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Gracefulness of families of spiders

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We say that a tree is a *spider* if it has at most one branch point. We prove the existence of a family of graceful labelings for spiders all of whose legs are equal in length.

1. Introduction

Let $G = (V, E)$ be a (simple, undirected) graph. A *labeling* of G is a map from the set V of vertices to the set of nonnegative integers. A labeling ϕ induces a labeling on the edge set E by assigning to $e = \{u, v\}$ the value $\phi(e) = |\phi(u) - \phi(v)|$.

A labeling is said to be *graceful* if its labels take values in $\{0, 1, \ldots, |V| - 1\}$, it has no repeated labels, and its induced edge labeling has no repeated labels.

A graph is graceful if there is some graceful labeling of its vertices. Graceful labelings were first defined by Rosa as he considered problems involving decompositions of graphs; see [Rosa 1967], in which various sorts of labelings are defined. Golomb [1972] was the first to use the term *graceful labeling*.

There is a long-standing conjecture that every tree — that is, every connected acyclic graph — is graceful. Known as the Ringel–Kotzig conjecture, it seems to have first been published as Problem 25, p. 162 in a collection of open problems in [Fiedler 1964]. See [Edwards and Howard 2006; Gallian 1997–2009] for more information on this conjecture and hundreds of related results. We note that proofs of gracefulness for general classes of trees are hard to come by.

We call the graph T a *spider* if it has at most one branch point — that is, at most one vertex v such that the degree $d(v)$ satisfies $d(v) \geq 3$. Let v^* denote the unique branch point of a spider T, if this point exists. We call this point the *center* of the graph T. A *leg* of the spider T is any one of the paths from v^* to a leaf of T. We will prove the following result in Section 2:

Theorem 1. *Let T be a spider with l legs, each of which has length in $\{m, m+1\}$ for some $m \geq 1$. Then T is graceful.*

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Theorem 1 is not a new result. It follows from [Poljak and Sûra 1982], but our proof also shows gracefulness for any tree formed by appending an extra leg of any length to an odd-legged spider with legs of lengths in $\{m, m+1\}$. A generalization of the construction, given in Section 3, leads to further interesting labelings: specifically, for spiders having an odd number of legs, all of equal length m, we construct for each positive divisor d of m a graceful labeling associated with d. This construction can be used to generate graceful labelings of many trees that are not spiders, as shown in [Bahls 2008].

2. Proof of the main theorem

We may assume that $l \geq 3$, as otherwise T is a path, which is known to be graceful. (For example, see [Aldred et al. 2003], in which an estimate is obtained for the number of graceful labelings on a path of a given length.)

Proof of Theorem 1 for l odd. Let $l = l_0 + l_1$, where l_i is the number of legs of length $m+i$ for $i \in \{0, 1\}$. Note that T has $n+1 = lm+l_1+1$ vertices, to be labeled by the set $\{0, 1, \ldots, n\}$. Label the legs by $L_1, L_2, \ldots, L_l$ so that $L_1, \ldots, L_{l_1}$ have length $m+1$ and $L_{l_1+1}, \ldots, L_l$ have length m. Let v^* denote the branch point of T and denote by $v_{i,j}$ the vertex in L_i of distance j from v^*.

Let ϕ be the labeling defined as follows:

(i) $\phi(v^*) = 0$;

(ii) if i and j are both odd,

$$\phi(v_{i,j}) = n - \frac{i-1}{2} - \frac{(j-1)l}{2};$$

(iii) if i and j are both even;

$$\phi(v_{i,j}) = n - \frac{l-1}{2} - \frac{i}{2} - \frac{(j-2)l}{2};$$

(iv) if i is even and j is odd,

$$\phi(v_{i,j}) = \frac{i}{2} + \frac{(j-1)l}{2};$$

(v) if i is odd and j is even,

$$\phi(v_{i,j}) = \frac{l-1}{2} + \frac{i+1}{2} + \frac{(j-2)l}{2}.$$

The labeling ϕ places 0 at the spider's center and, traversing the longer legs first, alternates between the highest and the lowest remaining unused labels, spiraling away from the center. This is illustrated in Figure 1, in which $l_0 = 2$, $l_1 = 3$, and $m = 4$.

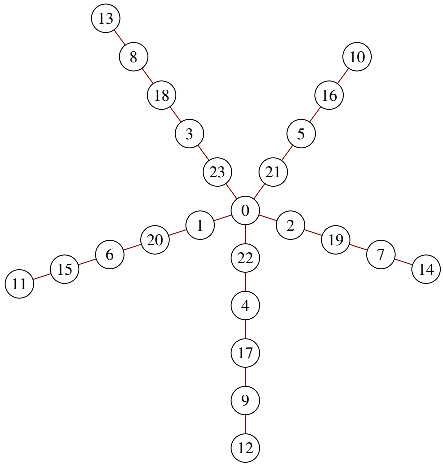

Figure 1. The labeling ϕ for $l_0 = 2$, $l_1 = 3$, and $m = 4$.

