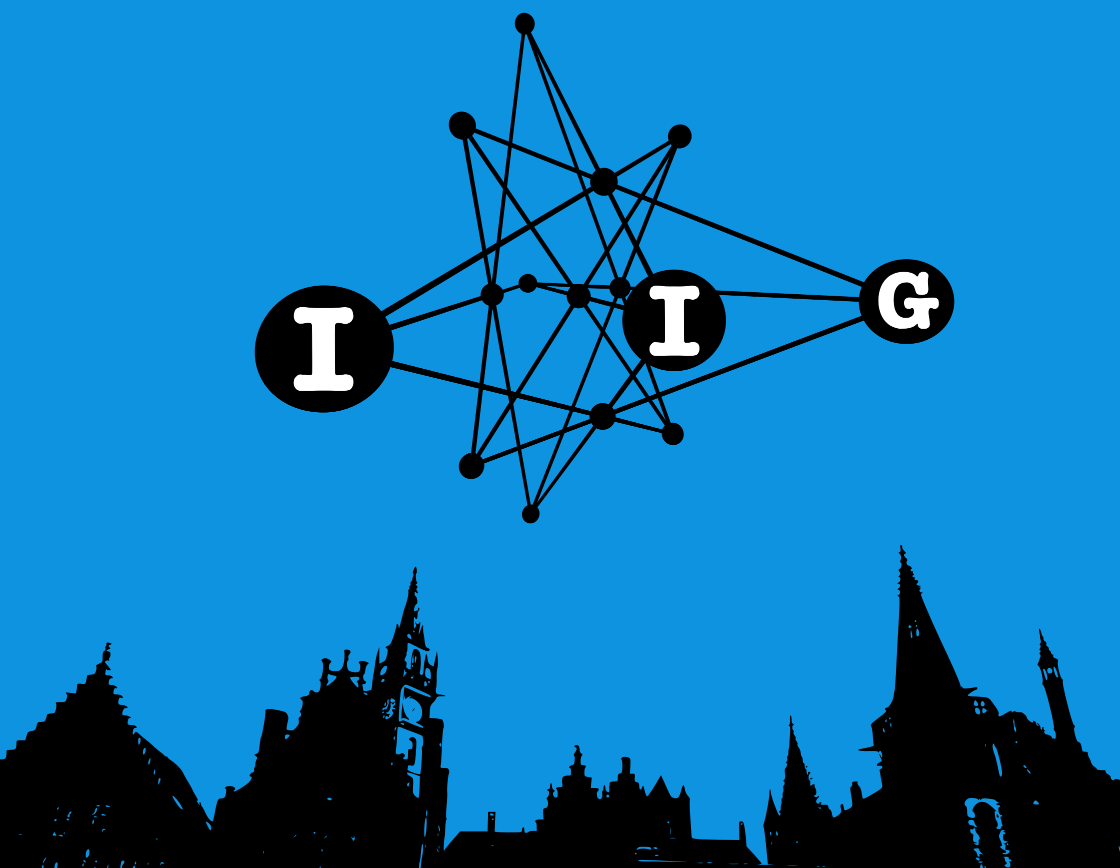
Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial



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We explore the minimal characteristic two parabolic geometries for the finite sporadic simple groups, as introduced by Ronan and Stroth. The chamber graphs of the geometries are studied, with the aid of Magma, focusing on their disc structure and geodesic closures. For the larger sporadic geometries which are beyond computational reach we give bounds on the diameter of their chamber graphs.

1. Introduction

In this paper, with the aid of computer programs [Kelsey and Rowley 2019], we investigate the chamber graphs of the characteristic two minimal parabolic geometries for the finite sporadic simple groups which are listed in [Ronan and Stroth 1984]. The motivation for the Ronan and Stroth catalogue was to obtain geometries which captured certain features seen in the buildings associated with the finite groups of Lie type.

The common thread of these geometries is a generalization of the idea of a minimal parabolic subgroup of a group of Lie type. We briefly review minimal parabolic subgroups, following Ronan and Stroth. Suppose G is a finite group, p a prime and $S \in Syl_p(G)$. Set $B = N_G(S)$. A subgroup P of G which properly contains B with $O_p(P) \neq 1$ and for which B is contained in a unique maximal subgroup of P is called a *minimal parabolic subgroup* of G with respect to B.

Let $P_1, \ldots, P_n$ be minimal parabolic subgroups of G with respect to B. Put $I = \{1, \ldots, n\}$. If $\langle P_i \mid i \in I \rangle = G$ and $\langle P_j \mid j \in J \rangle \neq G$ for all proper subsets J of I, we call $\{P_i \mid i \in I\}$ a *characteristic p minimal parabolic system of G of rank n*.

