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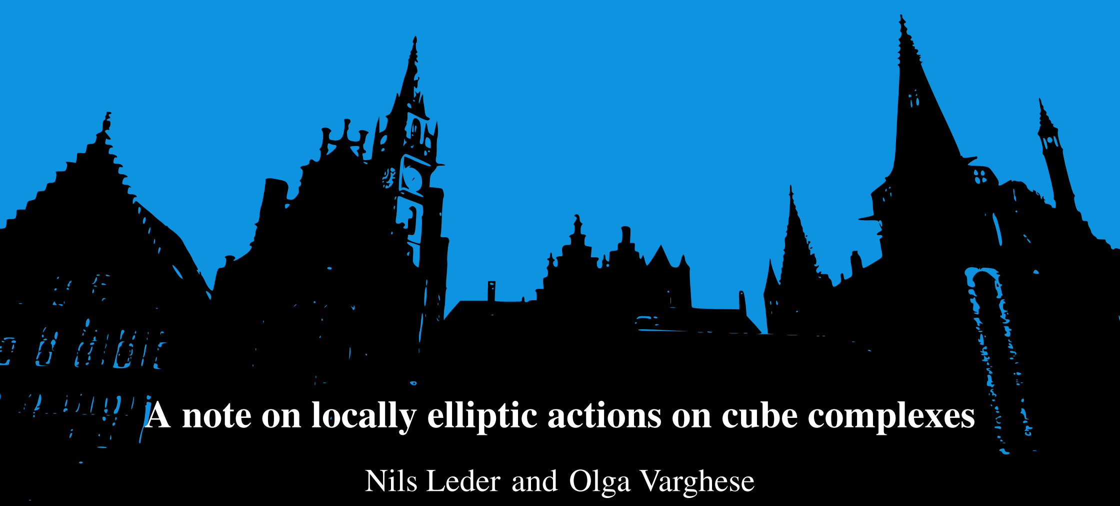

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A note on locally elliptic actions on cube complexes

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We deduce from Sageev's results that whenever a group acts locally elliptically on a finite-dimensional CAT(0) cube complex, then it must fix a point. As an application, we partially prove a conjecture by Marquis concerning actions on buildings and we give an example of a group G such that G does not have property (T), but G and all its finitely generated subgroups can not act without a fixed point on a finite-dimensional CAT(0) cube complex, answering a question by Barnhill and Chatterji.

1. Introduction

The questions we investigate in this note are concerned with fixed points on CAT(0) cube complexes. Roughly speaking, a cube complex is a union of cubes of any dimension which are glued together along isometric faces. Let $\mathcal{C}$ be a class of finite-dimensional CAT(0) cube complexes. A group G is said to have property F$\mathcal{C}$ if any simplicial action of G on any member of $\mathcal{C}$ has a fixed point. For a subclass $\mathcal{A}$ consisting of simplicial trees the study of property F$\mathcal{A}$ was initiated by Serre [1980].

Bass [1976] introduced a weaker property F$\mathcal{A}'$ for groups. A group has property F$\mathcal{A}'$ if any simplicial action of G on any member of $\mathcal{A}$ is locally elliptic, i.e. each $g \in G$ fixes some point on a tree. We define a generalization of property F$\mathcal{A}'$. A group G has property F$\mathcal{C}'$ if any simplicial action of G on any member of $\mathcal{C}$ is locally elliptic, i.e. each $g \in G$ fixes some point on a CAT(0) cube complex.

A finitely generated group which is acting locally elliptically on a simplicial tree has a global fixed point; see [Serre 1980, §6.5, Corollary 2]. The following result of Sageev is well known to the experts. It follows from the proof of Theorem 5.1 in [Sageev 1995].

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Theorem A. *Let G be a finitely generated group acting by simplicial isometries on a finite-dimensional* $\mathrm{CAT}(0)$ *cube complex. If the G-action is locally elliptic, then G has a global fixed point.*

In particular, a finitely generated group G has property $\mathrm{F}\mathcal{C}'$ *if and only if G has property* $\mathrm{F}\mathcal{C}$.

The result of Theorem A was also observed by Caprace and Lytchak in [Chatterji et al. 2016, Proposition B.8] and was proven for median spaces in [Fioravanti 2018, Theorem 3.1].

Before we state the corollaries of Theorem A, we observe that the result in Theorem A is not true for infinite-dimensional $\mathrm{CAT}(0)$ cube complexes. Let G be a finitely generated torsion group. Then, by the Bruhat–Tits fixed point theorem [Bridson and Haefliger 1999, Corollary II 2.8] follows, that G has property $\mathrm{F}\mathcal{C}'$ and thus by Theorem A the group G has property $\mathrm{F}\mathcal{C}$. Free Burnside groups are finitely generated torsion groups and thus these groups have always property $\mathrm{F}\mathcal{C}$, but many of these groups act without a fixed point on infinite-dimensional $\mathrm{CAT}(0)$ cube complexes; see [Osajda 2018, Theorem 1].

