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An elementary inequality about the Mahler measure

Konstantin Stulov and Rongwei Yang



# An elementary inequality about the Mahler measure

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(Communicated by Andrew Granville)

Let  $p(z)$  be a degree  $n$  polynomial with zeros  $z_j$ ,  $j = 1, 2, \dots, n$ . The total distance from the zeros of  $p$  to the unit circle is defined as  $\text{td}(p) = \sum_{j=1}^n ||z_j| - 1|$ . We show that up to scalar multiples,  $\text{td}(p)$  sits between  $M(p) - 1$  and  $m(p)$ . This leads to an equivalent statement of Lehmer's problem in terms of  $\text{td}(p)$ . The proof is elementary.

## 1. Introduction

Let  $p(z) = \sum_{j=0}^n a_j z^j$  be a polynomial with complex coefficients of degree  $n$ . The Mahler measure  $M(p)$  [Everest and Ward 1999] is defined as

$$M(p) = \exp\left(\int_0^{2\pi} \log |p(e^{i\theta})| \frac{d\theta}{2\pi}\right).$$

We denote  $\log M(p)$  by  $m(p)$ . Jensen's formula implies that

$$M(p) = |a_n| \prod_{|z_j| > 1} |z_j|,$$

where throughout this paper the  $z_j$ ,  $j = 1, 2, \dots, n$ , are the zeros of  $p(z)$ , counting multiplicity. We also assume that  $|a_n| = 1$ . It is then clear that  $M(p) \geq 1$ , and

$$0 \leq m(p) = \log((M(p) - 1) + 1) \leq M(p) - 1,$$

and when  $M(p)$  is close to 1,  $m(p)$  is close to  $M(p) - 1$ . Lehmer's problem is to verify that for integer-coefficient monic polynomials,  $m(p)$  is either 0 (for products of cyclotomic polynomials and possibly a factor of  $z^k$ ) or is bounded away from 0 by a fixed positive constant. This is a deep and unsolved problem.

For a polynomial  $p$  of degree  $n$ , the associated polynomial  $p^*(z)$  is defined as  $z^n \overline{p(1/\bar{z})}$ . We say  $p$  is reciprocal if  $p = cp^*$  for some complex number  $c$  of modulus 1. One sees that the zeros of a reciprocal  $p$  off the unit circle appear in conjugate reciprocal pairs. Interestingly, Lehmer's problem was unsolved only for reciprocal polynomials. A key ingredient of this paper is the total distance from the

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zeros of  $p$  to the unit circle  $T$  defined to be

$$\text{td}(p) = \sum_{j=1}^n \left| |z_j| - 1 \right|.$$

**Theorem.** For every complex polynomial  $p(z) = \sum_{j=0}^n a_j z^j$ , with  $|a_n| = |a_0| = 1$ , we have

$$m(p) \leq \text{td}(p) \leq 2(M(p) - 1).$$

If  $p$  is reciprocal, then  $2m(p) \leq \text{td}(p)$ . Further, the equalities hold only if  $\text{td}(p) = 0$ .

Therefore, Lehmer’s problem can be stated equivalently as follows: There is an  $\epsilon > 0$  such that if  $p$  has integer coefficients with  $|a_n| = |a_0| = 1$  and  $\text{td}(p) \neq 0$ , then  $\text{td}(p) \geq \epsilon$ .

### 2. Proof

**Lemma 1.** If  $t_j, j = 1, 2, \dots, k$  are numbers in the interval  $[0, 1]$ , then

$$\sum_{j=1}^k (1 - t_j) \leq \frac{1}{\prod_{j=1}^k t_j} - 1,$$

where equality holds only if  $t_j = 1$  for each  $j$ .

*Proof.* The inequality is trivial if one of the  $t_j$  is 0. Now, we assume  $t_j > 0$  for each  $j$ . We prove by induction. It is easy to see that the lemma is true for  $k = 1$ . Assume the lemma is true for  $k$ . For  $s$  and  $t$  in  $(0, 1]$ , one checks that

$$\frac{1}{ts} - \left( \frac{1}{t} + 1 - s \right) = \frac{(1-s)(1-ts)}{ts} \geq 0, \tag{2-1}$$

and hence

$$\frac{1}{ts} - 1 \geq \frac{1}{t} - s.$$

Therefore

$$\begin{aligned} \sum_{j=1}^k (1 - t_j) + (1 - t_{k+1}) &\leq \frac{1}{\prod_j^k t_j} - 1 + (1 - t_{k+1}) \\ &= \frac{1}{\prod_j^k t_j} - t_{k+1} \leq \frac{1}{\prod_j^{k+1} t_j} - 1. \quad \square \end{aligned}$$

If  $\{\lambda_j : j = 1, 2, \dots\}$  is a subset of the open unit disk  $\mathbb{D}$ , the associated Blaschke product is defined as

$$B(z) = \prod_{j=1}^{\infty} \frac{z - \lambda_j}{1 - \overline{\lambda_j}z}, \quad z \in \mathbb{D}.$$

Clearly, the product is convergent for each  $z$  if and only if  $\sum_{j=0}^{\infty} (1 - |\lambda_j|) < \infty$  [Garnett 2007]. In this case  $B(z)$  is a bounded analytic function on  $\mathbb{D}$ . It follows immediately from Lemma 1 that

$$\sum_{j=1}^{\infty} (1 - |\lambda_j|) \leq \frac{1}{|B(0)|} - 1.$$

*Proof of the Theorem.* For a polynomial  $p(z)$ , since  $\prod_{j=1}^n |z_j| = \frac{|a_0|}{|a_n|}$ , we have

$$\frac{M(p)}{|a_0|} - 1 = \frac{1}{\prod_{|z_j| \leq 1} |z_j|} - 1 \geq \sum_{|z_j| \leq 1} (1 - |z_j|) \tag{2-2}$$

by Lemma 1. On the other hand, inductively using that  $(a - 1) + (b - 1) < ab - 1$  for  $a, b > 1$ , we have

$$\sum_{|z_j| > 1} (|z_j| - 1) \leq \prod_{|z_j| > 1} |z_j| - 1 = \frac{M(p)}{|a_n|} - 1.$$

