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For $q \geq 27$ we determine the independence number $\alpha(\Gamma)$ of the Kneser graph Γ on plane-solid flags in PG(6, q). More precisely we describe all maximal independent sets of size at least q^{11} and show that every other maximal example has cardinality at most a constant times q^{10}.

1. Introduction

For integers $n \geq 2$ and prime powers q we denote by $\mathrm{PG}(n, q)$ the n-dimensional projective space over the finite field $\mathbb{F}_q$. A *flag* F of $\mathrm{PG}(n, q)$ is a set of nontrivial subspaces of $\mathrm{PG}(n, q)$ such that $U \subseteq U'$ or $U' \subseteq U$ for all $U, U' \in F$. Here nontrivial means different from $\varnothing$ and $\mathrm{PG}(n, q)$. The set $\{\dim(U) \mid U \in F\}$ is called the *type* of the flag F. Two flags F_1 and F_2 of $\mathrm{PG}(n, q)$ are said to be in *general position*, if for all subspaces $U_1 \in F_1$ and $U_2 \in F_2$ we have $U_1 \cap U_2 = \varnothing$ or $\langle U_1, U_2 \rangle = \mathrm{PG}(n, q)$.

If S is a nonempty subset of $\{0, 1, \ldots, n-1\}$, then the *Kneser graph* of flags of type S is the simple graph whose vertices are the flags of type S of $\mathrm{PG}(n, q)$ with two flags F and F' adjacent if and only if they are in general position. Note that this graph, among other generalizations of Kneser graphs, has already been defined in [Güven 2012].

For $|S| = 1$ the Kneser graph of type S is also known simply as q-Kneser graph and the size of maximal cocliques therein was determined in [Frankl and Wilson 1986]. Furthermore, for $|S| > 1$ maximal cocliques in this graph were studied in [Blokhuis and Brouwer 2017] for $S = \{1, 2\}$ and $n = 4$, in [Blokhuis, Brouwer and Güven 2014] for $S = \{0, n-1\}$ and $n \geq 2$, and in [Blokhuis, Brouwer and Szőnyi 2014] for $S = \{0, 2\}$ and $n = 4$. The result given in the second of these works was already conjectured in [Mussche 2009] and is also given in [Güven 2012].

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In this paper we determine the independence number $\alpha(\Gamma)$ for the Kneser graph Γ of type $\{2, 3\}$ in $\mathrm{PG}(6, q)$ for $q \geq 27$. We point out that a flag of type $\{2, 3\}$ of $\mathrm{PG}(6, q)$ is a self-dual object, hence any independent set of Γ can also be seen as an independent set of the Kneser graph of the same type in the dual space of $\mathrm{PG}(6, q)$. To simplify notation, we also denote a flag $\{E, S\}$ of type $\{2, 3\}$ by (E, S) where E is a plane and S is a solid. We first provide some examples.

Example 1.1. For a hyperplane H of $\mathrm{PG}(6, q)$ and a maximal set $\mathcal{E}$ of mutually intersecting planes of H, we denote by $\Lambda(H, \mathcal{E})$ the set of all flags (E, S) of type $\{2, 3\}$ of $\mathrm{PG}(6, q)$ that satisfy $S \subseteq H$ or $E \in \mathcal{E}$. Dually, for a point P of $\mathrm{PG}(6, q)$ and a maximal set $\mathcal{S}$ of 3-dimensional subspaces on P any two of which share at least a line, denote by $\Lambda(P, \mathcal{S})$, the set of all flags (E, S) of type $\{2, 3\}$ of $\mathrm{PG}(6, q)$ that satisfy $P \in E$ or $S \in \mathcal{S}$.

Indeed, the following special case of this example was already covered in a more general setting in [Blokhuis and Brouwer 2017].

Example 1.2. For an incident point-hyperplane pair (P, H) of $\mathrm{PG}(6, q)$, denote by $\Lambda(P, H)$ the set of all flags (E, S) of type $\{2, 3\}$ that satisfy $P \in E$ or $P \in S \subseteq H$ and let $\Lambda(H, P)$ be the set of all flags (E, S) of type $\{2, 3\}$ that satisfy $S \subseteq H$ or $P \in E \subseteq H$.

For an incident point-line pair (P, l) of $\mathrm{PG}(6, q)$, let $\Lambda(P, l)$ be the set of all flags (E, S) of type $\{2, 3\}$ that satisfy $P \in E$ or $l \subseteq S$.

For an incident pair (U, H) of a 4-dimensional space U and a hyperplane H of $\mathrm{PG}(6, q)$, let $\Lambda(H, U)$ be the set of all flags (E, S) of type $\{2, 3\}$ that satisfy $S \subseteq H$ or $E \subseteq U$.

We shall show in Proposition 3.2 that the sets described in Example 1.1 are maximal independent sets in the Kneser graph of flags of type $\{2, 3\}$ in $\mathrm{PG}(6, q)$. Notice that the condition on $\mathcal{E}$ means that $\mathcal{E}$ is an independent set of the Kneser graph of planes of $H \simeq \mathrm{PG}(5, q)$ and the condition on $\mathcal{S}$ means that $\mathcal{S}$ is an independent set of the Kneser graph of planes of the quotient space $\mathbb{P}/P \simeq \mathrm{PG}(5, q)$.

The sets constructed in Example 1.2 are special cases of the ones in Example 1.1 and hence also maximal independent sets. Here we use independent sets $\mathcal{E}$ and $\mathcal{S}$ of maximal size. Notice that the first and second examples as well as the third and forth examples in Example 1.2 are dual to each other.

In order to state our first theorem, we need the Gaussian coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, which are defined (for integer n and k) by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \prod_{i=1}^{k} \frac{q^{n+1-i} - 1}{q^i - 1} \quad \text{if } 0 \leq k \leq n$$

and $\begin{bmatrix} n \\ k \end{bmatrix}_q := 0$ otherwise.

Theorem 1.3. *For $q \geq 27$, the independence number of the Kneser graph of flags of type $\{2, 3\}$ of* PG(6, q) *is*

$$\begin{bmatrix}6\\4\end{bmatrix}_q \cdot \begin{bmatrix}4\\3\end{bmatrix}_q + \begin{bmatrix}5\\3\end{bmatrix}_q \cdot q^3$$

and the independent sets attaining this bound are precisely the four examples described in Example 1.2.

Remarks. 1. The independence number is of order q^{11}. As our proof of this theorem is geometric it also provides a stability result for independence sets. Essentially it says that, for large values of prime powers q, Example 1.1 describes all maximal independent sets with at least $27q^{10}$ elements. A precise formulation is given in Theorem 6.5.

