involve
a journal of mathematics

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msp

2013 vol. 6, no. 4

INVOLVE 6:4 (2013)
dx.doi.org/10.2140/involve.2013.6.393

An elementary inequality about the Mahler measure

Konstantin Stulov and Rongwei Yang

(Communicated by Andrew Granville)

Let $p(z)$ be a degree n polynomial with zeros z_j, $j = 1, 2, \dots, n$. The total distance from the zeros of p to the unit circle is defined as $\mathrm{td}(p) = \sum_{j=1}^n \big||z_j|-1\big|$. We show that up to scalar multiples, $\mathrm{td}(p)$ sits between $M(p)-1$ and $m(p)$. This leads to an equivalent statement of Lehmer's problem in terms of $\mathrm{td}(p)$. The proof is elementary.

1. Introduction

Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial with complex coefficients of degree n. The Mahler measure $M(p)$ [Everest and Ward 1999] is defined as

$$M(p) = \exp\left(\int_0^{2\pi} \log\left|p(e^{i\theta})\right| \frac{d\theta}{2\pi}\right).$$

We denote $\log M(p)$ by $m(p)$. Jensen's formula implies that

$$M(p) = |a_n| \prod_{|z_j|>1} |z_j|,$$

where throughout this paper the z_j, $j = 1, 2, \dots, n$, are the zeros of $p(z)$, counting multiplicity. We also assume that $|a_n| = 1$. It is then clear that $M(p) \geq 1$, and

$$0 \leq m(p) = \log\big((M(p)-1)+1\big) \leq M(p)-1,$$

and when $M(p)$ is close to 1, $m(p)$ is close to $M(p)-1$. Lehmer's problem is to verify that for integer-coefficient monic polynomials, $m(p)$ is either 0 (for products of cyclotomic polynomials and possibly a factor of z^k) or is bounded away from 0 by a fixed positive constant. This is a deep and unsolved problem.

For a polynomial p of degree n, the associated polynomial $p^*(z)$ is defined as $z^n \overline{p(1/\bar{z})}$. We say p is reciprocal if $p = cp^*$ for some complex number c of modulus 1. One sees that the zeros of a reciprocal p off the unit circle appear in conjugate reciprocal pairs. Interestingly, Lehmer's problem was unsolved only for reciprocal polynomials. A key ingredient of this paper is the total distance from the

MSC2010: 11CXX.
Keywords: Mahler measure, total distance.

zeros of p to the unit circle T defined to be

$$\operatorname{td}(p) = \sum_{j=1}^{n} \big||z_j| - 1\big|.$$

Theorem. *For every complex polynomial* $p(z) = \sum_{j=0}^{n} a_j z^j$, *with* $|a_n| = |a_0| = 1$, *we have*

$$m(p) \le \operatorname{td}(p) \le 2(M(p) - 1).$$

If p is reciprocal, then $2m(p) \le \operatorname{td}(p)$. *Further, the equalities hold only if* $\operatorname{td}(p) = 0$.

Therefore, Lehmer's problem can be stated equivalently as follows: There is an $\epsilon > 0$ such that if p has integer coefficients with $|a_n| = |a_0| = 1$ and $\operatorname{td}(p) \neq 0$, then $\operatorname{td}(p) \ge \epsilon$.

2. Proof

Lemma 1. *If t_j, $j = 1, 2, \ldots, k$ are numbers in the interval* $[0, 1]$, *then*

$$\sum_{j=1}^{k} (1 - t_j) \le \frac{1}{\prod_{j=1}^{k} t_j} - 1,$$

where equality holds only if $t_j = 1$ for each j.

Proof. The inequality is trivial if one of the t_j is 0. Now, we assume $t_j > 0$ for each j. We prove by induction. It is easy to see that the lemma is true for $k = 1$. Assume the lemma is true for k. For s and t in $(0, 1]$, one checks that

$$\frac{1}{ts} - \left(\frac{1}{t} + 1 - s\right) = \frac{(1-s)(1-ts)}{ts} \ge 0, \tag{2-1}$$

and hence

$$\frac{1}{ts} - 1 \ge \frac{1}{t} - s.$$

Therefore

$$\begin{aligned}\sum_{j=1}^{k} (1 - t_j) + (1 - t_{k+1}) &\le \frac{1}{\prod_j^k t_j} - 1 + (1 - t_{k+1}) \\ &= \frac{1}{\prod_j^k t_j} - t_{k+1} \le \frac{1}{\prod_j^{k+1} t_j} - 1.\end{aligned}$$ □

If $\{\lambda_j : j = 1, 2, \ldots\}$ is a subset of the open unit disk $\mathbb{D}$, the associated Blaschke product is defined as

$$B(z) = \prod_{j=1}^{\infty} \frac{z - \lambda_j}{1 - \overline{\lambda_j} z}, \quad z \in \mathbb{D}.$$

