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On (2, 3)-agreeable box societies

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The notion of a (k, m)-agreeable society was introduced by Berg, Norine, Su, Thomas and Wollan: a family of convex subsets of $\mathbb{R}^d$ is called (k, m)*-agreeable* if any subfamily of size m contains at least one nonempty k-fold intersection. In that paper, the (k, m)-agreeability of a convex family was shown to imply the existence of a subfamily of size βn with a nonempty intersection, where n is the size of the original family and $\beta \in [0, 1]$ is an explicit constant depending only on k, m and d. The quantity $\beta(k, m, d)$ is called the minimal *agreement proportion* for a (k, m)-agreeable family in $\mathbb{R}^d$.

If we assume only that the sets are convex, simple examples show that $\beta = 0$ for (k, m)-agreeable families in $\mathbb{R}^d$ where $k < d$. In this paper, we introduce new techniques to find positive lower bounds when restricting our attention to families of d-boxes, that is, cuboids with sides parallel to the coordinates hyperplanes. We derive explicit formulas for the first nontrivial case: (2, 3)-agreeable families of d-boxes with $d \geq 2$.

1. Introduction

Berg et al. [2010] introduced the concept of geometric approval voting, where a *platform* is a point in $\mathbb{R}^d$ and a *vote* can be any convex subset, representing all the platforms deemed acceptable by that particular voter. (The convexity assumption is a way to require our voters to be reasonable: the fact that all votes contain every point on a segment with both endpoints in the vote means that any platform obtained as a compromise between two acceptable positions is again deemed acceptable.) The main question addressed in [Berg et al. 2010] was, given a collection of votes, to find the largest number of overlapping votes, and thus the largest number of voters that could be satisfied by the adoption of any single platform.

More specifically, the authors concentrated on what they termed (k, m)*-agreeable societies*, where any group of m voters contains k or more who can agree on a common platform. Their main goal was to obtain lower bounds on the *agreement proportion* (the ratio of satisfied voters to total number of voters) in terms of k, m

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and d only. Using the version of the fractional Helly theorem due to Kalai [1984], they showed that if a (k, m)-agreeable society contains $n \geq m$ votes, all of which are convex subsets of $\mathbb{R}^d$, then there exists a platform contained in at least $\beta(k, m, d)\, n$ votes, where the *proportion* $\beta(k, m, d)$ satisfies:

$$\beta(k, m, d) \geq 1 - \left[1 - \frac{\binom{k}{d+1}}{\binom{m}{d+1}}\right]^{\frac{1}{d+1}}. \tag{1}$$

Given that the fractional Helly theorem makes no reference to k-fold intersections when $k \leq d$, it is no surprise that this lower bound is positive only when $k \geq d+1$.

If the general convex case requires detailed information about the whole intersection pattern of the arrangement of votes, the *intersection graph* does capture the structure of the whole arrangement in the special case when the votes are *boxes*, that is, cuboids with sides parallel to the coordinate axes. This case was also addressed in [Berg et al. 2010], and purely graph-theoretic considerations yielded a sharp bound of k/m for the agreement proportion in the *strong agreement case:* the situation of (k, m)-agreeability where $m \leq 2k-2$. (The result proved there for this case $m \leq 2k-2$ is in fact substantially stronger: if the number of boxes is n, there is an overlap of at least $n-m+k$ boxes, so the actual agreement proportion *starts* at k/m and *increases* to 1 as the number n of boxes goes to infinity.)

The case of $(2, m)$-agreeable societies of d-boxes does not fall in the strong agreement category, and it is left essentially open in [Berg et al. 2010]. In fact, it is not even clear at the outset that there is a positive agreement proportion for $(2, m)$-agreeable arrangements of d-boxes when $m \geq 3$ and $d \geq 2$, since the lower bound given by (1) is zero in that case. In this paper we tackle the (2, 3)-agreeable case and prove the following result.

Theorem 1.1. *For any $d \geq 1$, any* (2, 3)*-agreeable d-box society has an agreement proportion of at least* $(2d)^{-1}$.

The remainder of the paper is organized as follows.

The material in Section 2 is independent from the rest of the paper: it presents an elementary proof of the fact that (2, 3)-agreeable arrangements of intervals have an agreement proportion of at least $\frac{1}{2}$. Section 3 is devoted to preliminaries. It introduces notation and definitions regarding arrangements of boxes and their intersection graphs. Section 4 establishes upper and lower bounds on the degrees of vertices in a (2, 3)-agreeable graph G with bounded clique number. A classification of the small cases is given, and we prove that positive lower bounds do exist for all d. In Section 5, we establish the specific values of the lower bound stated in Theorem 1.1. The proof uses a lower bound on boxicity taken from [Adiga et al. 2008]. Section 6 relates (2, 3)-agreeability to Ramsey numbers and presents a few questions left open by our work.

