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Floer cohomology, multiplicity and the log canonical threshold

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Let f be a polynomial over the complex numbers with an isolated singularity at 0. We show that the multiplicity and the log canonical threshold of f at 0 are invariants of the link of f viewed as a contact submanifold of the sphere.

This is done by first constructing a spectral sequence converging to the fixed-point Floer cohomology of any iterate of the Milnor monodromy map whose E^1 page is explicitly described in terms of a log resolution of f. This spectral sequence is a generalization of a formula by A'Campo. By looking at this spectral sequence, we get a purely Floer-theoretic description of the multiplicity and log canonical threshold of f.

14J17, 32S55, 53D10, 53D40

1 Introduction

Let $f: \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial with an isolated singular point at 0 where $n \geq 1$. Let $S_\epsilon \subset \mathbb{C}^{n+1}$ be the sphere of radius ϵ centered at 0. The *link* of f at 0 is the submanifold $L_f \equiv f^{-1}(0) \cap S_\epsilon \subset S_\epsilon$, where $\epsilon > 0$ is sufficiently small. One can ask

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the following question: What is the relationship between the link of f and various algebraic properties of f? For instance, Zariski in [23] asked whether the multiplicity of f at 0 depends only on the embedding $L_f \subset S_\epsilon$. Another important invariant is the *log canonical threshold* (see Atiyah [3], Musta [30] or Definition 2.1). Again one can ask if this is an invariant of $L_f \subset S_\epsilon$ (see Budur [8, Section 1.6]). We will answer weaker versions of these questions. If $\epsilon > 0$ is small enough, it turns out that L_f is naturally a contact submanifold of S_ϵ (see Varchenko [42]). If $g: \mathbb{C}^{n+1} \to \mathbb{C}$ is another polynomial with isolated singularity at 0 then we say that f and g have *embedded contactomorphic links* if there is a contactomorphism $\Phi: S_\epsilon \to S_\epsilon$ sending L_f to L_g. Varchenko [42] showed that if there is a holomorphic change of coordinates sending f to g then they have embedded contactomorphic links. One of the goals of this paper is to prove the following theorem:

Theorem 1.1 *Suppose that $f, g: \mathbb{C}^{n+1} \to \mathbb{C}$ are two polynomials with isolated singular points at 0 with embedded contactomorphic links. Then the multiplicity and the log canonical threshold of f and g are equal.*

We will prove this theorem by finding formulas for the multiplicity and log canonical threshold in terms of a sequence Floer cohomology groups. The key technical result of this paper proving the above theorem will be a natural generalization of a formula by A'Campo [2].

For all $\epsilon > 0$ small enough, there is a smooth fibration

$$\arg(f): S_\epsilon - f^{-1}(0) \to \mathbb{R}/2\pi\mathbb{Z}, \quad \arg(f)(z) \equiv \arg(f(z)),$$

called the *Milnor fibration* associated to f (see Milnor [29, Chapter 4]). A fiber $M_f \equiv \arg(f)^{-1}(0)$ is called the *Milnor fiber* of f. By choosing an appropriate connection on this fibration, there is a natural compactly supported diffeomorphism $\phi: M_f \to M_f$ given by parallel transporting around the circle $\mathbb{R}/2\pi\mathbb{Z}$, called the *Milnor monodromy map*. The *Lefschetz number* $\Lambda(\phi^m)$ of ϕ^m is defined to be

$$\Lambda(\phi^m) \equiv \sum_{j=0}^{\infty} (-1)^j \operatorname{Tr}(\phi_*^m: H_j(M_f;\mathbb{Z}) \to H_j(M_f;\mathbb{Z})),$$

and this is an invariant of the embedding $L_f \subset S_\epsilon$ for each $m > 0$. A'Campo [2] computed these numbers in the following way. Let $\pi: Y \to \mathbb{C}^{n+1}$ be a log resolution of the pair $(\mathbb{C}^{n+1}, f^{-1}(0))$ at 0. Let $(E_j)_{j \in \check{S}}$ be the prime exceptional divisors of this resolution and define $E_{\star_S}$ to be the proper transform $\overline{\pi^{-1}(f^{-1}(0) - 0)}$ of $f^{-1}(0)$.

Let $S \equiv \check{S} \sqcup \{\star_S\}$. Define $E_j^0 \equiv E_j - \bigcup_{i \in S - \{j\}} E_i$ for all $j \in S$ and define $S_m \equiv \{i \in \check{S} : \operatorname{ord}_f(E_i) \text{ divides } m\}$ for all $m > 0$. A'Campo showed that

$$\Lambda(\phi^m) = \sum_{i \in S_m} \operatorname{ord}_f(E_i)\chi(E_i^o) \quad \text{for all } m > 0. \tag{1-1}$$

The key technical result of this paper (Theorem 1.2) is a spectral sequence converging to a group whose Euler characteristic is naturally equal to the left-hand side of (1-1) multiplied by $(-1)^n$ and such that the Euler characteristic of the E^1 page is naturally equal to the right-hand side multiplied by $(-1)^n$. We will now explain this result.

For $\epsilon > 0$ small enough, the Milnor fiber M_f is naturally a symplectic manifold and ϕ can be made to be a compactly supported symplectomorphism (see Section 3). For any compactly supported symplectomorphism ψ satisfying some additional properties, one can assign a group $\mathrm{HF}^*(\psi, +)$, called the *Floer cohomology* group of ψ (see Seidel [37, Section 4] or Section 4 of this paper). The Euler characteristic of this group is $(-1)^n$ multiplied by the Lefschetz number of ψ (see property (HF1) in Section 4). As a result, we have a sequence of groups $\mathrm{HF}^*(\phi^m, +)$ whose Euler characteristic is $(-1)^n \Lambda(\phi^m)$ for all $m > 0$. All of these groups are invariants of the link of f up to embedded contactomorphism (see Lemma B.17). The log resolution $\pi\colon Y \to \mathbb{C}^{n+1}$ is called a *multiplicity m separating resolution* if $\operatorname{ord}_f(E_i) + \operatorname{ord}_f(E_j) > m$ for all $i, j \in S$ satisfying $i \neq j$ and $E_i \cap E_j \neq \varnothing$.

Theorem 1.2 *Suppose that $\pi\colon Y \to \mathbb{C}^{n+1}$ is a multiplicity m separating resolution for some $m \in \mathbb{N}_{>0}$. Let $(w_i)_{i \in \check{S}}$ be positive integers such that $-\sum_{i \in \check{S}} w_i E_i$ is ample. Let a_i be the discrepancy of E_i (see Definition 2.1) and define $k_i \equiv m/\operatorname{ord}_f(E_i)$ for all $i \in S_m$. Then there is a cohomological spectral sequence converging to $\mathrm{HF}^*(\phi^m, +)$ with E^1 page*

$$E_1^{p,q} = \bigoplus_{\{i \in S_m : k_i w_i = -p\}} H_{n-(p+q)-2k_i(a_i+1)}(\widetilde{E}_i^o; \mathbb{Z}),$$

where $\widetilde{E}_i^o$ is an m_i–fold cover of E_i^o for all $i \in S_m$. The cover $\widetilde{E}_i^o$ is constructed as follows: Let U_i be a neighborhood of E_i^o inside $Y - \bigcup_{j \in S - i} E_i$ which deformation retracts onto E_i, let $\iota_i\colon U_i - E_i^o \to U_i$ be the natural inclusion map and define

$$f_i\colon U_i - E_i^o \to \mathbb{C}^*, \qquad f_i(x) \equiv f(\pi(x)).$$

Then $\widetilde{E}_i^o$ is a disjoint union of connected covers corresponding to a normal subgroup

$$G_i := (\iota_i)_*\big(\ker((f_i)_*)\big) \subset \pi_1(U_i) = \pi_1(E_i^o)$$

and the number of such covers is m_i divided by the index of G_i in $\pi_1(E_i^o)$.

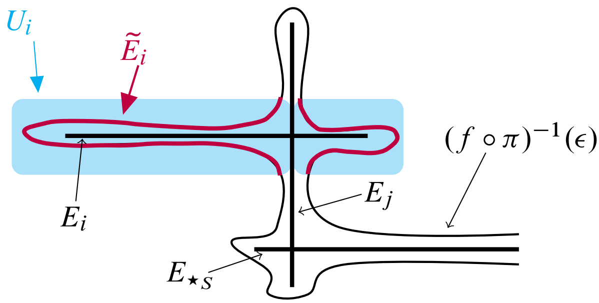

Figure 1

The covers $\widetilde{E}_i^o$ are described in an explicit algebraic way by Denef and Loeser [12, Section 2.3]. Intuitively, we should think of $\widetilde{E}_i^o$ in the following way: if U_i were a "nice" tubular neighborhood of E_i^o and we had a "nice" projection map $Q_i\colon U_i \to E_i^o$ then $Q_i\colon (f\circ\pi)^{-1}(\epsilon)\cap U_i \to E_i^o$ would be a covering map onto its image homotopic to $\widetilde{E}_i^o$ for $\epsilon > 0$ small enough (see the proof of Lemma 5.40). See Figure 1.

By Lemma 2.4 combined with Hironaka [20; 21], a resolution satisfying the properties stated in the above Theorem exists for each $m > 0$. By looking at this spectral sequence above, one gets the following corollary:

Corollary 1.3 *For each $m > 0$, define $\nu_m \equiv \sup\{\alpha : \mathrm{HF}^\alpha(\phi^m,+) \neq 0\}$ and define $\mu_m \equiv \inf\{k_i(a_i+1) : i \in S_m\}$, where k_i and a_i are defined as in Theorem 1.2 above. Then*

$$\nu_m = n - 2\mu_m \quad \text{for all } m > 0.$$

In particular, $\mathrm{HF}^*(\phi^m,+)$ vanishes if and only if $\mu_m = \infty$. Also, the numbers μ_m are invariants of the link up to embedded contactomorphism since the groups $\mathrm{HF}^*(\phi^m,+)$ are.

We will prove this corollary in Section 6. Note that the numbers μ_m have also appeared in Ein, Lazarsfeld and Musta [15, Corollary 2.4]. We have an immediate corollary of Corollary 1.3, proving a conjecture of Seidel [38] regarding the multiplicity of a singularity.

Corollary 1.4 *The multiplicity of f is the smallest $m > 0$ such that $\mathrm{HF}^*(\phi^m,+) \neq 0$. The log canonical threshold of f at 0 is*

$$\mathrm{lct}_0(f) = \liminf_{m\to\infty}\left(\inf\left\{-\frac{\alpha}{2m} : \mathrm{HF}^\alpha(\phi^m,+)\neq 0 \text{ or } -\frac{\alpha}{2m} = 1\right\}\right).$$

Since $\mathrm{HF}^*(\phi^m, +)$ is an invariant of the link of f up to embedded contactomorphism by Lemma B.17, we get that Theorem 1.1 follows immediately from Corollary 1.4.

For each $m > 0$, Denef and Loeser [12] constructed natural spaces $\chi_{m,1}$ whose Euler characteristic is $\Lambda(\phi^m)$. Therefore, it is natural to ask: What is the relationship between these spaces and the groups $\mathrm{HF}^*(\phi^m, +)$, if any? Such a question was considered by Seidel (see [12, Remark 2.7]). It might be interesting to see if there is a similar spectral sequence converging to $H_*(\chi_{m,1}; \mathbb{Z})$ since these spaces admit a natural stratification induced by the strata of the log resolution π. A possible proof would exploit the spectral sequence — see Petersen [33, Formula (3)] — combined with [12, Lemma 2.2] (see also the calculations in the proof of [12, Lemma 2.5]).

1.1 Sketch of the proof of Theorem 1.2

We will now state one of the key properties of the group $\mathrm{HF}^*(\psi, +)$ that will be used in this proof. This property is stated precisely in (HF3) in Section 4 and proven in Appendix C.

Spectral sequence property Suppose that the set of fixed points of ψ is a disjoint union of connected codimension 0 submanifolds $B_1, \dots, B_l$ with boundary and corners and suppose that ψ behaves in a particular way near the boundary of B_i for each i. Then there is a grading $\mathrm{CZ}(B_i) \in \mathbb{Z}$ for each B_i and there is a specific function $\iota\colon \{1, \dots, l\} \to \mathbb{N}$ such that there is a spectral sequence converging to $\mathrm{HF}^*(\psi, +)$ with E^1 page equal to

$$E_1^{p,q} = \bigoplus_{\{i \in \{1,\dots,l\} : \iota(i) = p\}} H_{n-(p+q)-\mathrm{CZ}(\phi, B_i)}(B_i; \mathbb{Z}).$$

The spectral sequence above is an example of a *Morse–Bott spectral sequence* (see Bott [5, Corollary 2] and Hutchings [22, Section 6.4] for other similar examples). Therefore, in order to prove Theorem 1.2 it would be sufficient for us to deform the monodromy symplectomorphism ϕ^m so that the set of fixed points is a union of codimension 0 submanifolds homotopic to $\widetilde{E}^o_{i_p}$ for each $i \in \{1, \dots, l\}$. The problem is that we cannot quite do this, but we can construct a new symplectomorphism with the required fixed-point sets without changing $\mathrm{HF}^*(\phi^m, +)$. Also, Theorem 1.2 really requires a specific ordering of the submanifolds $\widetilde{E}^o_{i_p}$ corresponding to the sequence of positive integers $(w_j)_{j \in S}$, but we will ignore this detail here, as the main applications of this paper do not need such an ordering. We will now explain how to modify ϕ^m without changing $\mathrm{HF}^*(\phi^m, +)$ so that it has this fixed-point property. This is done in Section 5.

We have a natural symplectic form ω_Y on Y that comes from the ample divisor $-\sum_{i\in S} w_i E_i$. This symplectic form gives us a natural Ehresmann connection on $\pi^* f$ away from $(\pi^* f)^{-1}(0)$ and hence gives us a monodromy map. First of all, we deform ω_Y so that it behaves well with respect to $\pi^* f$ (see Sections 5.1 and 5.2). The key idea is that since $\pi^* f$ locally looks like $\prod_{i=1}^m z_i^m$, we can deform ω_Y so that it basically looks like the standard symplectic form in these local charts (with a few modifications). The next step is to show that the corresponding monodromy map ψ satisfies $\mathrm{HF}^*(\psi^m,+) = \mathrm{HF}^*(\phi^m,+)$ (see Sections 5.3 and 5.4 and Appendix A). Here we are using the fact that these Floer cohomology groups are invariants of the mapping tori of ϕ^m and ψ^m, respectively, along with an additional contact structure on these tori and some additional data. Finally we need to compute the fixed points of the monodromy map, so that we can apply our spectral sequence property (see Section 5.5).

Plan of the paper

In Section 2 we construct algebraic invariants of $(\mathbb{C}^{n+1}, f^{-1}(0))$ which will be used to tell us the smallest nonvanishing degree of $\mathrm{HF}^*(\phi^m,+)$ for each m. These invariants are constructed by looking at the multiplicities and discrepancies of the prime exceptional divisors $(E_i)_{i\in S}$ of a resolution.

In Section 3 we give some basic definitions of the main objects in symplectic and contact geometry that will be used in this paper. These include Liouville domains, (abstract) open books, contact mapping cylinders and gradings. In Section 4 we give a definition via Floer cohomology of a symplectomorphism. We also state the three main properties (HF1)–(HF3) of the Floer cohomology group $\mathrm{HF}^*(\psi,+)$ that will be needed for this paper. These properties will be proven in Appendices B and C.

Section 5 is the largest section of the paper. This section is used to construct a monodromy symplectomorphism nice enough that we can use the properties from Section 4 to prove Theorem 1.2. This section heavily relies on results and notation from Tehrani, Mclean and Zinger [40]. Section 6 contains a proof of Theorem 1.2 and Corollary 1.3.

Appendix A deals with gradings. It enables us to compute the quantities $\mathrm{CZ}(B_i)$ stated in the spectral sequence property in the sketch of the proof of Theorem 1.2 earlier. Appendix B proves that the groups $\mathrm{HF}^*(\phi^m,+)$ only depend on the link $L_f \subset S_\epsilon$ as a contact submanifold. This relies heavily on results of McLean [27]. In Appendix C we prove the spectral sequence property of $\mathrm{HF}^*(\psi,+)$ described above.

Conventions

If (M,ω) is a symplectic manifold and θ is a 1–form then its *ω–dual* X^ω_θ is the unique vector field satisfying $\omega(X^\omega_\theta, Y) = \theta(Y)$ for all vectors Y. Sometimes we just write X_θ instead of X^ω_θ if it is clear from the context that the symplectic form we are using is ω. For a smooth function $H\colon M \to \mathbb{R}$, we define $X_H \equiv X_{-dH}$. The time t flow of X_{-dH} will be denoted by $\phi^H_t\colon M \to M$ (this is called the *time t Hamiltonian flow of H*).

Also if $f\colon B' \to B$ is a smooth map and $\pi\colon V \to B$ is a vector bundle then we will write elements of the pullback bundle $f^*(V)$ as pairs $(b', v) \in B' \times V$ satisfying $f(b') = \pi(v)$. For any fiber bundle $\pi\colon E \to B$ and any subsets $N \subset E$ and $C \subset B$, we define $N|_C \equiv N \cap \pi^{-1}(C)$. To avoid cluttered notation, we will not distinguish between an element of a set and a subset of size 1 when the context is clear (eg i will quite often mean $\{i\}$). We also write $\mathrm{Dom}(f)$ and $\mathrm{Im}(f)$ for the domain and image of a map f. For any set I, we define $\mathbb{N}^I_{>0}$ to be the set of tuples $(k_i)_{i\in I}$ where $k_i \in \mathbb{N}_{>0}$.

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2 Multiplicities and discrepancies of exceptional divisors

In this section we will introduce some of the basic tools that are needed from algebraic geometry. We will define the multiplicity and log canonical threshold of an isolated hypersurface singularity as well as some more general invariants. We will also explain how to compute these invariants in terms of certain resolutions, called multiplicity m separating resolutions, and show how such computational techniques do not depend on the choice of resolution.

Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial with an isolated singular point at 0.

Definition 2.1 A *log resolution at* 0 of the pair $(\mathbb{C}^{n+1}, f^{-1}(0))$ is a proper holomorphic map $\pi\colon Y \to \mathbb{C}^{n+1}$ from a complex manifold Y such that there is some open set $U \subset \mathbb{C}^{n+1}$ containing 0 satisfying:

(1) $\pi^{-1}(f^{-1}(0) \cap U)$ is a finite union of smooth transversally intersecting hypersurfaces $(E_i)_{i\in S}$. We will call such divisors *resolution divisors*. Each E_j

satisfying $\pi(E_j) = \{0\}$ is called a *prime exceptional divisor*. We require that the prime exceptional divisors be connected. We also require that there be a unique element $\star_S \in S$ where $E_{\star_S} = \overline{\pi^{-1}(f^{-1}(0) - 0)}$ ($E_{\star_S}$ need not be connected). We call $E_{\star_S}$ the *proper transform of* $f^{-1}(0)$.

(2) $\pi|_{\pi^{-1}(U \setminus \{0\})} \colon \pi^{-1}(U \setminus \{0\}) \to U \setminus \{0\}$ is a biholomorphism.

Since the only singularities in this paper will be at $0 \in \mathbb{C}^{n+1}$, we will just call a log resolution at 0 of f a *log resolution* of $(\mathbb{C}^{n+1}, f^{-1}(0))$.

The *multiplicity of f along E_j*, denoted by $\operatorname{ord}_f(E_j)$, is the order of $\pi^* f$ along E_j. In other words, choose some local coordinate chart $z_1, \dots, z_n$ centered at a generic point of E_j so that $E_j = \{z_1 = 0\}$ and define $\operatorname{ord}_f(E) \equiv k$, where $k \in \mathbb{Z}$ satisfies $\pi^* f = g z_1^k$ in this coordinate system for some holomorphic function g satisfying $g(0) \neq 0$.

The *discrepancy* of E_j, denoted by $a(E_j)$, is calculated as follows: Choose local holomorphic coordinates $y_1, \dots, y_n$ on Y centered at a point on E_j and holomorphic coordinates $x_1, \dots, x_n$ on $\mathbb{C}^{n+1}$ centered at 0. Then $a(E_j)$ is the order of the Jacobian determinant of f along E_j expressed in these coordinates. This quantity does not depend on the choices of such holomorphic coordinates. The *multiplicity* of f at 0 is $\min\{\operatorname{ord}_f(E_j) : j \in S - \star_S\}$ and the *log canonical threshold* is $\min\{(a(E_j)+1)/\operatorname{ord}_f(E_j) : j \in S\}$.

Throughout this paper, we will define E_I to be $\bigcap_{j \in I} E_j$ for each $I \subset S$. If I is the empty set then E_I is the entire manifold Y.

Definition 2.2 Let $\pi \colon Y \to \mathbb{C}^{n+1}$ be a log resolution of $(\mathbb{C}^{n+1}, f^{-1}(0))$ as above. For each $m \in \mathbb{N}_{>0}$, we define the *minimal multiplicity m discrepancy* to be

$$\operatorname{md}_m(\pi, f) \equiv$$
$$\inf\left\{ \sum_{j \in I} k_j a(E_j) : I \subset S,\ I \neq \star_S,\ (k_j)_{j \in I} \in \mathbb{N}^I_{>0},\ E_I \neq \varnothing,\ \sum_{j \in I} k_j \operatorname{ord}_f(E_j) = m \right\}.$$

Our convention here is that infimum of the empty set is ∞. Later on, in Lemma 2.6, we will show that $\operatorname{md}_m(\pi, f)$ does not depend on π and hence we can define $\operatorname{md}_m(f) \equiv \operatorname{md}_m(\pi, f)$ for some choice of log resolution π.

