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Joseph D'Avanzo and Nikolai A. Krylov

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$\zeta(n)$ via hyperbolic functions

Joseph D'Avanzo and Nikolai A. Krylov

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We present an approach to compute $\zeta(2)$ by changing variables in the double integral using hyperbolic trigonometric functions. We also apply this approach to present $\zeta(n)$, when $n > 2$, as a definite improper integral of a single variable.

1. Introduction

The Riemann zeta function is defined as the series

$$\zeta(n) = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{k^n} + \cdots$$

for any integer $n \geq 2$. Three centuries ago Euler found that $\zeta(2) = \pi^2/6$, which is an irrational number. The exact value of $\zeta(3)$ is still unknown, though it was proved by Apéry in 1979 that $\zeta(3)$ is also irrational [van der Poorten 1979]. The values of $\zeta(n)$, when n is even, are known and can be written in terms of Bernoulli numbers. We refer the interested reader to Chapter 19 of [Aigner and Ziegler 2001] for a "perfect" proof of the formula

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} B_{2k}}{(2k)!} \cdot \pi^{2k} \quad (k \in \mathbb{N}).$$

Notice that $\zeta(n)$ can be written as the following multivariable integral:

$$\zeta(n) = \int_0^1 \cdots \int_0^1 \frac{1}{1 - x_1 x_2 \cdots x_n} \, dx_1 \, dx_2 \ldots dx_n.$$

Indeed, each integral is improper at both ends and since the geometric series

$$\sum_{q \geq 0} x^q$$

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converges uniformly on the interval $|x| \le R$ for all $R \in (0, 1)$, we can write

$$\frac{1}{1 - x_1 x_2 \cdots x_n} = \sum_{q=0}^{\infty} (x_1 x_2 \cdots x_n)^q,$$

then interchange summation with integration, and then integrate $(x_1 x_2 \cdots x_n)^q$ for each q. Using the identities

$$\frac{1}{1-xy} + \frac{1}{1+xy} = \frac{2}{1-x^2y^2} \quad \text{and} \quad \frac{1}{1-xy} - \frac{1}{1+xy} = \frac{2xy}{1-x^2y^2}$$

and a simple change of variables, one can easily see that

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy = \frac{4}{3} \int_0^1 \int_0^1 \frac{1}{1-x^2y^2}\, dx\, dy.$$

By further generalizing this idea, one reaches

$$\zeta(n) = \frac{2^n}{2^n - 1} \int_0^1 \cdots \int_0^1 \frac{1}{1 - \prod_{i=1}^n x_i^2}\, dx_1 \ldots dx_n.$$

Notice that $(1, 1)$ is the only point in the square $[0, 1] \times [0, 1]$ that makes the integrand $1/(1-x^2y^2)$ singular. If we take another point on the graph of $1 = x^2y^2$, say, $(a, 1/a)$ with $a \in (0, \infty)$, it follows easily (see Lemma 1.1 below) that

$$\int_0^{1/a} \int_0^a \frac{1}{1-x^2y^2}\, dx\, dy = \int_0^1 \int_0^1 \frac{1}{1-x^2y^2}\, dx\, dy.$$

This result motivates the following definition:

Definition 1. For any point $(a_1, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}$ such that $a_i \in (0, +\infty)$, for all $i \in \{1, \ldots, n-1\}$ we define

$$I_n(a_1, \ldots, a_{n-1}) = \int_0^{1/(a_1 \cdots a_{n-1})} \cdots \int_0^{a_2} \int_0^{a_1} \frac{1}{1 - \prod_{i=1}^n x_i^2}\, dx_1\, dx_2 \ldots dx_n.$$

Lemma 1.1. *For any $a_i \in (0, +\infty)$, we have $I_n(a_1, \ldots, a_{n-1}) = I_n(1, 1, \ldots, 1)$.*

Proof. Simply observe that by using the change of variables

$$x_i = a_i u_i \quad \text{for all } i \in \{1, \ldots, n\}, \text{ where } a_n = 1/(a_1 a_2 \cdots a_{n-1}),$$

the Jacobian equals 1, and the integrand is unchanged. □

In this article we investigate $\zeta(n)$ following Beukers, Calabi and Kolk [1993], who used the change of variables

$$x = \frac{\sin u}{\cos v} \quad \text{and} \quad y = \frac{\sin v}{\cos u} \quad \text{to evaluate} \int_0^1 \int_0^1 \frac{1}{1-x^2y^2}\, dx\, dy.$$

A similar proof of the identity $\zeta(2) = \pi^2/6$ may also be found in [Aigner and Ziegler 2001, Chapter 6] and [Elkies 2003; Kalman 1993]. This last reference in addition to a few other proofs of the identity, contains a history of the problem and an extensive reference list.

