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The sharp log-Sobolev inequality on a compact interval

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We provide a proof of the sharp log-Sobolev inequality on a compact interval.

1. Introduction

The Gaussian log-Sobolev inequality, due to A. J. Stam [1959, Equation 2.3] or Paul Federbush [1969, Equation (14)], although often attributed to L. Gross [1975, Corollary 4.2], played a crucial role in Perelman's proof [2002] of the Poincaré conjecture. We consider log-Sobolev inequalities for finite Lebesgue measure. F. Maggi [Morgan 2009] observed that the sharp log-Sobolev inequality on the interval follows from an isoperimetric conjecture of Díaz et al. [2012], which remains open, but provided no proof. We found it in [Wang 1999], which cited Deuschel and Stroock [1990], who gave a proof of the sharp log-Sobolev inequality on the circle. We then traced this result back to [Émery and Yukich 1987, page 1; Rothaus 1980, Theorem 4.3; Weissler 1980, Theorem 1]. Our Theorem 2.2 shows that the interval case follows quickly from the circle case.

2. Log-Sobolev inequality on a compact interval

In considering the isoperimetric problem in sectors of the plane with density r^p, Díaz et al. [2012, Corollary 4.24, Conjecture 4.18] conjectured the inequality

$$\left[\int_0^1 r^q \, d\alpha\right]^{1/q} \leq \int_0^1 \sqrt{r^2 + (q-1)\frac{r'^2}{\pi^2}} \, d\alpha, \tag{1}$$

where $1 < q \leq 2$. F. Maggi [Morgan 2009] observed that (1) implies the log-Sobolev inequality of Theorem 2.2. Here we observe that Theorem 2.2 follows from a proposition of Weissler.

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Proposition 2.1 [Weissler 1980, Theorem 1]. *Let f be a nonnegative C^1 function on the circle S^1 of length* 1. *Suppose $\int_{S^1} f^2 = 1$. Then we have the sharp inequality*

$$4\pi^2 \int_{S^1} f^2 \log f \le \int_{S^1} f'^2.$$

Various proofs are discussed in Section 3.

Theorem 2.2. *Let f be a nonnegative C^1 function on the interval* [0, 1]. *Suppose $\int_0^1 f^2 = 1$. Then we have the inequality*

$$\pi^2 \int_0^1 f^2 \log f \le \int_0^1 f'^2. \tag{2}$$

Proof. Let f be any nonnegative C^1 function on [0, 1] such that $\int_0^1 f^2 = 1$. Define a nonnegative piecewise C^1 function g on S^1 such that

$$g(x) = \begin{cases} f(2x) & \text{if } 0 \le x \le \frac{1}{2}, \\ f(2-2x) & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Then $\int_{S^1} g^2 = 1$. By smoothing, Proposition 2.1 applies to g. By simple computation, we have that

$$\int_{S^1} g^2 \log g = \int_0^1 f^2 \log f \quad \text{and} \quad \int_{S^1} g'^2 = 4 \int_0^1 f'^2.$$

The conclusion follows. □

Remark 2.3. Feng-Yu Wang [1999, Example 1.2] suggested an alternative proof of (2), but we don't understand his proof. He considered densities $C_\epsilon \exp(\epsilon \cos \pi x)$ and functions $f_\epsilon = \exp(-\epsilon \cos \pi x)$, with C_ϵ chosen to make the integral of f_ϵ^2 equal to 1. Then f_ϵ satisfies the differential equation

$$f_\epsilon'' - \pi\epsilon \sin \pi x f_\epsilon' = -\pi^2 f_\epsilon \log f_\epsilon. \tag{3}$$

He said that it follows that (2) holds for those functions and densities with sharp constant π^2. This might follow if it were known that functions realizing equality exist, but Wang himself [1999, page 655] admits that "the author is not sure yet whether there always exists [such a function]." Indeed, in the case of the circle with unit density, there apparently is no such function. Of course, the sharp inequality for density 1 would follow as ϵ approaches 0.

A similar result holds on the interval $[a, b]$ for a function with root mean square m.

Corollary 2.4. *Let f be a nonnegative C^1 function on the interval $[a, b]$. Suppose*

$$\frac{1}{b-a} \int_a^b f^2 = m^2, \quad m > 0.$$

Then we have the inequality

$$\frac{\pi^2}{(b-a)^2}\left(\int_a^b f^2 \log f \; - (b-a)m^2 \log m\right) \le \int_a^b f'^2. \tag{4}$$

Proof. Let f be a nonnegative C^1 function on the interval $[a, b]$ such that

$$\frac{1}{b-a}\int_a^b f^2 = m^2 > 0, \quad m > 0.$$

Define a function g on the interval [0, 1] as

$$g(x) = \frac{1}{m} f\big((b-a)x + a\big).$$

Then g is nonnegative and C^1. Moreover, we have

$$\int_0^1 g(x)^2\,dx = \int_0^1 \frac{1}{m^2} f\big((b-a)x+a\big)^2\,dx = \frac{1}{(b-a)m^2}\int_0^a f(y)^2\,dy = 1.$$

Therefore, we can apply Theorem 2.2 to the function g. We have

$$\frac{\pi^2}{b-a}\int_0^1 g^2 \log g \le (b-a)\int_0^1 g'^2. \tag{5}$$

Note that

$$g'(x) = \frac{b-a}{m} f\big((b-a)x + a\big).$$

By direct calculation, we have

$$\int_0^1 g'(x)^2\,dx = \frac{(b-a)^2}{m^2}\int_0^1 f'\big((b-a)x+a\big)^2\,dx = \frac{(b-a)^2}{m^2}\int_a^b f'(x)^2\,dx.$$