2. Since we essentially show that any large independent set on the Kneser graph of plane-solid flags in PG(6, q) is given by Example 1.1, any Hilton-Milner type result for the Kneser graph of type $\{2\}$ in PG(5, q) translates to a Hilton-Milner type result for the Kneser graph of plane-solid flags in PG(6, q). In particular, in the main theorem of Section 6 of [Blokhuis, Brouwer and Szőnyi 2012], a Hilton-Milner type result for the Kneser graph of planes in PG(5, q) is given (the three largest examples are determined) and thus the second largest maximal EKR-set of plane-solid flags in PG(6, q) has size

$$\begin{bmatrix}6\\4\end{bmatrix}_q \cdot \begin{bmatrix}4\\3\end{bmatrix}_q + \left(\begin{bmatrix}5\\3\end{bmatrix}_q - (q^6 - q^3)\right) q^3.$$

and its structure can be derived from Example 1.1 and said Hilton-Milner result. However, note hat the flags provided by the sets $\mathcal{E}$ and $\mathcal{S}$ in Example 1.1 contribute only a small amount of flags (order q^9) to the total size (order q^{11}) of the maximal examples.

3. Every upper bound b for the independence number of a graph with n vertices leads to the lower bound $\chi \geq n/b$ for its chromatic number χ. In our situation this shows that the chromatic number of the Kneser graph of plane-solid flags of PG(6, q), $q \geq 27$, has chromatic number at least $q^4 - q^2 + 2q + 1$. On the other hand, if U is a 4-space, then the sets $\Lambda(P, \varnothing)$ with $P \in U$ are independent sets whose union covers every vertex, so the chromatic number is at most $q^4 + q^3 + q^2 + q + 1$. Using independent sets of the form $\Lambda(P, l)$ a simple construction given in Section 7 shows that this trivial upper bound can be slightly improved.

4. We keep all estimations in this paper as easy as possible and as such prove Theorem 1.3 only for $q > 27$. Only a more detailed approach, especially in Lemma 4.2, shows that Theorem 1.3 holds for $q = 27$. This will appear in the Ph.D. thesis of the second author.

2. Preliminaries

Let q be a prime power and $\mathbb{F}_q$ the finite field of order q. For integer $n, d \geq 0$, the number of d-dimensional subspaces of an $\mathbb{F}_q$-vector space of dimension n is given by the Gaussian coefficient $\begin{bmatrix} n \\ d \end{bmatrix}_q$ (see bottom of page 40 for the definition). If $0 \leq d \leq n$, and if D is a d-dimensional subspace of an n-dimensional $\mathbb{F}_q$ vector space V then D has exactly $q^{d(n-d)}$ complements in V. These two facts can be found in Section 3.1 of [Hirschfeld 1998]. We define

$$s_q(l, k, d, n) := q^{(l+1)(d-k)} \cdot \begin{bmatrix} n-k-l-1 \\ d-k \end{bmatrix}_q .$$

We also set $s_q(k, d, n) := s_q(-1, k, d, n)$, $s_q(d, n) := s_q(-1, d, n)$ and $s_q(n) := s_q(0, n)$ and omit the subscript q in the following.

Lemma 2.1. *Given two skew subspaces in* $\mathrm{PG}(n, q)$ *of dimensions k and l respectively and any integer d the number of d-subspaces of* $\mathrm{PG}(n, q)$ *that contain the k-subspace and are skew to the l-subspace is $s(l, k, d, n)$.*

Proof. We prove this for the underlying $\mathbb{F}_q$-vector space V of dimension $n+1$ and two skew subspaces K and L of dimension $k+1$ and $l+1$ respectively, where we have to count the number of subspaces D of dimension $d+1$ that contain K and are skew to L. Every such subspace D gives rise to a subspace $D+L$ of dimension $d+l+2$ of V. Going to the factor space $V/(K+L)$, we see that V has $\begin{bmatrix} n-k-l-1 \\ d-k \end{bmatrix}_q$ subspaces U of dimension $d+l+2$ that contain $K+L$. For such a subspace U we see in the quotient space U/K that U has $q^{(d-k)(l+1)}$ subspaces D of dimension $d+1$ with $U = L + D$. □

Lemma 2.2. *If $n \geq 5$ and if $\mathcal{E}$ is a set of planes of* $\mathrm{PG}(n, q)$ *such that any two distinct planes of $\mathcal{E}$ meet in a line, then $|\mathcal{E}| \leq s(n-2)$.*

Proof. If there exists a line contained in all planes of $\mathcal{E}$, then $|\mathcal{E}| \leq s(1, 2, n) = s(n-2)$. Otherwise there exist planes $E_1, E_2, E_3 \in \mathcal{E}$ such that $E_1 \cap E_3$ and $E_2 \cap E_3$ are distinct lines, which implies that E_3 is contained in the 3-space $U := \langle E_1, E_2 \rangle$. In this case, for every further plane E of $\mathcal{E}$ at least two of the lines $E \cap E_1$, $E \cap E_2$ and $E \cap E_3$ are distinct, so E is contained in U. Thus, in this case, every plane of $\mathcal{E}$ is one of the $s(2, 3)$ planes of U. □

The following result has been proven in Theorem 1.4 of [Blokhuis et al. 2010], where it was formulated in its dual version.

Result 2.3. *For $q \geq 3$ the independence number $\alpha(\Gamma)$ of the Kneser graph Γ of type $\{3\}$ in* $\mathrm{PG}(6, q)$ *is given by*

$$\alpha(\Gamma) = s(3, 5) = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1.$$

For each hyperplane H of PG$(6, q)$ *the set consisting of all solids of H is an independent set of Γ with $\alpha(\Gamma)$ vertices. Every other maximal independent set has cardinality at most $q^6 + 2q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1$.*

We shall conclude this section with the following result, which is one specific case of the main Theorem of [Frankl and Wilson 1986], which solves the Erdős–Ko–Rado problem for vector spaces in general.

Result 2.4. *If $\mathcal{E}$ is an independent set of the Kneser graph of type* $\{2\}$ *in* PG$(5, q)$, *then $|\mathcal{E}| \le s(1, 4)$ and equality holds if and only if $\mathcal{E}$ is the set of all planes on a point or the set of all planes in a hyperplane of* PG$(5, q)$.

3. Sets of flags of type {2, 3}

In this section we study sets of flags of type $\{2, 3\}$ of PG$(6, q)$. Recall that we also denote a flag $\{E, S\}$ of type $\{2, 3\}$ as the ordered pair (E, S) where E is the plane and S the solid of the flag. Note that two distinct such flags (E, S) and (E', S') are adjacent in Γ if and only if $E \cap S' = \varnothing = E' \cap S$. Let π_2 and π_3 be the maps from the set of all flags of type $\{2, 3\}$ to the set of subspaces of PG$(6, q)$ with $\pi_2(f) := E$ and $\pi_3(f) := S$ for all flags $f = (E, S)$ of type $\{2, 3\}$. For any set C of such flags, we define $\pi_i(C) := \{\pi_i(f) : f \in C\}$, $i = 2, 3$.