From now on we suppose $\{P_i \mid i \in I\}$ is a rank n minimal parabolic system. For nonempty $J \subseteq I$, we set $P_J = \langle P_j \mid j \in J \rangle$ and for $J = \varnothing$, $P_J = B$. If for all

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subsets $J, K \subseteq I$ we have

$$P_j \cap P_k = P_{J \cap K}$$

the minimal parabolic system $\{P_i \mid i \in I\}$ is called a *geometric system.*

We shall concentrate here on the case $p = 2$ with systems that are geometric. In fact, it is the chamber graph of these geometries we focus on. Chamber graphs were employed by Tits to give an alternative approach to buildings; see [Ronan 2009; Tits 1981]. They have proved to be a fruitful way of viewing buildings and so it is natural to study the chamber graphs of related geometries.

We recollect the salient features of chamber systems and chamber graphs that we need. Let Γ be the geometry associated with $\{P_i \mid i \in I\}$. In the group theory context, the chambers of the chamber system are $\{Bg \mid g \in G\}$. The chambers are the vertices of the chamber graph $\mathcal{C}(\Gamma)$.

Two (distinct) chambers Bg and Bh of $\mathcal{C}(\Gamma)$ are i-adjacent if $gh^{-1} \in P_i$, and two chambers are adjacent in the chamber graph, $\mathcal{C}(\Gamma)$, if they are i-adjacent for some $i \in I$. Since B is self-normalizing in G, $\mathcal{C}(\Gamma)$ may also be described as having $\{B^g \mid g \in G\}$ as its vertex set with B^g and B^h i-adjacent if $gh^{-1} \in P_i$.

All the chamber systems we consider here will be flag transitive. See [Buekenhout 1995, Chapter 3] for further background on group geometries.

In [Ronan and Stroth 1984] a dictionary of rank 2 subdiagrams is given, resulting in diagrams for these geometries analogous to the Dynkin diagrams of buildings. Usually these diagrams for the sporadic geometries have just one rank 2-subdiagram which is not associated with a crystallographic root system. So in this sense they look very close to buildings. This raises the question as to how chamber graphs of buildings and chamber graphs of the sporadic geometries compare. We recall that all essential properties of a building are encoded in its chamber graph (see [Tits 1981], for example) and so we cannot expect them to be too similar.

For γ a chamber of $\mathcal{C}(\Gamma)$ and $i \in \mathbb{N}$,

$$\Delta_i(\gamma) = \{\gamma' \in \mathcal{C}(\Gamma) \mid d(\gamma, \gamma') = i\},$$

where $d(\ ,\ )$ is the usual distance metric on the chamber graph $\mathcal{C}(\Gamma)$. We refer to $\Delta_i(\gamma)$ as the *i-th disc of γ*. For $\gamma, \gamma' \in \mathcal{C}(\Gamma)$ any path of shortest distance between them in $\mathcal{C}(\Gamma)$ is called a geodesic. The geodesic closure of a set of chambers X is defined to be the set $\overline{X}$ of all chambers lying on some geodesic of γ, γ', for any pair $\gamma, \gamma' \in X$. The graph theoretic structure and size of $\Delta_i(\gamma)$ tells us much about $\mathcal{C}(\Gamma)$. Suppose $d = \operatorname{Diam} \mathcal{C}(\Gamma)$, the diameter of $\mathcal{C}(\Gamma)$, then we call $\Delta_d(\gamma)$ the *last disc of γ*.

Assume that $\gamma \in \mathcal{C}(\Gamma)$ is such that $\operatorname{Stab}_G(\gamma) = B$. If G is a Lie type group and Γ its associated building, then the last disc of γ displays a number of interesting facets of Γ. Firstly, S acts simply transitively on the chambers in the last disc of γ (and so the size of this disc is $|S|$). More importantly if we choose any γ' in the last disc of γ, then the geodesic closure of γ and γ' gives the chambers of an apartment of Γ.

Accordingly, for the minimal parabolic sporadic geometries we investigate here we shall be looking for those with a small number of B-orbits in the last disc, and for these we shall also probe their geodesic closures. The minimal parabolic geometries of M_{12}, M_{24}, J_2, J_3, He, McL and Ru fall into this category.

2. Statement of results

Our first result concerns the diameter of $\mathcal{C}(\Gamma)$.