A morphism $\pi \colon Y \to \mathbb{C}^{n+1}$ is a *multiplicity m separating resolution* if it is a log resolution of $(\mathbb{C}^{n+1}, f^{-1}(0))$ such that for any two resolution divisors E and F of π satisfying $E \cap F \neq \varnothing$, the sum of the multiplicities of f along E and F is greater than m.

Multiplicity m separating resolutions make it much easier for us to compute the minimal multiplicity m discrepancy.

Lemma 2.3 *If $\pi\colon Y \to \mathbb{C}^{n+1}$ is a multiplicity m separating resolution of the pair $(\mathbb{C}^{n+1}, f^{-1}(0))$ and $(E_j)_{j\in S}$ are its resolution divisors, then*

$$\mathrm{md}_m(\pi, f) = \inf\{ka(E_j) : k \in \mathbb{N}_{>0},\ j \in S - \star_S,\ k\,\mathrm{ord}_f(E_j) = m\}.$$

Proof This follows from the fact that if $\sum_{j\in I} k_j \,\mathrm{ord}_f(E_j) = m$ and $E_I \neq \varnothing$ for some $I \subset S - \star_S$ and $(k_j)_{j\in I} \in \mathbb{N}^I_{>0}$, then $|I| = 1$. □

Lemma 2.4 *If we have any log resolution then we can blow such a resolution up along strata of $\bigcup_{i\in S} E_i$ inside $\pi^{-1}(0)$ so that it becomes a multiplicity m separating resolution.*

Proof Let $\pi\colon Y \to \mathbb{C}^{n+1}$ be a resolution with resolution divisors $(E_j)_{j\in S}$. Define

$$a_Y \equiv \min\Bigl\{\sum_{j\in I} k_j \,\mathrm{ord}_f(E_j) : I \subset S,\ |I| = 2,\ E_I \neq \varnothing,\ (k_j)_{j\in I} \in \mathbb{N}^I_{>0}\Bigr\}.$$

Let b_Y be the number of elements in the set

$$B_Y \equiv \Bigl\{I \subset S : |I| = 2,\ E_I \neq \varnothing,\ \sum_{j\in I} k_j \,\mathrm{ord}_f(E_j) = a_Y\Bigr\}.$$

Since $b_Y \geq 1$, choose $I \in B_Y$. Let Y' be the blowup of Y along E_I. Then $a_Y - b_Y$ is strictly smaller than $a_{Y'} - b_{Y'}$. Hence, by induction we can blow up Y along subsets of the form E_I until we get a log resolution $\pi''\colon Y'' \to \mathbb{C}$ of $(\mathbb{C}^{n+1}, f^{-1}(0))$ such that $a_{Y''} - b_{Y''} \geq m$. Since $b_{Y''} \geq 1$, we get that $a_{Y''} > m$. Hence, π'' is a multiplicity m separating resolution. □

Lemma 2.5 *Let $\pi\colon Y \to \mathbb{C}^{n+1}$ be a log resolution of $(\mathbb{C}^{n+1}, f^{-1}(0))$ and $I \subset S$ a subset satisfying $|I| \geq 2$. Let $\check{\pi}\colon \check{Y} \to \mathbb{C}^{n+1}$ be the log resolution of $(\mathbb{C}^{n+1}, f^{-1}(0))$ obtained by blowing up Y along E_I. Then*

$$\mathrm{md}_m(\check{\pi}, f) = \mathrm{md}_m(\pi, f).$$

Proof Let $(E_j)_{j\in S}$ be the resolution divisors of π. Let $\check{E}_j$ be the proper transform of E_j in $\check{Y}$ for all $j \in S$. Then

$$a(\check{E}_j) = a(E_j) \quad \text{and} \quad \mathrm{ord}_f(\check{E}_j) = \mathrm{ord}_f(E_j). \tag{2-1}$$

Let E be the exceptional divisor of the blowdown map $\check{Y} \to Y$. Then, by looking at a local model of the blowdown map and using the chain rule we get

$$a(E) = |I| - 1 + \sum_{j \in I} a(E_j), \quad \operatorname{ord}_f(E) = \sum_{j \in I} \operatorname{ord}_f(E_j). \tag{2-2}$$

Suppose, for some $\check{I} \subset S$ satisfying $E_{\check{I}} \neq \varnothing$, some $k \in \mathbb{N}_{\geq 0}$ and $(k_j)_{j \in \check{I}} \in \mathbb{N}_{>0}^{\check{I}}$, we have

$$k \operatorname{ord}_f(E) + \sum_{j \in \check{I}} k_j \operatorname{ord}_f(\check{E}_j) = m.$$

Then, by equations (2-1) and (2-2), we have

$$\begin{aligned} ka(E) + \sum_{j \in \check{I}} k_j a(\check{E}_j) \\ = k(|I| - 1) + \sum_{j \in I - \check{I}} ka(E_j) + \sum_{j \in I \cap I'} (k + k_j) a(E_j) + \sum_{j \in \check{I} - I} k_j a(E_j). \end{aligned} \tag{2-3}$$

Also, equations (2-1) and (2-2) tell us that

$$\begin{aligned} k \operatorname{ord}_f(E) + \sum_{j \in \check{I}} k_j \operatorname{ord}_f(\check{E}_j) \\ = \sum_{j \in I - \check{I}} k \operatorname{ord}_f(E_j) + \sum_{j \in I \cap \check{I}} (k + k_j) \operatorname{ord}_f(E_j) + \sum_{j \in \check{I} - I} k_j \operatorname{ord}_f(E_j). \end{aligned} \tag{2-4}$$

Equations (2-3) and (2-4) tell us that $\operatorname{md}_m(\check{\pi}, f) = \operatorname{md}_m(\pi, f)$. □

Lemma 2.6 *The minimal multiplicity m discrepancy does not depend on the choice of log resolution $\pi\colon Y \to \mathbb{C}^{n+1}$ of $(\mathbb{C}^{n+1}, f^{-1}(0))$.*

Proof Let $\pi\colon Y \to \mathbb{C}^{n+1}$ and $\check{\pi}\colon \check{Y} \to \mathbb{C}^{n+1}$ be two such resolutions. Lemma 2.5 tells us that blowing up along strata does not change the minimal multiplicity m discrepancy. Hence, by Lemma 2.4, we can assume that π and $\check{\pi}$ are multiplicity m separating resolutions.

Since π and $\check{\pi}$ are birational morphisms, we have that there is a birational morphism $\Phi\colon Y \dashrightarrow \check{Y}$ such that $\pi = \check{\pi} \circ \Phi$. Let $(E_j)_{j \in S}$ be the resolution divisors of π and $(\check{E}_j)_{j \in \check{S}}$ the resolution divisors of $\check{\pi}$. Suppose that $\operatorname{ord}_f(E_j)$ divides m for some $j \in S - \star_S$. Let $\check{I} \subset \check{S}$ be the largest subset satisfying $\Phi(E_j) \subset \check{E}_{\check{I}}$.

Since Φ is well defined outside a subvariety of codimension ≥ 2, we have a point $p \in E_j$ and holomorphic charts $y_1, \dots, y_n$ in Y and $x_1, \dots, x_n$ in $\check{Y}$ centered at p and $\Phi(p)$, respectively, where Φ is well defined. We will also assume that $E_j = \{y_1 = 0\}$ and $\check{E}_k = \{x_{\alpha(k)} = 0\}$ for all $k \in \check{I}$ and some $\alpha\colon \check{I} \to \{1, \dots, n\}$. Let J be the Jacobian of $\pi(y_1, \dots, y_n)$, $\check{J}$ the Jacobian of $\check{\pi}(x_1, \dots, x_n)$ and J_Φ the Jacobian of $\Phi(y_1, \dots, y_n)$. Then

$$\operatorname{ord}_f(E_j) = \sum_{k \in \check{I}} \operatorname{ord}_{x_{\alpha(k)} \circ \Phi}(E_k) \operatorname{ord}_f(\check{E}_k) \tag{2-5}$$

and

$$\operatorname{ord}_J(E_j) = \operatorname{ord}_{J_\Phi}(E_j) + \sum_{k \in \check{I}} \operatorname{ord}_{x_{\alpha(k)} \circ \Phi}(E_k) \operatorname{ord}_{\check{J}}(\check{E}_k).$$

By (2-5) combined with the fact that $\check{\pi}$ is a multiplicity m separating resolution, $\check{I} = \{k\}$ for some $k \in \check{S} - \star_{\check{S}}$. Hence, $\operatorname{ord}_f(E_j) = \kappa \operatorname{ord}_f(\check{E}_k)$ and $a(E_j) \geq \kappa a(\check{E}_k)$, where $\kappa = \operatorname{ord}_{x_{\alpha(k)} \circ \Phi}(E_j)$. Therefore, by Lemma 2.3, $\operatorname{md}_m(\pi, f) \geq \operatorname{md}_m(\check{\pi}, f)$. Similarly, $\operatorname{md}_m(\check{\pi}, f) \geq \operatorname{md}_m(\pi, f)$ and hence $\operatorname{md}_m(\pi, f) = \operatorname{md}_m(\check{\pi}, f)$. □

3 Liouville domains, symplectomorphisms and open books

In this section we give basic definitions of Liouville domains and graded symplectomorphisms and open books. We will also explain the correspondence between open book decompositions and graded symplectomorphisms of Liouville domains. All of the material here is contained in [17], with the exception of gradings, which is contained in [36]. For more details on open book decompositions see [13].

Definition 3.1 An *exact symplectic manifold* is a pair (M, θ_M) where M is a manifold and θ_M is a 1–form such that $\omega_M \equiv d\theta_M$ is symplectic. A *Liouville domain* is an exact symplectic manifold (M, θ_M) where M is a compact manifold with boundary and the ω_M–dual X_{θ_M} of θ_M points outwards along ∂M. The 1–form θ_M is called the *Liouville form*. The *contact boundary* of M is the pair $(\partial M, \alpha_M)$ where $\alpha_M \equiv \theta_M|_{\partial M}$. Here α_M is a contact form. Since X_{θ_M} points outwards along ∂M, we get that the backwards flow

$$(\phi_t\colon M \hookrightarrow M)_{t \in (-\infty, 0]}$$

of X_{θ_M} exists for all time t. By considering the smooth embeddings $\phi_{\ln(r_M)}|_{\partial M}$ for $r_M \in (0, 1]$, we can construct a *standard collar neighborhood* $(0, 1] \times \partial M \subset M$ of ∂M where

$$\theta_M|_{(0,1] \times \partial M} = r_M \alpha_M.$$

Here r_M is the coordinate given by the natural projection $r_M\colon (0,1]\times\partial M \twoheadrightarrow (0,1]$ and is called the *cylindrical coordinate* on M.

An *exact symplectomorphism* $\phi\colon M\to M$ is a diffeomorphism such that $\phi^*\theta_M = \theta_M + dF_\phi$ for some smooth function $F_\phi\colon M\to\mathbb{R}$. Technically we want to think of this as a pair (ϕ, F_ϕ), but we will suppress F_ϕ from the notation and just write ϕ. The *support* of such an exact symplectomorphism is the region

$$\{x\in M : \phi(x)\neq x \text{ or } dF_\phi(x)\neq 0\}.$$

We now need to define graded symplectomorphisms as in [36]. This is needed so that we can define their Floer cohomology groups in the next section.

Definition 3.2 We define $(\mathbb{R}^{2n},\Omega_{\mathrm{std}})$ to be the standard symplectic vector space. Let $\mathrm{Sp}(2n)$ be the space of linear symplectomorphisms of $(\mathbb{R}^{2n},\Omega_{\mathrm{std}})$ and $\widetilde{\mathrm{Sp}}(2n)$ its universal cover. Let $\pi\colon E\to V$ be a symplectic vector bundle with symplectic form Ω_E whose fibers have dimension $2n$. Sometimes we will write (E,Ω_E) or just E for such a symplectic vector bundle when the context is clear. Define the *symplectic frame bundle* $\mathrm{Fr}(E)$ to be an $\mathrm{Sp}(2n)$ bundle whose fiber at $x\in V$ is the space of linear symplectomorphisms from $(\mathbb{R}^{2n},\Omega_{\mathrm{std}})$ to $(\pi^{-1}(x),\Omega_E|_x)$. A *grading* on a symplectic vector bundle $\pi\colon E\to V$ is an $\widetilde{\mathrm{Sp}}(2n)$ bundle $\widetilde{\mathrm{Fr}}(E)\to V$ together with a choice of isomorphism of $\widetilde{\mathrm{Sp}}(2n)$ bundles

$$\iota\colon \widetilde{\mathrm{Fr}}(E)\times_{\widetilde{\mathrm{Sp}}(2n)}\mathrm{Sp}(2n)\cong \mathrm{Fr}(E). \tag{3-1}$$

This is just a choice of reduction of the structure group of E from $\mathrm{Sp}(2n)$ to $\widetilde{\mathrm{Sp}}(2n)$. A symplectic vector bundle with a choice of grading is called a *graded symplectic vector bundle*. Suppose that $\check\pi\colon \check E\to\check V$ is a symplectic vector bundle and $\widetilde\beta\colon\check E\to E$ is a bundle morphism covering a smooth map $\beta\colon\check V\to V$ such that $\widetilde\beta$ restricted to each fiber is a linear symplectomorphism. Let

$$\widehat\beta\colon \check E\to\beta^*(E),\qquad \widehat\beta(\check e)=(\check\pi(\check e),\widetilde\beta(\check e)),$$

be the natural isomorphism between this bundle and the pullback bundle. Then $\check E$ has a natural grading

$$\beta^*(\widetilde{\mathrm{Fr}}(E))\to\check V,$$
$$\check\iota\colon \beta^*(\widetilde{\mathrm{Fr}}(E))\times_{\widetilde{\mathrm{Sp}}(2n)}\mathrm{Sp}(2n)=\beta^*(\widetilde{\mathrm{Fr}}(E)\times_{\widetilde{\mathrm{Sp}}(2n)}\mathrm{Sp}(2n))\to\mathrm{Fr}(\check E),$$
$$\check\iota(\check v,w)\equiv\widehat\beta^{-1}\big((\check v,\iota(w))\big)\quad\text{for all } \check v\in\check V,\ w\in\widetilde{\mathrm{Fr}}(E)\times_{\widetilde{\mathrm{Sp}}(2n)}\mathrm{Sp}(2n),$$

called the *grading on E pulled back by $\widetilde\beta$*.

The simplest example of a grading is the natural grading on the trivial bundle $\check{V}\times\mathbb{C}^k\to\check{V}$ given by

$$\widetilde{\mathrm{Fr}}(\check{V}\times\mathbb{C}^k)\equiv\check{V}\times\widetilde{\mathrm{Sp}}(2n),$$

$$\iota\colon \widetilde{\mathrm{Fr}}(\check{V}\times\mathbb{C}^k)\times_{\widetilde{\mathrm{Sp}}(2n)}\mathrm{Sp}(2n)=\check{V}\times(\widetilde{\mathrm{Sp}}(2n)\times_{\widetilde{\mathrm{Sp}}(2n)}\mathrm{Sp}(2n))=\check{V}\times\mathrm{Sp}(2n)\\ \to\mathrm{Fr}(\check{V}\times\mathbb{C}^k).$$

We will call such a grading the *trivial grading*.

Remark 3.3 In this paper we are only interested in gradings up to isotopy and so we will sometimes regard isotopic gradings as the same grading. If we have a smooth connected family of symplectic vector bundles then, if one of them has a grading, then all of them have a natural choice of grading up to isotopy. For more information about this see Appendix A.

Definition 3.4 Suppose that

$$\pi_1\colon E_1\to V_1,\quad \pi_2\colon E_2\to V_2$$

are graded symplectic vector bundles and $F\colon E_1\to E_2$ is a symplectic vector bundle isomorphism covering a diffeomorphism $V_1\to V_2$. Then a *grading* on F is an $\widetilde{\mathrm{Sp}}(2n)$ bundle isomorphism

$$\widetilde{F}\colon\widetilde{\mathrm{Fr}}(E_1)\to\widetilde{\mathrm{Fr}}(E_2)$$

covering the $\mathrm{Sp}(2n)$ bundle morphism $\mathrm{Fr}(E_1)\to\mathrm{Fr}(E_2)$ induced by F.

Let (X,ω_X) be a $2n$–dimensional symplectic manifold. A *grading* on (X,ω) is a choice of grading on the symplectic vector bundle TX. A symplectic manifold with a choice of grading is called a *graded symplectic manifold*

A *grading* of a symplectomorphism ϕ between two graded symplectic manifolds (X_1,ω_{X_1}) and (X_2,ω_{X_2}) is a choice of grading for the symplectic bundle isomorphism $d\phi\colon TX_1\to TX_2$. A *graded symplectomorphism* is a symplectomorphism with a choice of grading.

We also wish to have a notion of grading for contact manifolds.

Definition 3.5 Recall that a *cooriented contact manifold* (C,ξ_C) is a manifold C of dimension $2n-1$ with a cooriented hyperplane distribution ξ_C with the property that there is a 1–form α_C whose kernel is ξ_C respecting this coorientation and such that $\alpha_C\wedge(d\alpha_C)^{n-1}$ is a volume form on C. The 1–form α_C is called a *contact* *form compatible with* ξ_C. A *coorientation-preserving contactomorphism* between two cooriented contact manifolds is a diffeomorphism preserving their respective hyperplane distributions and coorientations. A *contact submanifold* $B \subset C$ is a submanifold such that $(B, \xi_B \equiv \xi_C \cap TB)$ is a cooriented contact manifold with the induced coorientation from ξ_C.

A *grading* on a contact manifold (C, ξ_C) consists of a choice of contact form α_C compatible with ξ_C and a choice of grading on the symplectic vector bundle $(\xi_C, d\alpha_C|_{\xi_C})$. A cooriented contact manifold with a choice of grading is called a *graded contact manifold.* Since the space of contact forms compatible with ξ_C is contractible, we get an induced grading on $(\xi_C, d\alpha|_{\xi_C})$ for any other contact form α compatible with ξ_C, which is unique up to isotopy. Hence, from now on we will regard this as the same grading.

A *grading* of a coorientation-preserving contactomorphism ϕ between two graded contact manifolds (C_1, ξ_1) and (C_2, ξ_2) consists of a grading on the symplectic vector bundle isomorphism $d\phi|_{\xi_{C_1}}\colon \xi_{C_1} \to \xi_{C_2}$ where the symplectic forms on ξ_{C_1} and ξ_{C_2} come from contact forms α_{C_1} and α_{C_2} compatible with ξ_{C_1} and ξ_{C_2}, respectively, satisfying $\alpha_{C_1} = \phi^*\alpha_{C_2}$ with induced gradings. A *graded contactomorphism* is a coorientation-preserving contactomorphism with a choice of grading.

In this paper, we will only deal with cooriented contact manifolds and coorientation-preserving contactomorphisms. So, from now on, a *contact manifold* is a cooriented contact manifold and a *contactomorphism* is a coorientation-preserving contactomorphism.

Definition 3.6 Suppose that (C, ξ_C) is a contact manifold and B is a contact submanifold. The *normal bundle of* B is a symplectic vector bundle

$$\pi_{\mathcal{N}_C B}\colon \mathcal{N}_C B \equiv (TC|_B)/TB \twoheadrightarrow B$$

with symplectic form defined as follows: Choose a compatible contact form α_C on (C, ξ_C) and define

$$T^\perp B \equiv \{v \in \xi_C|_x : x \in B,\ d\alpha_C(v, w) = 0 \text{ for all } w \in \xi_C|_x \cap TB|_x\}.$$

Then $T^\perp B$ is a symplectic vector bundle with symplectic form $d\alpha_C|_{T^\perp B}$. The symplectic structure on $\mathcal{N}_C B$ is the pushforward of the above symplectic form under the natural bundle isomorphism $T^\perp B \to \mathcal{N}_C B$.

Since the space of compatible contact forms is contractible, the choice of symplectic form on the $\mathcal{N}_C B$ is unique up to symplectic bundle isomorphism and any two choices of such isomorphism are homotopic. As a result, we will refer to this bundle as *the* normal bundle as we are only concerned with isomorphisms of such bundles up to homotopy.

Definition 3.7 A *contact pair with normal bundle data* $(B \subset C, \xi_C, \Phi_B)$ consists of a contact manifold (C, ξ_C) where B is a codimension 2 contact submanifold along with a symplectic trivialization

$$\Phi_B\colon \mathcal{N}_C B \to B \times \mathbb{C}$$

of its normal bundle. A *contactomorphism* between two such triples

$$(B_1 \subset C_1, \xi_{C_1}, \Phi_{B_1}), \quad (B_2 \subset C_2, \xi_{C_2}, \Phi_{B_2}) \tag{3-2}$$

is a contactomorphism $\Psi\colon C_1 \to C_2$ sending B_1 to B_2 such that the composition

$$\mathcal{N}_{C_1} B_1 \xrightarrow{d\Psi|_{B_1}} \mathcal{N}_{C_2} B_2 \xrightarrow{\Phi_{B_2}} B_2 \times \mathbb{C} \xrightarrow{(\Psi|_{B_1})^{-1} \times \mathrm{id}_{\mathbb{C}}} B_1 \times \mathbb{C}$$

is homotopic through symplectic bundle trivializations to Φ_{B_1}. If there exists such a contactomorphism between the pairs as in (3-2) then we say that they are *contactomorphic*.