Clearly, the product is convergent for each z if and only if $\sum_{j=0}^{\infty}(1-|\lambda_j|)<\infty$ [Garnett 2007]. In this case $B(z)$ is a bounded analytic function on $\mathbb{D}$. It follows immediately from Lemma 1 that

$$\sum_{j=1}^{\infty}(1-|\lambda_j|)\le\frac{1}{|B(0)|}-1.$$

Proof of the Theorem. For a polynomial $p(z)$, since $\prod_{j=1}^{n}|z_j|=\frac{|a_0|}{|a_n|}$, we have

$$\frac{M(p)}{|a_0|}-1=\frac{1}{\prod_{|z_j|\le 1}|z_j|}-1\ge\sum_{|z_j|\le 1}(1-|z_j|) \tag{2-2}$$

by Lemma 1. On the other hand, inductively using that $(a-1)+(b-1)<ab-1$ for $a,b>1$, we have

$$\sum_{|z_j|>1}(|z_j|-1)\le\prod_{|z_j|>1}|z_j|-1=\frac{M(p)}{|a_n|}-1.$$

Here the equality is allowed only because there may not be a z_j with $|z_j|>1$. Combining with (2-2), we have $\operatorname{td}(p)\le M(p)(1/|a_n|+1/|a_0|)-2$. In the case $|a_0|=|a_n|=1$, we have

$$\operatorname{td}(p)\le 2(M(p)-1), \tag{2-3}$$

with equality occurring only if $\operatorname{td}(p)=0$. The dominance of $m(p)$ by $\operatorname{td}(p)$ is an easy consequence of the inequality $\log(1+t)\le t$. To be precise,

$$m(p)=\sum_{|z_k|>1}\log|z_k|\le\sum_{|z_k|>1}(|z_k|-1)\le\operatorname{td}(p).$$

We establish a stronger inequality for reciprocal polynomials with $|a_0|=|a_n|=1$. Let $z_1,z_2,\dots,z_k$ be the zeros of such a p that are outside of the unit circle, where $2k\le n$. Then $m(p)=\log|z_1|+\log|z_2|+\cdots+\log|z_k|$ and

$$\operatorname{td}(p)=\sum_{j=1}^{k}(|z_j|-1)+\Bigl(1-\frac{1}{|z_j|}\Bigr).$$

Let $f(t)=t-(1/t)-2\log t,\ t\ge 1$. One easily checks that f is strictly increasing and $f(1)=0$. It follows that $|z_j|-1/|z_j|>2\log|z_j|$ for each $1\le j\le k$, and hence $2m(p)\le\operatorname{td}(p)$, with equality precisely when $k=0$, which occurs if and only if $\operatorname{td}(p)=0$ since $|a_0|=|a_n|=1$. □

Example. Consider Lehmer's polynomial

$$G(z)=z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1.$$

It is well-known that eight of its zeros lie in the unit circle and the other two are real and form a reciprocal pair. Since $M(G) \approx 1.1763$, we have

$$\begin{aligned} \mathrm{td}(G) &\approx (1.1763-1)+(1-1/1.1763) \approx 0.3262, \\ 2m(G) &\approx 2\times 0.1624 = 0.3248, \\ 2(M(p)-1) &\approx 0.3526. \end{aligned}$$

Our Theorem has some interesting implications. We need two more definitions to state them. Define

$$\begin{aligned} \Delta(p) &= \max\bigl\{\bigl||\alpha|-1\bigr| : p(\alpha)=0\bigr\}, \\ \delta(p) &= \min\bigl\{\bigl||\alpha|-1\bigr| : p(\alpha)=0\bigr\}. \end{aligned}$$

Then it is clear that

$$\delta(p) \le \frac{\mathrm{td}(p)}{n} \le \Delta(p). \tag{2-4}$$

When p is reciprocal and α is a zero of p, $1/\alpha$ is also a zero. Since $t-1 \ge 1-1/t$ for $t \ge 1$, we have

$$\Delta(p) = \max\{|\alpha|-1 : p(\alpha)=0\} = \max\{|\alpha| : p(\alpha)=0\}-1.$$

Likewise

$$\delta(p) = 1-\max\{|\alpha| : |\alpha| \le 1,\ p(\alpha)=0\}.$$

For simplicity, we let

$$\lambda(p) = \max\{|\alpha| : p(\alpha)=0\}$$

and let

$$\lambda'(p) = \max\{|\alpha| : |\alpha| \le 1,\, p(\alpha)=0\}.$$

In [Smyth 2008], $\lambda(p)$ is called the house of the zeros of p. Geometrically, $\lambda(p)$ is the modulus of the zero that is the farthest from the unit circle, while $\lambda'(p)$ is the modulus of the zero that is the nearest to the unit circle. The next proposition then follows easily from (2-4).

Proposition. *For a reciprocal complex polynomial p of degree $n \ge 2$,*

$$\lambda(p) \ge 1+\frac{\mathrm{td}(p)}{n} \quad \textit{and} \quad \lambda'(p) \ge 1-\frac{\mathrm{td}(p)}{n}.$$

Regarding $\lambda(p)$, there is an unsolved conjecture by Schinzel and Zassenhaus that states that there is an absolute constant C so that if p is a monic irreducible polynomial of degree n with integer coefficients, then $\lambda(p) \ge 1+C/n$. This inequality will follow easily from a positive answer to Lehmer's problem. Indeed, one has $\lambda(p) \ge 1+m(p)/n$ [Smyth 2008]. But in view of Theorem, Proposition provides a better inequality for reciprocal polynomials.

Acknowledgment

The authors thank the referee for helpful and stimulating comments.

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Received: 2012-07-09 Revised: 2013-02-12 Accepted: 2013-02-16

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2013 vol. 6 no. 4