Throughout the paper, all arrangements of boxes are assumed to be (2, 3)-agreeable. Many of the definitions and results could easily be extended to the (k, m)-agreeable case; this level of generality was eschewed in order to keep notations simple and legible. The only step for which (2, 3)-agreeability is crucial is in establishing the lower bound of Section 4.

Notation 1.2. Throughout this paper, G denotes a simple, undirected graph. The sets $V(G)$ and $E(G)$ are respectively the sets of vertices and edges of G, and we let $n = \#V(G)$. Recall that any subset W of $V(G)$ gives rise to the subgraph $G[W]$ *induced* by W, which is the graph that has W as its set of vertices, and has for edges all the edges of $E(G)$ with both endpoints in W.

A *clique* in G is any subset of $V(G)$ that induces a complete subgraph, and the size of the largest clique is called the *clique number* of G and denoted by $\omega(G)$.

2. The linear case

The intersection graphs associated to arrangements of intervals in the line are *perfect graphs*. This allowed Berg et al. [2010] to prove the nontrivial fact: for any (k, m)-agreeable arrangement of intervals, the agreement number is at least $(n - R)/Q$, where Q and R denote respectively the quotient and the remainder of the Euclidean division of $m - 1$ by $k - 1$. This lower bound is sharp and it implies that any (k, m)-agreeable collection of intervals must have an agreement proportion

$$\beta(k, m, 1) \geq \frac{k-1}{m-1}.$$

In particular, the above implies that any (2, 3)-agreeable collection of intervals has agreement proportion at least $\frac{1}{2}$. This substantially improves the general case bound given in formula (1), which for $d = 1$ in the (2, 3)-agreeable setting yields an agreement proportion of

$$1 - \sqrt{\tfrac{2}{3}} \approx 0.1835.$$

We reprove the lower bound of $\frac{1}{2}$ using only elementary means. First we need to know when the agreement proportion equals 1.

Lemma 2.1. *A linear society has agreement proportion* 1 *if and only if every pair of votes intersects. In the terminology of* [Berg et al. 2010], *such an arrangement is called* super-agreeable.

Proof. This is a special case of Helly's theorem [Matoušek 2002], which states that for any arrangement of convex sets in $\mathbb{R}^d$, the sets have a nonempty intersection if and only if all $(d+1)$-fold intersections are nonempty. □

Theorem 2.2. [Berg et al. 2010, Theorem 1] *The minimal agreement proportion of a linear* $(2, 3)$*-agreeable society is* $\frac{1}{2}$.

Proof. If every pair of votes intersects, Helly's theorem for intervals implies that the agreement proportion is 1. So, without loss of generality, we can assume that in our one-dimensional $(2, 3)$-agreeable society, there are two nonintersecting intervals A (Alice's vote) and B (Bob's vote), with A to the left of B.

The remaining voters can be divided into three categories: those who agree only with Alice, those who agree only with Bob, and those who agree with both Alice and Bob. (There are no voters who agree with neither since that would violate $(2, 3)$-agreeability.) These three categories of voters — call them friends of Alice, friends of Bob and friends of both — form super-agreeable groups, where all voters can agree pairwise and thus, by Helly's theorem, all the votes in each group overlap. Indeed, friends of Alice must agree with each other, because if two of them did not agree, then taken together with Bob, we would have three votes containing no intersecting pair, violating the condition of $(2, 3)$-agreeability. Similarly, voters who only agree with Bob must also agree with each other. As for votes which overlap with both Alice and Bob's vote, they all meet in the interval $[\max(A), \min(B)]$ between A and B (Figure 1). If one of the three categories is empty, we have two super-agreeable groups, one of which must account for at least one half of the voters, and the result holds.

Suppose that all three categories are nonempty, and let C be a vote containing $[\max(A), \min(B)]$, D be a vote intersecting A but not B, and E be a vote intersecting B but not A. The three votes must share at least one intersection to respect the $(2, 3)$-agreeable condition; and note that if $D \cap E \neq \varnothing$, it implies that the two intersections with C are also nonempty (all meet in the middle region). If we can find a vote D from a friend of Alice such that $C \cap D = \varnothing$, then we must have $C \cap E \neq \varnothing$, and, replacing E by any other vote E' intersecting B, the same reasoning shows that $C \cap E' \neq \varnothing$, too. Thus any vote C bridging the gap between

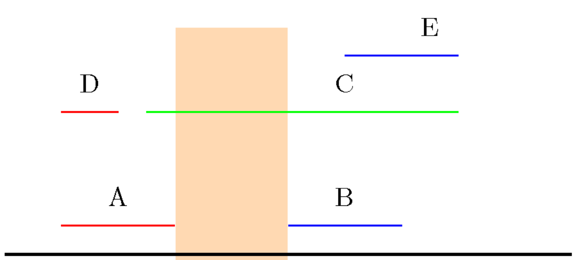

Figure 1. Alice, Bob, and their friends. The shaded area in the middle is shared by all the friends of Bob and Alice, such as C.