Here the equality is allowed only because there may not be a  $z_j$  with  $|z_j| > 1$ . Combining with (2-2), we have  $\text{td}(p) \leq M(p)(1/|a_n| + 1/|a_0|) - 2$ . In the case  $|a_0| = |a_n| = 1$ , we have

$$\text{td}(p) \leq 2(M(p) - 1), \tag{2-3}$$

with equality occurring only if  $\text{td}(p) = 0$ . The dominance of  $m(p)$  by  $\text{td}(p)$  is an easy consequence of the inequality  $\log(1 + t) \leq t$ . To be precise,

$$m(p) = \sum_{|z_k| > 1} \log |z_k| \leq \sum_{|z_k| > 1} (|z_k| - 1) \leq \text{td}(p).$$

We establish a stronger inequality for reciprocal polynomials with  $|a_0| = |a_n| = 1$ . Let  $z_1, z_2, \dots, z_k$  be the zeros of such a  $p$  that are outside of the unit circle, where  $2k \leq n$ . Then  $m(p) = \log |z_1| + \log |z_2| + \dots + \log |z_k|$  and

$$\text{td}(p) = \sum_{j=1}^k (|z_j| - 1) + \left(1 - \frac{1}{|z_j|}\right).$$

Let  $f(t) = t - (1/t) - 2 \log t$ ,  $t \geq 1$ . One easily checks that  $f$  is strictly increasing and  $f(1) = 0$ . It follows that  $|z_j| - 1/|z_j| > 2 \log |z_j|$  for each  $1 \leq j \leq k$ , and hence  $2m(p) \leq \text{td}(p)$ , with equality precisely when  $k = 0$ , which occurs if and only if  $\text{td}(p) = 0$  since  $|a_0| = |a_n| = 1$ . □

**Example.** Consider Lehmer’s polynomial

$$G(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

It is well-known that eight of its zeros lie in the unit circle and the other two are real and form a reciprocal pair. Since  $M(G) \approx 1.1763$ , we have

$$\begin{aligned} \text{td}(G) &\approx (1.1763 - 1) + (1 - 1/1.1763) \approx 0.3262, \\ 2m(G) &\approx 2 \times 0.1624 = 0.3248, \\ 2(M(p) - 1) &\approx 0.3526. \end{aligned}$$

Our **Theorem** has some interesting implications. We need two more definitions to state them. Define

$$\begin{aligned} \Delta(p) &= \max\{||\alpha| - 1| : p(\alpha) = 0\}, \\ \delta(p) &= \min\{||\alpha| - 1| : p(\alpha) = 0\}. \end{aligned}$$

Then it is clear that

$$\delta(p) \leq \frac{\text{td}(p)}{n} \leq \Delta(p). \tag{2-4}$$

When  $p$  is reciprocal and  $\alpha$  is a zero of  $p$ ,  $1/\alpha$  is also a zero. Since  $t - 1 \geq 1 - 1/t$  for  $t \geq 1$ , we have

$$\Delta(p) = \max\{|\alpha| - 1 : p(\alpha) = 0\} = \max\{|\alpha| : p(\alpha) = 0\} - 1.$$

Likewise

$$\delta(p) = 1 - \max\{|\alpha| : |\alpha| \leq 1, p(\alpha) = 0\}.$$

For simplicity, we let

$$\lambda(p) = \max\{|\alpha| : p(\alpha) = 0\}$$

and let

$$\lambda'(p) = \max\{|\alpha| : |\alpha| \leq 1, p(\alpha) = 0\}.$$

In [Smyth 2008],  $\lambda(p)$  is called the house of the zeros of  $p$ . Geometrically,  $\lambda(p)$  is the modulus of the zero that is the farthest from the unit circle, while  $\lambda'(p)$  is the modulus of the zero that is the nearest to the unit circle. The next proposition then follows easily from (2-4).

**Proposition.** *For a reciprocal complex polynomial  $p$  of degree  $n \geq 2$ ,*

$$\lambda(p) \geq 1 + \frac{\text{td}(p)}{n} \quad \text{and} \quad \lambda'(p) \geq 1 - \frac{\text{td}(p)}{n}.$$

Regarding  $\lambda(p)$ , there is an unsolved conjecture by Schinzel and Zassenhaus that states that there is an absolute constant  $C$  so that if  $p$  is a monic irreducible polynomial of degree  $n$  with integer coefficients, then  $\lambda(p) \geq 1 + C/n$ . This inequality will follow easily from a positive answer to Lehmer’s problem. Indeed, one has  $\lambda(p) \geq 1 + m(p)/n$  [Smyth 2008]. But in view of **Theorem**, **Proposition** provides a better inequality for reciprocal polynomials.

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[kstulov@stanford.edu](mailto:kstulov@stanford.edu)

*Institute for Computational and Mathematical Engineering,  
Stanford University, Stanford, NY 94305, United States*

[ryang@albany.edu](mailto:ryang@albany.edu)

*Department of Mathematics and Statistics,  
University of Albany, State University of New York,  
Albany, NY 12047, United States*

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