Theorem 2.1. *The diameter, or bounds for the diameter, of the chamber graphs of the minimal parabolic sporadic geometries are as shown in Table 1.*

In the table, the second column gives the set $\{P_i/O_2(P_i) \mid i \in I\}$, which we refer to as the set of *induced panel residues* of Γ. The third column gives the diameter of $\mathcal{C}(\Gamma)$, and the last gives the number n_{orbits} of B orbits of $\Delta_d(\gamma_0)$.The use of $-$ indicates we have no information.

In Theorem 2.1, M_{23} has two different minimal parabolic geometries whose induced panel residues are the same. They differ in the choice of $2^4 : L_3(2)$ ($=\langle P_1, P_3\rangle$ or $\langle P_3, P_4\rangle$ in [Ronan and Stroth 1984]) in $H = 2^4 : Alt(7)$. One choice leaves a 1-space of $O_2(H)$ invariant and the other a 3-space of $O_2(H)$ invariant. The former is called the 1-geometry and the latter the 3-geometry. Also in Theorem 2.1, to distinguish two of the McL geometries we use the same notation for minimal parabolic subgroups as in [Ronan and Stroth 1984].

Surveying the last column of Theorem 2.1 we see a number of geometries for which the last disc consists of relatively few B-orbits. These geometries certainly warrant further attention — indeed, those of M_{24} and He have been dissected in [Carr and Rowley 2018].

There has been considerable effort expended in collecting geometries, just as in [Ronan and Stroth 1984], which share properties similar to those in buildings. See [Buekenhout 1979a; 1979b; 1995; Kantor 1981; Ronan and Smith 1980; Tits 1980] for an overview of these. The, so-called, GABs which stands for geometries that are almost buildings are among this collection. Perversely, from the point of view of the number of B-orbits in the last disc these geometries are very different from buildings; see [Kelsey and Rowley 2019]. In this sense some of the sporadic geometries in Theorem 2.1 are more like buildings.

group	induced panel residues	$d = \mathrm{Diam}\,\mathcal{C}(\Gamma)$	n_{orbits}
M_{12}	$\{L_2(2),\ L_2(2)\}$	12	1
M_{22}	$\{L_2(2),\ Sym(5)\}$	5	12
M_{23}	$\{L_2(2),\ L_2(2),\ Sym(5)\}$ 1-geometry	7	228
	$\{L_2(2),\ L_2(2),\ Sym(5)\}$ 3-geometry	7	224
M_{24}	$\{L_2(2),\ L_2(2),\ L_2(2)\}$	17	2
J_2	$\{L_2(2),\ L_2(4)\}$	8	2
J_3	$\{L_2(2),\ L_2(4)\}$	14	1
J_4	$\{L_2(2),\ L_2(2),\ Sym(5)\}$	$12 \leq d \leq 75$	–
Co_3	$\{L_2(2),\ L_2(2),\ L_2(2)\}$	$13 \leq d$	–
Co_2	$\{L_2(2),\ L_2(2),\ Sym(5)\}$	15	86
Co_1	$\{L_2(2),\ L_2(2),\ L_2(2),\ L_2(2)\}$	$15 \leq d \leq 48$	–
HS	$\{L_2(2),\ Sym(5)\}$	8	39
He	$\{L_2(2),\ L_2(2),\ L_2(2)\}$	21	1
Ly	$\{L_2(2),\ Sym(9)\}$	$5 \leq d$	–
	$\{L_2(2),\ Sym(5)\}$	$15 \leq d$	–
McL	$\{L_2(2),\ L_2(2),\ L_2(2)\}$	20	4
	$\{L_2(2),\ L_2(2),\ Sym(5)\}$ $\{P_1,\ P_1^{\sigma},\ P_5\}$	11	1596
	$\{L_2(2),\ L_2(2),\ Sym(5)\}$ $\{P_1^{\sigma},\ P_2^{\sigma},\ P_5\}$	10	2042
	$\{L_2(2),\ L_2(2),\ L_2(2),\ L_2(2)\}$	14	881
$O'N$	$\{L_2(2),\ L_3(4).2\}$	$5 \leq d$	–
Ru	$\{L_2(2),\ Sym(5)\}$	12	3
Sz	$\{L_2(2),\ L_2(2),\ L_2(4)\}$	16	57
Fi_{22}	$\{L_2(2),\ L_2(2),\ Sym(5)\}$	$8 \leq d \leq 18$	–
Fi_{23}	$\{L_2(2),\ L_2(2),\ L_2(2),\ Sym(5)\}$	$11 \leq d \leq 32$	–
Fi_{24}'	$\{L_2(2),\ L_2(2),\ L_2(2),\ L_2(2)\}$	$21 \leq d \leq 90$	–
Th	$\{L_2(2),\ Alt(9)\}$	$9 \leq d \leq 11$	–
HN	$\{L_2(2),\ Alt(5) \wr \mathbb{Z}_2\}$	$9 \leq d \leq 11$	–
$\mathbb{B}$	$\{L_2(2),\ L_2(2),\ L_2(2),\ Sym(5)\}$	$17 \leq d \leq 64$	–
$\mathbb{M}$	$\{L_2(2),\ L_2(2),\ L_2(2),\ L_2(2),\ L_2(2)\}$	$42 \leq d \leq 344$	–