Alice and Bob must either meet all the votes that intersect A or all the votes that intersect B. Thus, we can assign those bridging votes to Alice or Bob, since they have to overlap with all of the friends of at least one. We can divide the votes into two super-agreeable groups once again. One of those must account for at least half the voters, proving the result. □

Remark 2.3. This theorem is sharp: any society formed by taking r copies of A and r copies of B is (2, 3)-agreeable with an agreement proportion of exactly $\frac{1}{2}$.

Remark 2.4. The result of the previous theorem only holds in dimension 1. For instance, Figure 2 shows five votes in dimension 2 arising from (2, 3)-agreeable voters, yet the agreement proportion is only $\frac{2}{5}$.

3. Boxes and agreeable graphs

We introduce some definitions and notation for the two main objects of study: arrangements of boxes and their associated intersection graphs.

3.1. ***Arrangements of boxes and intersection graphs.*** A d-*box* is a subset of $\mathbb{R}^d$ given by the Cartesian product of d closed intervals. A collection $\mathcal{B}$ of boxes gives rise to a graph in the following fashion.

Definition 3.1. The *intersection graph* $G_{\mathcal{B}}$ associated to an arrangement $\mathcal{B} = \{B_1, \ldots, B_n\}$ of d-boxes is the graph with vertices $V = \{1, \ldots, n\}$ such that $\{i, j\}$ is an edge if and only if $B_i \cap B_j \neq \varnothing$.

Conversely, given a simple, undirected graph G, we can define its *boxicity* $\text{box}(G)$: it is the smallest integer d such that there exists an arrangement of d-boxes $\mathcal{B}$ whose intersection graph is G.

Roberts [1969] showed that this number is always finite, and that $\text{box}(G) \leq \lfloor \#V/2 \rfloor$. (Graphs for which this bound is tight are classified in [Trotter 1979].)

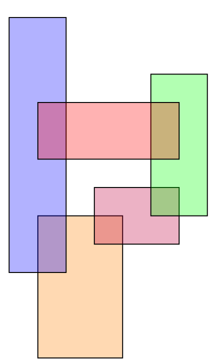

Figure 2. A (2, 3)-agreeable society of 2-boxes with agreement proportion $\frac{2}{5}$.

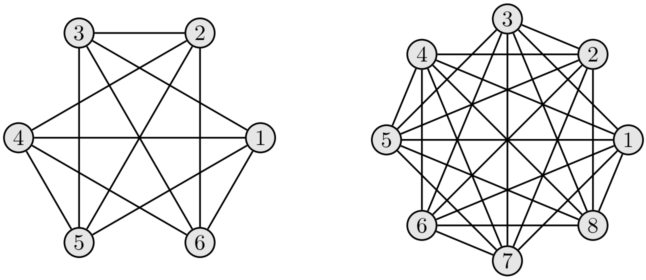

Figure 3. The complete partite graphs $K_3(2)$ (left) and $K_4(2)$ (right).

Remark 3.2. By convention we let $\mathrm{box}(K_n) = 0$ for all n (a 0-box would be a point). This shows that boxicity does not behave nicely with respect to taking subgraphs. On the other hand, it is clear that boxicity can only decrease when taking *induced subgraphs*, since for any arrangement of d-boxes $\mathcal{B} = \{B_1, \dots, B_n\}$ and any subset $I \subseteq \{1, \dots, n\}$, the intersection graph of the subarrangement $\{B_i \mid i \in I\}$ is simply the graph $G_{\mathcal{B}}[I]$ induced by the vertices I in $G_{\mathcal{B}}$.

Example 3.3. Note that the bound $\mathrm{box}(G) \le \lfloor \#V/2 \rfloor$ remains sharp, *even if we restrict our attention to* $(2, 3)$*-agreeable graphs.* Indeed, for any $d \ge 1$, let $K_d(2)$ be the complete d-partite graph on d pairs of vertices, that is, the graph with $V = \{1, 2, \dots, 2d\}$ and where E contains all possible edges except those of the form $\{i, i+1\}$ for i odd (see Figure 3). The graph $K_d(2)$ is $(2, 3)$-agreeable, and by [Roberts 1969, Theorem 7], we have $\mathrm{box}(K_d(2)) = d = \#V/2$.

Remark 3.4. Graphs with $\mathrm{box}(G) \le 1$ are *interval graphs*, which can be easily identified in linear time [Booth and Lueker 1976; Habib et al. 2000]. Algorithms exist to test if $\mathrm{box}(G) \le 2$ [Quest and Wegner 1990] or to compute boxicity in general [Cozzens and Roberts 1983], but they are a lot more cumbersome. The task of testing if $\mathrm{box}(G) \le d$ is known to be NP-complete for all $d \ge 2$ [Cozzens and Roberts 1983].

The definition of $(2, 3)$-agreeability as it appears in [Berg et al. 2010] can be reformulated in terms of intersection graphs.