We also have

$$\begin{aligned}\int_0^1 g(x)^2 \log g(x)\,dx &= \frac{1}{m^2}\int_0^1 f\big((b-a)x+a\big)^2 \log\frac{f\big((b-a)x+a\big)}{m}\,dx\\ &= \frac{1}{(b-a)m^2}\int_a^b f(x)^2 \log\frac{f(x)}{m}\,dx\\ &= \frac{1}{(b-a)m^2}\int_a^b f^2(\log f - \log m)\\ &= \frac{1}{(b-a)m^2}\left(\int_a^b f^2 \log f - (b-a)m^2 \log m\right).\end{aligned}$$

Therefore, by plugging these identities into (5), we have

$$\frac{\pi^2}{(b-a)m^2}\left(\int_a^b f^2 \log f - (b-a)m^2 \log m\right) \le \frac{b-a}{m^2}\int_a^b f'^2.$$

This is equivalent to the desired inequality (4). □

Corollary 2.4 can be written in the following form.

Corollary 2.5. *Let f be a nonnegative C^1 function on the interval $[a, b]$. Suppose*

$$\frac{1}{b-a}\int_a^b f = m > 0.$$

Then we have the inequality

$$\frac{2\pi^2}{(b-a)^2}\left(\int_a^b f\log f \ - m\log m\right) \leq \int_a^b \frac{f'^2}{f}.$$

Proof. Define a nonnegative piecewise C^1 function g on the interval $[a, b]$ as $g = \sqrt{f}$. Plugging g into Corollary 2.4 yields the desired result. □

Proposition 2.6. *In Theorem 2.2, π^2 is the best possible constant.*

Proof. For any $0 < \epsilon < 1$, define

$$f_\epsilon(x) = \sqrt{1-\epsilon^2} + \sqrt{2}\epsilon\cos\pi x.$$

Then by direct computation, we have

$$\lim_{\epsilon\to 0^+}\frac{\int_0^1 f_\epsilon'^2}{\int_0^1 f_\epsilon^2\log f_\epsilon} = \pi^2.$$

Therefore, the constant π^2 cannot be replaced by a larger constant. □

Remark 2.7. The function $\cos\pi x$ comes from the equality case of a Wirtinger inequality which follows from the log-Sobolev inequality [Morgan 2009].

3. Proofs of the sharp log-Sobolev inequality on the circle

We summarize three proofs of Proposition 2.1 given by Rothaus [1980, Theorem 4.3], Weissler [1980, Theorem 1], Émery and Yukich [1987, page 1], and Deuschel and Stroock [1990, Remark 1.14 (i)].

3.1. *Weissler's proof.* Weissler proved a stronger result than Proposition 2.1 by Fourier expansion of functions of period 2π.

Proposition 3.1 [Weissler 1980, Theorem 1]. *Let $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be in L^2 and suppose $f(\theta) \geq 0$ almost everywhere. Then*

$$\int f^2\log f \leq \sum_{n=-\infty}^{\infty}|n||a_n|^2 + \|f\|_2^2\log\|f\|_2$$

in the sense that if the right-hand side is finite, then so is the left-hand side and the inequality holds. ($0^2\log 0$ *is taken to be* 0.)

Obviously the above inequality is stronger than the inequality

$$\int f^2 \log f \leq \sum_{n=-\infty}^{\infty} |n|^2 |a_n|^2 + \|f\|_2^2 \log\|f\|_2,$$

which is equivalent to Proposition 2.1 by change of variables as in Corollary 2.4.

Weissler [1980] cited [Rothaus 1978] but did not have [Rothaus 1980], where Rothaus gave his proof of Proposition 2.1.

3.2. ***Rothaus's proof.*** Rothaus proved Proposition 2.1 by a variational method. (References in this section are relative to [Rothaus 1980].) He considered an equivalent problem with a positive parameter ρ in Section 4. If a related constant b_ρ is zero, then the log-Sobolev inequality on the circle with the constant $2/\rho$ holds. For each b_ρ, he showed in Theorem 4.2 that a minimizing function exists, is positive and satisfies a related differential equation. Moreover, for $\rho > 1/2\pi^2$, the only positive solution to the differential equation is the constant function 1 (Theorem 4.3) and hence b_ρ is zero. Therefore in the limit $b_{1/2\pi^2}$ is zero, and our Proposition 2.1 follows.

Rothaus cited [Weissler 1980], saying that "a result related to Theorem 6.3 appears in" that paper.

3.3. ***Émery and Yukich's proof.*** Proposition 2.1 was proved by Émery and Yukich [1987, page 1] by using estimates deploying the Brownian motion semigroup.

Émery and Yukich [1987] cited both Weissler [1980] and Rothaus [1980].

3.4. ***Deuschel and Stroock's proof.*** Deuschel and Stroock considered the log-Sobolev inequality in general spaces with densities. As a special case, they proved [Deuschel and Stroock 1990, Remark 1.14 (i)] that the log-Sobolev constant for the circle of length 1 with Lebesgue measure is the first eigenvalue of the Laplacian, namely $4\pi^2$ (corresponding to the first eigenfunction $\sin 2\pi x$).

Deuschel and Stroock [1990] cited [Émery and Yukich 1987].

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