Lemma 3.1. *Let Γ be the Kneser graph of flags of type* $\{2, 3\}$ *of* PG$(6, q)$, *let C be an independent set of Γ, let H be a hyperplane and let P be a point of* PG$(6, q)$.

(i) *Let $\mathcal{E}$ be the set whose elements are the planes E of H for which there exists a solid S with $(E, S) \in C$ and $E = H \cap S$. Then $E \cap E' \ne \varnothing$ for all $E, E' \in \mathcal{E}$, that is, $\mathcal{E}$ is an independent set of the Kneser graph of planes of H. Hence $|\mathcal{E}| \le s(1, 4)$.*

(ii) *Let $\mathcal{S}$ be the set whose elements are the solids S for which there exists a flag $(E, S) \in C$ with $P \in S \setminus E$. Then $|\mathcal{S}| \le s(1, 4)$.*

Proof. (i) For $E, E' \in \mathcal{E}$ let (E, S) and (E', S') be flags of C with $S \cap H = E$ and $S' \cap H = E'$. Then $S' \cap E = E' \cap E$ and $S \cap E' = E' \cap E$. Since C is independent, it follows that $E \cap E' \ne \varnothing$. Thus $\mathcal{E}$ is an independent set of the Kneser graph of planes of H and Result 2.4 shows $|\mathcal{E}| \le s(1, 4)$.

(ii) This is a special case of the dual statement of part (i). □

In the following proposition we investigate the sets constructed in Example 1.1 up to duality.

Proposition 3.2. *Let H be a hyperplane of* PG$(6, q)$ *and let Γ be the Kneser graph of flags of type* $\{2, 3\}$ *of* PG$(6, q)$.

(i) $\Lambda(H, \varnothing)$ *is an independent set of* Γ.

(ii) *The maximal independent sets of* Γ *that contain* $\Lambda(H, \varnothing)$ *are the sets* $\Lambda(H, \mathcal{E})$ *for maximal independent sets* $\mathcal{E}$ *of planes of* H.

(iii) *For every maximal independent set* $\mathcal{E}$ *of the Kneser graph of planes of* H *we have*

$$|\Lambda(H, \mathcal{E})| = s(3,5) \cdot s(3) + |\mathcal{E}| \cdot q^3$$
$$\leq q^{11}+2q^{10}+5q^9+7q^8+10q^7+11q^6+11q^5+9q^4+7q^3+4q^2+2q+1.$$

If equality holds, that is, if $|\mathcal{E}| = s(1,4)$, *then either there exists a point* P *in* H *such that* $\mathcal{E}$ *consists of all planes of* H *that contain* P, *or there exists a* 4*-dimensional subspace of* H *such that* $\mathcal{E}$ *consists of all planes of this* 4*-dimensional subspace.*

Proof. (i) This follows from the fact that every solid of H meets every plane of H.

(ii) If $\mathcal{E}$ is an independent set of planes in the Kneser graph of planes of H, then every solid of H meets every plane of $\mathcal{E}$ nontrivially and every two planes of $\mathcal{E}$ meet nontrivially. Therefore $\Lambda(H, \mathcal{E})$ is an independent set of Γ. In order to prove the assertion, it therefore suffices to consider an independent set C of Γ with $\Lambda(H, \varnothing) \subseteq C$ and to show that C is contained in $\Lambda(H, \mathcal{E})$ for a set $\mathcal{E}$ of mutually intersecting planes of H.

Let C be an independent set with $\Lambda(H, \varnothing) \subseteq C$. Let $\mathcal{E}$ be the set of all planes E of H such that C contains a flag (E, S) with $E = S \cap H$. Lemma 3.1 shows that the planes of $\mathcal{E}$ are mutually intersecting. It remains to show that $C \subseteq \Lambda(H, \mathcal{E})$. Suppose on the contrary that there exists a flag $(E, S) \in C$ with $S \not\subseteq H$ and $H \cap S \neq E$. Then $S \cap H$ is a plane and $E \cap H$ is a line of this plane and H contains a solid S' that is skew to the line $E \cap H$. This implies that S' meets the plane $S \cap H$ in a point and therefore S' contains a plane E' with $E' \cap S \cap H = \varnothing$. Then $(E', S') \in \Lambda(H, \varnothing) \subseteq C$ with $S' \cap E = \varnothing = S \cap E'$ and since C is independent this is a contradiction.

(iii) Since H contains $s(3,5)$ solids all of which contain $s(2,3) = s(3)$ planes, we have $|\Lambda(H, \varnothing)| = s(3,5) \cdot s(3)$. Every plane E of H lies on $s(2,3,6) - s(2,3,5) = q^3$ solids S with $S \cap H = E$. Hence $|\Lambda(H, \mathcal{E})| = |\Lambda(H, \varnothing)| + |\mathcal{E}| \cdot q^3$. Result 2.4 shows $|\mathcal{E}| \leq s(1,4)$ with equality if and only if all planes of $\mathcal{E}$ contain a common point of H or lie in a common 4-subspace of H. □

Lemma 3.3. *Let* $\mathcal{C}$ *be an independent set of the Kneser graph of type* $\{2,3\}$ *in* PG$(6, q)$ *and let* $\xi \in \mathbb{N}$ *be such that every solid of* PG$(6, q)$ *occurs in at most* ξ *flags of* C. *Let* (E, S) *be an element of* C. *Then there are at most*

$$s(2) \cdot s(1,4) \cdot \xi = (q^8 + 2q^7 + 4q^6 + 5q^5 + 6q^4 + 5q^3 + 4q^2 + 2q + 1) \cdot \xi$$

flags $(E', S') \in \mathcal{C}$ *with* $E' \cap E = \varnothing$ *and* $S' \cap E \neq \varnothing$.

Proof. Since C is independent, every flag $(E', S') \in \mathcal{C}$ with $E' \cap E = \varnothing$ and $S' \cap E \neq \varnothing$ has the property that $S' \cap E$ is a point P with $P \notin E'$. Hence for every such flag there exists a point P in E with $P \in S' \setminus E'$. Since E has $s(2)$ points and since every solid occurs in at most ξ flags of C Lemma 3.1(ii) proves the statement. □

We now proceed to prove our theorem in three steps, where we consider two special cases in the first two steps: In the first step we only consider independent sets C in which no plane or solid occurs in more than $s(1)$ flags of C and in the second step we consider independent sets C in which no plane or solid occurs in more than $s(2)$ flags of C.

4. The first special case

In this section we consider an independent set C of the Kneser graph of type $\{2, 3\}$ in PG$(6, q)$ that has the property that every plane and every solid of PG$(6, q)$ occurs in at most $q + 1$ flags of C. Our aim is to prove an upper bound for $|C|$. For every point P we denote the set of all flags $(E, S) \in C$ with $P \in E$ by $\Delta_P(C)$.