To help compute the induced edge labels, we note that the local maxima of ϕ occur at $v_{i,j}$ for which i and j have the same parity — that is, $i \equiv j \pmod 2$, For such i and j, we have

$$\phi(v_{i,j}) - \phi(v_{i,j+1}) = n - \frac{l-1}{2} - i + (1-j)l > 0, \tag{1}$$

$$\phi(v_{i,j}) - \phi(v_{i,j-1}) = n - \frac{l-1}{2} - i + (2-j)l > 0. \tag{2}$$

Suppose, to obtain a contradiction, that there are two distinct edges that share the same label. By considering the indexes of the vertices at both ends end of these edges, we see that we can choose distinct pairs of indexes (i, j) and (i', j') such that i and j have the same parity, i' and j' likewise have the same parity, and an edge incident on $v_{i,j}$ shares the same label as a different edge incident on $v_{i',j'}$, that is, one of these three cases occur:

$$\phi(v_{i,j}) - \phi(v_{i,j+1}) = \phi(v_{i',j'}) - \phi(v_{i',j'+1}), \tag{3}$$

$$\phi(v_{i,j}) - \phi(v_{i,j+1}) = \phi(v_{i',j'}) - \phi(v_{i',j'-1}), \tag{4}$$

$$\phi(v_{i,j}) - \phi(v_{i,j-1}) = \phi(v_{i',j'}) - \phi(v_{i',j'-1}). \tag{5}$$

Consider first the case where (3) holds. From (1), we obtain $i - i' + (j - j')l = 0$, which shows that $j \neq j'$, since otherwise $i = i'$ as well, contrary to the assumption that $(i, j) \neq (i', j')$. We therefore can write

$$l = \frac{i - i'}{j' - j}.$$

Thus $|i-i'| < l$ and $|j-j'| \geq 1$, and

$$l = \left| \frac{i-i'}{j'-j} \right| < \frac{l}{1} = l,$$

a contradiction.

Similar contradictions arise when (4) or (5) hold. Thus no two distinct edges bear the same labels, and ϕ is graceful. □

Proof of Theorem 1 for l even. Without loss of generality assume L_l is a leg of length m. Remove it, resulting in a tree T_0 with an odd number of legs, $l-1$. The construction above yields a graceful labeling ϕ_0 of T_0 such that $\phi_0(v^*) = 0$. Let $|V(T_0)| = n'+1$. We define a new graceful labeling, ϕ_0', on T_0 by $\phi_0'(v) = n' - \phi_0(v)$ for all v.

Construct a new tree T_1 by appending a new vertex, w_1, to T_0's center. Define ϕ_1 on $V(T_1)$ by $\phi_1(w_1) = 0$ and $\phi_1(v) = \phi_0'(v) + 1$ for all $v \in V(T_0)$. Define ϕ_1' on T_1 by $\phi_1'(v) = n' + 1 - \phi_1(v)$ for all v; note that $\phi_1'(w_1) = n' + 1$.

We now append a vertex w_2 to w_1 and construct graceful labelings ϕ_2 from ϕ_1', ϕ_2' from ϕ_2, and so forth, until we have reconstructed $L_l = \{w_1, w_2, \dots, w_m\}$, recovering T. □

The argument in the case of l even actually shows this:

Theorem 2. *Let T be a spider with l legs, where l is even. Suppose each leg, except possibly one, has length in $\{m, m+1\}$ for some $m \geq 1$. Then T is graceful.*

3. A family of graceful labelings

Now assume that T is a spider with an odd number l of legs, each of length m. Let d be any fixed positive divisor of m; we define a graceful labeling ϕ_d corresponding to d.

We retain the notation $v_{i,j}$ from the previous section. Given a pair (i, j), set $t = \lceil j/d \rceil$ and $r = j - (t-1)d$. Roughly, t gives the "tier" of length d inside the i-th leg in which the vertex $v_{i,j}$ lies, and r gives its position relative to that tier. The value of $\phi_d(v_{i,j})$ will depend on the parity of each of d, i, t, and r, so we consider the vector $\vec{v}_{i,j} = (d, i, t, r)$ as an element of $\mathbb{Z}_2^4$ by reducing all coordinates modulo 2.

Let $\phi_d(v^*) = 0$, as before. The following formula gives $\phi_d(v_{i,j})$:

(i) if $\vec{v}_{i,j} \in \{(0, 1, 1, 1), (1, 1, 1, 1)\}$,

$$\phi_d(v_{i,j}) = ml - \frac{(t-1)ld}{2} - \frac{(i-1)d}{2} - \frac{r-1}{2};$$

(ii) if $\vec{v}_{i,j} \in \{(0,1,1,0),(1,1,1,0)\}$,

$$\phi_d(v_{i,j}) = \frac{(t-1)ld}{2} + \frac{(i-1)d}{2} + \frac{r}{2};$$

(iii) if $\vec{v}_{i,j} \in \{(0,0,1,1),(1,0,1,1)\}$,

$$\phi_d(v_{i,j}) = \frac{(t-1)ld}{2} + \frac{id}{2} - \frac{r-1}{2};$$

(iv) if $\vec{v}_{i,j} \in \{(0,0,1,0),(1,0,1,0)\}$,

$$\phi_d(v_{i,j}) = ml - \frac{(t-1)ld}{2} - \frac{id}{2} + \frac{r}{2};$$

(v) if $\vec{v}_{i,j} \in \{(1,1,0,1),(0,1,0,0)\}$,

$$\phi_d(v_{i,j}) = \left\lceil \frac{ld}{2} \right\rceil + \frac{(t-2)ld}{2} + \frac{(i-1)d}{2} + \left\lfloor \frac{r}{2} \right\rfloor;$$