Definition 3.5. An arrangement $\mathcal{B} = \{B_1, \dots, B_n\}$ of d-boxes is $(2, 3)$-agreeable if and only if any one of the three equivalent properties holds:

(1) For any $1 \le i < j < k \le n$, one at least of the intersections $B_i \cap B_j$, $B_i \cap B_k$ or $B_j \cap B_k$ is nonempty.

(2) For any three vertices in the intersection graph $G_{\mathcal{B}}$, the graph induced by these vertices contains at least one edge.

(3) The graph complement $\overline{G_{\mathcal{B}}}$ of the intersection graph satisfies $\omega(\overline{G_{\mathcal{B}}}) \le 2$.

3.2. ***Agreement number and agreement proportion.*** Since any simple, undirected graph can be realized as the intersection graph of an arrangement of boxes, it will be convenient to blur the distinction between the two notions. In particular, we can use properties (2) and (3) in Definition 3.5 to define (2, 3)-agreeability for graphs rather than arrangements.

Another good reason to identify arrangements and their graphs is that the intersection graph encodes all the information about arrangements of boxes (this fails for arrangements of more general convex sets). Indeed, in such an arrangement, having nonempty pairwise intersection and having a point common to all the boxes are equivalent. In particular, the maximal number of overlapping boxes (or *agreement number* of the society) is simply the clique number $\omega(G_{\mathcal{B}})$ of the intersection graph.

Notation 3.6. We denote by $\mathcal{G}$ the set of all (2, 3)-agreeable graphs, and, for any $d \geq 0$, denote by $\mathcal{G}_d$ the subset of those graphs with boxicity at most d. Given $r \geq 1$, we let $\mathcal{G}(r)$ and $\mathcal{G}_d(r)$ respectively be the subsets of $\mathcal{G}$ and $\mathcal{G}_d$ formed by graphs G with $\omega(G) \leq r$. Note that for any $G \in \mathcal{G}_d(r)$ and any subset of vertices $W \subseteq V(G)$, the subgraph $G[W]$ induced by W is also in $\mathcal{G}_d(r)$: (2, 3)-agreeability is preserved by taking induced graphs, and both clique size and boxicity can only decrease (Remark 3.2).

We define the associated vertex sizes for all $r \geq 1$ and all $d \geq 0$,

$$\eta(r, d) = \max\{\#V(G) \mid G \in \mathcal{G}_d(r)\},$$
$$\eta(r) = \max\{\#V(G) \mid G \in \mathcal{G}(r)\}.$$

These quantities are related by the inequalities

$$2r = \eta(r, 1) \leq \eta(r, 2) \leq \cdots \leq \eta(r).$$

We will show in Proposition 4.5 that $\eta(r) \leq r(r+3)/2$ for all $r \geq 1$, and thus all sets $\mathcal{G}_d(r)$ are finite. This is not a surprising result since it is the expected behavior brought on by (k, m)-agreeability, but note that in our case of interest, (2, 3)-agreeability, the very existence of a positive lower bound on the agreement proportion was left open in [Berg et al. 2010].

For any graph, the *agreement proportion* is defined as $\omega(G)/\#V(G)$. Once we prove that the set $\mathcal{G}_d(r)$ is finite for all $r \geq 1$ and $d \geq 1$, we can define

$$\rho(r, d) = \min\{\omega(G)/\#V(G) \mid G \in \mathcal{G}_d(r)\}, \tag{2}$$

that is, the *minimal agreement proportion* that can be obtained from a (2, 3)-agreeable graph with boxicity at most d and clique number at most r.

4. Upper and lower bounds on degrees

Throughout this section, $G = (V, E)$ denotes a $(2, 3)$-agreeable graph on n vertices. We show that a $(2, 3)$-agreeable graph with low clique number must have many edges. The results obtained here are purely combinatorial: in this section, we ignore the geometry of the problem and the boxicity of G.

4.1. ***Lower bound on the degree.*** The following trivial observation is the key to establishing lower bounds on the degrees of vertices.

Lemma 4.1. *If G is a $(2, 3)$-agreeable graph, then for any vertex $v \in V$, we have* $\deg(v) \geq n - \omega(G) - 1$.

Note that the inequality in this lemma may be strict, even if v is of minimal degree. We can see this by considering $G = W_4$, the wheel with four spokes, which is a $(2, 3)$-agreeable graph with $n = 5$ and $\omega(G) = 3$.

Proof. The vertex $v \in V$ is connected to $\deg(v)$ vertices. The other $n - \deg(v) - 1$ vertices must form a clique W. Indeed, if W were not a clique, it would contain two nonadjacent vertices, u and w. The subgraph induced by the three vertices $\{u, v, w\}$ would be empty, which would contradict the fact that G is $(2, 3)$-agreeable. Thus, $\omega(G) \geq |W| = n - \deg(v) - 1$, and the result follows. □

Using the formula

$$|E| = \tfrac{1}{2} \sum_{v \in V} \deg(v),$$

Lemma 4.1 yields the following lower bound on $|E|$.