Here we will be changing variables too, but in the integrals $I_n(a_1, \ldots, a_{n-1})$ and using the hyperbolic trigonometric functions sinh and cosh instead of sin and cos. Such a change of variables was considered independently by [Silagadze 2010].

2. Hyperbolic change of variables

The change of variables

$$x_i = \frac{\sin u_i}{\cos u_{i+1}}, \qquad \text{for all } i \in \mathbb{N} \bmod n$$

reduces the integrand in $I_n(1, \ldots, 1)$ to 1 only when n is even. The region of integration $\Phi_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_1, \ldots, x_n < 1\}$ becomes the one-to-one image of the n-dimensional polytope

$$\Pi_n := \left\{(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n : u_i > 0,\ u_i + u_{i+1} < \frac{\pi}{2},\ 1 \le i \le n\right\}$$

(note that $u_{n+1} = u_1$).

We suggest here a different change of variables that will produce an integrand of 1 for all values of n in $I_n(a_1, \ldots, a_{n-1})$. But first we define the corresponding region.

Definition 2. For any point $(a_1, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}$ such that $a_i \in (0, +\infty)$, for all $i \in \{1, \ldots, n-1\}$, we define

$$\Phi_n(a_1, a_2, \ldots, a_{n-1}) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 < x_i < a_i \text{ for all } i \in \{1, \ldots, n\}\},$$

where $a_n = 1/(a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1})$.

Lemma 2.1. *The change in variables*

$$x_i = \frac{\sinh u_i}{\cosh u_{i+1}} \qquad \textit{for all } i \in \mathbb{N} \bmod n$$

reduces the integrand of $I_n(a_1, \ldots, a_{n-1})$ to 1 for all values of $n \ge 2$. It also gives a one-to-one differentiable map between the region $\Phi_n(a_1, a_2, \ldots, a_{n-1})$ and the set $\Gamma_n \subset \mathbb{R}^n$ described by the n inequalities

$$0 < u_i < \operatorname{arcsinh}(a_i \cosh u_{i+1}), \qquad \textit{for all } i \in \mathbb{N} \bmod n.$$

The set Γ_2 is illustrated in Figure 1.

Proof. The inequalities for Γ_n follow trivially from the corresponding inequalities $0 < x_i < a_i$ and the facts that $\cosh x > 0$ and $\operatorname{arcsinh} x$ is increasing everywhere. Injectivity and smoothness of the map may be proven by writing down formulas, which express each u_i in terms of all x_j. For example, here are the corresponding formulas for the set Γ_3:

$$u_i = \operatorname{arcsinh}\left(x_i \sqrt{\frac{1 + x_{i+1}^2 + x_{i-1}^2 x_{i+1}^2}{1 - x_1^2 x_2^2 x_3^2}} \right), \quad i \in \mathbb{N} \bmod 3.$$

The Jacobian is the determinant of the matrix

$$A = \begin{pmatrix} \dfrac{\cosh u_1}{\cosh u_2} & \dfrac{-\sinh u_1 \sinh u_2}{\cosh^2 u_2} & 0 & \dots & 0 \\ 0 & \dfrac{\cosh u_2}{\cosh u_3} & \dfrac{-\sinh u_2 \sinh u_3}{\cosh^2 u_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{-\sinh u_n \sinh u_1}{\cosh^2 u_1} & 0 & 0 & \dots & \dfrac{\cosh u_n}{\cosh u_1} \end{pmatrix}.$$

To compute this determinant we observe that the first column expansion reduces the computation to two determinants of the upper and lower triangular matrices. This results in the formula below, where the first term comes from the upper triangular matrix and the second from the lower triangular matrix (recall that $u_{n+1} = u_1$):

$$\det A = \prod_{i=1}^{n} \frac{\cosh u_i}{\cosh u_{i+1}} + (-1)^{n-1} \cdot \prod_{i=1}^{n} \frac{-\sinh u_i \sinh u_{i+1}}{\cosh^2 u_{i+1}} = 1 - \prod_{i=1}^{n} \tanh^2 u_i .$$

When using the above change in variables, the denominator of the integrand $1 - \prod_{i=1}^{n} x_i^2$ becomes $1 - \prod_{i=1}^{n} \tanh^2 u_i$, which we just proved to be the Jacobian. □

3. Computation of $\zeta(2)$

We begin with $\zeta(2)$, which is a rational multiple of $I_2(1)$. Lemma 1.1 implies that it's enough to compute

$$I_2(a) = \int_0^{1/a} \int_0^{a} \frac{1}{1 - x^2 y^2} \, dx \, dy \qquad \text{for arbitrary } a > 0.$$