Table 1. Information on the the diameter of the chamber graphs of the minimal parabolic sporadic geometries. The second column gives the set $\{P_i/O_2(P_i) \mid i \in I\}$, the third gives the diameter of $\mathcal{C}(\Gamma)$, and the last gives the number n_{orbits} of B orbits of $\Delta_d(\gamma_0)$.The use of – indicates we have no information.

Our second result describes the disc structure of some of the minimal parabolic sporadic geometries.

Theorem 2.2. *Let G denote one of the sporadic simple groups M_{12}, M_{22}, M_{23}, J_2, J_3, Co_2, HS, McL and Ru. Let Γ denote a minimal parabolic geometry associated to one of these groups. Set $\mathcal{C} = \mathcal{C}(\Gamma)$, and let γ_0 be a fixed chamber of $\mathcal{C}$. Put $B = \mathrm{Stab}_G(\gamma_0)$ and let n_{orbits} be the number of B orbits of $\Delta_d(\gamma_0)$.*

(i) *If $G \cong M_{12}$ and Γ has induced panel residues $\{L_2(2), L_2(2)\}$, then $\mathcal{C}$ has* 1485 *chambers*, 44 *B-orbits*, *diameter* 12 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert\Delta_i(\gamma_0)\rvert$	4	8	16	32	64	128	256	384	320	192	64	16
n_{orbits}	2	2	2	2	3	4	6	6	6	6	3	1

(ii) *If $G \cong M_{22}$ and Γ has induced panel residues $\{L_2(2), L_2(2)\}$, then $\mathcal{C}$ has* 3465 *chambers*, 60 *B-orbits*, *diameter* 5 *and this disc structure*:

i-th disc	1	2	3	4	5
$\lvert\Delta_i(\gamma_0)\rvert$	16	56	432	1040	1920
n_{orbits}	4	6	15	17	17

(iii) *If $G \cong M_{23}$ and Γ has induced panel residues $\{L_2(2), L_2(2), Sym(5)\}$, the* 1*-geometry*, *then $\mathcal{C}$ has* 79,695 *chambers*, 835 *B-orbits*, *diameter* 7 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7
$\lvert\Delta_i(\gamma_0)\rvert$	18	92	664	3104	10,728	36,032	29,056
n_{orbits}	5	13	32	81	157	318	228

(iv) *If $G \cong M_{23}$ and Γ has induced panel residues $\{L_2(2), L_2(2), Sym(5)\}$, the* 3*-geometry*, *then $\mathcal{C}$ has* 79,695 *chambers*, 835 *B-orbits*, *diameter* 7 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7
$\lvert\Delta_i(\gamma_0)\rvert$	18	92	664	3104	10,728	36,544	28,544
n_{orbits}	5	13	32	81	157	322	224

(v) *If $G \cong J_2$ and Γ has induced panel residues $\{L_2(2), L_2(4)\}$, then $\mathcal{C}$ has* 1575 *chambers*, 20 *B-orbits*, *diameter* 8 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7	8
$\lvert\Delta_i(\gamma_0)\rvert$	6	16	48	128	384	640	288	64
n_{orbits}	2	2	2	2	3	3	3	2

(vi) *If* $G \cong J_3$ *and* Γ *has induced panel residues* $\{L_2(2), L_2(4)\}$, *then* $\mathcal{C}$ *has* 130,815 *chambers*, 370 B*-orbits*, *diameter* 14 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\lvert\Delta_i(\gamma_0)\rvert$	6	16	48	128	384	1024	3072	7936	20,736	42,240	42,432	10,944	1656	192
n_{orbits}	2	2	2	2	3	4	10	22	55	114	115	30	7	1