Corollary 4.2. *For any $(2, 3)$-agreeable graph G, we have*

$$|E| \geq \tfrac{n}{2}(n - \omega(G) - 1).$$

Equality can occur in Corollary 4.2, for example, for the 5-cycle that appeared in Remark 2.4.

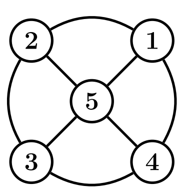

Figure 4. The wheel with four spokes W_4 is an example of a graph for which the inequality in Lemma 4.1 is strict.

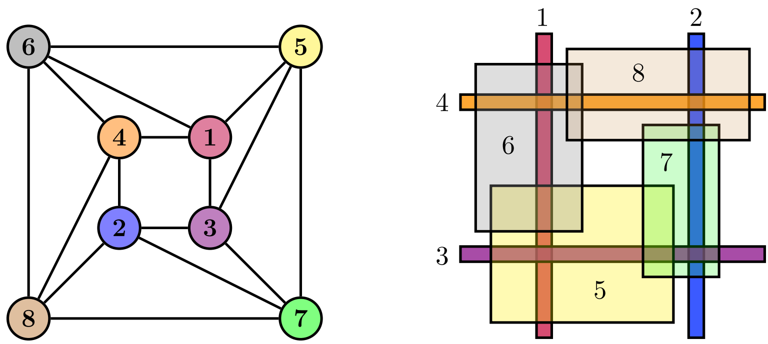

Figure 5. A (2, 3)-agreeable graph G with $|V(G)| = 8$, $\omega(G) = 3$ and $\text{box}(G) = 2$, together with a family of 2-boxes whose intersection graph is G. This graph is 4-regular, $|E| = 16$.

4.2. ***Examples with low agreement proportion.*** The conclusion of [Berg et al. 2010] mentioned the existence of (2, 3)-agreeable families of 2-boxes with agreement $\frac{3}{8}$. (The example, credited to Rajneesh Hegde, was not given in the paper.) We give a few examples.

Case $n = 8$**,** $\text{box}(G) = 2$**.** Figures 5 and 6 give two nonisomorphic examples of families of eight 2-boxes with no more than triple intersections. The corresponding intersection graphs have respectively 8 and 10 triangles.

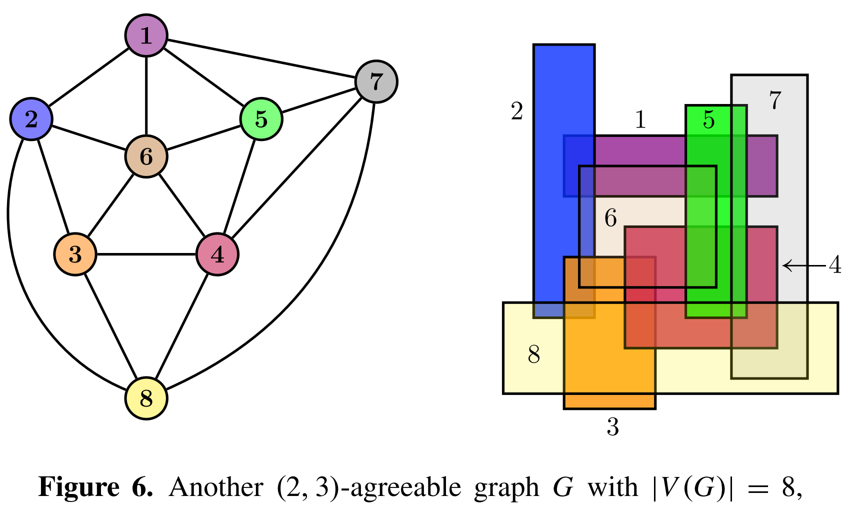

Figure 6. Another (2, 3)-agreeable graph G with $|V(G)| = 8$, $\omega(G) = 3$ and $\text{box}(G) = 2$. Like the example in the previous figure, this graph also has boxicity 2, but $|E| = 17$.

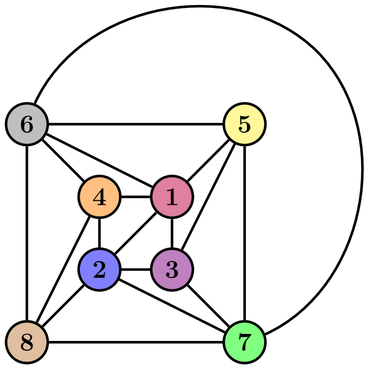

Figure 7. A (2, 3)-agreeable graph with $|V(G)| = 8$, $\omega(G) = 3$, but $\mathrm{box}(G)$ unknown. Modifying the arrangement of Figure 5 so $B_1 \cap B_2 \neq \varnothing$ and $B_6 \cap B_7 \neq \varnothing$ creates more intersections, so it is not obvious whether a 2-box arrangement can realize this graph.

Case $n = 8$, box(G) = ?. Figure 7 presents a third example of a (2, 3)-agreeable graph with agreement proportion $\frac{3}{8}$, obtained from Figure 5 by adding two edges. This graph has 12 triangles; its boxicity may be more than 2 (and we conjecture that it is).