A contact pair with normal bundle data $(B \subset C, \xi_C, \Phi_B)$ is *graded* if there is a choice of grading on $C - B$. A *graded contactomorphism* between two graded contact pairs with normal bundle data as in (3-2) consists of a contactomorphism Ψ between these contact pairs and a choice of grading of the contactomorphism $\Psi|_{C_1 - B_1}\colon C_1 - B_1 \to C_2 - B_2$. If a graded contactomorphism exists between two graded contact pairs with normal bundle data then we say that they are *graded contactomorphic*.

The main example of a contact pair with normal bundle data comes from singularity theory.

Example 3.8 Fix $n > 0$. Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function with an isolated singularity at 0 and let $J_0\colon T\mathbb{C}^{n+1} \to T\mathbb{C}^{n+1}$ be the standard complex structure on $\mathbb{C}^{n+1}$. Let $S_\epsilon \subset \mathbb{C}^{n+1}$ be the sphere of radius $\epsilon > 0$ and let $\xi_{S_\epsilon} = TS_\epsilon \cap J_0 TS_\epsilon$ be the standard contact structure on S_ϵ. Define $L_f \equiv f^{-1}(0) \cap S_\epsilon$. Then a result by Varchenko [42] tells us that for all sufficiently small $\epsilon > 0$, $L_f \subset S_\epsilon$

is a contact submanifold, called the *link* of f at 0. Also, df gives us an induced map $\overline{df}\colon \mathcal{N}_{S_\epsilon} L_f \to \mathbb{C}$ and hence a trivialization

$$\Phi_f \equiv (\mathrm{id}_{L_f}, \overline{df})\colon \mathcal{N}_{S_\epsilon} L_f \to L_f \times \mathbb{C}.$$

The contact pair with normal bundle data $(L_f \subset S_\epsilon, \xi_{S_\epsilon}, \Phi_f)$ is called the *contact pair associated to* f.

We also need a grading on this contact pair. It turns out, by the discussion in Definition A.7, that every contact manifold with trivial first and second homology groups has a unique grading up to homotopy. This means that ξ_{S_ϵ} has a grading, giving us an induced grading on the contact pair associated to f. We will call this the *standard grading*.

We will now define open book decompositions and also graded open book decompositions. Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk with polar coordinates (r, ϑ).

Definition 3.9 An *open book* is a pair (C, π) where

- C is a smooth manifold,
- $\pi\colon C - B \to \mathbb{R}/\mathbb{Z}$ is a smooth fibration where B is a codimension 2 submanifold, and
- there is a tubular neighborhood $B \times \mathbb{D} \subset C$ of $B = B \times \{0\}$ in C such that π satisfies

$$\pi|_{B \times (\mathbb{D} - 0)}\colon B \times (\mathbb{D} - 0) \to \mathbb{R}/\mathbb{Z}, \quad \pi(x, (r, \theta)) = 2\pi\theta.$$

The submanifold B is called the *binding* of the open book and a *page* of the open book is the closure of a fiber of π which is a submanifold with boundary equal to B.

We now want open books to be compatible with contact pairs.

Definition 3.10 A contact pair with normal bundle data $(B \subset C, \xi_C, \Phi_B)$ is *supported by an open book* (C, π) if

(1) there is a contact form α_C compatible with ξ_C such that $d\alpha_C|_{\pi^{-1}(t)}$ is a symplectic form for all $t \in \mathbb{R}/\mathbb{Z}$,

(2) the trivialization of $\mathcal{N}_C B$ induced by the choice of tubular neighborhood from Definition 3.9 is homotopic through orientation-preserving bundle trivializations to Φ_B (this does not depend on the choice of such a tubular neighborhood).

We will write (C, ξ_C, π) for a contact pair supported by an open book and we will call it a *contact open book*. Note that B and Φ_B are not included in the notation as $B = C - \mathrm{Dom}(\pi)$ and Φ_B is determined by the open book due to the fact that the space of orientation-preserving trivializations of $\mathcal{N}_C B$ is weakly homotopic to the space of symplectic trivializations of $\mathcal{N}_C B$. The contact pair $(B \subset C, \xi_C, \Phi_B)$ is called the *contact pair associated to* (C, ξ_C, π). If the contact pair associated to (C, ξ_C, π) is graded then we call this a *graded contact open book.*

An *isotopy* between two contact open books (C_1, ξ_{C_1}, π_1) and (C_2, ξ_{C_2}, π_2) is a contactomorphism $\Phi\colon C_1 \to C_2$ between the respective contact pairs with normal bundle data together with a smooth family of maps $(\check{\pi}_t\colon \mathrm{Dom}(\pi_1) \to \mathbb{R}/\mathbb{Z})_{t\in[1,2]}$ joining π_1 and $\pi_2 \circ \Phi$ such that $(C_1, \xi_{C_1}, \check{\pi}_t)$ is a contact open book for all $t \in [1,2]$. Such an isotopy is *graded* if we have a smooth family of graded contact open books and Φ is a graded contactomorphism.

The main example of a contact open book will come from singularity theory.

Example 3.11 Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$, $n > 1$ be a holomorphic function with an isolated singularity at 0 and let $(L_f \subset S_\epsilon, \xi_{S_\epsilon}, \Phi_f)$ be the contact pair associated to f as in Example 3.8. Let $\arg(f)\colon \mathbb{C}^{n+1} - f^{-1}(0) \to \mathbb{R}/2\pi\mathbb{Z}$ be the argument of f. Then, by [9, Proposition 3.4], we have that $\left(S_\epsilon, \xi_{S_\epsilon}, \frac{1}{2\pi}\arg(f)|_{S_\epsilon}\right)$ is a contact open book for all $\epsilon > 0$ small enough. This open book is supported by the contact pair associated to f from Example 3.8 and it has a grading coming from the standard grading. This is a graded contact open book, called the *Milnor open book of* f.

Definition 3.12 An *abstract contact open book* consists of a triple (M, θ_M, ϕ) where (M, θ_M) is a Liouville domain and ϕ is an exact symplectomorphism with support in the interior of M. A *graded abstract contact open book* is an abstract contact open book (M, θ_M, ϕ) with a choice of grading on $(M, d\theta_M)$ and a graded symplectomorphism ϕ.

An *isotopy* between abstract contact open books $(M_1, \theta_{M_1}, \phi_1)$ and $(M_2, \theta_{M_2}, \phi_2)$ consists of a diffeomorphism $\Phi\colon M_1 \to M_2$, a smooth family of 1–forms $(\theta_t)_{t\in[0,1]}$ joining θ_{M_1} and $\Phi^*\theta_{M_2}$ and a smooth family of diffeomorphisms $(\check{\phi}_t)_{t\in[1,2]}$ joining ϕ_1 and $\Phi^{-1} \circ \phi_2 \circ \Phi$ such that $(M_1, \theta_t, \check{\phi}_t)$ are all abstract contact open books and the support of $\check{\phi}_t$ is contained inside a fixed compact subset of the interior of M. If both of our abstract contact open books are graded then such an isotopy is a *graded isotopy* if all the abstract contact open books $(M_1, \theta_t, \check{\phi}_t)$ are graded so that these gradings smoothly depend on $t \in [0,1]$ and the grading on $(M_0, \theta_0, \check{\phi}_0)$ coincides with the grading on $(M_1, \theta_{M_1}, \phi_1)$ and the grading on $(M_1, \theta_1, \check{\phi}_1)$ coincides with the grading on $(M_2, \theta_{M_2}, \phi_2)$ pulled back by Φ.

From an abstract contact open book (M, θ_M, ϕ), we wish to construct a contact open book. This construction is referred to as a *generalized Thurston–Winkelnkemper construction* in [13, Section 2.2.1]. To do this we need the following definition:

Definition 3.13 Let (M, θ_M, ϕ) be an abstract contact open book. Let $F_\phi\colon M \to \mathbb{R}$ be the smooth function with support in the interior of M satisfying $\phi^*\theta_M = \theta_M + dF_\phi$. Let $\rho\colon [0,1] \to [0,1]$ be a smooth function equal to 0 near 0 and 1 near 1.

The *mapping torus* of (M, θ_M, ϕ) is a smooth map

$$\pi_{T_\phi}\colon T_\phi \to \mathbb{R}/\mathbb{Z}, \quad T_\phi \equiv M \times [0,1]/\sim,$$

together with a contact form α_{T_ϕ} on T_ϕ, where

- $\sim$ identifies $(x, 1)$ with $(\phi(x), 0)$,
- $\alpha_{T_\phi} \equiv \theta_M + d(\rho(t)F_\phi) + C\,dt$, where $C > 0$ is large enough to ensure that α_{T_ϕ} is a contact form, and
- $\pi_{T_\phi}(x,t) \equiv t$ for all $(x,t) \in T_\phi = M \times [0,1]/\sim$.

For $\delta > 0$ small enough, we have that the subset

$$(1-\delta, 1] \times \partial M \subset (0,1] \times \partial M$$

of the collar neighborhood of ∂M is disjoint from the support of ϕ. This means that there is a natural embedding

$$(1-\delta, 1] \times \partial M \times \mathbb{R}/\mathbb{Z} \subset T_\phi,$$

which we will call the *standard collar neighborhood of* ∂T_ϕ. Note that α_{T_ϕ} is equal to $r_M \alpha_M + C\,dt$ in the standard collar neighborhood where r_M (resp. t) is the natural projection map to $(1-\delta, 1]$ (resp. $\mathbb{R}/\mathbb{Z}$) and $\alpha_M = \theta_M|_{\partial M}$.

If (M, θ_M, ϕ) is a graded abstract contact open book then we get a natural grading on $(T_\phi, \ker(\alpha_{T_\phi}))$ as follows: Since the kernel of $d\alpha_{T_\phi}$ is transverse to the fibers of T_ϕ, $\xi_{T_\phi} \equiv \ker(\alpha_{T_\phi})$ is isotopic through hyperplane distributions H_t for $t \in [0,1]$ to the vertical tangent space $T^{\mathrm{ver}}T_\phi \equiv \ker(D\pi_\phi)$ of π_{T_ϕ} with the property that $d\alpha_{T_\phi}|_{H_t}$ is nondegenerate for all $t \in [0,1]$. Therefore, it is sufficient to construct a grading for the symplectic vector bundle $(T^{\mathrm{ver}}T_\phi, d\alpha_{T_\phi})$. Consider the symplectic vector bundle $(\mathrm{pr}^*TM, \mathrm{pr}^*(d\theta_M))$ where

$$\mathrm{pr}\colon M \times [0,1] \to M$$

is the natural projection map. We have that the symplectic vector bundle $(T^{\mathrm{ver}}T_\phi, d\alpha_{T_\phi})$ is isomorphic to the symplectic vector bundle on $(\mathrm{pr}^*TM/{\sim}, \mathrm{pr}^*(d\theta_M))$ where $\sim$ identifies $\mathrm{pr}^*TM|_{M\times\{1\}}$ with $\mathrm{pr}^*TM|_{M\times\{0\}}$ using the map

$$D\phi\colon TM = \mathrm{pr}^*TM|_{M\times\{1\}} \to TM = \mathrm{pr}^*TM|_{M\times\{0\}}.$$

Since $(TM, d\theta_M)$ has a natural grading, we get that $(\mathrm{pr}^*TM, \mathrm{pr}^*(d\theta_M))$ has an induced grading

$$\widetilde{\mathrm{Fr}}(\mathrm{pr}^*TM) \times_{\widetilde{\mathrm{Sp}}(2n)} \mathrm{Sp}(2n) \cong \mathrm{Fr}(\mathrm{pr}^*TM)$$

pulled back via pr. The map $D\phi$ gives us an induced map

$$\mathrm{Fr}(\mathrm{pr}^*TM)|_{M\times\{1\}} \to \mathrm{Fr}(\mathrm{pr}^*TM)|_{M\times\{0\}}$$

and, since ϕ is a graded symplectomorphism, the map above lifts to a map

$$\widetilde{\mathrm{Fr}}(\mathrm{pr}^*TM)|_{M\times\{1\}} \to \widetilde{\mathrm{Fr}}(\mathrm{pr}^*TM)|_{M\times\{0\}}.$$

Hence, by using the above gluing map, we get a grading on the quotient

$$(\mathrm{pr}^*TM/{\sim}, \mathrm{pr}^*(d\theta_M))$$

and therefore a grading on $(T^{\mathrm{ver}}T_\phi, d\alpha_{T_\phi})$. In turn this gives us a grading on the contact manifold $(T_\phi, \ker(\alpha_{T_\phi}))$. We will call this the *induced grading on* T_ϕ.

We will now construct a contact open book from an abstract contact open book. Let $\mathbb{D}(\rho) \subset \mathbb{C}$ be the closed disk of radius $\rho > 0$ with polar coordinates (r, ϑ).

Definition 3.14 Let (M, θ_M, ϕ) be an abstract contact open book decomposition and let

$$\pi_{T_\phi}\colon T_\phi \to \mathbb{R}/\mathbb{Z}$$

be the associated mapping torus with contact form α_ϕ and standard collar neighborhood

$$(1-\delta, 1] \times \partial M \times \mathbb{R}/\mathbb{Z} \subset T_\phi$$

as in Definition 3.13. Define $C_\phi \equiv (\partial M \times \mathbb{D}(\delta)) \sqcup T_\phi/{\sim}$, where $\sim$ identifies $(x, (r, \vartheta)) \in \partial M \times \mathbb{D}(\delta)$ with

$$\left(1 - r, x, \frac{\vartheta}{2\pi}\right) \in (1-\delta, 1] \times \partial M \times \mathbb{R}/\mathbb{Z} \subset T_\phi.$$

We define $B_\phi \equiv \partial M \times \{0\} \subset \partial M \times \mathbb{D}(\delta) \subset C_\phi$ and

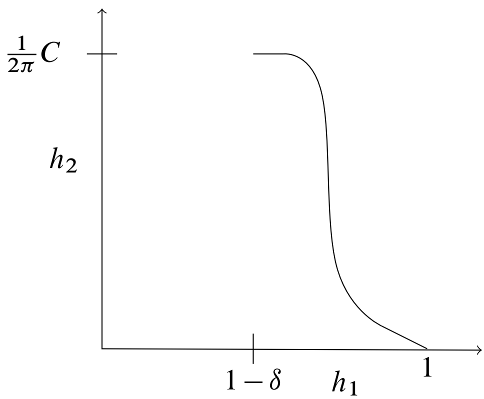
$$\pi_{C_\phi}\colon C_\phi - B_\phi = T_\phi - \partial T_\phi \to \mathbb{R}/\mathbb{Z}, \quad \pi_{C_\phi} = \pi_{T_\phi}|_{T_\phi - \partial T_\phi}.$$

Let

$$h_1, h_2\colon [0,\delta) \to \mathbb{R} \tag{3-3}$$

be a pair of smooth functions such that

- $h_1'(r) < 0$, $h_2'(r) \geq 0$ for all $r > 0$,
- $h_1(r) = 1 - r^2$ and $h_2(r) = \frac{1}{2}r^2$ for small r, and
- $h_1(r) = 1 - r$ and $h_2(r) = \frac{1}{2\pi}C$ for $r \in \left[\frac{1}{2}\delta, \delta\right)$:



The above conditions ensure that

$$\alpha_{C_\phi} \equiv \begin{cases} h_1(r)\alpha_M + h_2(r)\,d\vartheta & \text{inside } \partial M \times \mathbb{D}\left(\frac{1}{2}\delta\right), \\ \alpha_{T_\phi} & \text{inside } T_\phi - \partial M \times \mathbb{D}\left(\frac{1}{2}\delta\right), \end{cases}$$

is a contact form whose restriction to B_ϕ is also a contact form and that the restriction of $d\alpha_{C_\phi}$ to $\pi^{-1}(t)$ is symplectic for all $t \in \mathbb{R}/\mathbb{Z}$. Define $\xi_{C_\phi} \equiv \ker(\alpha_{C_\phi})$. The tubular neighborhood $\partial M \times \mathbb{D}\left(\frac{1}{2}\delta\right)$ of B_ϕ gives us a trivialization Φ_{B_ϕ} of its normal bundle and hence we get a contact pair with normal bundle data $(B_\phi \subset C_\phi, \xi_{C_\phi}, \Phi_{B_\phi})$, which we will call the *contact pair associated to* (M, θ_M, ϕ). This contact pair with normal bundle data is supported by the open book (C_ϕ, π_{C_ϕ}). Hence,

$$\mathrm{OBD}(M, \theta_M, \phi) \equiv (C_\phi, \xi_{C_\phi}, \pi_{C_\phi})$$

is a contact open book, which we call the *open book associated to the abstract contact open book* (M, θ_M, ϕ).

Now suppose that (M, θ_M, ϕ) is a graded abstract contact open book. Then, since the contact manifold $(C_\phi - B_\phi, \xi_{C_\phi}|_{C_\phi - B_\phi})$ is isotopic through contact manifolds to T_ϕ, we get that the induced grading on T_ϕ gives us a grading on $(C_\phi - B_\phi, \xi_{C_\phi}|_{C_\phi - B_\phi})$. Hence, we have a relative grading on $(B_\phi \subset C_\phi, \xi_{C_\phi}, \Phi_\phi)$, which we will call the *induced grading*.

It is fairly straightforward to show that if two abstract contact open books are (graded) isotopic then their respective contact open book decompositions are (graded) isotopic. Hence, we get a map

$$\mathrm{OBD}\colon \{\text{(graded) abstract contact open books}\}/\text{isotopy} \\ \to \{\text{(graded) open books}\}/\text{isotopy}.$$

The theorem below is a result of Giroux [17].

Theorem 3.15 *The above map* OBD *is a bijection.*

A detailed construction of the inverse of OBD is contained in the proof of [13, Theorem 3.1.22]. As a result of this theorem, we have the following definition:

Definition 3.16 The *monodromy map* of a (graded) contact open book (C, ξ_C, π) is defined to be the (graded) contactomorphism $\phi\colon M \to M$, where (M, θ_M, ϕ) is the abstract contact open book $\mathrm{OBD}^{-1}((C, \xi_C, \pi))$.

Technically, this monodromy map is only defined up to isotopy, and so the monodromy map is really just a choice of representative in this isotopy class.

4 Fixed-point Floer cohomology definition

Definition 4.1 Let (M, θ_M) be a Liouville domain. An almost complex structure J on M is *cylindrical near* ∂M if it is compatible with the symplectic form $d\theta_M$ (ie $d\theta_M(\cdot, J(\cdot))$ is a Riemannian metric) and if $dr_M \circ J = -\alpha_M$ near ∂M inside the standard collar neighborhood $(0, 1] \times \partial M$.

An exact symplectomorphism $\phi\colon M \to M$ is *nondegenerate* if for every fixed point p of ϕ the linearization of ϕ at p has no eigenvalue equal to 1. It has *small positive slope* if it is equal to the time 1 Hamiltonian flow of δr_M near ∂M, where $\delta > 0$ is smaller than the period of the smallest periodic Reeb orbit of α_M (this means that it corresponds to the time δ Reeb flow near ∂M). If ϕ is an exact symplectomorphism, then a *small* *positive slope perturbation* $\check{\phi}$ of ϕ is an exact symplectomorphism $\check{\phi}$ equal to the composition of ϕ with a C^∞ small Hamiltonian symplectomorphism of small positive slope. The *action* of a fixed point p is $-F_\phi(p)$, where F_ϕ is a function satisfying $\phi^*\theta_M = \theta_M + dF_\phi$. The action depends on a choice of F_ϕ, which has to be fixed when ϕ is defined, although usually F_ϕ is chosen so that it is zero near ∂M (if possible). All of the symplectomorphisms coming from isolated hypersurface singularities will have such a unique F_ϕ. An *isolated family of fixed points* is a path-connected compact subset $B \subset M$ consisting of fixed points of ϕ of the same action and for which there is a neighborhood $N \supset B$ where $N \setminus B$ has no fixed points. Such an isolated family of fixed points is called a *codimension* 0 *family of fixed points* if in addition there is an autonomous Hamiltonian $H\colon N \to (-\infty, 0]$ such that $H^{-1}(0) = B$ is a connected codimension 0 submanifold with boundary and corners, the time t flow of X_H is well defined for all $t \in \mathbb{R}$ and $\phi|_N\colon N \to N$ is equal to the time 1 flow of H. The *action* of an isolated family of fixed points $B \subset M$ is the action of any point $p \in B$.

Before we define Floer cohomology, we need some definitions, so that we can give it a grading. To any path of symplectic matrices $(A_t)_{t\in[a,b]}$ we can assign an index $\mathrm{CZ}(A_t)$, called its *Conley–Zehnder index*. The Conley–Zehnder index of certain paths of symplectic matrices A_t was originally defined in [11]. It was defined for a general path of symplectic matrices in [34] and also in [19]. We will not define it here as we will only use the following properties (see [19, Proposition 8]):

(CZ1) $\mathrm{CZ}((e^{it})_{t\in[0,2\pi]}) = 2$.