The next corollary follows from Theorem A and is known in the case of trees by a result of Tits [1970, Proposition 3.4].

Corollary B. *Let G be a group acting by simplicial isometries on a finite-dimensional* $\mathrm{CAT}(0)$ *cube complex X. If the G-action is locally elliptic, then G has a global fixed point in $X \cup \partial X$, where ∂X denotes the visual boundary of X.*

Proof. For the proof we need the following result by Caprace [2010, Theorem 1.1]:

Let X be a finite-dimensional $\mathrm{CAT}(0)$ cube complex and $\{X_\alpha\}_{\alpha\in A}$ be a filtering family of closed convex nonempty subsets. Then either the intersection $\bigcap_{\alpha\in A} X_\alpha$ is nonempty or the intersection of the visual boundaries $\bigcap_{\alpha\in A} \partial X_\alpha$ is a nonempty subset of ∂X.

Recall that a family $\mathcal{F}$ of subsets of a given set is called *filtering* if for all E, F in $\mathcal{F}$ there exists $D \in \mathcal{F}$ such that $D \subseteq E \cap F$.

Let X be a finite-dimensional $\mathrm{CAT}(0)$ cube complex and Φ a simplicial action of G on X. For $S \subseteq G$ we define the set $\mathrm{Fix}(S) = \{x \in X \mid \Phi(s)(x) = x \text{ for all } s \in S\}$. It is closed and convex. If S is a finite set, it follows by Theorem A that $\mathrm{Fix}(S)$ is nonempty. Further, we define $\mathrm{Fix}(G)^\partial = \{\xi \in \partial X \mid \Phi(g)(\xi) = \xi \text{ for all } g \in G\}$.

Now we consider the following family $\mathcal{F} = \{\mathrm{Fix}(S) \mid S \subseteq G \text{ and } \#S < \infty\}$. If $S, T \subseteq G$ are finite subsets, we have $\mathrm{Fix}(S \cup T) \subseteq \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ and thus $\mathcal{F}$ is a filtering. The result of Caprace stated above implies that

$$\bigcap \mathcal{F} = \mathrm{Fix}(G) \text{ is nonempty}$$

or

$$\bigcap \{\partial \mathrm{Fix}(S) \mid S \subseteq G \text{ and } \#S < \infty\} \subseteq \mathrm{Fix}(G)^\partial \text{ is nonempty.}$$

□

Since the Davis realization of a right-angled building carries the structure of a finite-dimensional CAT(0) cube complex, we can apply Corollary B to confirm the following conjecture by Marquis [2015, Conjecture 2] in the special case of right-angled buildings.

Conjecture. *Let G be a group acting by type-preserving simplicial isometries on a building Δ. If the G-action on the Davis realization X of Δ is locally elliptic, then G has a global fixed point in $X \cup \partial X$.*

Another fixed point property of interest is Kazhdan's property (T). Niblo and Reeves [1997, Theorem B] proved in that if a group G has Kazhdan's property (T), then G also has property F$\mathcal{C}$. Barnhill and Chatterji raised the following question [2008, Question 5.3]:

Question. *Is* F$\mathcal{C}$ *equivalent to* (T), *or does there exist a group G such that G does not have property* (T), *but G and all its finite-index subgroups have property* F$\mathcal{C}$?

With the next result we can answer this question in the negative.

Corollary C. *Let G be the first Grigorchuk group. Then G and all its finitely generated subgroups have property* F$\mathcal{C}$, *but G doesn't have property* (T). *In particular, all finite-index subgroups of G also have property* F$\mathcal{C}$.

Proof. The first Grigorchuk group G is a finitely generated infinite torsion group (see [Grigorchuk 1980]) and thus G and all its finitely generated subgroups have property F$\mathcal{C}$. But G does not have property (T) since G is amenable, see [Grigorchuk 1984]. □

Further, many free Burnside groups have property F$\mathcal{C}$, but don't have property (T), see [Osajda 2018, Theorem 1]. Other examples of groups with property F$\mathcal{C}$ and without property (T) were given by Cornulier in [Cornulier 2015] and by Genevois in [Genevois 2019].

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2. Proof of Theorem A

In this section we give the proof of Theorem A, which is hidden in the proof of Theorem 5.1 in [Sageev 1995] by Sageev. For definitions and properties of CAT(0) cube complexes see [Sageev 1995].