Case $n = 13$, $\omega(G) = 4$. Figure 8 presents (the complement of) a (2, 3)-agreeable graph on 13 vertices with unknown boxicity and agreement proportion $\frac{4}{13} \approx 0.31$. There are 39 cliques of size 4 in that example. We prove in Proposition 4.4 that 13 is the maximum number of vertices for a (2, 3)-agreeable graph with no 5-cliques.

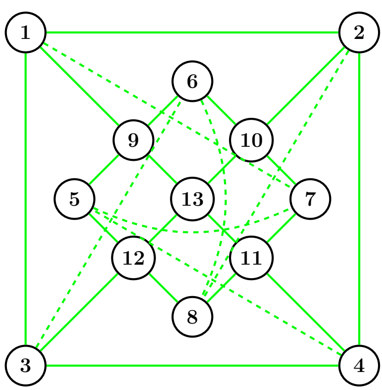

Figure 8. A graph whose *complement* is a (2, 3)-agreeable 8-regular graph with $\omega = 4$.

4.3. ***Upper bounds on degree and graph size.*** We now give upper bounds on the degrees of vertices in (2, 3)-agreeable graphs, and deduce an upper bound on the number of vertices of such graphs with a given clique number.

Lemma 4.3. *Let $G \in \mathcal{G}_d(r)$, where $r \geq 2$ and $d \geq 1$. Then for any $v \in V$, we have*

$$\deg(v) \leq \eta(r-1, d) \leq \eta(r-1).$$

Proof. The neighbors of v induce a (2, 3)-agreeable graph H. If there are more than $\eta(r-1, d)$ vertices in the graph H, it must contain an r-clique, which together with v forms an $(r+1)$-clique in G, contradicting the hypothesis $\omega(G) \leq r$. □

The inequality in the lemma can be sharp, but it is not always so, even if G has the maximum possible number of vertices $\eta(r)$: in the case $r = 3$, we have $\eta(r-1) = 5$ and $\eta(r) = 8$ (see Proposition 4.4). The graphs in Figures 5 and 6 both have the maximum number of vertices (8) for their clique number of 3, but the maximum degree is 4 in the first graph, and 5 (the maximum possible) in the second example.

With Lemma 4.1 giving a lower bound on the number of edges which increases with the number of vertices, and Lemma 4.3 giving an upper bound which depends only on the clique number, this suggests that the graphs which maximize $\eta(r)$ must be regular or almost-regular. We can use this idea to establish step-by-step the first few values of $\eta(r)$.

Proposition 4.4. *We have the following table for the maximal size of* (2, 3)*-agreeable graphs with $\omega(G) = r$.*

r	1	2	3	4	5
$\eta(r)$	2	5	8	13	≤ 18

Proof. By definition of (2, 3)-agreeability, any graph with at least three vertices must have an edge, and thus $\eta(1) = 2$. The examples we've seen so far give the following lower bounds:

$$\eta(2) \geq 5, \quad \eta(3) \geq 8, \quad \eta(4) \geq 13.$$

Suppose that one of these lower bounds is not sharp: in other words, there exists at least one (2, 3)-agreeable graph with one of the following.

$\omega(G)$	2	3	4	5
$\#V$	6	9	14	19

Let $\delta(G)$ and $\Delta(G)$ denote respectively the minimum and the maximum degree for vertices in G. The case $|V(G)| = 6$ and $\omega(G) = 2$ is clearly impossible, since Lemma 4.1 implies $\delta(G) \geq 3$, and Lemma 4.3 implies $\Delta(G) \leq \eta(1) = 2$, giving the contradiction $\delta(G) > \Delta(G)$.

Thus we have proved that $\eta(2) = 5$, which combined with Lemma 4.3 implies that for any $G \in \mathscr{G}$ with $\omega(G) = 3$ we must have $\Delta(G) \leq 5$. In the case $|V(G)| = 9$ with $\omega(G) = 3$, Lemma 4.1 yields $\delta(G) \geq 5$. Since the graph G cannot be 5-regular (the sum of all degrees must be even), this yields in turn $\Delta(G) \geq 6$, which is again a contradiction.

This proves $\eta(3) = 8$, which implies that $\Delta(G) \leq 8$ for any $G \in \mathscr{G}$ with $\omega(G) = 4$. The other cases are similar. □

The method used in the proof of the above proposition could be extended indefinitely, provided one can construct examples that provide lower bounds on η. Even without a battery of examples, we can prove that the function $\eta(r)$ has at most quadratic growth, and thus that the sets $\mathscr{G}_d(r)$ are finite for any $d \geq 1$ and $r \geq 1$.

Proposition 4.5. *For all* $r \geq 1$, *the maximal number of vertices* $\eta(r)$ *for a* $(2, 3)$-*agreeable graph* G *with* $\omega(G) \leq r$ *satisfies* $\eta(r) \leq r(r+3)/2$.