(CZ2) $(-1)^{n-\mathrm{CZ}((A_t)_{t\in[0,1]})} = \operatorname{sign}\det_{\mathbb{R}}(\mathrm{id} - A_1)$ for any path of symplectic matrices $(A_t)_{t\in[0,1]}$.

(CZ3) $\mathrm{CZ}(A_t \oplus B_t) = \mathrm{CZ}(A_t) + \mathrm{CZ}(B_t)$.

(CZ4) The Conley–Zehnder index of the catenation of two paths is the sum of their Conley–Zehnder indices.

(CZ5) If A_t and B_t are two paths of symplectic matrices which are homotopic relative to their endpoints then they have the same Conley–Zehnder index. Also, such an index only depends on the path up to orientation-preserving reparametrization.

Definition 4.2 Let (M, θ_M, ϕ) be a graded abstract contact open book. The *Conley–Zehnder index* $\mathrm{CZ}(p)$ of a fixed point p of the graded symplectomorphism ϕ is defined as follows: Since $(TM, d\theta_M)$ is a graded symplectic vector bundle, we have an associated $\widetilde{\mathrm{Sp}}(2n)$ bundle

$$\widetilde{\mathrm{Fr}}(TM) \to M$$

together with a choice of isomorphism of $\widetilde{\mathrm{Sp}}(2n)$ bundles

$$\iota\colon \widetilde{\mathrm{Fr}}(TM) \times_{\widetilde{\mathrm{Sp}}(2n)} \mathrm{Sp}(2n) \cong \mathrm{Fr}(TM).$$

Now choose an identification of $\widetilde{\mathrm{Sp}}(2n)$ torsors

$$\widetilde{\mathrm{Sp}}(2n) = \widetilde{\mathrm{Fr}}(TM)|_p. \tag{4-1}$$

The symplectomorphism ϕ has a choice of grading, giving us a natural map

$$\tilde{\phi}\colon \widetilde{\mathrm{Fr}}(TM)|_p \to \widetilde{\mathrm{Fr}}(TM)|_p$$

and hence, by (4-1), a map

$$\tilde{\phi}\colon \widetilde{\mathrm{Sp}}(2n) \to \widetilde{\mathrm{Sp}}(2n).$$

Since $\widetilde{\mathrm{Sp}}(2n)$ is the universal cover of $\mathrm{Sp}(2n)$, its elements correspond to paths of symplectic matrices starting from the identity up to homotopy fixing their endpoints and so let β be the path corresponding to $\tilde{\phi}(\mathrm{id}) \in \widetilde{\mathrm{Sp}}(2n)$. We define $\mathrm{CZ}(\phi, p) \equiv \mathrm{CZ}(\beta)$.

If we have an isolated family of fixed points $B \subset X$, then we define its *Conley–Zehnder index* $\mathrm{CZ}(\phi, B)$ to be the Conley–Zehnder index of some $b \in B$. Since B is path-connected, this does not depend on the choice of $b \in B$ by property (CZ5) above.

In Appendix A, we also have a different way of computing the Conley–Zehnder index by looking at the mapping torus T_ϕ of ϕ. This will be useful in the proof of Theorem 5.41.

Definition 4.3 Let (M, θ_M, ϕ) be an abstract contact open book. Let $(J_t)_{t \in [0,1]}$ be a smooth family of almost complex structures with the property that $\phi^* J_0 = J_1$. A *Floer trajectory* of $(\phi, J_t)_{t \in [0,1]})$ joining $p_-, p_+ \in M$ is a smooth map $u\colon \mathbb{R} \times [0,1] \to M$ such that $\partial_s u + J_t \partial_t u = 0$, where (s,t) parametrizes $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$, $u(s,0) = \phi(u(s,1))$ and such that $\lim_{s \to \pm\infty} u(s,t) = p_\pm$ for all $t \in [0,1]$. We write $\mathcal{M}(\phi, J_t, p_-, p_+)$ for the set of such Floer trajectories and define $\overline{\mathcal{M}}(\phi, J_t, p_-, p_+) \equiv \mathcal{M}(\phi, J_t, p_-, p_+)/\mathbb{R}$, where $\mathbb{R}$ acts by translation in the s coordinate.

Let (M, θ_M, ϕ) be a graded abstract contact open book. We will now give a sketch of the definition of the *Floer cohomology* group $\mathrm{HF}^*(\phi, +)$ (see [37]). Let $\check{\phi}$ be a small positive slope perturbation of ϕ. This can be done so that $\check{\phi}$ is C^∞ close to ϕ and so that the fixed points of $\check{\phi}$ are nondegenerate (see [35, Theorem 9.1] in the case where $\check{\phi}$ is Hamiltonian; the general case is similar [14, page 586]). We can also ensure that $\check{\phi}$ is a graded symplectomorphism due to the fact that it is isotopic to ϕ through symplectomorphisms.

We now choose a C^∞ generic family $(J_t)_{t\in[0,1]}$ of cylindrical almost complex structures satisfying $\phi^* J_0 = J_1$. The genericity property then tells us that $\overline{\mathcal{M}}(\check{\phi}, J_t, p_-, p_+)$ is a compact oriented manifold of dimension 0 for all fixed points p_- and p_+ of ϕ satisfying $\mathrm{CZ}(p_-) - \mathrm{CZ}(p_+) = 1$ [14, Theorem 3.2]. We define $\#^{\pm}\overline{\mathcal{M}}(\check{\phi}, J_t, p_-, p_+)$ to be the signed count of elements of $\overline{\mathcal{M}}(\check{\phi}, J_t, p_-, p_+)$. Let $\mathrm{CF}^*(\check{\phi})$ be the free abelian group generated by fixed points of ϕ and graded by the Conley–Zehnder index *taken with negative sign*. The differential $\partial_{\check{\phi},(J_t)_{t\in[0,1]}}$ on $\mathrm{CF}^*(\check{\phi})$ is a $\mathbb{Z}$–linear map satisfying $\partial_{\check{\phi},(J_t)_{t\in[0,1]}}(p_+) = \sum_{p_-} \#^{\pm}\overline{\mathcal{M}}(J_t, p_-, p_+) p_-$ for all fixed points p_+ of $\check{\phi}$, where the sum is over all fixed points p_- satisfying $(-\mathrm{CZ}(p_-)) - (-\mathrm{CZ}(p_+)) = 1$. Because $(J_t)_{t\in[0,1]}$ is C^∞ generic, one can show [14, Theorem 3.3(1)] that

$$\partial^2_{\check{\phi},(J_t)_{t\in[0,1]}} = 0$$

and we define the resulting homology group to be $\mathrm{HF}^*(\check{\phi}, (J_t)_{t\in[0,1]})$. We define $\mathrm{HF}^*(\phi, +) \equiv \mathrm{HF}^*(\check{\phi}, (J_t)_{t\in[0,1]})$. This does not depend on the choice of perturbation $\check{\phi}$ or on the choice of almost complex structure $(J_t)_{t\in[0,1]}$ [14, Theorem 3.3(2)]. Our conventions then tell us that, if $\phi\colon M \to M$ is the identity map with the trivial grading and $\dim(M) = n$, then $\mathrm{HF}^k(\phi, +) = H^{n+k}(M; \mathbb{Z})$.

We will *only* use the following properties of these Floer cohomology groups:

(HF1) For a graded abstract contact open book (M, θ_M, ϕ), the Lefschetz number $\Lambda(\phi)$ of ϕ is equal to the Euler characteristic of $\mathrm{HF}^*(\phi, +)$ multiplied by $(-1)^n$ (which follows from (CZ2)).

(HF2) Suppose that $(M_1, \theta_{M_1}, \phi_1)$, $(M_2, \theta_{M_2}, \phi_2)$ are graded abstract contact open books such that the graded contact pairs associated to them are graded contactomorphic. Then $\mathrm{HF}^*(\phi_0, +) = \mathrm{HF}^*(\phi_1, +)$ (see Appendix A).

(HF3) Let (M, θ_M, ϕ) be a graded abstract contact open book where $\dim(M) = 2n$. Suppose that the set of fixed points of a small positive slope perturbation $\check{\phi}$ of ϕ is a disjoint union of codimension 0 families of fixed points $B_1, \dots, B_l$ and let $\iota\colon \{1, \dots, l\} \to \mathbb{N}$ be a function where

- $\iota(i) = \iota(j)$ if and only if the action of B_i equals the action of B_j, and
- $\iota(i) < \iota(j)$ if the action of B_i is less than the action of B_j.

Then there is a cohomological spectral sequence converging to $\mathrm{HF}^*(\phi,+)$ whose E_1 page is equal to

$$E_1^{p,q} = \bigoplus_{\{i\in\{1,\dots,l\}:\iota(i)=p\}} H_{n-(p+q)-\mathrm{CZ}(\phi,B_j)}(B_p;\mathbb{Z})$$

(see Appendix C).

5 Constructing a well-behaved contact open book

The aim of this section is to modify the Milnor monodromy map so that the set of fixed points is a union of codimension 0 families of fixed points, so that we can apply (HF3) above. We will do this by constructing such a nice symplectomorphism whose mapping torus is isotopic to the mapping torus of the Milnor monodromy map.

5.1 Some preliminary definitions

The aim of this section is to construct a symplectic form on the resolution which behaves well with respect to the resolution divisors. To do this we need a purely symplectic notion of smooth normal crossing divisor. We will introduce some notation from [40].

If $V \subset X$ is a submanifold of a manifold X then we will use the notation

$$\pi_{\mathcal{N}_X V}\colon \mathcal{N}_X V \equiv \frac{TX|_V}{TV} \twoheadrightarrow V \tag{5-1}$$

for the normal bundle of V. If $(V_i)_{i\in S}$ is a finite collection of submanifolds of X and $I \subset S$, let

$$V_I \equiv \bigcap_{i\in I} V_i.$$

Also, by convention we define $V_\varnothing \equiv X$. The collection $(V_i)_{i\in S}$ intersects *transversally* if for every subset $I \subset S$ and every $x \in V_I$,

$$\mathrm{codim}_{T_xX}\Big(\bigcap_{i\in I} T_xV_i\Big) = \sum_{i\in I} \mathrm{codim}_{T_xX}(T_xV_i).$$

If $V \subset X$ is a submanifold and X is oriented then an orientation on V induces an orientation on $\mathcal{N}_X V$ and conversely an orientation on $\mathcal{N}_X V$ induces an orientation on V using (5-1) (if V is odd-dimensional, this depends on a sign convention, but we will not need this since the manifolds that we will be dealing with are even-dimensional). If X is oriented and $(V_i)_{i\in S}$ is a collection of oriented transversally intersecting submanifolds, then the submanifold V_I has a natural orientation called the *intersection orientation* since $\mathcal{N}_X V_I = \bigoplus_{i \in I} \mathcal{N}_X V_i$ is oriented.

Definition 5.1 Suppose that (X, ω) is a symplectic manifold. Then $(V_i)_{i \in S}$ is called a *symplectic crossings divisor* or *SC divisor* if $(V_i)_{i \in S}$ are transversally intersecting codimension 2 symplectic submanifolds such that V_I is also symplectic and such that the symplectic orientation on V_I agrees with its corresponding intersecting orientation for all $I \subset S$. We will also assume that S is a finite set.

We now want to define SC divisors with particularly nice neighborhoods. We call

$$\pi\colon (L, \rho, \nabla) \to V$$

a *Hermitian line bundle* if $\pi\colon L \to V$ is a complex line bundle, ρ is a Hermitian metric and ∇ is a ρ–compatible Hermitian connection on L. We define $\rho^{\mathbb{R}}$ and $\rho^{i\mathbb{R}}$ to be the real and complex parts of the Hermitian metric ρ. We will also use the notation ρ to denote square of the norm function on L. If we view L as an oriented real vector bundle then we can recover the complex structure $\boldsymbol{i}_\rho$ from the metric $\rho^{\mathbb{R}}$ using the fact that for all $x \in V$ and $W \in L|_x - 0$, $\boldsymbol{i}_\rho(W)$ is the unique vector making $W, \boldsymbol{i}_\rho(W)$ into an oriented orthonormal basis of $L|_x$. Hence, we can define a *Hermitian structure* (ρ, ∇) on any oriented 2–dimensional real vector bundle $L \twoheadrightarrow V$ to be a pair (ρ, ∇) making $(L, \boldsymbol{i}_\rho)$ into a Hermitian line bundle.

For any such triple (L, ρ, ∇) we have an associated Hermitian connection 1–form $\alpha_{\rho,\nabla} \in \Omega^1(L - V)$. This is the pullback of the associated principal $U(1)$–connection on the unit circle bundle of (L, ρ) (see [4] or [40, Appendix A]). If $(L_i, \rho_i, \nabla^{(i)})_{i \in I}$ is a finite collection of Hermitian line bundles over a symplectic manifold (V, ω) and $\mathrm{pr}_{I;i}\colon \bigoplus_{i \in I} L_i \to L_i$ is the natural projection map then we define

$$\omega_{(\rho_i, \nabla^{(i)})_{i \in I}} \equiv \pi^*\omega + \frac{1}{2} \sum_{i \in I} \mathrm{pr}_{I;i}^* \, d(\rho \alpha_{\rho_i, \nabla^{(i)}}). \tag{5-2}$$

This is a symplectic form in some small neighborhood of the zero section. Given a 2–dimensional symplectic vector bundle $L \twoheadrightarrow V$ with symplectic form Ω, an Ω–*compatible Hermitian structure* on L is a Hermitian structure (ρ, ∇) where the complex structure $\boldsymbol{i}_\rho$ is compatible with Ω.

Suppose that $\Psi\colon \check{V} \to V$ is a diffeomorphism and suppose $\pi\colon (L_i, \rho_i, \nabla^{(i)})_{i \in I} \to V$ and $\check{\pi}\colon (\check{L}_i, \check{\rho}_i, \check{\nabla}^{(i)})_{i \in I} \to \check{V}$ are two collections of Hermitian line bundles; then a *product Hermitian isomorphism* is a vector bundle isomorphism

$$\widetilde{\Psi}\colon \bigoplus_{i\in I} \check{L}_i \to \bigoplus_{i\in I} L_i$$

covering Ψ and respecting the direct sum decomposition such that the induced map $\widetilde{\Psi}\colon (\check{L}_i, \check{\rho}_i, \check{\nabla}^{(i)}) \to (L_i, \rho_i, \nabla^{(i)})$ is an isomorphism of Hermitian line bundles for all $i \in \check{I}$.

Definition 5.2 Let $V \subset X$ be a submanifold of a manifold X. A *regularization* is a diffeomorphism $\Psi\colon \check{\mathcal{N}} \to X$ from a neighborhood $\check{\mathcal{N}} \subset \mathcal{N}_X V$ of the zero section onto its image such that $\Psi(x) = x$ for all $x \in V$ and such that the map

$$d_x\Psi\colon \mathcal{N}_X V|_x \to \mathcal{N}_X V|_x, \quad d_x\Psi(v) \equiv Q\Big(D\Psi\Big(\frac{d}{dt}(tv)\Big|_{t=0}\Big)\Big),$$

is the identity map, where $Q\colon TX|_V \to \mathcal{N}_X V$ is the natural quotient map.

We also need a notion of regularization compatible with the symplectic form. Because of this we first need to talk about connections induced by closed 2–forms. Recall that an *Ehresmann connection* on a smooth submersion $\pi\colon E \to B$ is a distribution $H \subset TE$ such that $D\pi|_{H_x}\colon H_x \to T_{\pi(x)}B$ is an isomorphism for all $x \in E$.

Definition 5.3 Let $\pi\colon E \to B$ be a smooth submersion and let Ω be a 2–form on E whose restriction to each fiber is nondegenerate. Then the *symplectic connection associated to* Ω is an Ehresmann connection $H \subset TE$, where H is the set of vectors which are Ω–orthogonal to the fibers of π. In other words,

$$H = \{V \in T_x E : x \in E,\ \Omega(V, W) = 0 \text{ for all } W \in T_x^{\mathrm{ver}} E\},$$

where $T_x^{\mathrm{ver}} E \equiv \ker(D\pi)$ is the *vertical tangent bundle*.

If (X,ω) is a symplectic manifold and $V \subset X$ is a symplectic submanifold then $\mathcal{N}_X V$ is a symplectic vector bundle on V. We write $\omega|_{\mathcal{N}_X V}$ for the symplectic form on this vector bundle and $\omega|_L$ for the restriction of $\omega|_{\mathcal{N}_X V}$ to L, where L is any subbundle $L \subset \mathcal{N}_X V$. The following definition differs in notation from [39, Definition 2.8] but it defines the same object.

Definition 5.4 Let (X,ω) be a symplectic manifold, V a symplectic submanifold and let

$$\mathcal{N}_X V \equiv \bigoplus_{i\in I} L_i \tag{5-3}$$

be a splitting into 2–dimensional subbundles such that $\omega|_{L_i}$ is nondegenerate for all $i \in I$. A tuple $((\rho_i)_{i\in I}, \Psi)$ is called an *ω–regularization for V in X* if Ψ is a regularization for V in X and ρ_i is a map $\rho_i\colon \operatorname{Im}(\Psi) \to [0,\infty)$ such that there is an $\omega|_{L_i}$–compatible Hermitian structure $(\widetilde{\rho}_i, \nabla^{(i)})$ on L_i satisfying

- $\widetilde{\rho}_i|_{\operatorname{Dom}(\Psi)} = \rho_i \circ \Psi$ for all $i \in I$ where $\widetilde{\rho}_i$ is (by abuse of notation) the pullback of $\widetilde{\rho}_i$ by the natural projection map $\bigoplus_{j\in I} L_j \to L_i$,
- $\nabla^{(i)}$ restricted to $\operatorname{Dom}(\Psi) \cap L_i$ coincides with the symplectic connection associated to $\omega|_{L_i}$ for all $i \in I$, and

$$\Psi^*\omega = \omega_{(\widetilde{\rho}_i, \nabla^{(i)})_{i\in I}}|_{\operatorname{Dom}(\Psi)} \tag{5-4}$$

for each $i \in I$.

The splitting (5-3) is called the *associated splitting* and the $\omega|_{L_i}$–compatible Hermitian structure $(\widetilde{\rho}_i, \nabla^{(i)})$ is called the *associated Hermitian structure on L_i*. This Hermitian structure is uniquely determined by ρ_i and Ψ. Also, since $\widetilde{\rho}_i$ gives us a complex structure on L_i for each $i \in I$, we get a natural complex structure on $\mathcal{N}_X V$ which we will call the *complex structure associated to the ω–regularization* $((\rho_i)_{i\in I}, \Psi)$.

We wish to extend Definitions 5.2 and 5.4 to transverse collections of submanifolds and SC divisors, respectively. To do this we need some preliminary notation. Let X be a manifold and $(V_i)_{i\in S}$ transversely intersecting submanifolds. We have a canonical identification

$$\mathcal{N}_X V_I = \pi^*_{I;I'}(\mathcal{N}_{V_{I'}} V_I) \tag{5-5}$$

for each $I' \subset I$, where

$$\pi_{I;I'}\colon \mathcal{N}_{V_{I-I'}} V_I \to V_I$$

is the natural projection map. Note that (5-5) is not an identification of bundles since the base of the left-hand bundle is V_I whereas the base of the right-hand bundle is $\mathcal{N}_{V_{I-I'}} V_I$.

Definition 5.5 Let X be a manifold and $(V_i)_{i\in S}$ a transverse collection of submanifolds. A *system of regularizations* is a tuple $(\Psi_I)_{I\subset S}$, where Ψ_I is a regularization for V_I such that

$$\Psi_I(\mathcal{N}_{V_{I'}} V_I \cap \operatorname{Dom}(\Psi_I)) = V_{I'} \cap \operatorname{Im}(\Psi_I) \tag{5-6}$$

for all $I' \subset I \subset S$.

Define

$$\iota\colon \pi^*_{I;I'}(\mathcal{N}_{V_{I'}}V_I) \hookrightarrow T\pi^*_{I;I'}(\mathcal{N}_{V_{I'}}V_I) \overset{(5\text{-}5)}{=} T\mathcal{N}_X V_I, \quad \iota(x,v) \equiv \frac{d}{dt}(x,tv)\Big|_{t=0}.$$

Using the inclusion map ι above, define

(5-7)
$$\mathfrak{D}\Psi_{I;I'}\colon \pi^*_{I;I'}(\mathcal{N}_{V_{I'}}V_I)|_{\Psi_I^{-1}(V_{I'})} \to \mathcal{N}_X(V_{I'}\cap \mathrm{Im}(\Psi_I)),$$
$$\mathfrak{D}\Psi_{I;I'}(w) \equiv Q\big(D\Psi_I(\iota(w))\big),$$

where $Q\colon TX|_{V_{I'}\cap\mathrm{Im}(\Psi_I)} \to \mathcal{N}_X(V_{I'}\cap\mathrm{Im}(\Psi_I))$ is the natural projection map.