(vi) if $\vec{v}_{i,j} \in \{(1,0,0,1),(0,0,0,0)\}$,

$$\phi_d(v_{i,j}) = ml - \left\lfloor \frac{ld}{2} \right\rfloor - \frac{(t-2)ld}{2} - \frac{id}{2} + \left\lfloor \frac{r}{2} \right\rfloor.$$

(vii) if $\vec{v}_{i,j} \in \{(1,1,0,0),(0,1,0,1)\}$,

$$\phi_d(v_{i,j}) = ml - \left\lceil \frac{ld}{2} \right\rceil - \frac{(t-2)ld}{2} - \frac{(i-1)d}{2} - \left\lceil \frac{r}{2} \right\rceil + 1;$$

(viii) if $\vec{v}_{i,j} \in \{(1,0,0,0),(0,0,0,1)\}$,

$$\phi_d(v_{i,j}) = \left\lfloor \frac{ld}{2} \right\rfloor + \frac{(t-2)ld}{2} + \frac{id}{2} - \left\lceil \frac{r}{2} \right\rceil + 1.$$

That this yields a graceful labeling can be proved in a manner similar to the proof of Theorem 1.

Like the labeling introduced in the proof of Theorem 1, this labeling proceeds by alternating between the greatest and least labels yet unused, spiraling outward from the center. Now, however, d vertices on each leg are labeled before proceeding to the next leg, and the direction in which the labeling proceeds within this length-d segment (inward or outward relative to the center) alternates from one leg to the next. An example is shown in Figure 2.

In the special case $d = 1$, we obtain the labeling constructed in the proof of Theorem 1. In this case $t = j$ and $r = 1$, so our labeling depends only on the parities of i and j, and indeed after reduction the corresponding formulas in the above list, namely (i), (iii), (v), and (vi), coincide precisely with those in the proof of Theorem 1.

The labelings ϕ_d have the property that the edges

$$\{v^*, v_{i,1}\}, \{v_{i,d}, v_{i,d+1}\}, \{v_{i,2d}, v_{i,2d+1}\}, \ldots, \{v_{i,m-d}, v_{i,m-d+1}\}$$

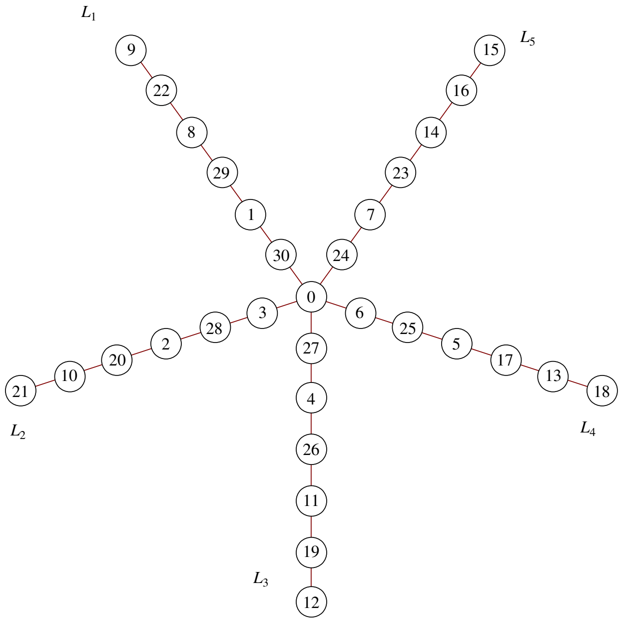

Figure 2. The labeling ϕ_d for $l = 5$, $m = 6$, and $d = 3$.

have labels divisible by d. This fact enables us to "deflate" the labeling ϕ_d and obtain a labeling ϕ'_d on the spider T' with l legs, each of length m/d. This new labeling is defined inductively as follows, spiraling outward from the center v' of T', where we denote by $v'_{i,j}$ the vertex in T' in position (i, j) as before and let $v_{i,0} = v^*$, $v'_{i,0} = v'$:

(i) $\phi'_d(v') = 0$;

(ii) $\phi'_d(v'_{i,1}) = \phi_d(v_{i,1})/d$;

(iii) assuming $\phi'_d(v'_{i,j})$ has been defined, let

$$\phi'_d(v'_{i,j+1}) = \phi'_d(v'_{i,j}) + (-1)^{l+j+1} \frac{\phi_d(\{v_{i,jd}, v_{i,jd+1}\})}{d}.$$

This process acts as an inverse to the process of edge subdivision considered in [Bahls 2008], in which each edge of a given gracefully labeled tree is subdivided a fixed number of times, yielding a new graph that can be gracefully labeled by making use of the labeling on the original tree.

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