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An unexpected discovery

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One summer, I chose two undergraduate students to work with me on a research project. Our goal was specifically to find a new approach to proving a theorem I had already proved several years before. We were looking for a new approach because the proof I had written is too long for publication, but the result itself is interesting. As is common in mathematics, our work led us to an unexpected discovery. This article leads the reader through our journey.

In 2010, I enlisted two students, Trevor J. Gionet Jr. and Yixiao Sha, to work with me for eight weeks in the summer on a research project funded through our provost's office. What follows is the story of our research journey. Our project was in the field of graph theory. I will use terminology from graph theory, though it is not necessary for you to know it to follow our adventure.

At the start of the project, Trevor had just completed a first graph theory course with me. Yixiao had never had graph theory, but had read through portions of the textbook [Chartrand and Zhang 2005] that had been used for Trevor's graph theory course. Our goal was to take a long proof I had written about ten years previously and make it shorter. The proof I had was over 350 pages long and therefore virtually unpublishable. However, some graph theorists thought the result it proved was interesting enough that they encouraged me to rework the proof and get it published. I hoped that if some students worked with me, they could offer some fresh ideas to help create a lemma or two (or five!) that would shorten the proof.

The result in question classifies a set of graphs with certain properties. I gave Trevor and Yixiao the statement of the theorem.

Theorem 1. *There are precisely seven connected*, 4*-regular*, *claw-free*, *well-dominated graphs.*

However, I did not give them the list of those seven graphs. In order to gain an understanding of what graphs with these properties look like, I wanted them to experiment and try to find the seven graphs on their own. So their first assignment was to find as many graphs as they could with these four properties.

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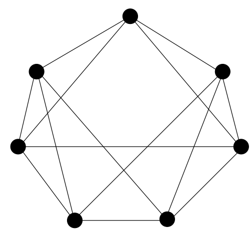
At our next meeting I asked them what they had found and whether they could justify that their graphs had the properties we were looking for. Yixiao and Trevor both had graphs to share. For one of his examples, Trevor drew this graph on the board:

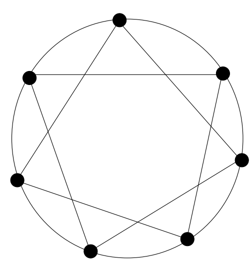


I asked him to be more and more specific about why this graph fit into our class of graphs. We went on to discuss other graphs and ideas. After a while, we finished our meeting. They left to do more exploring. I stared at my board. I had not told them, but this graph was not on my list of seven. As you can see, it is a small graph, so checking the properties was not difficult, but I did it several times before I sat down to write Michael Plummer and Bert Hartnell.

Trevor's graph is not just connected, it is 4-connected. My long proof had attacked this classification by breaking the possibilities into cases by connectivity. Since the graphs are 4-regular (meaning each vertex has a degree of four) and connected (meaning there exists a path between every pair of vertices in the graph), we know they are at least 1-connected and no more than 4-connected (we need to delete at least one vertex, but no more than four vertices, to disconnect the graph). This approach had given me four cases. More importantly, there was already a result published in *Discrete Applied Mathematics* by Hartnell and Plummer [1996] classifying 4-connected, 4-regular, claw-free, well-covered graphs. This was helpful since Finbow, Hartnell and Nowakowski [Finbow et al. 1988] showed that well-dominated graphs are a subclass of well-covered graphs. This meant that for one of my cases (the 4-connected one) I needed only to determine which well-covered graphs in Hartnell and Plummer's classification were also well-dominated. But since Trevor's graph was 4-connected, it not only should have appeared in my result, it should have appeared in Hartnell and Plummer's well-covered result as well. It did not. Hence, my impulse to write graph theorists Michael Plummer and Bert Hartnell.

Both Hartnell and Plummer responded quickly that we were correct and the graph should have been in their characterization. Furthermore, Plummer realized that the error was actually in an even earlier paper he had written on classes of claw-free graphs [Plummer 1995]. In the earlier paper, Plummer characterized 4-regular, 4-connected, claw-free graphs. He and Hartnell then used that characterization to determine the well-covered ones. However, Plummer had neglected to consider a case which included some graphs with an odd number of vertices. This was somewhat understandable given the topic of his paper primarily concerned graphs with an even number of vertices. However there were now two results based on his incomplete characterization and those results together with his characterization needed revising.

I shared the news with the students and their surprise and excitement were palpable. The goal of our summer project shifted to working on trying to complete Plummer's original characterization of the 4-regular, 4-connected, claw-free graphs, and to revising the Hartnell and Plummer result on well-covered graphs that followed. The students worked hard and we discovered that Plummer's characterization actually omitted an *infinite number* of graphs of odd order! However, Plummer's proof was still quite good and we needed only to rework a small portion of it to incorporate the missing graphs. In the process, Trevor realized that all the graphs in the characterization, including the original graphs and the new ones, were in a class of graphs we had read about in our textbook [Chartrand and Zhang 2005]. The textbook authors, Chartrand and Zhang, called this class Harary graphs. These graphs are very symmetric. Here is a redrawing of Trevor's graph, which is called $H_{4,7}$ in the class of Harary graphs, that makes the connection to Chartrand and Zhang's construction clearer.



$H_{4,7}$

By the end of the summer, we were able to complete Plummer's characterization of 4-regular, 4-connected, claw-free graphs, and revise Hartnell and Plummer's result about which of those are well-covered. Interestingly, even though there were an infinite number of odd graphs omitted from Plummer's characterization, *only two* of those graphs are well-covered ($H_{4,7}$ and $H_{4,11}$)! Both of the new well-covered graphs are also well-dominated, so now the theorem I first handed the students has been revised as follows:

Theorem 2. *There are precisely nine connected*, 4-*regular*, *claw-free*, *well-dominated graphs.*

We published a paper with our results, including the two revised results and a theorem classifying 4-connected, 4-regular, claw-free, well-dominated graphs, in *Discrete Applied Mathematics* [Gionet et al. 2011]. It was an exciting adventure and taught the students that although we start working on one problem, our research may lead us down a different path. Now back to figuring out how to shorten this 350 page proof!

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