Using the equality (5-5), $\mathfrak{D}\Psi_{I;I'}$ identifies the normal bundle of $\mathcal{N}_{V_{I-I'}}V_I$ inside $\mathcal{N}_X V_I$ near 0 with the normal bundle of $V_{I'}$ near V_I using the derivative of the regularization Ψ_I. The map $\mathfrak{D}\Psi_{I;I'}$ is also a bundle isomorphism covering the diffeomorphism

$$\Psi_I|_{\Psi_I^{-1}(V_{I'})}\colon \Psi_I^{-1}(V_{I'}) \to V_{I'}\cap\mathrm{Im}(\Psi_I).$$

The definition below tells us that $\Psi_{I'}$ and Ψ_I should be equal under the identification (5-7).

Definition 5.6 Let X be a manifold and $(V_i)_{i\in S}$ a transverse collection of submanifolds of X. Then a *regularization for* $(V_i)_{i\in S}$ is a system of regularizations $(\Psi_I)_{I\subset S}$ for $(V_i)_{i\in S}$ such that

(5-8) $$\mathfrak{D}\Psi_{I;I'}(\mathrm{Dom}(\Psi_I)) = \mathrm{Dom}(\Psi_{I'})|_{V_{I'}\cap\mathrm{Im}(\Psi_I)}, \quad \Psi_I = \Psi_{I'}\circ\mathfrak{D}\Psi_{I;I'}|_{\mathrm{Dom}(\Psi_I)}.$$

The following definition differs from [39, Definition 2.11] for the same reasons that Definition 5.4 differs from [39, Definition 2.8]. Apart from that, this definition is exactly the same. This structure also appears in [26, Lemma 5.14] although the regularization maps have particular domains and it is defined in a slightly different way.

Definition 5.7 Let (X,ω) be a symplectic manifold and $(V_i)_{i\in S}$ an SC divisor. An *ω–regularization* is a pair of tuples

$$((\rho_i)_{i\in S}, (\Psi_I)_{I\subset S}),$$

where

(1) $(\Psi_I)_{I\subset S}$ is a regularization for $(V_i)_{i\in S}$ as in Definition 5.6 and

$$\rho_i\colon \bigcup_{i\in I\subset S} \mathrm{Im}(\Psi_I) \to [0,\infty)$$

is a smooth map,

(2) $((\rho_i|_{\mathrm{Im}(\Psi_I)})_{i\in I}, \Psi_I)$ is an ω–regularization for V_I in X for each $I \subset S$ as in Definition 5.4, and

(3) the maps $\mathfrak{D}\Psi_{I;I'}$ from (5-7) are product Hermitian isomorphisms for all $I' \subset I \subset S$ with respect to the natural splittings

$$\pi^*_{I;I'}(\mathcal{N}_{V_{I'}} V_I)|_{\Psi_I^{-1}(V_{I'})} = \bigoplus_{i\in I'} \pi^*_{I;I'}(\mathcal{N}_X V_i|_{V_I})|_{\Psi_I^{-1}(V_{I'})},$$
$$\mathcal{N}_X(V_{I'} \cap \mathrm{Im}(\Psi_I)) = \bigoplus_{i\in I'} \mathcal{N}_X V_i|_{V_{I'}\cap \mathrm{Im}(\Psi_I)}.$$

We are only interested in regularizations near $(V_i)_{i\in S}$ and so we want a notion of equivalence to reflect this.

Definition 5.8 Two ω–regularizations

$$((\rho_i)_{i\in S}, (\Psi_I)_{I\subset S}), \quad ((\check{\rho}_i)_{i\in S}, (\check{\Psi}_I)_{I\subset S})$$

for $(V_i)_{i\in S}$ are *germ equivalent* if there is an open set

$$U_I \subset \mathrm{Dom}(\Psi_I) \cap \mathrm{Dom}(\check{\Psi}_I)$$

containing V_I such that $\Psi_I|_{U_I} = \check{\Psi}_I|_{U_I}$ and $\rho_i|_{\Psi_I(U_I)} = \check{\rho}_i|_{\Psi_I(U_I)}$ for each $i \in I \subset S$.

A real codimension 2 submanifold with an oriented normal bundle should be thought of as the differential geometric analogue of a smooth divisor in algebraic geometry. We wish to construct complex line bundles from such submanifolds in the same way that line bundles are constructed from Cartier divisors in algebraic geometry. The following line bundle associated to a codimension 2 submanifold V of a manifold X will depend on a choice of regularization $\Psi\colon \check{\mathcal{N}} \to X$ of V and a complex structure $\boldsymbol{i}$ on $\mathcal{N}_X V$ and it will come with a canonical section $s_V\colon V \to \mathcal{O}_X(V)$ whose zero set is V. We define

$$\mathcal{O}_X(V) = \big((\pi^*_{\mathcal{N}_X V}\mathcal{N}_X(V))|_{\mathrm{Dom}(\Psi)} \sqcup (X-V)\times\mathbb{C}\big)/\sim, \tag{5-9}$$

where

$$(\pi^*_{\mathcal{N}_X V}\mathcal{N}_X(V))|_{\mathrm{Dom}(\Psi_I)} \ni (v, cv) \sim (\Psi(v), c) \in (X-V)\times\mathbb{C}$$
$$\text{for all } v \in \mathcal{N}_X V - V,\ c\in\mathbb{C}.$$

The corresponding fibration is defined in the following natural way:

$$\begin{aligned} &\pi_{\mathcal{O}_X(V)}\colon \mathcal{O}_X(V) \to X, \\ &\pi_{\mathcal{O}_X(V)}(v,w) \equiv \Psi(v) \quad \text{for all } (v,w) \in \pi^*_{\mathcal{N}_X V}(\mathcal{N}_X V), \\ &\pi_{\mathcal{O}_X(V)}(x,c) \equiv x \qquad\ \ \text{for all } (x,c) \in (X-V)\times\mathbb{C}. \end{aligned}$$

This is a line bundle satisfying the following important canonical identities:

$$\mathcal{O}_X(V)|_V = \mathcal{N}_X(V), \quad \mathcal{O}_X(V)|_{X-V} = (X-V)\times\mathbb{C}. \tag{5-10}$$

We will call Ψ the *associated regularization*. The *canonical section* $s_V\colon X \to \mathcal{O}_X(V)$ of this line bundle is defined as follows:

$$s_V(x) \equiv \begin{cases} (\Psi^{-1}(x), \Psi^{-1}(x)) \in (\pi^*_{\mathcal{N}_X V}\mathcal{N}_X(V))|_{\mathrm{Dom}(\Psi_I)} & \text{if } x \in \mathrm{Im}(\Psi_I), \\ (x,1) \in (X-V)\times\mathbb{C} & \text{if } x \in (X-V). \end{cases} \tag{5-11}$$

We also define $\mathcal{O}_X(0) \equiv \mathcal{O}_X \equiv X\times\mathbb{C}$ to be the trivial bundle. This is also $\mathcal{O}_X(\varnothing)$, where $\varnothing$ is the empty submanifold.

5.2 Trivializing line bundles

In the previous section, we constructed a line bundle from any codimension 2 submanifold with oriented boundary. As a result, we can construct line bundles from any SC divisor. In this subsection we show that if such a line bundle is trivial and if our SC divisor admits a regularization, then our line bundle admits a trivialization which behaves well with respect to this regularization. This trivialization will be used later to construct a map from a neighborhood of our SC divisor to $\mathbb{C}$ with nice parallel transport maps away from the singularities.

Let (X,ω) be a symplectic manifold and $(V_i)_{i\in S}$ an SC divisor on X admitting an ω–regularization

$$\mathcal{R} \equiv ((\rho_i)_{i\in S}, (\Psi_I)_{I\subset S})$$

as in Definition 5.7. Let $(m_i)_{i\in S}$ be natural numbers indexed by S. For all $i\in S$, let $\mathcal{O}_X(V_i)$ be the line bundle with associated regularization Ψ_i and complex structure associated to the ω–regularization (ρ_i,Ψ_i). Recall that these have natural sections $s_{V_i}\colon X \to \mathcal{O}_X(V_i)$ as in (5-11). Define $\mathcal{O}_X\big(\sum_i m_i V_i\big) \equiv \bigotimes_{i\in S}\mathcal{O}_X(V_i)^{\otimes m_i}$ and let

$$s_{(m_i)_{i\in S}} \equiv \bigotimes_{i\in S} s_{V_i}^{\otimes m_i} \tag{5-12}$$

be the canonical section of this bundle. Using the identity (5-10) we have

$$\mathcal{O}_X(V_i)|_{V_I} = \mathcal{N}_X V_i|_{V_I} \quad \text{for all } i \in I \subset S,$$

and hence we get the following maps:

$$\Pi_{V_i;I}\colon \mathcal{N}_X V_I = \bigoplus_{j\in I} \mathcal{N}_X V_j|_{V_I} \to \mathcal{O}_X(V_i)|_{V_I},$$

$$(v_j)_{j\in I} \to \begin{cases} v_i & \text{if } i \in I, \\ s_{V_i}(x) & \text{if } i \notin I, \end{cases} \qquad \text{for all } (v_j)_{j\in I} \in \mathcal{N}_X V_I|_x,\ x \in V_I.$$

Therefore, we get a (not necessarily fiberwise linear) map

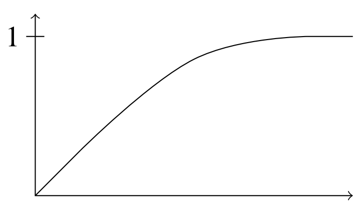
$$\Pi_{(m_i)_{i\in S},I}\colon \mathcal{N}_X V_I \to \mathcal{O}_X\Big(\sum_i m_i V_i\Big)\Big|_{V_I},$$

(5-13)

$$\Pi_{(m_i)_{i\in S},I}(v) \equiv \bigotimes_{i\in S} \Pi_{V_i;I}(v)^{\otimes m_i} \quad \text{for all } v \in \mathcal{N}_X V_I.$$

One can think of the above map as a section of $\mathcal{O}_X\big(\sum_i m_i V_i\big)\big|_{V_I}$ along with nontrivial infinitesimal information in the normal direction of V_I.

Below is a definition of a trivialization of $\mathcal{O}\big(\sum_i m_i V_i\big)$ with the property that locally around each point of V_I, the canonical section $s_{(m_i)_{i\in S}}$ of $\mathcal{O}_X\big(\sum_i m_i V_i\big)$ looks approximately like the map $(z_1,\dots,z_n) \to \prod_{i\in I}(z_i a(|z_i|)/|z_i|)^{m_i}$ in some local chart $z_1,\dots,z_n$, where $I \subset \{1,\dots,n\}$ and $a\colon \mathbb{R}\to\mathbb{R}$ has the following graph:



We need to use the above function a to ensure that we have good dynamical properties (see Section 5.5). See [40, Definition 3.8] for a related definition.

Definition 5.9 For each $r > 0$, we define the *radius r tube of V_I* to be the set

$$T_{r,I} \equiv \bigcap_{i\in I}\{x \in \operatorname{Im}(\Psi_I) : \rho_i(x) \le r\} \tag{5-14}$$

over V_I. Let $B \subset X$ be any set. The *tube radius of $\mathcal{R}$ along B* is the largest radius r tube of V_I along B that can "fit" inside the image of Ψ_I for each $I \subset S$. More precisely, it is the supremum of all $r \geq 0$ with the property that $T_{r,I} \cap (\mathrm{Im}(\Psi_I)|_x)$ is a compact subset of X for all $x \in B \cap V_I$ and $I \subset S$.

Let $U \subset X$ be an open set. Now suppose that the tube radius of $\mathcal{R}$ along U is positive and let $R > 0$ be any constant smaller than the tube radius and also smaller than 1. We let $a_R \colon [0,\infty) \to [0,\infty)$ be a smooth function satisfying

(1) $a'_R(x) > 0$ for $x \in \bigl[0, \frac{3}{4}R\bigr)$,

(2) $a_R(x) = x$ for $x \leq \frac{1}{4}R$,

(3) $a_R(x) = 1$ for $x \geq \frac{3}{4}R$.

A bundle trivialization

$$\Phi \equiv (\pi, \Phi_2) \colon \mathcal{O}_X\Bigl(\sum_i m_i V_i\Bigr) \to X \times \mathbb{C}$$

is *radius R compatible with $\mathcal{R}$ along $U \cap V_I$* if

$$\Phi_2(s_{(m_i)_{i\in S}}(x)) = \Phi_2\bigl(\Pi_{(m_i)_{i\in S},I}(\Psi_I^{-1}(x))\bigr)\prod_{i\in I}\left(\frac{\sqrt{a_R(\rho_i(x))}}{\sqrt{\rho_i(x)}}\right)^{m_i} \tag{5-15}$$
$$\text{for all } x \in T_{R,I} \cap (\mathrm{Im}(\Psi_I)|_{V_I \cap U - \bigcup_{i\in S-I} T_{3R/4,i}}).$$

and where the norm of the linear map

$$\Phi_2|_{\bigotimes_{i\in I}(\mathcal{N}_X V_i|_x)^{\otimes m_i}} \colon \Bigl(\bigotimes_{i\in I} \mathcal{N}_X V_i|_x\Bigr)^{\otimes m_i} \to \mathbb{C} \tag{5-16}$$

is equal to 1 for all $x \in V_I \cap U - \bigcup_{i\in S-I} T_{3R/4,i}$ using the identification (5-10).

We say that Φ is *radius R compatible with $\mathcal{R}$ along U* if it is radius R compatible with $\mathcal{R}$ along $U \cap V_I$ for each $I \subset S$. It is *compatible with $\mathcal{R}$ along U* if it is radius R compatible with $\mathcal{R}$ along U for some R smaller than the tube radius of $\mathcal{R}$.

One should think of (5-15) as saying that the trivialization Φ identifies the canonical section of $\mathcal{O}_X\bigl(\sum_i m_i V_i\bigr)$ with the "infinitesimal" section $\Pi_{(m_i)_{i\in S},I}$ multiplied by a particular factor near V_I. This particular factor that we are multiplying by is actually equal to 1 if we are very near V_I. Note also that (5-15) tells us that the norm of $\Phi_2(s_{(m_i)_{i\in S}}(x))$ only depends on $(\rho_i(x))_{i\in I}$ and if $\rho_i(x) \geq \frac{3}{4}R$ for some $i \in I$ then this norm does not depend on $\rho_i(x)$ for all $x \in \mathrm{Im}(\Psi_I) - \bigcup_{i\in S-I} T_{3R/4,i}$. As a result, the above definition is consistent with the stated norm property of the linear map (5-16).

Lemma 5.10 *Suppose $\mathcal{O}_X\left(\sum_i m_i V_i\right)$ admits a trivialization Φ and let $U \subset X$ be a relatively compact open set. Then there is a trivialization of $\mathcal{O}_X\left(\sum_i m_i V_i\right)$ compatible with $\mathcal{R}$ along U which is homotopic to Φ through trivializations of $\mathcal{O}_X\left(\sum_i m_i V_i\right)$.*

Proof Just as in the proof of [40, Definition 3.9] we will proceed by induction on the strata of $\bigcup_j V_j$. Before we do this though we will need to construct certain natural maps that identify $\Pi_{(m_i)_{i\in S};I}$ with $s_{(m_i)_{i\in S}}$ near V_I. Define

$$\begin{aligned} N_{i,I} &\equiv \mathrm{Dom}(\Psi_i)|_{V_i\cap \mathrm{Im}(\Psi_{I\cup i})} \overset{(5\text{-}8)}{=} \mathfrak{D}\Psi_{I\cup i;i}(\mathrm{Dom}(\Psi_{I\cup i})),\\ W_{i;I} &\equiv \Psi_I(\mathrm{Dom}(\Psi_I)|_{V_I - V_i}),\\ D_{i;I} &\equiv W_{i;I} \cup \Psi_i(N_{i;I}) \end{aligned}$$

for all $i \in S$ and $I \subset S$. By (5-9), we have the natural identification

$$\mathcal{O}_X(V_i)|_{\Psi_i(N_{i;I})} = \pi^*_{\mathcal{N}_X V_i}\mathcal{N}_X(V_i)|_{N_{i;I}} \tag{5-17}$$

and by (5-6) and (5-10) we have the identity

$$\mathcal{O}_X(V_i)|_{W_{i;I}} = W_{i;I} \times \mathbb{C} \tag{5-18}$$

for all $i \in S$ and $I \subset S$. By using the natural projections

$$\mathrm{pr}_{I;I'}\colon \mathcal{N}_X V_I \to \mathcal{N}_X V_{I'}|_{V_I} \quad \text{for all } I' \subset I \subset S,$$

we have a map

$$\widehat{\Pi}_{V_i;I}\colon \mathcal{O}_X(V_i)|_{D_{I;i}} \to \mathcal{O}_X(V_i)|_{V_I}$$

for all $i \in S$ and $I \subset S$, whose restriction to $\Psi_i(N_{i;I})$ is defined by the equation

$$\widehat{\Pi}_{V_i;I}(v,w) \equiv \left(\mathrm{pr}_{I\cup i;i}(\mathfrak{D}\Psi^{-1}_{I\cup i;i}(v)), \mathrm{pr}_{I\cup i;i}(\mathfrak{D}\Psi^{-1}_{I\cup i;i}(w))\right) \quad \text{for all } (v,w) \in (\pi^*_{\mathcal{N}_X V_i}\mathcal{N}_X(V_i))|_{N_{i;I}}$$

by using the identity (5-17) and whose restriction to $W_{i;I}$ is defined by

$$\widehat{\Pi}_{V_i;I}(x,c) \equiv \left(\pi_{\mathcal{N}_X V_I}(\Psi_I^{-1}(x)), c\right) \in (V_I - V_i) \times \mathbb{C} \quad \text{for all } (x,c) \in W_{i;I} \times \mathbb{C}$$

by using the identity (5-18). One should think of this map as a way of canonically identifying the bundle $\mathcal{O}_X(V_i)$ near V_I with its pullback along the natural projection map from $D_{i;I}$ to V_I induced by $\pi_{\mathcal{N}_X V_I} \circ \Psi_I^{-1}$. By (5-9) and (5-11),

$$\Pi_{V_i;I}|_{\Psi_I^{-1}(D_{i;I})} = \widehat{\Pi}_{V_i;I} \circ s_{V_i} \circ \Psi_I \quad \text{for all } i \in S,\ I \subset S. \tag{5-19}$$

Also, by (5-8) and the fact that $\mathfrak{D}\Psi_{I;I'}$ is a product Hermitian isomorphism for all $I' \subset I \subset S$,

$$\widehat{\Pi}_{V_i;I} \circ \widehat{\Pi}_{V_i;I'}|_{D_{I;i}} = \widehat{\Pi}_{V_i;I} \quad \text{for all } i \in S,\ I' \subset I \subset S. \tag{5-20}$$

We can define similar maps for the line bundle $\mathcal{O}_X\big(\sum_i m_i V_i\big)$ in the following way:

$$\widehat{\Pi}_{(m_i)_{i\in S};I}\colon \mathcal{O}_X\Big(\sum_i m_i V_i\Big)\Big|_{\bigcap_{i\in S} D_{i;I}} \to \mathcal{O}_X\Big(\sum_i m_i V_i\Big)\Big|_{V_I},$$

$$\widehat{\Pi}_{(m_i)_{i\in S};I}\Big(\bigotimes_{i\in S,\ j\in\{1,\dots,m_i\}} v_{i,j}\Big) = \bigotimes_{i\in S,\ j\in\{1,\dots,m_i\}} \widehat{\Pi}_{V_i;I}(v_{i,j}) \quad \text{for all } I \subset S.$$

Equations (5-19) and (5-20) give us the equations

$$\Pi_{(m_i)_{i\in S};I}|_{\Psi_I^{-1}(\bigcap_{i\in S} D_{i;I})} = \widehat{\Pi}_{(m_i)_{i\in S};I} \circ s_{(m_i)_{i\in S}} \circ \Psi_I|_{\bigcap_{i\in S} D_{i;I}} \quad \text{for all } I \subset S \tag{5-21}$$

and

$$\widehat{\Pi}_{(m_i)_{i\in S};I} \circ \widehat{\Pi}_{(m_i)_{i\in S};I'}|_{\bigcap_{i\in S} D_{i;I}} = \widehat{\Pi}_{(m_i)_{i\in S};I} \quad \text{for all } I' \subset I \subset S. \tag{5-22}$$

Using these equations, we will now prove our lemma by induction on the set of subsets of S. Let $\preceq$ be a total order on the set of subsets of S with the property that if $|I'| < |I|$ then $I \preceq I'$. We write $I \prec I'$ when $I \preceq I'$ and $I \neq I'$. Since U is relatively compact, the tube radius of $\mathcal{R}$ along U is positive and hence we can choose any constant $R > 0$ smaller than this tube radius.

Suppose that there is some $I^* \subset S$ and a trivialization

$$\Phi^{\prec} \equiv (\pi, \Phi_2^{\prec})\colon \mathcal{O}_X\Big(\sum_i m_i V_i\Big) \to X \times \mathbb{C}$$

which is radius R compatible with $\mathcal{R}$ along $U \cap V_I$ for all $I \prec I^*$ and which is isotopic to Φ through trivializations of $\mathcal{O}_X\big(\sum_i m_i V_i\big)$. We now wish to modify the trivialization $\Phi^{\prec}$ so that these properties hold for all $I \preceq I^*$. Let $T_{r,I}$ be the radius r tube of V_I as in (5-14) and define $L_r^{\prec} \equiv \bigcup_{I \prec I^*} T_{r,I}$.