Proof. Let G be a $(2, 3)$-agreeable graph such that $\omega(G) = r$ and $|V(G)| = \eta(r)$. If v is a vertex of G, Lemmas 4.1 and 4.3 imply the inequalities

$$\eta(r) - r - 1 \leq \deg(v) \leq \eta(r - 1).$$

Solving the recurrence $\eta(r) - r - 1 - \eta(r-1) \leq 0$ with the initial condition $\eta(1) = 2$ gives the result. □

5. Lower bound on boxicity and the main result

Given a simple graph G on n vertices, call a vertex $v \in V(G)$ *universal* if $\deg(v) = n - 1$. [Adiga et al. 2008] presents several lower bounds on the boxicity of a graph; we need the following.

Theorem 5.1. [Adiga et al. 2008, Theorem 9] *Let* G *be a graph with no universal vertices and minimum degree* δ. *Then the boxicity of* G *has the lower bound*:

$$\operatorname{box}(G) \geq \frac{n}{2\,(n-\delta-1)}.$$

The theorem above only applies to graphs with no universal vertices. Fortunately the lemma below shows that graphs minimizing the agreement proportion satisfy this property. Recall that for all $r \geq 1$ and $d \geq 1$, the quantity $\rho(r, d)$ denotes the minimum agreement proportion that can be achieved by a graph $G \in \mathscr{G}_d(r)$.

Lemma 5.2. *Given* $r \geq 1$ *and* $d \geq 1$, *consider a graph* $G \in \mathscr{G}_d(r)$ *such that the agreement proportion of* G *is equal to* $\rho(r, d)$. *Then* G *has no universal vertices.*

Proof. Suppose $G \in \mathcal{G}_d(r)$ is a graph with universal vertices, $G \neq K_n$. We construct from G a graph $\widehat{G} \in \mathcal{G}_d(r)$ without universal vertices and with a lower agreement proportion. Let Ω be the set of universal vertices,

$$\Omega = \{v \in V(G) \mid \deg(v) = n-1\};$$

define $W = V(G) \backslash \Omega$, and let $\widehat{G} = G[W]$ be the graph induced by W. Since we assumed $G \neq K_n$, the graph $\widehat{G}$ is nonempty. Note that $\text{box}(\widehat{G}) \leq \text{box}(G) \leq d$, since boxicity can only decrease when considering induced graphs (Remark 3.2). Letting $k = |\Omega|$, we have for any vertex in $w \in W$,

$$\deg_{\widehat{G}}(w) = \deg_G(w) - k < n - 1 - k = |W| - 1,$$

so that no vertex in $\widehat{G}$ is universal. Moreover, we have

$$\omega(\widehat{G}) = \omega(G) - k,$$

since any maximal clique in G must contain all the vertices in Ω. Thus, the agreement proportion for $\widehat{G}$ is

$$\frac{\omega(\widehat{G})}{\#V(\widehat{G})} = \frac{\omega(G) - k}{n-k} < \frac{\omega(G)}{n};$$

thus, any graph which minimizes agreement proportion does not have any universal vertices. □

Proof of Theorem 1.1. Consider a graph $G \in \mathcal{G}_d(r)$ on n vertices such that the agreement proportion of G is equal to the minimum $\rho(r,d)$. By Lemma 5.2, G does not contain a universal vertex. Theorem 5.1 applies so that

$$d \geq \text{box}(G) \geq \frac{n}{2\,(n-\delta-1)},$$

where δ denotes the minimum degree in G. Since G is (2, 3)-agreeable, Lemma 4.1 yields

$$\omega(G) \geq n - \delta - 1.$$

Combining the two inequalities, we get

$$\rho(r,d) = \frac{\omega(G)}{n} \geq \frac{n-\delta-1}{n} \geq \frac{1}{2\,d}.$$

This completes the proof of the main theorem. □

6. Discussion

6.1. ***Ramsey numbers and agreement.*** Recall that the Ramsey number $R(k,m)$ is the smallest number such that any simple undirected graph G with $|V(G)| \geq R(k,m)$ contains either a clique of size at least k or an independent set of vertices of size at least m. Ramsey numbers are notoriously difficult to study, and few precise results are known about them. Fortunately, one of the sharper known bounds applies directly to the study of (2, 3)-agreeable graphs.