Lemma 4.1. *Let P_1, P_2 and P_3 be noncollinear points of* PG$(6, q)$.

(i) *If*

$$|\Delta_{P_1}(C)| > (q + 1)(6q^6 + 10q^5 + 17q^4 + 15q^3 + 15q^2 + 9q + 5), \qquad (1)$$

then there are flags $f_i = (E_i, S_i) \in C$ for $i \in \{1, 2, 3\}$ with $\dim(\langle E_1, E_2, E_3\rangle) \geq 5$, *$P_2, P_3 \notin S_1, S_2, S_3$ as well as $E_i \cap E_j = P_1$ and $P_2, P_3 \notin \langle E_i, E_j\rangle$ for all distinct $i, j \in \{1, 2, 3\}$.*

(ii) *If there are flags f_1, f_2 and f_3 with the properties stated in* (i) *and if*

$$|\Delta_{P_2}(C)| > (q + 1)(6q^6 + 10q^5 + 17q^4 + 18q^3 + 15q^2 + 9q + 5),$$

then there are flags $f_i' = (E_i', S_i') \in C$ for $i \in \{1, 2, 3\}$ with $\dim(\langle E_1', E_2', E_3'\rangle) \geq 5$, *$P_1, P_3 \notin S_1', S_2', S_3'$,* $\dim(S_i \cap S_j') \leq 1$ *for all $i, j \in \{1, 2, 3\}$ as well as $E_i' \cap E_j' = P_2$ and $P_1, P_3 \notin \langle E_i', E_j'\rangle$ for all distinct $i, j \in \{1, 2, 3\}$.*

Proof. (i) We frequently make use of the fact that every plane and every solid occurs in at most $q + 1$ flags of C. We also make use of the following properties:

(Q1) There are $2 \cdot s(1, 3, 6) - s(2, 3, 6) = 2 \cdot s(1, 4) - s(3)$ solids that contain P_1 and a point of $\{P_2, P_3\}$.

(Q2) If E is a plane on P_1 and P is a point not contained in E, then every plane E' on P_1 with $E' \cap E \neq P_1$ or $P \in \langle E, E'\rangle$ meets the solid $\langle P, E\rangle$ in at least a line and hence there are at most $s(0, 1, 3) \cdot s(1, 2, 6) = s(2) \cdot s(4)$ such planes E'.

(Q3) If E_1 and E_2 are planes with $E_1 \cap E_2 = P_1$, then there exist less than $s(1, 0, 2, 4) = q^4$ planes in $\langle E_1, E_2\rangle$ with $E \cap E_1 = E \cap E_2 = P_1$.

From (Q1) and the bound in (1) we see that there exists a flag (E_1, S_1) in C with $P_1 \in E_1$ and $P_2, P_3 \notin S_1$. According to (Q1) and (Q2) the number of flags $(E, S) \in \Delta_{P_1}(C)$ for which $E_1 \cap E \neq P_1$ or for which $\langle E_1, E\rangle$ or S contains a point of $\{P_2, P_3\}$ is at most

$$(q+1)(2s(1, 4)-s(3)+2s(2)s(4)) = (q+1)(4q^6+6q^5+10q^4+9q^3+9q^2+5q+3)$$

which is smaller than the right-hand side of (1). Therefore, we find a flag $(E_2, S_2) \in \Delta_{P_1}(C)$ such that $E_1 \cap E_2 = P_1$ and neither of the spaces $\langle E_1, E_2\rangle$ or S_2 contains one of the points P_2 and P_3. Notice that $\dim(\langle E_1, E_2\rangle) = 4$, so for the remaining flag (E_3, S_3) we need that E_3 is not contained in $\langle E_1, E_2\rangle$. Using (Q1), (Q2) and (Q3) a similar argument shows that at most

$$(q+1)(2\cdot s(1, 4) - s(3) + 4\cdot s(2)\cdot s(4) + q^4)$$
$$= (q+1)(6q^6 + 10q^5 + 17q^4 + 15q^3 + 15q^2 + 9q + 5) \quad (2)$$

flags of $\Delta_{P_1}(C)$ do not satisfy all of the properties we want for the final flag (E_3, S_3). Since this is the right-hand side of (1) and thus smaller than $|\Delta_{P_1}(C)|$ we find a flag (E_3, S_3) with the desired properties.

(ii) We can argue analogously to the proof of (i). However, when choosing the flags (E_i', S_i') for $i \in \{1, 2, 3\}$ we additionally have to avoid all flags $(E, S) \in \Delta_{P_2}(C)$ for which S meets one of the solids S_1, S_2 and S_3 in a plane π with $P_1 \notin \pi$. For $j \in \{1, 2, 3\}$ each S_j has q^3 planes that do not contain P_1, so in total there are at most $3q^3$ solids S that must not appear in any of our desired flags (E_i', S_i') for $i \in \{1, 2, 3\}$ and were not considered before. Therefore, it is sufficient to check that the sum of the number in (2) and the number $3q^3(q + 1)$ is the right-hand side of (1) and thus smaller than $|\Delta_{P_2}(C)|$, which is obviously true. □

Lemma 4.2. *Let P_1 and P_2 be two distinct points of* PG$(6, q)$ *and let E_1, E_2 and E_3 be planes such that $E_i \cap E_j = P_1$ and $P_2 \notin \langle E_i, E_j\rangle$ for all distinct $i, j \in \{1, 2, 3\}$. Furthermore, let $\mathcal{S}$ be the set of all solids of* PG$(6, q)$ *with $P_2 \in S$ and $S \cap E_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$. Then we have $|\mathcal{S}| \leq 3q^6 + 6q^5 + 7q^4 + 4q^3 + 2q^2 + q + 1$.*