First of all, let

$$g\colon V_{I^*} - L_{3R/4}^{\prec} \to (0,\infty)$$

be a smooth function whose value at $x \in V_{I^*} - L_{3R/4}^{\prec}$ is equal to the norm of the linear map

$$\Phi_2^{\prec}|_{\bigotimes_{i\in I^*}(\mathcal{N}_X V_i|_x)^{\otimes m_i}}\colon \Big(\bigotimes_{i\in I^*} \mathcal{N}_X V_i|_x\Big)^{\otimes m_i} \to \mathbb{C}.$$

Since the map $\Pi_{V_i;I}$ restricted to each fiber of $\mathrm{pr}_{I;I-i}$ is an isometry for all $i \in I \subset S$, we get that $g(x) = 1$ for all $x \in V_{I^*} \cap (L^{\prec}_R - L^{\prec}_{3R/4})$ by our induction hypothesis. Combining this with the fact that $a_R(s) = 1$ for all $s \geq \frac{3}{4}R$, we can choose a smooth function $f\colon X \to (0,\infty)$ which is equal to 1 in the region $L^{\prec}_R$ and equal to $(\pi_{\mathcal{N}_X V_{I^*}} \circ \Psi^{-1}_{I^*})^*\big(\frac{1}{g}\big)$ inside $T_{R,I^*} \cap \Psi_{I^*}(\mathrm{Dom}(\Psi_{I^*})|_{V_{I^*}-L^{\prec}_{3R/4}})$. This implies that the norm of the linear map

$$f(x)\Phi^{\prec}_2|_{\bigotimes_{i\in I}(\mathcal{N}_X V_i|_x)^{\otimes m_i}}\colon \Big(\bigotimes_{i\in I} \mathcal{N}_X V_i|_x\Big)^{\otimes m_i} \to \mathbb{C} \tag{5-23}$$

is 1 for all $I \preceq I^*$. Hence, (5-16) holds for $f\Phi^{\prec}_2$ for all $I \preceq I^*$.

We now wish to modify $f\Phi^{\prec}$ so that (5-15) for this new trivialization holds as well. We let

$$\Phi^{=} \equiv (\pi, \Phi^{=}_2)\colon \mathcal{O}_X\Big(\sum_i m_i V_i\Big)\Big|_{\mathrm{Im}(\Psi_{I^*})} \to X \times \mathbb{C},$$

given by

$$\Phi^{=}(v) \equiv f(x)\Phi^{\prec}(\widehat{\Pi}_{(m_i)_{i\in S};I^*}(v)) \prod_{i\in I^*}\left(\frac{\sqrt{a_R(\rho_i(x))}}{\sqrt{\rho_i(x)}}\right)^{m_i} \quad \text{for all } v \in \mathcal{O}_X\Big(\sum_i m_i V_i\Big)\Big|_x,\ x \in \mathrm{Im}(\Psi_{I^*}) \tag{5-24}$$

be a smooth trivialization. Equation (5-22) combined with the fact that $\Phi^{\prec}_2$ is radius R compatible with $\mathcal{R}$ along $U \cap V_I$ for all $I \prec I^*$ implies that $\Phi^{\prec}_2$ is radius R compatible with $\mathcal{R}$ along $U \cap (L^{\prec}_R - L^{\prec}_{3R/4}) \cap V_{I^*}$. Hence, by (5-24), we have

$$\Phi^{\prec}_2(v) = \Phi^{=}_2(v) \quad \text{for all } v \in T_{R,I^*} \cap (L^{\prec}_R - L^{\prec}_{3R/4}).$$

Combining this with the fact that T_{R,I^*} deformation retracts onto $V_{I^*} \cup (T_{R,I^*} \cap L^{\prec}_R)$, we can construct a smooth trivialization

$$\Phi^{\preceq} \equiv (\pi, \Phi^{\prec}_2)\colon \mathcal{O}_X\Big(\sum_i m_i V_i\Big) \to X \times \mathbb{C}$$

homotopic to $\Phi^{\prec}$ so that

$$\Phi^{\preceq}|_{L^{\prec}_R} = \Phi^{\prec}|_{L^{\prec}_R} \quad \text{and} \quad \Phi^{\preceq}|_{T_{R,I^*}} = \Phi^{=}|_{T_{R,I^*}}. \tag{5-25}$$

Equations (5-21), (5-24) and (5-25) tell us that $\Phi^{\preceq}$ is radius R compatible with $\mathcal{R}$ along $U \cap V_{I^*}$. Also, since $\Phi^{\preceq} = \Phi^{\prec}$ along $L^{\prec}_R$, we get that $\Phi^{\preceq}$ is radius R compatible with $\mathcal{R}$ along $U \cap V_I$ for all $I \prec I^*$. Hence, $\Phi^{\preceq}$ is radius R compatible with $\mathcal{R}$ along $U \cap V_I$ for all $I \preceq I^*$.

Because the norm of the linear map (5-23) is 1 for all $I \preceq I^*$, we have by (5-24) and (5-25) that the norm of the linear map

$$f\Phi_2^{\preceq}|_{\bigotimes_{i\in I}(\mathcal{N}_X V_i|_x)^{\otimes m_i}} \colon \Big(\bigotimes_{i\in I} \mathcal{N}_X V_i|_x\Big)^{\otimes m_i} \to \mathbb{C}$$

is equal to 1 for all $I \preceq I^*$. Hence, we are done by induction. □

5.3 Links of divisors and open books

In this subsection, we first give a purely symplectic definition of a divisor which looks like the resolution divisors of a log resolution of an isolated hypersurface singularity. We then construct the "link" of this resolution divisor, which corresponds to the embedded link of our isolated singularity. Finally we construct an open book decomposition of this resolution divisor corresponding to the Milnor open book of our hypersurface singularity.

We have the following definition from [26]:

Definition 5.11 Let (X,ω) be a symplectic manifold of dimension $2n$ and let $\theta \in \Omega^1(X-K)$ satisfy $d\theta = \omega|_{X-K}$ for some compact $K \subset X$. Suppose that K admits an open neighborhood U which deformation retracts onto K. Let $\rho\colon X \to [0,1]$ be a smooth function equal to 0 along K and equal to 1 outside a compact subset of U. Then the *dual*

$$c(\omega,\theta) \in H_{2n-2}(K;\mathbb{R}) = H_{2n-2}(U;\mathbb{R})$$

of (ω,θ) is defined to be the Lefschetz dual of $(\omega - d(\rho\theta))|_U \in H^2_c(U)$.

Now suppose that $K = \bigcup_{i\in S} V_i$, where $(V_i)_{i\in S}$ is an SC divisor and each V_i is connected and compact. Then a Mayer–Vietoris argument tells us that $H_{2n-2}\big(\bigcup_{i\in S} V_i;\mathbb{R}\big)$ is freely generated by the fundamental classes $[V_i]$ and hence there are unique real numbers $(w_i)_{i\in S}$ such that $c(\omega,\theta) = -\sum_i w_i[V_i]$. The *wrapping number* of θ around V_j is defined to be $w(\theta,V_j) \equiv w_j$.

The wrapping number does not depend on the choice of neighborhood U or bump function ρ. We can calculate the wrapping number $w(\theta,V_j)$ in the following way (see the discussion after Definition 5.4 in [28]): Let $D \subset X$ be a small symplectically embedded disk with polar coordinates (r,ϑ) which intersects V_i positively once at $0 \in D$ and does not intersect V_j for all $j \in S - i$. Then the wrapping number w_i is the unique number such that $(w_i/2\pi)\,d\vartheta$ is cohomologous to

$$\theta|_{D-0} - \tfrac{1}{2}r^2\,d\vartheta$$

inside $H^1(D-0;\mathbb{R})$. The computation of the wrapping number using the disk D above enables us to define wrapping numbers in the case when each V_i is properly embedded but not necessarily compact. We will need this broader definition of wrapping number in the proof of Lemma 5.23 below.

The next definition is supposed to be a way of describing a log resolution of a pair $(\mathbb{C}^{n+1}, f^{-1}(0))$ (as in Definition 2.1) in a symplectic way. The key motivating example for such a definition is given in Example 5.14 below.

Definition 5.12 A *resolution divisor* is a pair $(X, (V_i)_{i\in S})$ where $(V_i)_{i\in S}$ are transversally intersecting properly embedded codimension 2 submanifolds of a manifold X such that there is a unique element $\star_S \in S$ with the property that $V_{\star_S}$ is noncompact and V_i is compact for all $i \in S - \star_S$. We also require that $\bigcup_i V_i$ is connected and that V_i is connected for each $i \in S - \star_S$ (although $V_{\star_S}$ is allowed to be disconnected).

A *model resolution* is a triple $\left(\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right), \Phi, \theta\right)$ where X is a manifold, $(V_i)_{i\in S}$ are codimension 2 submanifolds, $\theta \in \Omega^1\left(X - \bigcup_{i\in S-\star_S} V_i\right)$ and Φ is a trivialization of

$$\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right) \equiv \bigotimes_{i\in S} \mathcal{O}_X(V_i)^{\otimes m_i},$$

where $(m_i)_{i\in S}$ are positive integers satisfying:

1. $(X, (V_i)_{i\in S})$ is a resolution divisor as above.
2. $d\theta = \omega|_{X-\bigcup_{i\in S-\star_S} V_i}$ for some symplectic form ω on X.
3. $(V_i)_{i\in S}$ is an SC divisor with respect to ω.
4. $m_{\star_S} = 1$ and the wrapping number $w(\theta, V_i)$ is positive for all $i \in S - \star_S$.

The form ω is called the *symplectic form associated to* $\left(\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right), \Phi, \theta\right)$. A *grading* on this model resolution is a grading on $\left(X - \bigcup_{i\in S-\star_S} V_i, \omega\right)$.

Definition 5.13 Two model resolutions

$$Y \equiv \left(\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right), \Phi, \theta\right), \quad \hat{Y} \equiv \left(\mathcal{O}_{\hat{X}}\left(\sum_{i\in \hat{S}} \hat{m}_i \hat{V}_i\right), \hat{\Phi}, \hat{\theta}\right)$$

are *isotopic* if there is

- a bundle isomorphism $\tilde{\Psi}\colon \mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right) \to \mathcal{O}_{\hat{X}}\left(\sum_{i\in \hat{S}} \hat{m}_i \hat{V}_i\right)$ covering a diffeomorphism $\Psi\colon X \to \hat{X}$,

- a bijection $\iota\colon S \to \widehat{S}$ sending $\star_S$ to $\star_{\widehat{S}}$, and
- a smooth family of 1–forms $\bigl(\theta_t \in \Omega^1\bigl(X - \bigcup_{i \in S - \star_S} V_i\bigr)\bigr)_{t\in[0,1]}$ joining θ and $\Psi^*\widehat{\theta}$ and trivializations $(\Phi_t)_{t\in[0,1]}$ of $\mathcal{O}_X\bigl(\sum_{i\in S} m_i V_i\bigr)$ joining Φ and $\widehat{\Phi} \circ \widetilde{\Psi} \circ (\Psi^{-1} \times \mathrm{id}_{\mathbb{C}})$

such that $m_i = \widehat{m}_{\iota(i)}$ and $\Psi(V_i) = \widehat{V}_{\iota(i)}$ for all $i \in S$ and $Y_t \equiv (\mathcal{O}_X\bigl(\sum_{i\in S} m_i V_i\bigr), \Phi_t, \theta_t)$ is a model resolution for all $t \in [0,1]$.

These model resolutions are *graded isotopic* if they are isotopic as above with the additional property that the model resolutions Y_t all admit gradings that smoothly depend on $t \in [0,1]$ and where the grading on Y_0 coincides with the grading on Y and the grading on Y_1 coincides with the grading on $\widehat{Y}$ under the identification Ψ.

We will now give an example of a model resolution.

Example 5.14 Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial with an isolated singularity at 0 and let $U \subset \mathbb{C}^{n+1}$ be an open set containing 0 such that $f|_{U-0}$ is regular. Let $\pi\colon Y \to \mathbb{C}^{n+1}$ be a log resolution of $(\mathbb{C}^{n+1}, f^{-1}(0))$ obtained by a sequence of blowups along smooth loci and define $X \equiv \pi^{-1}(U)$ (such a resolution exists by [20; 21]). Let $(E_j)_{j\in S}$ be the resolution divisors of this resolution and let $E_{\star_S} \subset X$ be the proper transform of $f^{-1}(0) \cap U$. Because such a resolution is obtained by a sequence of blowups along smooth loci, there are positive integers $(w_j)_{j \in S - \star_S}$ such that $A \equiv -\sum_{i\in S-\star_S} w_i E_i$ is ample on X. By the divisor line bundle correspondence, let $L \to X$ be the corresponding ample line bundle with a meromorphic section satisfying $(s) = A$. Choose a Hermitian metric $\|\cdot\|$ on L whose curvature form is a positive one-to-one form. Define $\theta = -d^c \ln(\|s\|)$. The 2–form $d\theta$ extends uniquely to a Kähler form ω on X. The wrapping number of θ around E_j is w_j for all $j \in S - \star_S$.

We also let $m_i \in \mathbb{N}_{>0}$ be the multiplicity of f along E_i for all $i \in S$. Since $\sum_{i\in S} m_i E_i$ is the divisor defined by $f \circ \pi|_X$ we get, by the divisor line bundle correspondence, that the holomorphic line bundle $\mathcal{O}_X\bigl(\sum_{i\in S} m_i E_i\bigr)$ has an induced trivialization $\Phi\colon \mathcal{O}_X\bigl(\sum_{i\in S} m_i E_i\bigr) \cong X \times \mathbb{C}$ such that the section s corresponding to the holomorphic function $f \circ \pi|_X$ satisfies $\mathrm{pr}_2 \circ \Phi \circ s = f \circ \pi|_X$, where $\mathrm{pr}_2\colon X \times \mathbb{C} \to \mathbb{C}$ is the natural projection map. Then $\bigl(\mathcal{O}_X\bigl(\sum_{i\in S} m_i E_i\bigr), \Phi, \theta\bigr)$ is a model resolution, called a *model resolution associated to* f.

Such a resolution also has a grading as follows: Let $\check{X} \equiv X - \bigcup_{i\in S-\star_S} E_i$. Since

$$\pi|_{\check{X}}\colon \check{X} \to U - 0 \subset \mathbb{C}^{n+1}$$

is a biholomorphism, we get a canonical holomorphic trivialization

$$\Phi\colon T\check{X} \to \check{X} \times \mathbb{C}^{n+1}$$

as a unitary vector bundle coming from the trivialization on $T\mathbb{C}^{n+1}$. The grading on $T\check{X}$ is equal to the trivial grading on $\check{X} \times \mathbb{C}^{n+1}$ pulled back by Φ. We will call this the *standard grading*.

We now wish to associate a pair of contact manifolds with normal bundle data to a model resolution. In the case of Example 5.14, this is the link of our singularity (as defined in the introduction) with some additional data.

Definition 5.15 Let $(X, (V_i)_{i \in S})$ be a resolution divisor. A tuple of regularizations

$$(\Psi_i)_{i \in S - \star_S}$$

is *compatible with* $V_{\star_S}$ if Ψ_i is a regularization of V_i for each $i \in S - \star_S$ and

$$\Psi_i(\mathrm{Dom}(\Psi_i)|_{V_{\star_S} \cap V_i}) \subset V_{\star_S} \quad \text{for all } i \in S - \star_S.$$

In other words, Ψ_i restricted to $\mathrm{Dom}(\Psi_i)|_{V_{\star_S} \cap V_i}$ is a regularization of $V_i \cap V_{\star_S}$ inside $V_{\star_S}$ for all $i \in S - \star_S$.

Definition 5.16 Let Ψ and $\check{\Psi}$ be regularizations of a submanifold V of a manifold X. Then a smooth family of regularizations Ψ_t of V *connects* Ψ *and* $\check{\Psi}$ if Ψ is germ equivalent to Ψ_0 and $\check{\Psi}$ is germ equivalent to Ψ_1.

Lemma 5.17 *Let $(X, (V_i)_{i \in S})$ be a resolution divisor. For any two tuples of regularizations $(\Psi_i)_{i \in S - \star_S}$ and $(\check{\Psi}_i)_{i \in S - \star_S}$ compatible with $V_{\star_S}$, there is a smooth family of such regularizations*

$$(\Psi_i^t)_{i \in S - \star_S}, \quad t \in [0, 1],$$

compatible with $V_{\star_S}$ which connects $(\Psi_i)_{i \in S - \star_S}$ and $(\check{\Psi}_i)_{i \in S - \star_S}$.

Proof Choose a metric making $V_{\star_S}$ into a totally geodesic submanifold. Define $T^r X \subset TX$ to be the set of tangent vectors of length at most r. Fix $i \in S$. Choose a relatively compact neighborhood W_i of V_i in X and let $\delta > 0$ be small enough that the exponential map restricted to $T^\delta_w X$ is a diffeomorphism onto its image for all w in W_i. Let $\widetilde{W}_i \subset \Psi_i^{-1}(W_i) \cap (\check{\Psi}_i)^{-1}(W_i)$ be a small enough neighborhood of V_i that the distance $d(v)$ between $\Psi_i(v)$ and $\check{\Psi}_i(v)$ is less than δ for all $v \in \widetilde{W}_i$. Now let

$\gamma_v\colon [0, d(v)] \to X$ be the unique geodesic of length $< \delta$ joining $\Psi_i(v)$ and $\check{\Psi}_i(v)$ for all $v \in \widetilde{W}_i$. Define

$$\widetilde{\Psi}_i^t\colon \widetilde{W}_i \to X, \quad \widetilde{\Psi}_i^t(v) \equiv \gamma_v(t d(v)) \quad \text{for all } t \in [0,1].$$

Since $d_v\Psi$ and $d_v\check{\Psi}$ from Definition 5.2 are both the identity map, we get that $d_v\widetilde{\Psi}_i^t$ is also the identity map for all $v \in V_i$. Hence, there is a neighborhood $\widetilde{W}_i' \subset \widetilde{W}_i$ of V_i such that $\Psi_i^t \equiv \widetilde{\Psi}_i^t|_{\widetilde{W}'}$ is a diffeomorphism onto its image for all $t \in [0,1]$. Hence, $(\Psi_i^t)_{t\in[0,1]}$ is a smooth family of regularizations compatible with $V_{\star_S}$ which connects Ψ_i and $\check{\Psi}_i$. Therefore, $(\Psi_i^t)_{i\in S-\star_S}$ for $t \in [0,1]$ connects $(\Psi_i)_{i\in S-\star_S}$ and $(\check{\Psi}_i)_{i\in S-\star_S}$. □

Definition 5.18 Let X be a smooth manifold and let $(V_i)_{i\in\check{S}}$ be transversely intersecting compact codimension 2 submanifolds of X. A smooth function

$$f\colon X - \bigcup_{i\in\check{S}} V_i \to \mathbb{R}$$

is *compatible with* $(V_i)_{i\in\check{S}}$ if there is

- a regularization Ψ_i of V_i,
- a real number $b_i > 0$, and
- a smooth function $q_i\colon \mathrm{Dom}(\Psi_i) \to [0,1]$ equal to the square of some norm on $\mathcal{N}_X V_i$ near V_i, equal to 1 outside a compact subset of $\mathrm{Dom}(\Psi_i)$ and nonzero on $\mathrm{Dom}(\Psi_i) - V_i$

for each $i \in S$ and a smooth function $\tau\colon X \to \mathbb{R}$ such that

$$f = \sum_{i\in\check{S}} b_i \log(q_i \circ \Psi_i^{-1}) + \tau,$$

where $\log(q_i \circ \Psi_i^{-1})$ is defined to be 0 outside $\mathrm{Im}(\Psi_i)$ for each $i \in S$. We will call the regularizations $(\Psi_i)_{i\in S-\star_S}$ *associated regularizations of* f.

Now suppose that we have an additional smooth submanifold $V_{\star_S}$ of X such that $(V_i)_{i\in S}$ becomes a resolution divisor, where $S = \check{S} \sqcup \{\star_S\}$. We say that f is *compatible with* $(V_i)_{i\in S}$ if it is compatible with $(V_i)_{i\in S-\star_S}$ as above with the additional property that the associated regularizations of f are compatible with $V_{\star_S}$. As a consequence of this we have that $f|_{V_{\star_S}}$ is compatible with $(V_i \cap V_{\star_S})_{i\in S-\star_S}$.

We say that f is *strongly compatible* with $(V_i)_{i\in S}$ if in addition $\tau = 0$.

Lemma 5.19 *Let $(X,(V_i)_{i\in S})$ be a resolution divisor and let*

$$f,g\colon X-\bigcup_{i\in S-\star_S} V_i\to\mathbb{R}$$

be a pair of smooth functions compatible with $(V_i)_{i\in S}$. Then there is a smooth family of functions

$$f_t\colon X-\bigcup_{i\in S-\star_S} V_i\to\mathbb{R},\quad t\in[0,1],$$

compatible with $(V_i)_{i\in S}$ such that $f_0=f$ and $f_1=g$.