We first need the following result.

Proposition. *Let X be a d-dimensional* CAT(0) *cube complex and $\mathcal{S}$ be a finite set of hyperplanes in X. If $\#\mathcal{S} \geq d + d \cdot (d+1)$, then there exist three hyperplanes in $\mathcal{S}$ that do not intersect pairwise.*

Proof. Let $\mathcal{T} = \{J_1, \ldots, J_k\} \subseteq S$ be a maximal set of pairwise intersecting hyperplanes. Then by Helly's Theorem for CAT(0) cube complexes or [Sageev 1995, Theorem 4.14] follows that $\bigcap \mathcal{T}$ is not empty. Further, since the dimension of X is d we have: $k \leq d$. By maximality of $\mathcal{T}$, for each hyperplane $J \in \mathcal{S} - \mathcal{T}$ there exists $i = 1, \ldots, k$ such that $J \cap J_i = \varnothing$. This yields a well-defined map

$$q : \mathcal{S} - \mathcal{T} \to \{1, \ldots, k\},\ J \mapsto \min\{i \mid J \cap J_i = \varnothing\}.$$

Let B_i denote the preimage $q^{-1}(i)$ for $i = 1, \ldots, k$. Since $\#\mathcal{S} \geq d + d \cdot (d+1)$ and $k \leq d$, we have $\#(\mathcal{S} - \mathcal{T}) \geq d \cdot (d+1)$. Thus, by the pigeon-hole principle there exists $j \in \{1, \ldots, k\}$ such that $\#B_j \geq d+1$. By maximality of $\mathcal{T}$, not all hyperplanes of B_j intersect pairwise, i.e there are $H_1, H_2 \in B_j$ such that $H_1 \cap H_2 = \varnothing$. Then, J_j, H_1, H_2 are three hyperplanes that do not intersect each other. □

Proof of Theorem A. Let G be a finitely generated group with a symmetric generating set $Y = \{g_1, \ldots, g_n\}$. Let X be a d-dimensional CAT(0) cube complex, $v \in X$ be a vertex and $G \to \mathrm{Isom}(X)$ be a simplicial locally elliptic action.

For $i = 1, \ldots, n$ we choose a combinatorial geodesic λ_i from v to $g_i(v)$. Further, we denote by $\mathcal{S}_i$ the set of hyperplanes crossed by λ_i. We have $\#\mathcal{S}_i = D(v, g_i(v))$, where we denote by D the metric on the 1-skeleton of X. Hence the union $\mathcal{S} := \bigcup_{i=1}^n \mathcal{S}_i$ is a finite set.

Let us assume that the action has no global fixed point. Then the Bruhat–Tits fixed point theorem implies that the orbit of v is unbounded. Thus, there exists $g \in G$ such that

$$N := D(v, g(v)) \geq \#\mathcal{S} \cdot (d + d(d+1)).$$

Since Y generates G, we can write $g = g_{i_1} \ldots g_{i_l}$ with $g_{i_j} \in Y$ for $i = 1, \ldots, l$. We define

$$v_j := g_{i_1} \ldots g_{i_j}(v) \text{ and } \gamma_j := g_{i_1} \ldots g_{i_j}(\lambda_{i_{j+1}}).$$

The map γ_j is a combinatorial geodesic from v_j to v_{j+1}. Hence $\alpha := \gamma_l \ldots \gamma_1 \lambda_{g_{i_1}}$ is a combinatorial path from v to $g(v)$. Since $D(v, g(v)) = N$, there exists a set of hyperplanes $\mathcal{T} = \{K_1, \ldots, K_N\}$ such that α crosses each hyperplane in T.

By construction, for each K_i in $\mathcal{T}$ there exists $J \in \mathcal{S}$ such that $K_i = hJ$ for some $h \in G$. By pigeon-hole principle there exists a hyperplane $J \in \mathcal{S}$ such that

$$\#\{K \in \mathcal{T} \mid \exists h \in G : K = hJ\} \geq d + d(d+1).$$

By the Proposition there exist three hyperplanes $h_1 J$, $h_2 J$ and $h_3 J$ in

$$\{K \in \mathcal{T} \mid \exists h \in G : K = hJ\}$$

whose pairwise intersection is empty. But each of these hyperplanes is crossed precisely once by a combinatorial geodesic from v to $g(v)$. Therefore one of these hyperplanes separates the other two.

It is not difficult to verify the following: If there exist a hyperplane $J \subseteq X$ and $g, h \in G$ such that J, gJ, hJ do not intersect pairwise and gJ separates J and hJ, then g, h or hg^{-1} is hyperbolic.

This completes the proof. □

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