu

*Department of Economics, University of Pennsylvania,
Philadelphia, PA 19104, United State*

enw27@cornell.edu

*Center for Applied Mathematics, Cornell University,
Ithaca, NY 14853, United States*

westona@canisius.edu

*Faculty of Arts and Sciences, Australian Catholic University,
North Sydney, NSW 2060, Australia*

and

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