Proof For all $i\in S-\star_S$, there are regularizations Ψ_i and $\check{\Psi}_i$ of V_i, smooth functions

$$q_i\colon \mathrm{Dom}(\Psi_i)\to[0,1],\quad \check{q}_i\colon \mathrm{Dom}(\check{\Psi}_i)\to[0,1]$$

equal to the square of a norm near V_i, equal to 1 outside a compact set and nonzero outside V_i, real numbers $b_i,\check{b}_i>0$ and smooth functions $\tau,\check{\tau}\colon X\to\mathbb{R}$ such that

$$f=\sum_{i\in S-\star_S} b_i\log(q_i\circ\Psi_i^{-1})+\tau\quad\text{and}\quad g=\sum_{i\in S-\star_S}\check{b}_i\log(\check{q}_i\circ\check{\Psi}_i^{-1})+\check{\tau}.$$

First of all, we can smoothly deform f and g through smooth functions compatible with $(V_i)_{i\in S}$ by changing q_i, $\check{q}_i$, b_i, $\check{b}_i$ and τ and $\check{\tau}$ such that $q_i=\check{q}_i$, $b_i=\check{b}_i$ and $\tau=\check{\tau}=0$. Hence, we can assume that

$$f=\sum_{i\in S-\star_S} b_i\log(q_i\circ\Psi_i^{-1})\quad\text{and}\quad g=\sum_{i\in S-\star_S} b_i\log(q_i\circ\check{\Psi}_i^{-1}).$$

Lemma 5.17 tells us that there is a smooth family of regulations $(\Psi_i^t)_{i\in S-\star_S}$ for $t\in[0,1]$ compatible with $V_{\star_S}$ connecting $(\Psi_i)_{i\in S-\star_S}$ and $(\check{\Psi}_i)_{i\in S-\star_S}$ and hence we get a smooth family of functions

$$f_t=\sum_{i\in S-\star_S} b_i\log(q_i\circ(\Psi_i^t)^{-1})$$

compatible with $(V_i)_{i\in S}$ $\big($after possibly shrinking the region on which q_i is not equal to 1 so that it fits inside $\bigcap_{t\in[0,1]}\mathrm{Dom}(\Psi_t)\big)$. This is a smooth family of functions compatible with $(V_i)_{i\in S}$ joining f and g. □

Lemma 5.20 *Let (X,ω) be a symplectic manifold with a choice of grading and $C\subset X$ a contact hypersurface with a contact form α compatible with the contact structure satisfying $d\alpha=\omega|_C$. Then $(C,\ker(\alpha))$ has a natural induced choice of grading.*

We will call such a grading the *induced grading on C*.

Proof Let X_α be a smooth section of the bundle $TX|_C \to C$ equal to the ω–dual of α. Since $d\alpha = \omega|_C$ and α is a contact form, we get that X_α is transverse to C. Let R be the Reeb vector field of α and define $\xi_C \equiv \ker(\alpha)$. Let $H \subset TX|_C$ be the 2–dimensional symplectic vector subbundle spanned by X_α and R. Then we have the direct sum decomposition of symplectic vector bundles

$$(TM,\omega)|_C \cong (\xi_C, d\alpha) \oplus (H, \omega|_H).$$

Since X_α, R is a symplectic basis for H at each point of C, we have a natural symplectic trivialization of $(H,\omega|_H)$ sending X_α and R to the standard symplectic basis vectors on $\mathbb{C}$. Hence, we have a natural isomorphism

$$(TM,\omega)|_C \cong (\xi_C, d\alpha) \oplus (\mathbb{C}, \omega_{\mathrm{std}}). \tag{5-26}$$

Now choose an almost complex structure J on M compatible with ω such that its restriction to $TM|_C$ is equal to $J_C \oplus i$ with respect to the splitting (5-26), where J_C is an almost complex structure on ξ_C compatible with $d\alpha|_{\xi_C}$ and i is the standard complex structure on $\mathbb{C}$.

Now we will use the natural correspondence between gradings and trivializations of the canonical bundle as stated in Appendix A. Let $\Phi\colon \kappa_J \to X \times \mathbb{C}$ be the choice of trivialization of the canonical bundle of (TM, J) associated to the grading on (X,ω) (see Definition A.7). Since $J|_C = J_C \oplus i$ under the splitting (5-26), we get a natural trivialization $\Phi_C\colon \kappa_{J_C} \to C \times \mathbb{C}$ induced from the trivialization $\Phi|_C$. The induced grading on (C, ξ_C) is then the grading associated to the trivialization Φ_C as in Definition A.7. □

We wish to use functions compatible with a resolution divisor to construct its "link". The following proposition tells us how to at least start doing this.

Proposition 5.21 *Let $\left(\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right), \Phi, \theta\right)$ be a model resolution. Define $K \equiv \bigcup_{i \in S - \star_S} V_i$. Let $f\colon X - K \to \mathbb{R}$ be a smooth function compatible with $(V_i)_{i\in S}$. Then there is a smooth function $g\colon X - K \to \mathbb{R}$ and an open neighborhood U of K such that $df(X^{\omega}_{\theta+dg})|_U > 0$ and $df_\star(X^{\omega_\star}_{\theta_\star + dg_\star})|_{U\cap V_{\star_S}} > 0$, where ω is the symplectic form associated to our model resolution, $\omega_\star \equiv \omega|_{V_{\star_S}}$, $f_\star \equiv f|_{V_{\star_S} - K}$ and $g_\star \equiv g|_{V_{\star_S} - K}$.*

The proof of this proposition is almost exactly the same as the proof of Proposition 5.8 of [28]. The only difference is that we have to take into account the additional submanifold $V_{\star_S}$. For the sake of completeness we have produced the proof below. We also have a parametrized version of the proposition above.

Proposition 5.22 *Let $\mathcal{M}_t \equiv \big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta_t\big)$ for $t \in [0,1]$ be a smooth family of model resolutions. Define $K \equiv \bigcup_{i\in S-\star_S} V_i$. Let $f_t\colon X-K \to \mathbb{R}$ for $t \in [0,1]$ be a smooth family of functions compatible with $(V_i)_{i\in S}$. Then there is a smooth family of functions $g_t\colon X-K \to \mathbb{R}$ for $t\in[0,1]$ and an open neighborhood U of K such that $df_t(X^{\omega_t}_{\theta+dg_t})|_U > 0$ and $df_{\star,t}(X^{\omega_{\star,t}}_{\theta_\star+dg_{\star,t}})|_{U\cap V_{\star_S}} > 0$, where ω_t is the symplectic form associated to $\mathcal{M}_t$, $\omega_{\star,t} \equiv \omega_t|_{V_{\star_S}}$, $f_{\star,t} \equiv f_t|_{V_{\star_S}-K}$ and $g_{\star,t} \equiv g_t|_{V_{\star_S}-K}$ for all $t\in[0,1]$.*

The proof of this proposition is almost exactly the same as the proof of Proposition 5.21 except that all variables are now parametrized by t. Therefore, for notational simplicity we will just prove Proposition 5.21. Before we prove this we need a few preliminary technical lemmas. The following lemma is very similar to [28, Lemma 5.12]. This lemma should be thought of as a local version of Proposition 5.21.

Lemma 5.23 *Let $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ be a model resolution and fix a metric $\|\cdot\|$ on X. Define $K \equiv \bigcup_{i\in S-\star_S} V_i$. Fix $I \subset S$ and let ω be the symplectic form associated to our model resolution. Let $U \subset K$ be an open set with the property that $\overline{U} \cap V_{I'}$ is contained inside a contractible Darboux chart of $V_{I'}$ for all $I' \subset S$ and such that $\overline{U} \cap V_i = \varnothing$ for all $i \in S-I$.*

Then there is a smooth function $g\colon X-K \to \mathbb{R}$ such that for any function $f\colon X-K\to\mathbb{R}$ compatible with $(V_i)_{i\in S}$, we have

(1) *$df(X^{\omega}_{\theta+dg})|_{W_f} > c_f\|\theta+dg\|\|df\|\big|_{W_f}$ for some constant $c_f>0$ and some small neighborhood W_f of $\overline{U}\cap V_I$,*

(2) *$df_\star(X^{\omega_\star}_{\theta_\star+dg_\star})|_{W_{f,\star}} > c_f\|\theta_\star + dg_{p,\star}\|\|df_\star\|\big|_{W_{f,\star}}$, where $f_\star \equiv f|_{V_{\star_S}-K}$, $W_{f,\star} \equiv W_{f,\star}\cap V_{\star_S}$, $\theta_\star \equiv \theta|_{V_{\star_S}-K}$ and $g_\star \equiv g|_{V_{\star_S}-K}$, and*

(3) *$a_1\|df\| < \|\theta+dg\| < a_2\|df\|$ inside W_f for some constants $a_1,a_2>0$.*

Proof Define $\widehat{I} \equiv I - \star_S$ and let n be the dimension of X divided by 2. Since $\overline{U}\cap V_i$ is contained inside a contractible Darboux chart we have, by a Moser argument, symplectic coordinates $x^i_1, y^i_1, \dots, x^i_n, y^i_n$ defined on some neighborhood W_i of $\overline{U}\cap V_i$ in X such that $V_i \cap W_i = \{x^i_1 = y^i_1 = 0\}$ for each $i \in \widehat{I}$. We can also choose W_i so that $W_i \cap V_j = \varnothing$ for all $j \in S-I$ and so that $\overline{W}_i$ is a compact contractible codimension 0 submanifold of X with boundary such that $\partial\overline{W}_i$ and $(V_j)_{j\in S}$ are transversely intersecting. Define $W'_i \equiv W_i - V_i$ and let $P_i\colon \widetilde{W}'_i \to W'_i$ be the cover corresponding to the subgroup of $\pi_1(W'_i)$ generated by loops wrapping around V_i

near V_i for each $i \in \hat{I}$. In other words, the cover corresponding to the image of $\pi_1(T_i - V_i)$ in $\pi_1(W_i')$, where T_i is a small tubular neighborhood of $V_i \cap W_i$ in W_i. Let $r_i \colon W_i' \to \mathbb{R}$ and $\vartheta_i \colon W_i' \to \mathbb{R}/2\pi\mathbb{Z}$ be functions satisfying $x_1^i = r_i \cos(\vartheta_i)$ and $y_1^i = r_i \sin(\vartheta_i)$. Define $\rho_i \equiv \frac{1}{2}(r_i)^2$, $\tilde{\rho}_i \equiv P_i^* \rho_i$, $\tilde{y}_j^i \equiv P_i^* y_j^i$ and $\tilde{x}_j^i \equiv P_i^* x_j^i$ for all $i \in \hat{I}$ and $j \in \{1, \dots, n\}$. Let $\tilde{\vartheta}_i \colon \widetilde{W}_i' \to \mathbb{R}$ be a smooth function whose value mod 2π is equal to $P_i^* \vartheta_i$. Also let $\tilde{\omega} \equiv P_i^*(\omega|_{W_i'})$ and $\tilde{\theta} \equiv P_i^*(\theta|_{W_i'})$.

Then $\tilde{\omega} = d(\tilde{\rho}_i) \wedge d(\tilde{\vartheta}^i) + \sum_{j=2}^n d(\tilde{x}_j^i) \wedge d(\tilde{y}_j^i)$. Therefore, there is a natural symplectic embedding of $\iota_i \colon \widetilde{W}_i' \hookrightarrow \mathbb{C}^{n+1}$ such that the standard symplectic coordinates $x_1, y_1, \dots, x_n, y_n$ in $\mathbb{C}^n$ restricted to $\widetilde{W}_i'$ are $\tilde{\rho}_i, \tilde{\vartheta}_i, \tilde{x}_2^i, \tilde{y}_2^i, \dots, \tilde{x}_n^i, \tilde{y}_n^i$, respectively. Let $\widehat{W}_i' \subset \mathbb{C}^{n+1}$ be the closure of $P_i^{-1}(W_i')$ inside $\mathbb{C}^{n+1}$ and let $\hat{V}_{I-i}'$ be the closure of $P_i^{-1}(V_{I-i})$ inside $\mathbb{C}^{n+1}$. Then $\widehat{W}_i'$ is a codimension 0 submanifold of $\mathbb{C}^{n+1}$ with boundary and corners and $\hat{V}_{I-i}'$ is a codimension $2(|I|-1)$ submanifold of $\widehat{W}_i'$ with boundary and corners where one part of the boundary is $\check{V}_{I-i} \equiv \{x_1 = 0\} \cap \hat{V}_{I-i}'$. Let $\overline{W}_i$ be the closure of W_i in X. The map P_i extends to a map $\overline{P}_i \colon \widehat{W}_i' \to \overline{W}_i$ whose fibers over $\overline{W}_i \cap V_i$ are 1–dimensional. Also $\check{V}_{I-i}$ is equal to $\overline{P}_i^{-1}(V_I)$ and $\hat{V}_{I-i}' = \overline{P}_i^{-1}(V_{I-i})$. See Figure 2.

Let $w_i > 0$ be the wrapping number of θ around V_i. Let $H \subset T\mathbb{C}^{n+1}|_{\check{V}_{I-i}}$ be a 2–dimensional symplectic subbundle over $\check{V}_{I-i}$ containing

$$\ker D\overline{P}_i|_{\check{V}_{I-i}} = \operatorname{Span}\Big(\frac{\partial}{\partial y_1}\Big)\Big|_{\check{V}_{I-i}}$$

such that H is contained in $T\hat{V}_{I-i}'$. Let $T^\perp \hat{V}_{I-i}'$ be the set of vectors which are symplectically orthogonal to $T\hat{V}_{I-i}'$ and define $\hat{H} \equiv H \oplus T^\perp \hat{V}_{I-i}'|_{\check{V}_{I-i}}$.

Choose a smooth function $\tilde{g}_i \colon \mathbb{C}^{n+1} \to \mathbb{R}$ so that

(a) $\tilde{g}_i(x_1, y_1, \dots, x_n, y_n) = \tilde{g}_i(x_1, y_1 + 2\pi, x_2, y_2, \dots, x_n, y_n) + w_i$,

(b) $dx_1(X_{d\tilde{g}_i}) > 0$ at each point of $\check{V}_{I-i}$, and

(c) the $\omega_{\mathbb{C}^{n+1}}|_{\hat{H}}$–dual of $d\tilde{g}_i|_{\hat{H}}$ is tangent to $\hat{V}_{I-i}'$ at each point of $\check{V}_{I-i}$, where $\omega_{\mathbb{C}^{n+1}}$ is the standard symplectic structure on $\mathbb{C}^{n+1}$.

Condition (c) implies that $X_{d\tilde{g}_i}$ is tangent to $\hat{V}_{I-i}'$ at each point of $\check{V}_{I-i}$. By (a) there is a closed 1–form $\beta_i \in \Omega^1(W_i')$ whose pullback to $\widetilde{W}_i'$ is equal to $d\tilde{g}_i|_{\widetilde{W}_i'}$.

Define $W' \equiv \bigcap_{i \in \hat{I}} W_i$. Let $\theta_1 \in \Omega^1(W')$ be any 1–form of bounded norm satisfying $d\theta_1 = \omega|_{W'}$. Define

$$\Theta \in \Omega^1(W' - K), \quad \Theta \equiv \theta_1 + \sum_{i \in \hat{I}} \beta_i.$$

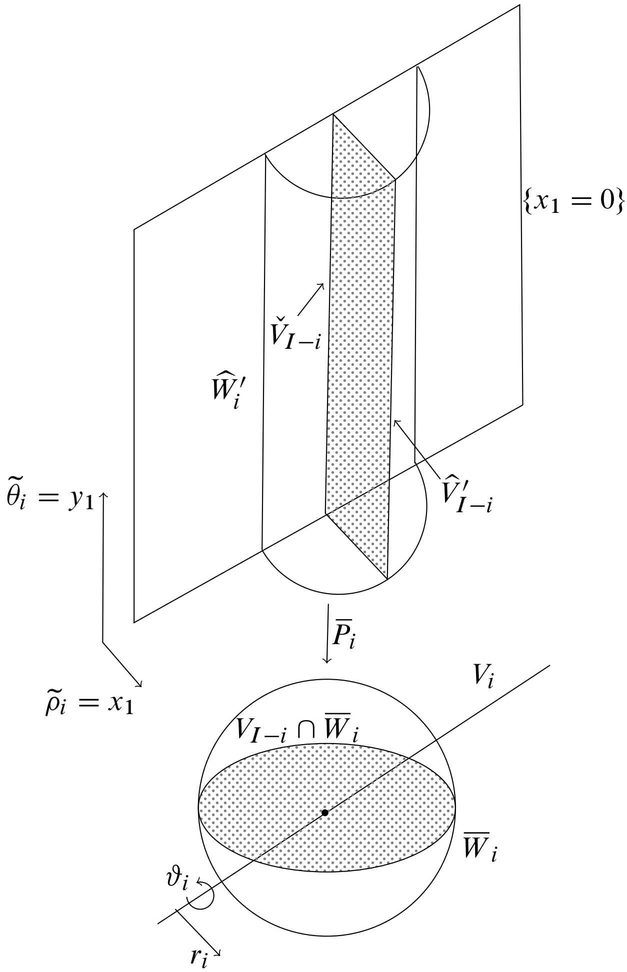


Figure 2

By (a), we get that the wrapping number of Θ around $V_i \cap W'$ is w_i for all $i \in \widehat{I}$. Hence (after shrinking $(W_i)_{i\in\widehat{I}}$ so that W' deformation retracts onto $W' \cap K$), there is a function $g\colon X - K \to \mathbb{R}$ such that $(\theta + dg)|_{W'} = \Theta|_{W'}$.

Let $f\colon X - K \to \mathbb{R}$ be compatible with $(V_i)_{i\in S}$ and let $(\Psi_i)_{i\in S-\star_S}$ be its associated regularizations. Then, by definition, $f|_{W'} = \sum_{i\in\widehat{I}} b_i \log(q_i \circ \Psi_i^{-1}) + \tau$, where

- $\tau\colon W' \to \mathbb{R}$ is a smooth function and $b_i > 0$ are constants for all $i \in \widehat{I}$,
- $q_i\colon \mathrm{Dom}(\Psi_i) \to \mathbb{R}$ is equal to a square norm near V_i, equal to 1 outside a compact subset of $\mathrm{Dom}(\Psi_i)$ and nonzero on $\mathrm{Dom}(\Psi_i) - V_i$, and
- $\log(q_i \circ \Psi_i^{-1})$ is defined to be zero outside $\mathrm{Im}(\Psi_i)$.

Define $s_i \equiv \log(q_i \circ \Psi_i^{-1})$ and $s_{i,\star} \equiv s_i|_{V_{\star_S}-K}$. Define $\beta_{i,\star} \equiv \beta_i|_{V_{\star_S}\cap(W'-K)}$. Since

$$\frac{c_1}{r_i} < \|\beta_i\| < \frac{c_2}{r_i}, \qquad \frac{c_1}{r_i} < \|ds_i\| < \frac{c_2}{r_i} \tag{5-27}$$

for some $c_1, c_2 > 0$ and $\|\theta_1\|$ is bounded, part (3) of our lemma holds.

Since $\|\tau\|$ and $\|\theta_1\|$ are bounded and since (5-27) holds, it is sufficient for us to prove the following statements:

(1) $ds_i\big(\sum_{j\in\hat I} X_{\beta_i}\big) > c_f/(r_i)^2$ inside some small neighborhood W_f of $W' \cap V_I$ and some constant $c_f > 0$ for all $i \in \hat I$.

(2) $ds_{i,\star}\big(\sum_{j\in\hat I} X_{\beta_{j,\star}}\big) > c_f/(r_i)^2$ inside $W_f \cap V_{\star_S}$ for all $i \in \hat I$.

Let $\hat V_{\star_S}$ be closure of $P_i^{-1}(V_{\star_S})$ inside $\mathbb{C}^{n+1}$. Let $|\cdot|$ be the standard norm on $\mathbb{C}^{n+1}$. Since there is a diffeomorphism $\Phi_i\colon W_i \to W_i$ which is the identity on $V_i \cap W_i$, fixing $V_j \cap W_i$, for all $j \in I - i$, which is also isotopic to the identity through such diffeomorphisms and, pulling back s_i to $\log(\rho_i)$, we have that inequalities (1) and (2) above can be deduced from

(1) $d\tilde\rho_i(D\tilde\Phi_i(X_{d\tilde g_i})) > c'_f$ inside $\widetilde W_f - \bar P_i^{-1}(V_i)$ for some small neighborhood $\widetilde W_f$ of $\check V_{I-i}$ and some constant $c'_f > 0$, where $\tilde\Phi_i\colon \widetilde W'_i \to \widetilde W'_i$ is a lift of $\Phi|_{W'_i}\colon W'_i \to W'_i$,

(2) $d\tilde\rho_i(D\tilde\Phi_i(X_{P_i^*\beta_j})) \to 0$ as we approach $\check V_{I-i}$ for all $j \in \hat I - i$, and

(3) $X_{d\tilde g_i}$ is tangent to $\hat V_{\star_S}$ along $\check V_{I-i}$ for all $i \in \hat I$.