Proof. Let $\mathcal{E}$ be the set of all planes that contain P_2 but not P_1 and that meet all the planes E_1, E_2 and E_3. There are $s(1, 3, 6)$ solids on P_2 that contain P_1. If a solid on P_2 does not contain P_1 nor any plane of $\mathcal{E}$, then it meets all the planes E_1, E_2 and E_3 in unique points (different from P_1) and these three intersection points together with P_2 span the solid. Hence, there are at most $(q^2 + q)^3$ such solids. Finally, each plane of $\mathcal{E}$ lies in at most $s(0, 2, 3, 6)$ solids that do not contain P_1, which shows that the number of solids on P_2 that meet E_1, E_2 and E_3 is at most

$$s(1,3,6)+(q^2+q)^3+|\mathcal{E}|\cdot s(0,2,3,6). \tag{3}$$

It remains to determine an upper bound on $|\mathcal{E}|$. We put $U := \langle E_1, E_2\rangle$ with $P_2 \notin U$ and if $E \in \mathcal{E}$ we know from $P_2 \in E$ and $P_1 \notin E$ that $E \cap U$ is a line. We show that every point of $E_3 \setminus \{P_1\}$ lies on at most q planes of $\mathcal{E}$. To see this, let Q be a point of $E_3 \setminus \{P_1\}$ and suppose that Q lies on at least one plane E of $\mathcal{E}$. Since the lines $\langle P_2, Q\rangle$ and $E \cap U$ of E are distinct, they meet in a unique point R, and $E \cap U$ is a line on R. Since P_2 is not contained in $\langle E_1, E_3\rangle$ nor in $\langle E_2, E_3\rangle$ we have $R \notin E_1$ and $R \notin E_2$. This implies that R lies on exactly q lines of U that meet E_1 and E_2 but do not contain P_1. Since every plane of $\mathcal{E}$ on Q is generated by P_2 and such a line, we see that Q lies on at most q planes of $\mathcal{E}$. As there are q^2+q choices for Q, we find $|\mathcal{E}| \le (q^2+q)q$. Using this upper bound for $|\mathcal{E}|$, the statement follows from (3). □

Lemma 4.3. *Let P be a point and suppose that there are flags $(E_i, S_i) \in \Delta_P(C)$, for $i \in \{1,2,3\}$, such that $E_i \cap E_j = P$ for distinct $i, j \in \{1,2,3\}$. Then every point Q with $Q \notin \langle E_i, E_j\rangle$ and $Q \notin S_i$ for all $i, j \in \{1,2,3\}$ satisfies*

$$|\Delta_Q(C)| \le 3q^8+12q^7+21q^6+28q^5+26q^4+18q^3+12q^2+8q+4.$$

Proof. For $i \in \{1,2,3\}$ exactly $n := s(0,2,6)-s(3,0,2,6)$ planes on Q meet S_i. Since every plane lies in at most $q+1$ flags of C, it follows that there exists at most $3n(q+1)$ flags $(E, S) \in \Delta_Q(C)$ such that E has nonempty intersection with at least one of the solids S_1, S_2 or S_3.

Every other flag $f = (E, S) \in \Delta_Q(C)$ has the property that its solid S meets E_1, E_2 and E_3. Lemma 4.2 shows that there at most $n' := 3q^6+6q^5+7q^4+4q^3+2q^2+q+1$ such solids. Since each solid lies in at most $q+1$ flags of C, there are at most $n'(q+1)$ such flags. Therefore $|\Delta_Q(C)| \le (3n+n')(q+1)$ proving the desired bound. □

Lemma 4.4. *Let S_1 and S_2 be solids of* PG$(6, q)$ *with* $\dim(S_1 \cap S_2) \le 1$ *and let P be a point that is not contained in $S_1 \cup S_2$. Then the number of planes that contain P and meet S_1 and S_2 nontrivially is at most* $2q^6+2q^5+3q^4+2q^3+2q^2+q+1$.

Proof. We have $d := \dim(S_1 \cap S_2) \in \{0,1\}$. A line through P meets S_1 and S_2 if and only if it meets one and hence both of the subspaces $U_1 := \langle P, S_2\rangle \cap S_1$ and $U_2 := \langle P, S_1\rangle \cap S_2$, that is, if the line is contained in the subspace $V := \langle U_1, P\rangle$. The subspaces U_1 and U_2 have the same dimension u where $u = 1$ if $d = 0$ and $u \in \{1,2\}$ when $d = 1$. We have $\dim(V) = u+1$.

A plane on P that meets V only in P is spanned by P, a point of $S_1 \setminus U_1$ and a point of $S_2 \setminus U_2$, so there are $(s(3)-s(u))^2$ such planes. The number of planes on P that meet V in a line is equal to the number $s(0,1,u+1)$ of lines of V on

P multiplied with the number $s(1,2,6)-s(1,2,u+1)$ of planes that meet V in a given line. Finally there are $s(0,2,u+1)$ planes on P that are contained in V. Hence, the total number of planes on P that meet S_1 and S_2 nontrivially is

$$(s(3)-s(u))^2+s(0,1,u+1)(s(1,2,6)-s(1,2,u+1))+s(0,2,u+1)$$

The larger value occurs for $u=2$ and gives the bound in the lemma. $\square$

Lemma 4.5. *Let P_1, P_2 and P_3 be noncollinear points of* $\mathrm{PG}(6,q)$. *Suppose that for $i\in\{1,2\}$ and $r\in\{1,2,3\}$ there exist flags $f_{i,r}=(E_{i,r},S_{i,r})\in\Delta_{P_i}(C)$ such that*

- $\forall r,s\in\{1,2,3\}:\dim(S_{1,r}\cap S_{2,s})\le 1$ *and*
- $\forall i\in\{1,2\},\forall\{r,s,t\}=\{1,2,3\}:P_{3-i},P_3\notin\langle E_{i,r},E_{i,s}\rangle\cup S_{i,r}$ *and* $E_{i,r}\cap E_{i,s}=P_i$.

Then $|\Delta_{P_3}(C)|\le 24q^7+54q^6+71q^5+67q^4+48q^3+33q^2+22q+11$.

Proof. Because C is independent we know that for every $(E,S)\in C$ and every $i\in\{1,2\}$ we have $S\cap E_{i,r}\ne\varnothing$ for all $r\in\{1,2,3\}$ or $E\cap S_{i,r}\ne\varnothing$ for at least one $r\in\{1,2,3\}$. For $i\in\{1,2\}$ Lemma 4.2 shows that the number of solids of $\mathrm{PG}(6,q)$ that contain P_3 and meet $E_{i,1}$, $E_{i,2}$ and $E_{i,3}$ is at most

$$m:=3q^6+6q^5+7q^4+4q^3+2q^2+q+1. \tag{4}$$

For every flag $(E,S)\in\Delta_{P_3}(C)$ for which S is not such a solid we know that E is a plane that meets $S_{1,r}$ and $S_{2,s}$ for some $r,s\in\{1,2,3\}$. For any choice of $r,s\in\{1,2,3\}$, Lemma 4.4 shows that there exist at most

$$n:=2q^6+2q^5+3q^4+2q^3+2q^2+q+1$$

planes on P_3 that meet $S_{1,r}$ and $S_{2,s}$. Since every plane and every solid occurs in at most $q+1$ flags of C, it follows that $|\Delta_{P_3}(C)|\le(2m+9n)(q+1)$, as claimed. $\square$