(vii) *If* $G \cong Co_3$ *and* Γ *has induced panel residues* $\{L_2(2).L_2(2), L_2(2)\}$, *then* $\mathcal{C}$ *has* 484,147,125 *chambers*, 484,680 B*-orbits and this disc structure as far as* $i = 14$ (note incomplete data here):

i-th disc	1	2	3	4	5	6	7	8	9
$\lvert\Delta_i(\gamma_0)\rvert$	6	24	84	258	792	2344	6976	19,552	53,728
n_{orbits}	3	6	12	20	34	56	100	162	281

i-th disc	10	11	12	13	14
$\lvert\Delta_i(\gamma_0)\rvert$	144,960	382,464	1,006,720	2,567,232	6,494,720
n_{orbits}	512	999	1991	3963	8133

(viii) *If* $G \cong Co_2$ *and* Γ *has induced panel residues* $\{L_2(2), L_2(2), Sym(5)\}$, *then* $\mathcal{C}$ *has* 161,382,375 *chambers*, 2791 B*-orbits*, *diameter* 15 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7	8	9
$\lvert\Delta_i(\gamma_0)\rvert$	18	92	664	3104	11,264	46,912	159,360	5,501,44	1,597,952
n_{orbits}	5	11	28	53	83	139	187	265	303

i-th disc	10	11	12	13	14	15
$\lvert\Delta_i(\gamma_0)\rvert$	4,143,104	11,051,008	27,033,600	47,185,920	47,054,848	22,544,384
n_{orbits}	338	377	365	347	203	86

(ix) *If* $G \cong HS$ *and* Γ *has induced panel residues* $\{L_2(2), Sym(5)\}$, *then* $\mathcal{C}$ *has* 86,625 *chambers*, 270 B*-orbits*, *diameter* 8 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7	8
$\lvert\Delta_i(\gamma_0)\rvert$	16	56	440	1312	7872	17,664	40,448	18816
n_{orbits}	4	6	15	19	47	50	89	39

(x) *If* $G \cong McL$ *and* Γ *has induced panel residues* $\{L_2(2), L_2(2), L_2(2)\}$, *then* $\mathcal{C}$ *has* 7,016,625 *chambers*, 57,866 B*-orbits*, *diameter* 20 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7	8	9	10	11	12
$\|\Delta_i(\gamma_0)\|$	6	20	56	144	376	936	2210	5124	11,656	26,640	60,544	136,032
n_{orbits}	3	5	8	13	24	45	82	135	216	383	714	1408

i-th disc	13	14	15	16	17	18	19	20
$\|\Delta_i(\gamma_0)\|$	284,880	588,800	1,162,272	1,934,416	2,019,280	745,408	37,568	256
n_{orbits}	2638	5033	9432	15,379	16,026	6002	315	4

(xi) *If $G \cong McL$ and Γ has induced panel residues* $\{L_2(2), L_2(2), Sym(5)\}$, $\{P_1, P_1^\sigma, P_5\}$, *then $\mathcal{C}$ has* 7,016,625 *chambers*, 57,866 *B-orbits*, *diameter* 11 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7	8	9	10	11
$\|\Delta_i(\gamma_0)\|$	18	112	770	3964	17400	71440	294760	1078784	2789696	2555840	203840
n_{orbits}	5	16	52	138	358	998	3037	9182	22326	20157	1596

(xii) *If $G \cong McL$ and Γ has induced panel residues* $\{L_2(2), L_2(2), Sym(5)\}$, $\{P_1^\sigma, P_2^\sigma, P_5\}$, *then $\mathcal{C}$ has* 7,016,625 *chambers*, 57,866 *B-orbits*, *diameter* 10 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7	8	9	10
$\|\Delta_i(\gamma_0)\|$	18	116	880	5288	28,062	154,772	711,008	2,560,688	3,296,208	259,584
n_{orbits}	5	16	53	162	518	1814	6418	20769	26068	2042

(xiii) *If $G \cong McL$ and Γ has induced panel residues* $\{L_2(2), L_2(2), L_2(2), L_2(2)\}$, *then $\mathcal{C}$ has* 7,016,625 *chambers*, 57,866 *B-orbits*, *diameter* 14 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7	8	9
$\|\Delta_i(\gamma_0)\|$	8	40	176	704	2384	7936	26,048	79,616	238,720
n_{orbits}	4	11	26	66	134	253	560	1228	2651

i-th disc	10	11	12	13	14
$\|\Delta_i(\gamma_0)\|$	661,632	1,581,184	2,658,560	1,646,848	112768
n_{orbits}	5844	12,564	20,777	12,866	881

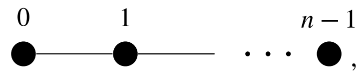
(xiv) *If $G \cong Ru$ and Γ has induced panel residues* $\{L_2(2), Sym(5)\}$, *then $\mathcal{C}$ has* 8,906,625 *chambers*, 847 *B-orbits*, *diameter* 12 *and this disc structure*:

i-th disc	1	2	3	4	5	6	7	8	9	10	11	12
$\|\Delta_i(\gamma_0)\|$	16	56	440	1344	10560	32000	231936	647168	3588096	3997696	385024	12288
n_{orbits}	4	6	11	12	27	33	65	94	304	250	37	3

3. Diameters and geodesic closures

We first give three results concerning the diameter of chamber graphs. For Γ a geometry and $x \in \Gamma$, the residue of x, denoted Γ_x, is the subgeometry consisting of all $y \in \Gamma$ incident with x.