We now perform the change in variables

$$x = \frac{\sinh u}{\cosh v}, \qquad y = \frac{\sinh v}{\cosh u}.$$

As we proved above, our integrand reduces to 1 and all we must do is worry about the limits. If $x = 0$, then clearly $u = 0$; the same is true for y and v. If $x = a$ then $a \cosh v = \sinh u$, so $v = \operatorname{arccosh}(\sinh(u)/a)$, and if $y = 1/a$, then

$$\frac{1}{a} \cosh u = \sinh v,$$

so $v = \operatorname{arcsinh}(\cosh(u)/a)$ — thus describing our region of integration, depicted in Figure 1. We then write the integral $I_2(a)$ as

$$\int_0^{\operatorname{arcsinh} a} \operatorname{arcsinh} \frac{\cosh u}{a}\, du + \int_{\operatorname{arcsinh} a}^{\infty} \left(\operatorname{arcsinh} \frac{\cosh u}{a} - \operatorname{arccosh} \frac{\sinh u}{a} \right) du.$$

Lemma 3.1. $$\lim_{a\to 0} \int_0^{\operatorname{arcsinh} a} \operatorname{arcsinh} \frac{\cosh u}{a}\, du = 0.$$

Proof. We have $\cosh \operatorname{arcsinh} z = \sqrt{1+z^2}$. Therefore

$$\operatorname{arcsinh} \frac{\cosh \operatorname{arcsinh} a}{a} = \operatorname{arcsinh} \sqrt{\frac{1}{a^2} + 1}\,.$$

Since $\operatorname{arcsinh}(\cosh(u)/a)$ is convex, we can take the area of the rectangle with vertices at $(0, 0)$, $(\operatorname{arcsinh} a, 0)$, and $(\operatorname{arcsinh} a, \operatorname{arcsinh}(\cosh(\operatorname{arcsinh}(a))/a)$ as an overestimate of the integral — that is,

$$\operatorname{arcsinh} a \ \operatorname{arcsinh} \sqrt{\frac{1}{a^2} + 1} \geq \int_0^{\operatorname{arcsinh} a} \operatorname{arcsinh} \frac{\cosh u}{a}\, du \geq 0.$$

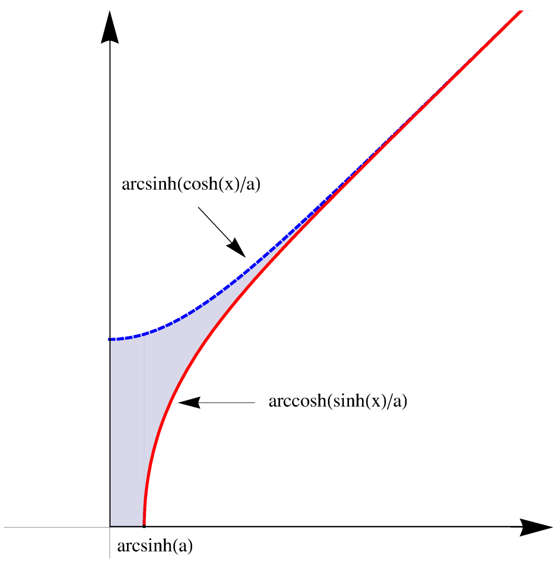

Figure 1. The set $\Gamma_2 \subset \mathbb{R}^2$, for all $a > 0$.

Then by applying L'Hospital's rule one can deduce

$$\lim_{a\to 0} \operatorname{arcsinh} a \cdot \operatorname{arcsinh} \sqrt{\frac{1}{a^2}+1} = 0. \qquad \square$$

Now, since $I_2(a) = I_2(1)$, for all $a > 0$, we conclude that $I_2(1) = \lim_{a\to 0} I_2(a)$. Therefore we have

$$I_2(1) = \lim_{a\to 0} \int_{\operatorname{arcsinh} a}^{\infty} \left(\operatorname{arcsinh} \frac{\cosh u}{a} - \operatorname{arccosh} \frac{\sinh u}{a} \right) du,$$

which, after taking the limit as $a \to 0$, gives

$$I_2(1) = \int_0^{\infty} \ln \frac{\cosh x}{\sinh x}\, dx.$$

Using integration by parts with $u = \ln \dfrac{\cosh x}{\sinh x}$ and $v = dx$ one obtains

$$I_2(1) = x \ln \frac{\cosh x}{\sinh x}\Big|_0^{\infty} + \int_0^{\infty} \frac{2x}{\sinh 2x}\, dx.$$