Proposition 4.6. *Let C be an independent set of the Kneser graph of type* $\{2,3\}$ *in* $\mathrm{PG}(6,q)$ *with $q\ge 7$ that has the property that every plane and every solid of* $\mathrm{PG}(6,q)$ *is contained in at most $s(1)=q+1$ flags of C. Then*

$$|C|\le 24q^{10}+79q^9+155q^8+210q^7+216q^6+187q^5+140q^4+93q^3+51q^2+22q+5.$$

Proof. Let P_1 and P_2 be distinct points of $\mathrm{PG}(6,q)$ such that $|\Delta_{P_1}(C)|,|\Delta_{P_2}(C)|\ge|\Delta_P(C)|$ for all points $P\ne P_1$. If every flag $(E,S)\in C$ satisfies $\langle P_1,P_2\rangle\cap E\ne\varnothing$, then

$$\begin{aligned}|C|&\le(s(2,6)-s(1,-1,2,6))\cdot s(1)\\&=q^{10}+3q^9+5q^8+7q^7+8q^6+8q^5+7q^4+5q^3+3q^2+2q+1\end{aligned}$$

since there are $s(2,6)-s(1,-1,2,6)$ planes that meet the line $\langle P_1,P_2\rangle$ and since every plane lies in at most $q+1$ flags of C. Therefore, we may assume that C contains a flag $f=(E,S)$ with $\langle P_1,P_2\rangle\cap E=\varnothing$ and thus $\dim(S\cap\langle P_1,P_2\rangle)\le 0$.

Every flag $(E', S') \in C$ either satisfies $E' \cap S \neq \varnothing$ or $E' \cap S = \varnothing \neq S' \cap E$. Lemma 3.3 shows that at most

$$(q^8 + 2q^7 + 4q^6 + 5q^5 + 6q^4 + 5q^3 + 4q^2 + 2q + 1) \cdot s(1) \tag{5}$$

flags (E', S') of C satisfy $E' \cap S = \varnothing \neq S' \cap E$. Before we count all flags $f' = (E', S')$ with $E' \cap S \neq \varnothing$ we note that we either have

$$|\Delta_P(C)| \leq |\Delta_{P_2}(C)| \leq 6q^7 + 16q^6 + 27q^5 + 35q^4 + 33q^3 + 24q^2 + 14q + 5 \tag{6}$$

for all $P \in \mathrm{PG}(6, q) \setminus \langle P_1, P_2 \rangle$ or

$$|\Delta_{P_1}(C)| \geq |\Delta_{P_2}(C)| > 6q^7 + 16q^6 + 27q^5 + 35q^4 + 33q^3 + 24q^2 + 14q + 5.$$

If the second situation occurs, then Lemma 4.1 provides flags $f_{i,j} \in C$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ required to apply Lemma 4.5 proving

$$|\Delta_P(C)| \leq 24q^7 + 54q^6 + 71q^5 + 67q^4 + 48q^3 + 33q^2 + 22q + 11 \tag{7}$$

for all $P \in \mathrm{PG}(6, q) \setminus \langle P_1, P_2 \rangle$. Since the bound in (7) is weaker than the bound given in (6) we know that it also holds in the first case. In particular, (7) holds for all $P \in S \setminus (S \cap \langle P_1, P_2 \rangle)$. Note that we chose f such that $S \cap \langle P_1, P_2 \rangle$ is at most a point. Now, if $\hat{P} := S \cap \langle P_1, P_2 \rangle \neq \varnothing$, then, since P_1 and P_2 are distinct, there is an index $i \in \{1, 2\}$ such that $\hat{P} \neq P_i$ and, using the flags $f_{i,1}$, $f_{i,2}$ and $f_{i,3}$, we may apply Lemma 4.3 to see that

$$|\Delta_{\hat{P}}(C)| \leq 3q^8 + 12q^7 + 21q^6 + 28q^5 + 26q^4 + 18q^3 + 12q^2 + 8q + 4,$$

which is weaker than the bound in (7) for $q \geq 7$. Therefore, the number of all flags (E', S') of C with $E' \cap S \neq \varnothing$ is at most

$$\begin{aligned}
&(s(3) - 1) \cdot (24q^7 + 54q^6 + 71q^5 + 67q^4 + 48q^3 + 33q^2 + 22q + 11) \\
&\quad + 3q^8 + 12q^7 + 21q^6 + 28q^5 + 26q^4 + 18q^3 + 12q^2 + 8q + 4 \\
&\quad = 24q^{10} + 78q^9 + 152q^8 + 204q^7 + 207q^6 + 176q^5 + 129q^4 + 84q^3 + 45q^2 + 19q + 4.
\end{aligned}$$

Together with the upper bound in (5) for the remaining flags of C, this provides the claimed upper bound on the cardinality of C. □

5. The second special case

In this section we generalize the results of Section 4 to Kneser graphs of type {2, 3} in PG(6, q) with the property that every plane and every solid of PG(6, q) occurs in at most $q^2 + q + 1$ flags of C. Let Γ be a Kneser graph with that property.

Lemma 5.1. *Let E be a plane and suppose that the solids S with $(E, S) \in C$ span a subspace H of dimension at least* 5*. Suppose also that every plane of* $\mathrm{PG}(6, q)$ *occurs in at most $s(2) = q^2 + q + 1$ flags of C. Then the number of flags $(E', S') \in C$ with $E' \cap E = \varnothing$ is at most $s(1, 4) \cdot s(2) \cdot (s(2) + 1)$.*

Proof. Let M be the set consisting of all flags (E', S') of C such that $S' \cap E = \varnothing$ and let N be the set consisting of all flags (E', S') of C such that $S' \cap E \neq \varnothing$ and $E' \cap E = \varnothing$. Every flag $(E', S') \in C$ with $E' \cap E = \varnothing$ lies in $M \cup N$. Lemma 3.3 applied with $\xi = s(2)$ shows that $|N| \leq s(1, 4) \cdot s(2)^2$. For an upper bound on the number of flags in M, we let $\mathcal{E}$ denote the set of all planes that occur in a flag of M. The hypothesis of this lemma shows that $|M| \leq |\mathcal{E}| \cdot s(2)$. In order to prove the statement, it remains to show that $|\mathcal{E}| \leq s(1, 4)$.

Consider $E' \in \mathcal{E}$. Let S' be a solid with $(E', S') \in M$. Then $S' \cap E = E' \cap E = \varnothing$. Since C is independent and since $S' \cap E = \varnothing$ the plane E' meets every solid S for which $(E, S) \in C$. Then every such solid S is spanned by E and a point of E', so $H \subseteq \langle E, E' \rangle$. Since H has dimension at least 5, it follows that H has dimension 5 and that $H = \langle E, E' \rangle$ for all $E' \in \mathcal{E}$. Lemma 3.1 shows that $|\mathcal{E}| \leq s(1, 4)$. □

Proposition 5.2. *Let C be an independent set of the Kneser graph of type* $\{2, 3\}$ *in* $\mathrm{PG}(6, q)$ *with* $q \geq 8$ *that has the property that every plane and every solid of* $\mathrm{PG}(6, q)$ *occurs in at most $s(2) = q^2 + q + 1$ flags of C. Then*

$$|C| \leq 24q^{10} + 79q^9 + 155q^8 + 210q^7 + 218q^6 + 189q^5 + 142q^4 + 95q^3 + 53q^2 + 22q + 5.$$

Proof. Let $\mathcal{E}$ be the set consisting of all planes of $\mathrm{PG}(6, q)$ that lie in at least $q + 2$ flags of C, and let $\mathcal{S}$ be the set consisting of all solids of $\mathrm{PG}(6, q)$ that lie in at least $q + 2$ flags of C. We distinguish three cases.