Lemma 3.1. *Suppose that* Γ *is a string geometry with diagram*



where the type 0 *and type* 1 *objects are, respectively, the points and lines of* Γ. *Let* $\mathcal{G}(\Gamma)$ *be the point-line collinearilty graph of* Γ. *Assume that*

(i) $G = Aut(\Gamma)$ *acts flag transitively on* Γ;

(ii) *for x a point of* Γ, *the chamber graph* $\mathcal{C}(\Gamma_x)$ *is connected with* $\operatorname{Diam} \mathcal{C}(\Gamma_x) = e$; *and*,

(iii) $\mathcal{G}(\Gamma)$ *is connected with* $\operatorname{Diam} \mathcal{G}(\Gamma) = f$.

Then

$$\operatorname{Diam} \mathcal{C}(\Gamma) \leq f(1+e).$$

Proof. Let $\gamma_1 = \{x_1, x_2, \ldots, x_n\}$ be a chamber of Γ with $x = x_1$, a point and $\ell = x_2$ a line. Note that x and ℓ are incident. Let y be a point incident with ℓ and $y \neq x$. Since Γ is a string geometry $\gamma_2 = \{y, \ell, x_3, \ldots, x_n\}$ is a chamber of Γ. Moreover, in $\mathcal{C}(\Gamma)$, $d(\gamma_1, \gamma_2) = 1$. Also $\{\ell, x_3, \ldots x_n\}$ is a chamber in Γ_y. Hence for any chamber γ of Γ which contains y, we have $d(\gamma_1, \gamma) \leq 1 + e$. Let γ_0 be a chamber of Γ. Because, by assumption, $\mathcal{G}(\Gamma)$ is connected, a straight forward induction argument shows $d(\gamma_0, \gamma) \leq f(1+e)$ for any chamber γ of Γ. Hence, as G is flag transitive on Γ, we deduce that $\operatorname{Diam} \mathcal{C}(\Gamma) \leq f(1+e)$. □

Lemma 3.2. *Suppose* $\Gamma = \{P_1, \ldots, P_n\}$ *is a minimal parabolic geometry*, *and set* $a_i = [P_i : B]$, *for* $i = 1, \ldots, n$. *Let*

$$a = \sum_{i=1}^{n} (a_i - 1) \quad \textit{and} \quad b = \sum_{i=1}^{n} \big((a_i - 1)(a - (a_i - 1))\big).$$

Then

$$\operatorname{Diam} \mathcal{C}(\Gamma) \geq \left\lceil \log_{a-1} \left(\frac{a-2}{b} \big(|\mathcal{C}(\Gamma)| - (1+a)\big) + 1 \right) \right\rceil + 1.$$

Proof. Let γ be a type i neighbour of γ_0, then γ is i-adjacent to all other type i neighbours of γ_0. And so γ is joined to at least $a_i - 1$ chambers in $\Delta_1(\gamma_0) \cup \{\gamma_0\}$. Hence γ has at most $a - (a_i - 1)$ neighbours in $\Delta_2(\gamma_0)$. There are $(a_i - 1)$ chambers of type i in $\Delta_1(\gamma_0)$, and so there are at most $\sum_{i=1}^{n} \big((a_i - 1)[a - (a_i - 1)]\big)$ chambers in the second disc.