Ajtai et al. [1980] (upper bound) and Kim [1995] (lower bound) proved the existence of positive constants c_1 and c_2 such that, for all $k \geq 2$,

$$c_1 \frac{k^2}{\log k} \leq R(k, 3) \leq c_2 \frac{k^2}{\log k}. \tag{3}$$

The connection to (2, 3)-agreeability is the following: recall that we denote by $\eta(r)$ the maximum number of vertices for a (2, 3)-agreeable graph with clique number r. Thus any graph G with $\eta(r)+1$ vertices either contains a clique of size $r+1$ or is not (2, 3)-agreeable, that is, $\overline{G}$ contains a triangle. It follows that we have, for all $r \geq 2$,

$$\eta(r) + 1 = R(r+1, 3). \tag{4}$$

The quantity $r/\eta(r)$ is the minimum agreement proportion for (2, 3)-agreeable graphs G with $\omega(G) \leq r$. Since $\eta(r)$ can be expressed in terms of $R(r+1, 3)$, Kim's lower bound allows us to conclude that the agreement proportion satisfies

$$\lim_{r\to\infty} \frac{r}{\eta(r)} = 0.$$

6.2. ***Agreement proportion and boxicity.*** The above argument indicates that, for any infinite family of (2, 3)-agreeable graphs that minimizes the agreement proportion, the boxicity must go to infinity, since Theorem 1.1 shows that the agreement proportion is bounded away from zero when the boxicity is bounded.

We do not know of any explicit version of this result. Constructing a family of (2, 3)-agreeable graphs whose agreement proportion goes to zero would be of great interest.

6.3. ***Asymptotics of $\eta(r)$ and* $\mathbf{box}(G)$.** In Proposition 4.5 we showed that $\eta(r) \leq r(r+3)/2$. Equation (4) shows that $\eta(r)$ follows an inequality similar to (3), and in particular, it does grow almost quadratically. We can use Theorem 1.1 to relate these estimates to the boxicity of the graphs: if G is a graph of boxicity at most d, with $\omega(G) = r$ and $|V(G)| = \eta(r)$, then we must have

$$d \geq \frac{\eta(r)}{2r} \geq c \frac{r}{\log r} \tag{5}$$

for some positive constant c. Having sharp bounds for the value of c could be of considerable practical interest.

6.4. ***Exposed boxes.*** Our work originally established weaker lower bounds by borrowing the exposed boxes techniques used by Eckhoff [1988]. These allowed us to establish inequalities between face vectors of arrangements of d-boxes and $(d-1)$-boxes. The bounds were much weaker than Theorem 1.1 for $d \geq 4$, but these ideas might still produce interesting results for $(2, m)$-agreeability with $m \geq 4$.

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References

[Adiga et al. 2008] A. Adiga, L. S. Chandran, and N. Sivadasan, "Lower bounds for boxicity", Preprint, 2008. arXiv 0806.3175v1

[Ajtai et al. 1980] M. Ajtai, J. Komlós, and E. Szemerédi, "A note on Ramsey numbers", *J. Combin. Theory Ser. A* **29**:3 (1980), 354–360. MR 82a:05064 Zbl 0455.05045

[Berg et al. 2010] D. E. Berg, S. Norine, F. E. Su, R. Thomas, and P. Wollan, "Voting in agreeable societies", *Amer. Math. Monthly* **117**:1 (2010), 27–39.

[Booth and Lueker 1976] K. S. Booth and G. S. Lueker, "Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms", *J. Comput. System Sci.* **13**:3 (1976), 335–379. MR 55 #6932 Zbl 0367.68034

[Cozzens and Roberts 1983] M. B. Cozzens and F. S. Roberts, "Computing the boxicity of a graph by covering its complement by cointerval graphs", *Discrete Appl. Math.* **6**:3 (1983), 217–228. Zbl 0524.05059

[Eckhoff 1988] J. Eckhoff, "Intersection properties of boxes. I. An upper-bound theorem", *Israel J. Math.* **62**:3 (1988), 283–301. MR 90a:52016

[Habib et al. 2000] M. Habib, R. McConnell, C. Paul, and L. Viennot, "Lex-BFS and partition refinement, with applications to transitive orientation, interval graph recognition and consecutive ones testing", *Theoret. Comput. Sci.* **234**:1-2 (2000), 59–84. MR 2000m:68131 Zbl 0945.68189

[Kalai 1984] G. Kalai, "Intersection patterns of convex sets", *Israel J. Math.* **48**:2-3 (1984), 161–174. MR 86c:52002 Zbl 0557.52005

[Kim 1995] J. H. Kim, "The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$", *Random Structures Algorithms* **7**:3 (1995), 173–207. MR 96m:05140 Zbl 0832.05084

[Matoušek 2002] J. Matoušek, *Lectures on discrete geometry*, Springer, New York, 2002. Zbl 0999.52006

[Quest and Wegner 1990] M. Quest and G. Wegner, "Characterization of the graphs with boxicity ≤ 2", *Discrete Math.* **81**:2 (1990), 187–192. Zbl 0725.05070

[Roberts 1969] F. S. Roberts, "On the boxicity and cubicity of a graph", pp. 301–310 in *Recent progress in combinatorics: proceedings of the 3rd Waterloo Conference on Combinatorics* (Waterloo, ON, 1968), edited by W. T. Tutte, Academic Press, New York, 1969. MR 40 #5489 Zbl 0193.24301

[Trotter 1979] W. T. Trotter, Jr., "A characterization of Roberts' inequality for boxicity", *Discrete Math.* **28**:3 (1979), 303–313. MR 81a:05118

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