Case 1. We assume that $|\mathcal{E}| \leq s(4)$ and $|\mathcal{S}| \leq s(4)$. In this case we choose a subset C' of C such that every plane and every solid of C' lies in at most $q + 1$ flags of C'. Since every plane and solid lies in at most $q^2 + q + 1$ flags of C, we can find such a subset with $|C'| \geq |C| - (|\mathcal{E}| + |\mathcal{S}|)q^2$ and then $|C| \leq |C'| + 2 \cdot s(4) \cdot q^2$. Now the statement follows by applying Proposition 4.6 to C'.

Case 2. We assume that $|\mathcal{E}| > s(4)$. Lemma 2.2 proves the existence of planes $E_1, E_2 \in \mathcal{E}$ satisfying $\dim(E_1 \cap E_2) \leq 0$. From Lemma 5.1 we know that at most

$$2 \cdot s(1, 4) \cdot s(2) \cdot (s(2) + 1) \tag{8}$$

flags $(E, S) \in C$ satisfy $E \cap E_1 = \varnothing$ or $E \cap E_2 = \varnothing$. It remains to find an upper bound on the number of flags in C whose planes meet both E_1 and E_2. Therefore, we count the number of planes of $\mathrm{PG}(6, q)$ that meet E_1 and E_2. First consider the case that $E_1 \cap E_2$ is a point Q. In this case there are $s(0, 2, 6)$ planes on Q, there are $(s(2) - 1)^2(s(1, 2, 6) - (2 \cdot s(0, 1, 2) - 1))$ planes that do not contain Q and meet both E_1 and E_2 in exactly one point and there are $2 \cdot s(0, -1, 1, 2)(s(2) - 1)$

planes that do not contain Q and meet E_1 or E_2 in a line and the other plane in a point. Thus, in this case the number of planes that meet E_1 and E_2 is equal to

$$n := 2q^8 + 4q^7 + 6q^6 + 4q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1.$$

If E_1 and E_2 are skew than a similar calculation shows that there are even less than n planes that meet E_1 and E_2, so that n is an upper bound for the number of planes that meet E_1 and E_2 in both situations. Since every plane lies in at most $s(2)$ flags of C, it follows that there are at most $n \cdot s(2)$ flags $(E, S) \in C$ such that E meets E_1 and E_2. Together with the count in (8) we find $|C| \le n \cdot s(2) + 2 \cdot s(1, 4) \cdot s(2) \cdot (s(2) + 1)$ and this bound is better than the one in the statement.

Case 3. We assume that $|\mathcal{S}| > s(4)$. This is dual to Case 2. $\square$

6. Proof of the theorem

In this section, Γ denotes the Kneser graph of plane-solid flags in PG(6, q) and C denotes a maximal independent set of Γ.

Lemma 6.1. (i) *Every solid S of* PG(6, q) *has a subspace U with the following property*: *For every plane E of S we have $(E, S) \in C$ if and only if $U \subseteq E$.*

(ii) *For every plane E of* PG(6, q) *there exists a subspace U containing E with the following property*: *For every solid S on E we have $(E, S) \in C$ if and only if $S \subseteq U$.*

Proof. Since the two statements are dual to each other, it suffices to prove the first statement. Thus consider a plane E and let $\mathcal{S}$ be the set of solids S satisfying $E \subseteq S$ and $(E, S) \in C$. In the quotient space PG(6, q)$/E$ the set $\{S/E \mid S \in \mathcal{S}\}$ is a set of points and we have to show that this set is a subspace of PG(6, q)$/E$. In that regard, it is sufficient to show for any two distinct solids $S_1, S_2 \in \mathcal{S}$ and every solid S with $E \subseteq S \subseteq \langle S_1, S_2\rangle$ we have $S \in \mathcal{S}$. Let S be such a solid. If (E', S') is any flag of C then either $S' \cap E \neq \varnothing$ or E' meets $S_1 \setminus E$ and $S_2 \setminus E$. In the second case E' meets $\langle S_1, S_2\rangle$ in a line and hence E' meets S. Thus for every $(E', S') \in C$ we have $E \cap S' \neq \varnothing$ or $E' \cap S \neq \varnothing$. This shows that $C \cup \{(E, S)\}$ is an independent set of Γ and since C is a maximal independent set we have $(E, S) \in C$, that is, $S \in \mathcal{S}$. $\square$

Definition 6.2. A plane E will be called *saturated* (for C) if $(E, S) \in C$ for all solids S of PG(6, q) that contain E. Dually, a solid S will be called *saturated* (for C), if $(E, S) \in C$ for all planes E of S.

Lemma 6.3. (i) *For every saturated solid S and every flag $(E', S') \in C$ we have $E' \cap S \neq \varnothing$.*

(ii) *If S is a solid with $S \cap E' \neq \varnothing$ for all flags (E', C') of C, then S is saturated.*

(iii) *If S and S' are saturated solids, then* $\dim(S \cap S') \geq 1$.

(iv) *Let H be a hyperplane of* $\mathrm{PG}(6, q)$ *and suppose that $E \subseteq H$ for all flags $(E, S) \in C$. Then every solid of H is saturated.*

Proof. (i) Suppose that there is a flag $(E', S') \in C$ with $E' \cap S = \varnothing$. Since $\mathrm{PG}(6, q)$ has dimension 6, it follows $S' \cap S$ is a point P with $P \notin E'$. Let E be a plane of S with $P \notin E$. Then $E \cap S' = \varnothing$. As S is a saturated solid we have $(E, S) \in C$. But then (E, S) and (E', S') are flags of the independent set C with $E \cap S' = \varnothing$ and $E' \cap S = \varnothing$, a contradiction.

(ii) Let E be a plane of S. We have to show that $(E, S) \in C$. Since $S \cap E' \neq \varnothing$ for every flag (E', S) of C, the set $C \cup \{(E, S)\}$ is independent. Maximality of C implies $(E, S) \in C$.

(iii) Assume to the contrary that S and S' only meet in a point P. Choose planes E of S and E' of S' with $P \notin E, E'$. Then $S \cap E' = \varnothing = S' \cap E$. Hence (E, S) and (E', S') are adjacent elements of the Kneser graph Γ. As C is independent, this is a contradiction.