For $i \geq 2$, each chamber in $\Delta_i(\gamma_0)$ has at most $a-1$ neighbours in $\Delta_{i+1}(\gamma_0)$. Consequently the number of chambers in $\Delta_{i+1}(\gamma_0)$ is at most $(a-1)|\Delta_i(\gamma_0)|$. Hence summing across the discs up to and including $\Delta_{k+2}(\gamma_0)$, there are at most $1+a+b+b(a-1)+\cdots+b(a-1)^k$ chambers. Set $d = \operatorname{Diam}\mathcal{C}(\Gamma)$. Then

$$|\mathcal{C}(\Gamma)| \leq 1+a+b+b(a-1)+\cdots+b(a-1)^{d-2} = 1+a+\frac{b\big((a-1)^{d-1}-1\big)}{a-2}$$

and hence

$$(a-1)^{d-1} \geq \frac{a-2}{b}\big(|\mathcal{C}(\Gamma)|-(1+a)\big)+1.$$

Taking log base $a-1$ gives the inequality in the lemma. □

Lemma 3.3. *Suppose* Γ *is a rank* 2 *geometry with point-line collinearity graph* $\mathcal{G}(\Gamma)$. *If* $\operatorname{Diam}\mathcal{G}(\Gamma) = f$, *then* $2f-1 \leq \operatorname{Diam}\mathcal{C}(\Gamma) \leq 2f+1$.

Proof. Given a path $\{x_0, x_1, \ldots, x_\ell\}$ with lines l_{i+1} joining x_i to x_{i+1} for $0 \leq i \leq \ell-1$ in $\mathcal{G}(\Gamma)$, there is a corresponding path in $\mathcal{C}(\Gamma)$ given by

$$\{(x_0, l_1), (x_1, l_2), (x_1, l_2), \ldots, (x_\ell, l_\ell)\}.$$

If the path in $\mathcal{G}(\Gamma)$ is a geodesic then so is the corresponding path in $\mathcal{C}(\Gamma)$, as any shorter path in $\mathcal{C}(\Gamma)$ results in a shorter path in $\mathcal{G}(\Gamma)$.

Hence the longest geodesic in $\mathcal{G}(\Gamma)$ of length f gives rise to a geodesic of length $2f-1$ in $\mathcal{C}(\Gamma)$. If there is a vertex x_{-1} joined to x_0 by l_0 such that $d(x_0, x_f) = d(x_{-1}, x_f)$ then prepending (x_0, l_0) to the induced path in $\mathcal{C}(\Gamma)$ creates a geodesic of length $2f$. The same situation occurring at x_f can result in a geodesic of length $2f+1$. □

Proof of Theorem 1.2. The combined efforts of Magma [Cannon and Playoust 1997], and the code used in [Carr and Rowley 2018] or [Kelsey and Rowley 2019] yield the data on disc structure given in Theorem 2.2. □

Proof of Theorem 1.1. The diameters for the geometries associated with M_{12}, M_{22}, M_{23}, J_2, J_3, Co_2, HS, McL and Ru follow from Theorem 2.2. For the geometries associated with M_{24} and He see [Carr and Rowley 2018] and for Suz see [Kelsey and Rowley 2019]. The bounds for the Th and HN geometries follow from [Rowley and Taylor 2011] and Lemma 3.3. Now let Γ be the characteristic two minimal parabolic geometry for one of the groups J_4, Co_1, Fi_{22}, Fi_{23}, Fi'_{24}, $\mathbb{B}$ and $\mathbb{M}$ given in [Ronan and Stroth 1984]. These are all string geometries. Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for Γ, where we will nominate in each case which objects play the role of points. Set $f = \operatorname{Diam}\mathcal{G}(\Gamma)$ and for x a point of Γ let e denote the diameter of $\mathcal{C}(\Gamma_x)$. We aim to determine, or obtain bounds for, e and f, first looking at Γ for J_4. Call those objects whose stabilizer in J_4 has shape $2^{1+12}3M_{22}2$ and $2^{3+12+2}(Sym(3) \times Sym(5))$ points and lines respectively. Now subgroups H

of J_4 of shape $2^{2+12}2M_{22}2$ have $|Z(H)| = 2$ and are self normalizing (H is in fact a maximal subgroup, see [Conway et al. 1985]). Thus we may identify the points of Γ with the $2A$ conjugacy class of J_4. Let x be a point of Γ and l a line incident with x. Now l is incident with seven points and under this identification they correspond to the seven involutions in the minimal normal subgroup of the stabilizer of l of order 2^3. Since the stabilizer of x is transitive on the lines incident with x and the first disc of the commuting involution graph of $2A$ has size 194106, we conclude that $\mathcal{G}(\Gamma)$ is the same as the commuting involution graph for $2A$. Therefore, by [Bates et al. 2007, Theorem 1.1] $\mathcal{G}(\Gamma)$ has diameter 3. From [Rowley 2010] the diameter of the chamber graph of the $3^{\cdot}M_{22}.2$ geometry is 24. Thus $f = 3$ and $e = 24$ for J_4. Now using [Segev 1988], [Rowley and Walker 1996, 2011; 2012b; 2012a; 2016; 2004a; 2004b] and [Rowley 2019] we have the values for f in the table below. (For Co_1, Fi_{23}, Fi'_{24} and $\mathbb{M}$ we note the given reference deals with the point-line collinearity graph for their maximal parabolic geometries which is the same as that for its minimal parabolic geometries.) The values given for e are obtained from Theorem 2.2 except for $\mathbb{M}$, where $e \leq 3(17+1) = 48$ follows from Lemma 3.1, using the data for Co_1.