By examining the limits of the first half of the formula as x goes to 0 and ∞ we are left with only the integral

$$I_2(1) = \int_0^{\infty} \frac{2x}{\sinh 2x}\, dx.$$

By applying the change in variables $u = 2x$ our formula becomes

$$I_2(1) = \frac{1}{2} \int_0^{\infty} \frac{u}{\sinh u}\, du.$$

Now we use the method of differentiation under the integral sign and consider the function

$$F(\alpha) = \frac{1}{2} \int_0^{\infty} \frac{\operatorname{arctanh}(\alpha \tanh x)}{\sinh x}\, dx.$$

One should consider the function F at the points $\alpha = 1$ and $\alpha = 0$. $F(1)$ is clearly the integral we are trying to find and $F(0)$ is 0. Thus, by differentiating under the integral with respect to α and using some algebra, we obtain

$$F'(\alpha) = f(\alpha) = \frac{1}{2} \int_0^{\infty} \frac{\cosh x}{1 + (1-\alpha^2)\sinh^2 x}\, dx.$$

Then, by performing the change of variables $u = \sqrt{1-\alpha^2} \cdot \sinh x$, the integral becomes

$$f(\alpha) = \frac{1}{2\sqrt{1-\alpha^2}} \int_0^{\infty} \frac{1}{1+u^2}\, du,$$

which is simply
$$\frac{\operatorname{arctanh} u}{2\sqrt{1-\alpha^2}}\bigg|_0^\infty = \frac{\pi}{4\sqrt{1-\alpha^2}}.$$

Since we took the derivative with respect to α we must take the integral with respect to alpha, so we have
$$\int_0^1 f(\alpha)\, d\alpha = F(1) - F(0) = F(1) - 0 = F(1),$$
which, as stated above is our goal. So
$$I_2(1) = \int_0^1 f(\alpha)\, d\alpha = \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1-\alpha^2}}\, d\alpha = \frac{\pi}{4} \arcsin\alpha\bigg|_0^1 = \frac{\pi^2}{8},$$
and, hence,
$$\zeta(2) = \frac{4}{3}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

4. A formula for $\zeta(n)$, $n \geq 2$

One can try to use a similar approach to compute $\zeta(n)$, for $n > 2$, however the computations become too long. Instead, we present an elementary proof of the following theorem, which generalizes our formula for $\zeta(2)$ from Section 3.

Theorem 4.1. *Let $n \geq 2$ be a natural number. Then*
$$\int_0^1 \cdots \int_0^1 \frac{1}{1-\prod_{i=1}^n x_i^2}\, dx_1 \ldots dx_n = \frac{1}{(n-1)!}\cdot \int_0^\infty \ln^{n-1}(\coth x)\, dx.$$

We start with the following lemma, which can be easily proved by using induction on k, integration by parts and L'Hospital's rule.

Lemma 4.2. $$\int_0^1 \ln^k(z)\, z^{2q} dz = \frac{(-1)^k k!}{(2q+1)^{k+1}} \quad \textit{for all } k \in \mathbb{N} \textit{ and } q \geq 0.$$

Proof of the theorem. Applying the substitution $z = \tanh x$ to the integral
$$\frac{1}{(n-1)!}\int_0^\infty \ln^{n-1}(\coth x)\, dx,$$
gives
$$\frac{1}{(n-1)!}\int_0^1 \frac{(-\ln z)^{n-1}}{1-z^2}\, dz = \frac{1}{(n-1)!}\int_0^1 (-\ln z)^{n-1}\left(\sum_{q\geq 0} z^{2q}\right) dz.$$

Since the integral is improper at both ends and the geometric series $\sum_{q\geq 0} z^{2q}$ converges uniformly on the interval $|z| \leq R$, for all $R \in (0, 1)$, the last integral equals
$$\frac{1}{(n-1)!}\sum_{q\geq 0} (-1)^{n-1}\int_0^1 \ln^{n-1}(z) z^{2q} dz,$$

which, by Lemma 4.2, is equal to

$$\sum_{q\geq 0}\frac{1}{(2q+1)^n}.$$

Using the geometric series expansion one can easily show that we also have

$$\int_0^1\cdots\int_0^1\frac{1}{1-\prod_{i=1}^n x_i^2}\,dx_1\ldots dx_n=\sum_{q\geq 0}\frac{1}{(2q+1)^n}. \qquad \square$$

Corollary 4.3. *For any integer $n\geq 2$,*

$$\zeta(n)=\frac{2^n}{(2^n-1)(n-1)!}\int_0^\infty \ln^{n-1}(\coth x)\,dx.$$

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jt17dava@siena.edu *Siena College, Department of Mathematics, 515 Loudon Road, Loudonville, NY 12211, United States*

nkrylov@siena.edu *Siena College, Department of Mathematics, 515 Loudon Road, Loudonville, NY 12211, United States*

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