(iv) Let S be a solid of H. The dimension formula shows that $S \cap E \neq \varnothing$ for all planes E of H. Therefore part (ii) shows that S is saturated. $\square$

Lemma 6.4. *Let C be a maximal independent set of* Γ. *If there are more than $c := q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$ saturated solids for C, then $C = \Lambda(H, \mathcal{E})$ for some hyperplane H and some maximal independent set $\mathcal{E}$ of the Kneser graph of planes of H (cf. Example 1.1).*

Proof. Let $\mathcal{S}$ be the set of saturated solids in $\Pi_3(C)$. We have $c > q^6 + 2q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1$ and according to Lemma 6.3(iii) we have $\dim(S_1 \cap S_2) \geq 1$ for all $S_1, S_2 \in \mathcal{S}$. Result 2.3 shows that there exists a hyperplane H containing all saturated solids. If there would be a flag $(E, S) \in C$ such that $E \not\subseteq H$, then according to Lemma 6.3(i) all solids of $\mathcal{S}$ would have nonempty intersection with the line $E \cap H$ and thus

$$|\mathcal{S}| \leq s(3, 5) - s(1, -1, 3, 5) = q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1,$$

which is a contradiction. Hence $E \subseteq H$ for all planes $E \in \Pi_2(C)$. Lemma 6.3(iv) shows that all solids of H are saturated. This means that $\Lambda(H, \varnothing) \subseteq C$. Proposition 3.2 now proves the statement. $\square$

Theorem 6.5. *Suppose that $q \geq 8$ and that C is a maximal independent set in* Γ *with*

$$|C| > 26q^{10} + 83q^9 + 159q^8 + 216q^7 + 222q^6 + 193q^5 + 144q^4 + 97q^3 + 53q^2 + 22q + 5.$$

Then $C = \Lambda(H, \mathcal{E})$ for a hyperplane H and a maximal set $\mathcal{E}$ of mutually intersecting planes of H, or $C = \Lambda(P, \mathcal{S})$ for a point P and a maximal set $\mathcal{S}$ of solids on P any two of which share at least a line.

Proof. The class of examples described in Example 1.1 is closed under duality. In view of Lemma 6.4 we may assume that there exists at most $c := q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$ saturated solids, and, dually, that there are at most c saturated planes. For every saturated plane E choose one hyperplane H_E on E, and for every saturated solid S choose a point P_S of S. Let C' be the subset of C that is obtained from C by removing all flags (E, S) such that E is saturated and S is not contained in H_E and by removing all flags (E, S) such that S is saturated and E does not contain P_S. Then $|C'| \geq |C| - 2cq^3$, that is

$$|C| \leq 2(q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1) \cdot q^3 + |C'|. \tag{9}$$

Lemma 6.1 shows that every plane E which is not saturated for C has the property that the solids S with $(E, S) \in C$ span a proper subspace of PG(6, q). Therefore the construction of C' implies that every plane E has the property that the solids S with $(E, S) \in C'$ span a proper subspace of PG(6, q). Consequently every plane of PG(6, q) lies in at most $q^2 + q + 1$ flags of C'. Dually, every solid S of PG(6, q) lies in at most $q^2 + q + 1$ flags of C'. Therefore Proposition 5.2 proves an upper bound for $|C'|$. Now (9) proves the bound for $|C|$ that is given in the statement. □

Corollary 6.6. *For $q > 27$ the maximal independent set in the Kneser graph of flags of type $\{2, 3\}$ in* PG(6, q) *with $|C| \geq q^{11} + 2q^{10}$ are the independent sets described in Example 1.1.*

Theorem 1.3 follows from this corollary and Proposition 3.2(ii) for $q > 27$ and for $q = 27$ consider Remark 4 on page 41.

7. Bounds on the chromatic number of Γ

Let Γ be the Kneser graph of flags of type $\{2, 3\}$ in PG(6, q). The chromatic number of Γ is the smallest number χ such that the vertex set can be represented as the union of χ independent sets. Using the upper bound α for the size of such an independent set this immediately gives the bound $\chi \geq \frac{n}{\alpha}$. With the upper bound from Theorem 1.3 we find

Proposition 7.1. *For $q \geq 27$, the chromatic number of Γ is at least $q^4 - q^2 + 2q + 1$.*

On the other hand, if V is a subspace of dimension 4 of PG(6, q), then the independent sets $\Lambda(P, \varnothing)$ with $P \in V$ comprise all vertices of Γ, so we have the trivial upper bound $\chi \leq s(4) = q^4 + q^3 + q^2 + q + 1$. We can slightly improve this bound using the following construction.

Proposition 7.2. *The chromatic number χ of Γ satisfies $\chi \leq q^4 + q^3 + q^2 + 1$.*

Proof. Consider a point P, a line l, a plane E and a 4-space V that are mutually incident. Let Q be a point of V that is not in E. Let $l_1, \dots, l_q$ be the lines of plane $\langle l, Q\rangle$ with $P \in l_i$ and $Q \notin l_i$, let $E_1, \dots, E_q$ be the planes of $\langle E, Q\rangle$ with $l \subseteq E_i$ and $Q \notin E_i$, and let $S_1, \dots, S_q$ be the solids of V with $E \subseteq S_i$ and $Q \notin S_i$. For $i \in \{1, \dots, q\}$ put $M_i := l_i \cup (E_i \setminus l) \cup (S_i \setminus E)$. Then $|M_i| = q^3 + q^2 + q + 1$ with $M_i \cap M_j = P$ for distinct $i, j \in \{1, \dots, q\}$ and the union of the sets $M_1, \dots, M_q$ is $\{P\} \cup V \setminus \langle P, Q\rangle$. Let $\{Q_1, \dots, Q_q\} = \langle P, Q\rangle \setminus \{P\}$ and consider the independent set $\Lambda(X, \langle X, Q_i\rangle)$ for $X \in M_i$ and $i \in \{1, \dots, q\}$. Then for $i \in \{1, \dots, q\}$ all lines of V on Q_i occur in one of these sets and every solid that contains Q_i contains a line $\langle X, Q_i\rangle$ with $X \in M_i$. Therefore the union of the sets $\Lambda(X, \langle X, Q_i\rangle)$ for $i \in \{1, \dots, q\}$ covers all vertices of Γ. □

In some situations, having a Hilton-Milner result for the size of the independent sets used (here these are Erdős–Ko–Rado sets in PG$(6, q)$) is a good tool to determine the chromatic number of a graph exactly and with little effort. However, we are convinced that this is not the case in this situation. The reason is, that the second largest independent sets are still almost as large as the largest independent sets, as we have stated in Remark 2 on page 41.

However, one could use the fact that every independent set which is essentially different from the largest examples (that is, different from those given in Example 1.1) is much smaller. Indeed, we have given this some thought, but are convinced that this is not quite simple and would go far beyond the scope of this work.

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