Group	e	f	point-stabilizer
J_4	24	3	$2^{1+12}.3^{\cdot}M_{22}.2$
Co_1	17	3	$2^{11}.M_{24}$
Fi_{22}	5	3	$2^{10}.M_{22}$
Fi_{23}	7	4	$2^{11}.M_{23}$
Fi'_{24}	17	5	$2^{11}.M_{24}$
$\mathbb{B}$	15	4	$2^{1+22}.Co_2$
$\mathbb{M}$	≤ 48	≤ 6	$2^{1+24}.Co_1$

Applying Lemma 3.1 yields the bounds for $\mathcal{C}(\Gamma)$ as stated in Theorem 2.1. The given lower bounds for $\operatorname{Diam}\mathcal{C}(\Gamma)$ may be obtained using Lemma 3.2. □

We single out for special attention those chamber graphs having few B-orbits in the last disc.

Theorem 3.4. *Let γ_i be B-orbit representatives for the chambers in the disc γ_0. The geodesic closure of B-orbit representatives of the last disc are given below.*

(i) *If $G \cong M_{12}$ and Γ has induced panel residues $\{L_2(2), L_2(2)\}$, then $\mathcal{C}$ has the following geodesic closure*:

disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert\overline{\{\gamma_0, \gamma_1\}} \cap \Delta_i(\gamma_0)\rvert$	1	4	8	12	16	16	16	16	16	12	8	4	1

(ii) *If* $G \cong J_2$ *and* Γ *has induced panel residues* $\{L_2(2), L_2(4)\}$, *then for* $i = 1, 2$, *the two* B*-orbits have the following geodesic closure data*:

disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8
$\lvert\{\overline{\gamma_0, \gamma_i}\} \cap \Delta_i(\gamma_0)\rvert$	1	5	8	8	8	8	8	5	1

(iii) *If* $G \cong J_3$ *and* Γ *has induced panel residues* $\{L_2(2), L_2(4)\}$, *then* $\mathcal{C}$ *has the following geodesic closure*:

disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert\{\overline{\gamma_0, \gamma_1}\} \cap \Delta_i(\gamma_0)\rvert$	1	6	16	40	52	56	56	56	52	48	40	16	6	1

(iv) *If* $G \cong McL$ *and* Γ *has induced panel residues* $\{L_2(2), L_2(2), L_2(2)\}$, *then, for* $i = 1, 2$, *the four* B*-orbits have the following geodesic closure data*:

disc i *of* $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8	9	10
$\lvert\{\overline{\gamma_0, \gamma_i}\} \cap \Delta_i(\gamma_0)\rvert$	1	5	14	28	32	38	44	46	52	46	48
$\lvert\{\overline{\gamma_0, \gamma_3}\} \cap \Delta_i(\gamma_0)\rvert$	1	5	15	28	34	32	30	32	36	36	32
$\lvert\{\overline{\gamma_0, \gamma_4}\} \cap \Delta_i(\gamma_0)\rvert$	1	5	15	28	32	32	36	38	36	34	32
disc i *of* $\mathcal{C}(\Gamma)$	11	12	13	14	15	16	17	18	19	20	
$\lvert\{\overline{\gamma_0, \gamma_i}\} \cap \Delta_i(\gamma_0)\rvert$	46	52	46	44	38	32	28	14	5	1	
$\lvert\{\overline{\gamma_0, \gamma_3}\} \cap \Delta_i(\gamma_0)\rvert$	34	36	38	36	32	32	28	15	5	1	
$\lvert\{\overline{\gamma_0, \gamma_4}\} \cap \Delta_i(\gamma_0)\rvert$	36	36	32	30	32	34	28	15	5	1	

(v) *If* $G \cong Ru$ *and* Γ *has induced panel residues* $\{L_2(2), Sym(5)\}$, *then for* $i = 1, 2, 3$, *the three* B*-orbits have the following geodesic closure data*:

disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert\{\overline{\gamma_0, \gamma_i}\} \cap \Delta_i(\gamma_0)\rvert$	1	14	40	40	40	40	40	40	40	40	40	14	1

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