These properties follow from (a)–(c) above. □

Lemma 5.24 *Let $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ be a model resolution and fix a metric $\|\cdot\|$ on X. Define $K \equiv \bigcup_{i\in S-\star_S} V_i$ and let $f\colon X - K \to \mathbb{R}$ be a smooth function compatible with $(V_i)_{i\in S}$. Then there is a smooth function $h\colon X - K \to \mathbb{R}$ and constants $a_1, a_2 > 0$ such that $a_1\|df\| < \|\theta + dh\| < a_2\|df\|$ near K.*

Proof We will use Lemma 5.23(3) and an induction argument to do this. Choose open sets $U_1, \dots, U_m$ in X along with subsets $I_1, \dots I_m \subset S$ such that

- $\bigcup_{j=1}^m (U_j \cap V_{I_j}) = K$,
- $\bar U_j \cap V_{I'}$ is contained inside a contractible Darboux chart of $V_{I'}$ for all $I' \subset S$ and $j \in \{1, \dots, m\}$, and
- $\bar U_j \cap V_k = \varnothing$ for all $k \in S - I_j$ and all $j \in \{1, \dots, m\}$.

Let $\check{U}_1, \dots, \check{U}_m$ be open sets of X such that the closure of $\check{U}_i$ is contained inside U_i for all $i \in \{1, \dots, m\}$ and $\bigcup_{j=1}^m (\check{U}_j \cap V_{I_j}) = K$. Define $U_{<k} \equiv \bigcup_{j<k} U_j$ and $\check{U}_{<k} \equiv \bigcup_{j<k} \check{U}_j$.

Suppose, by induction, there is a smooth function $h_\prec \colon X - K \to \mathbb{R}$ and constants $a_1^\prec, a_2^\prec > 0$ such that

$$a_1^\prec \|df\| < \|\theta + dh_\prec\| < a_2^\prec \|df\| \tag{5-28}$$

on a neighborhood $N \subset U_{<k}$ of the closure of $\check{U}_{<k} \cap K$ for some $k \in \{1, \dots, m\}$. By Lemma 5.23, there is a function $h_= \colon X - K \to \mathbb{R}$ and constants $a_1^=, a_2^= > 0$ such that

$$a_1^= \|df\| < \|\theta + dh_\prec + dh_=\| < a_2^= \|df\| \tag{5-29}$$

on a neighborhood W_k of $\overline{U_k \cap K}$ in X. Now let $\rho \colon X \to [0,1]$ be a smooth function equal to 0 on a neighborhood of $\check{U}_{<k} \cap K$ and which is 1 outside a compact subset of N. Define $h_\preceq \equiv h_\prec + \rho h_=$. Equations (5-28) and (5-29) tell us that $\|dh_=\| \le (a_2^\prec + a_2^=)\|df\|$ near $U_k \cap (N - \check{U}_{<k}) \cap K$, which implies that $|h_=| < C|f| + \check{C}$ near $(N - \check{U}_{<k}) \cap K$ for some $C, \check{C} > 0$. Hence,

$$a_1^\preceq \|df\| < \|\theta + dh_\preceq\| < a_2^\preceq \|df\|$$

near $\overline{\check{U}_{<k+1}} \cap K$ for some $a_1^\preceq, a_2^\preceq > 0$ and so we are finished by induction. □

Proof of Proposition 5.21 By Lemma 5.24 we can add an exact 1–form to θ so that

$$b_1 \|df\| < \|\theta\| < b_2 \|df\| \tag{5-30}$$

inside a neighborhood N of K for some constants $b_1, b_2 > 0$. By Lemma 5.23 we can find open sets $W_1, \dots, W_m$ of X covering K, smooth functions $g_i \colon X - K \to \mathbb{R}$ for $i = 1, \dots, m$, and a constant $c > 0$ such that

(1) $df(X^\omega_{\theta + dg_i})|_{W_i} > c\|\theta + dg_i\| \|df\| \big|_{W_i}$,
(2) $df_\star(X^{\omega_\star}_{\theta_\star + dg_i})|_{W_\star} > c\|\theta + dg_{i,\star}\| \|df_\star\| \big|_{W_{i,\star}}$, where $f_\star \equiv f|_{V_{\star S} - K}$, $W_{i,\star} = W_i \cap V_{\star S}$, $\theta_\star = \theta|_{V_{\star S} - K}$ and $g_{i,\star} = g_i|_{V_{\star S} - K}$, and
(3) $c\|df\| < \|\theta + dg_i\| < \check{c}\|df\|$ inside W_i for some constants $c, \check{c} > 0$.

Now choose smooth functions ρ_i for $i = 1, \dots, m$ so that $\sum_{i=1}^m \rho_i|_K = 1$ and $\rho_i = 0$ outside a compact subset of W_i for each $i = 1, \dots, m$. We define

$$g \colon X - K \to \mathbb{R}, \quad g \equiv \sum_{i=1}^m \rho_i g_i.$$

We define $g_\star \equiv g|_{V_{\star_S}-K}$. The inequality (5-30) combined with property (3) above tells us that $|g_i| < C|f| + \check{C}$ for some $C, \check{C} > 0$ near $W_i \cap K$. This means that $df(X^\omega_{\theta+dg}) > c\|\theta + dg\|\|df\|$ near K, $df(X^\omega_{\theta_\star+dg_\star}) > c\|\theta_\star + dg_\star\|\|df\|$ near $V_{\star_S} \cap K$ and $a_1\|df\| < \|\theta + dg\| < a_2\|df\|$ near K for some constants $a_1, a_2 > 0$. □

Definition 5.25 The *link* of a model resolution $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ is a contact pair with normal bundle data $(B \subset C, \xi_C, \Phi_B)$ defined as follows: By [39, Lemma 4.1], there is a tuple of regularizations $(\Psi_i)_{i\in S-\star_S}$ compatible with $V_{\star_S}$ as in Definition 5.15. Define $K \equiv \bigcup_{i\in S-\star_S} V_i$. Let $f: X - K \to \mathbb{R}$ be a smooth function compatible with $(V_i)_{i\in S}$ such that $(\Psi_i)_{i\in S-\star_S}$ are associated regularizations of f. Then $f_\star \equiv f|_{V_{\star_S}}$ is a smooth function compatible with $(V_i \cap V_{\star_S})_{i\in S-\star_S}$. Hence, by Proposition 5.21 we have that $df(X_{\theta+dg}) > 0$ and $df_\star(X_{\theta_\star+dg_\star}) > 0$ in an open neighborhood U of K for some smooth function $g: X - K \to \mathbb{R}$, where $g_\star \equiv g|_{V_{\star_S}-K}$ and $\theta_\star \equiv \theta|_{V_{\star_S}-K}$. Let $c \ll -1$ be a constant satisfying $f^{-1}(c) \subset U$. Define

$$C \equiv f^{-1}(c), \quad B \equiv C \cap V_{\star_S}, \quad \xi_C \equiv \ker(\theta + dg)|_C.$$

Finally, the trivialization Φ_B of the normal bundle of B in C is induced from the trivialization Φ since the normal bundle of B is naturally isomorphic as an oriented vector bundle to $\mathcal{N}_X V_{\star_S}|_B$, which in turn is naturally isomorphic to $\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big)\big|_B$ since $m_{\star_S} = 1$.

If $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ has a grading then this gives us an induced grading on the link as in the proof of Lemma 5.20 since $C - B$ is a contact hypersurface of $\big(X - \bigcup_{i\in S-\star_S} V_i, \omega\big)$, where ω is the symplectic form associated to our model resolution. We will call this the *induced grading* on $(B \subset C, \xi_C, \Phi_B)$.

The link does not depend on the choice of neighborhood U, constant C or function f by the following lemma:

Lemma 5.26 *Suppose that* $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ *and* $\big(\mathcal{O}_{\hat{X}}\big(\sum_{i\in \hat{S}} \hat{m}_i \hat{V}_i\big), \hat{\Phi}, \hat{\theta}\big)$ *are (graded) isotopic. Then their links are also (graded) isotopic for any choice of neighborhood U, constant C or function f chosen for each of these two model resolutions.*

Proof This follows from Lemma 5.19 and Proposition 5.22. □

5.4 Constructing a contact open book from a model resolution

The aim of this section is to construct a contact open book for each model resolution such that the contact pair associated to this open book is the link of our model resolution.

Definition 5.27 Let $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ be a model resolution with associated symplectic structure ω. Suppose $(V_i)_{i\in S}$ admits an ω–regularization

$$\mathcal{R} \equiv ((\rho_i)_{i\in S}, (\Psi_I)_{I\subset S}) \tag{5-31}$$

with associated Hermitian structures $(\rho_{I;i}, \nabla^{(I;i)})$ on $\mathcal{N}_X V_i|_{V_I}$ for each $i \in I \subset S$. Let $\alpha_{I;i} \equiv \alpha_{\rho_{I;i},\nabla^{(I;i)}} \in \Omega^1(\mathcal{N}_X V_i|_{V_I} - V_I)$ be the associated Hermitian connection 1–form on $\mathcal{N}_X V_i|_{V_I}$. Define $K \equiv \bigcup_{i\in S-\star_S} V_i$. Let w_i be the wrapping number of θ around V_i for each $i \in S - \star_S$ and define $a_{\star_S} \equiv 0$. Let

$$\mathrm{pr}_{I;i}\colon \mathcal{N}_X V_I \to \mathcal{N}_X V_i|_{V_I} \tag{5-32}$$

be the natural projection map for all $I \subset S$. We say that θ is *compatible with* $\mathcal{R}$ if the restriction of

$$(\Psi_I)^*\theta - \sum_{i\in I} \mathrm{pr}^*_{I;i}\left(\left(\rho_{I;i} + \frac{w_i}{2\pi}\right)\alpha_{I;i}\right) \tag{5-33}$$

to each fiber of $\pi_{\mathcal{N}_X V_I}|_{\Psi_I^{-1}(X-K)}$ is 0 for every $I \subset S$.

Lemma 5.28 *Let $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ be a model resolution such that $(V_i)_{i\in S}$ admits an ω–regularization $\mathcal{R}$. Define $K \equiv \bigcup_{i\in S-\star_S} V_i$. Then there is a smooth function $g\colon X - K \to \mathbb{R}$ such that $\theta + dg$ is compatible with $\check{\mathcal{R}}$ for some regularization $\check{\mathcal{R}}$ which is germ equivalent to $\mathcal{R}$.*

Proof This is done by induction on the strata of $\bigcup_i V_i$. We will use the notation from Definition 5.27 above. Let $\preceq$ be a total order on the set of subsets of S with the property that if $|I'| < |I|$ then $I \preceq I'$. We write $I \prec I'$ when $I \preceq I'$ and $I \neq I'$. Suppose, for some $I^* \subset S$, we have constructed open sets $U_I^{\prec}$ inside $\mathrm{Dom}(\Psi_I)$ containing V_I for all $I \prec I^*$ and a smooth function $g^{\prec}\colon X - K \to \mathbb{R}$ with the property that

$$(\Psi_I)^*(\theta + dg^{\prec}) - \sum_{i\in I} \mathrm{pr}^*_{I;i}\left(\left(\rho_{I;i} + \frac{w_i}{2\pi}\right)\alpha_{I;i}\right)$$

vanishes along each fiber of $\pi_{\mathcal{N}_X V_I}|_{U_I^{\prec}\cap \mathrm{Dom}(\Psi_I)}$ for all $I \prec I^*$.

We now want these properties to hold for all $I \preceq I^*$. For each $I \preceq I^*$, let $U_I^{\preceq} \subset \mathrm{Dom}(\Psi_I)$ be an open set containing V_I whose closure is compact such that the closure of $U_I^{\preceq}$ is contained in $U_I^{\prec}$ when $I \prec I^*$ and such that $\pi_{\mathcal{N}_X V_I}|_{U_I^{\preceq}}$ has contractible fibers for all $I \preceq I^*$. Since the wrapping number of θ around V_i is w_i for all $i \in I$ and since $(\Psi_I)^*(d\theta) = \omega_{(\rho_{I;i}, \nabla^{(I;i)})_{i\in I}}|_{\mathrm{Dom}(\Psi_I)}$, we get that the restriction of

$$A \equiv (\Psi_{I^*})^*(\theta + dg^{\prec}) - \sum_{i\in I^*} \mathrm{pr}^*_{I^*;i}\left(\left(\rho_{I^*;i} + \frac{w_i}{2\pi}\right)\alpha_{I^*;i}\right)$$

to each fiber of $\pi_{\mathcal{N}_X V_I}|_{\mathrm{Dom}(\Psi_{I^*})}$ is exact. Also, by our induction hypothesis, the restriction of A to the fibers of $\pi_{\mathcal{N}_X V_{I^*}}|_{(\Psi_{I^*})^{-1}(\Psi_I(U_I^{\prec}))}$ is 0 for all $I \subsetneq I^*$. This means that there is a smooth function $g^{=}\colon X - K \to \mathbb{R}$ such that $g^{=}$ restricted to a small neighborhood of the closure of $\Psi_I(U_I^{\preceq})$ is 0 for all $I \subsetneq I^*$ and such that $A + (\Psi_{I^*})^* dg^{=}$ restricted to each fiber of $\pi_{\mathcal{N}_X V_{I^*}}|_{U_{I^*}^{\preceq} \cap \mathrm{Dom}(\Psi_{I^*})}$ is 0. Define $g^{\preceq} \equiv g^{\prec} + g^{=}$. Then

$$(\Psi_I)^*(\theta + dg^{\preceq}) - \sum_{i\in I} \mathrm{pr}^*_{I;i}\left(\left(\rho_{I;i} + \frac{w_i}{2\pi}\right)\alpha_{I;i}\right)$$

vanishes along each fiber of $\pi_{\mathcal{N}_X V_I}|_{U_I^{\preceq} \cap \mathrm{Dom}(\Psi_I)}$ for all $I \preceq I^*$. Hence, by induction we have shown that there is a smooth function $g\colon X - K \to \mathbb{R}$ and open subsets $U_I \subset \mathrm{Im}(\Psi_I)$ containing V_I such that

$$(\Psi_I)^*(\theta + dg) - \sum_{i\in I} \mathrm{pr}^*_{I;i}\left(\left(\rho_{I;i} + \frac{w_i}{2\pi}\right)\alpha_{I;i}\right)$$

vanishes along each fiber of $\pi_{\mathcal{N}_X V_I}|_{U_I \cap \mathrm{Dom}(\Psi_I)}$ for all $I \subset S$. By [39, Lemma 5.5], we can shrink these open subsets U_I so that $\check{\mathcal{R}} \equiv ((\rho_i|_{\Psi_I(U_I)})_{i\in S}, (\Psi_I|_{U_I})_{I\subset S})$ is a regularization. □

Definition 5.29 Let

$$\mathcal{M} \equiv \left(\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right), \Phi, \theta\right)$$

be a model resolution with associated symplectic form ω and let $U \subset X$ be an open set. A *regularization of* $\mathcal{M}$ *of radius* R *along* U for some $R < 1$ is an ω–regularization $\mathcal{R}$ of $(V_i)_{i\in S}$ as in (5-31) such that

1. the line bundle $\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right)$ is also defined using the regularization maps $(\Psi_i)_{i\in S}$ from $\mathcal{R}$,
2. Φ is radius R compatible with $\mathcal{R}$ along U as in Definition 5.9, and
3. θ is compatible with $\mathcal{R}$.

A *regularization of* $\mathcal{M}$ *along* U is a regularization of $\mathcal{M}$ of radius R along U for some $R < 1$ smaller than the tube radius of $\mathcal{R}$ along U.

We wish to show that every model resolution is isotopic to one admitting a regularization as above. Before we do this we need a preliminary lemma.

Lemma 5.30 *Let X be a smooth manifold with a smooth family of cohomologous symplectic forms $(\omega_t)_{t\in[0,1]}$ and $(V_i)_{i\in S}$ a compact SC divisor on X with respect to ω_t for all $t\in[0,1]$. Define $K\equiv\bigcup_{i\in S}V_i$ and let θ be a 1–form on $X-K$ satisfying $d\theta=\omega_0|_{X-K}$. Then there exists a smooth family of 1–forms $(\theta_t)_{t\in[0,1]}$ on $X-K$ such that $d\theta_t=\omega_t|_{X-K}$ for all $t\in[0,1]$ and such that the wrapping number of θ_t around V_j does not depend on $t\in[0,1]$ for all $j\in S$.*

Proof Since $\omega_t-\omega_0$ is exact for all $t\in[0,1]$, there is (by exploiting the Hodge decomposition theorem for differential forms) a smooth family of 1–forms $(\beta_t)_{t\in[0,1]}$ on X such that $d\beta_t=\omega_t-\omega_0$. Let $U\subset X$ be a neighborhood of K whose closure is a compact manifold with boundary which deformation retracts onto K and $\rho\colon X\to[0,1]$ a smooth function equal to 0 near K and equal to 1 outside a compact subset of U. Define $\check\beta_t\equiv\theta+\beta_t\in\Omega^1(X-K)$ for all $t\in[0,1]$. Since $\rho\check\beta_t=0$ near K, we can think of this as a smooth 1–form on X by defining it to be 0 along K. Let $c_t\equiv[\omega_t-d(\rho\check\beta_t)]\in H^2(X,X-U;\mathbb{R})$ for all $t\in[0,1]$ and define $c\equiv[\omega_0-d(\rho\theta)]\in H^2(X,X-U;\mathbb{R})$. Since $H^2(X,X-U;\mathbb{R})=H^2(X,X-K;\mathbb{R})$, we have the long exact sequence

$$H^1(X-K;\mathbb{R})\xrightarrow{\alpha}H^2(X,X-U;\mathbb{R})\xrightarrow{\check\alpha}H^2(X;\mathbb{R})\to H^2(X-K;\mathbb{R}).$$

Since $\check\alpha(c_t)=\check\alpha(c)=[\omega_0]$ for all $t\in[0,1]$, we have a smooth family closed 1–forms $b_t\in\Omega^1(X-K)$ such that $\alpha(b_t)=c-c_t$ for all $t\in[0,1]$. Let $\theta_t=\check\beta_t+b_t$. Then $[\omega_t|_U-d(\rho\theta_t)]\in H^2_c(U;\mathbb{R})$ is independent of t, which proves our lemma. □

Lemma 5.31 *Let $\big(\mathcal{O}_X\big(\sum_{i\in S}m_iV_i\big),\Phi,\theta\big)$ be a model resolution and let $U\subset X$ be a relatively compact open set. Then $\big(\mathcal{O}_X\big(\sum_{i\in S}m_iV_i\big),\Phi,\theta\big)$ is isotopic to a model resolution $\big(\mathcal{O}_X\big(\sum_{i\in S}m_iV_i\big),\widehat\Phi,\widehat\theta\big)$ admitting a regularization along U.*

Proof By [39, Theorem 2.17], there a smooth family of cohomologous symplectic forms $(\omega_t)_{t\in[0,1]}$ such that $\omega_0=\omega$ and $(V_i)_{i\in S}$ admits an ω_1–regularization

$$\mathcal{R}\equiv((\rho_i)_{i\in S},(\Psi_I)_{I\subset S}).$$

Lemma 5.30 then tells us that there is a smooth family of 1–forms $(\theta_t)_{t\in[0,1]}$ on $X-\bigcup_{i\in S-\star_S}V_i$ such that $\theta_0=\theta$ and $d\theta_t=\omega|_{X-\bigcup_{i\in S-\star_S}V_i}$ for all $t\in[0,1]$ and such that the wrapping number of θ_t around V_i is independent of t for each

$i \in S - \star_S$. We can assume that $\mathcal{O}_X\big(\sum_{i \in S} m_i V_i\big)$ is defined using the regularizations $(\Psi_i)_{i \in S}$ as changing the regularization needed to define a line bundle as in (5-9) creates an isomorphic line bundle. Now we isotope Φ through trivializations to a trivialization $\hat{\Phi}$ such that $\hat{\Phi}$ is compatible with $\mathcal{R}$ along U by Lemma 5.10. By Lemma 5.28 we have, after replacing $\mathcal{R}$ with a germ equivalent regularization, that $\hat{\theta} \equiv \theta_1 + dg$ is compatible with $\mathcal{R}$ for some $g \in C^\infty\big(X - \bigcup_{i \in S - \star_S} V_i\big)$.

Hence, $\big(\mathcal{O}_X\big(\sum_{i \in S} m_i V_i\big), \Phi, \theta\big)$ is isotopic to $\big(\mathcal{O}_X\big(\sum_{i \in S} m_i V_i\big), \hat{\Phi}, \theta_1\big)$, which in turn is isotopic to $\big(\mathcal{O}_X\big(\sum_{i \in S} m_i V_i\big), \hat{\Phi}, \hat{\theta}\big)$, which admits a regularization along U. □

Lemma 5.32 *Let $\big(\mathcal{O}_X\big(\sum_{i \in S} m_i V_i\big), \Phi, \theta\big)$ be a model resolution admitting a regularization along U for some relatively compact open set $U \subset X$ as in Definition 5.29. Let Φ_2 be the composition of Φ with the natural projection $X \times \mathbb{C} \twoheadrightarrow \mathbb{C}$. Define*

$$\pi_\Phi \colon U \to \mathbb{C}, \quad \pi_\Phi \equiv \Phi_2 \circ s_{(m_i)_{i \in S}}|_U,$$

where $s_{(m_i)_{i \in S}}$ is the canonical section of $\mathcal{O}_X\big(\sum_i m_i V_i\big)$ as defined in (5-12). Then there is some $\epsilon > 0$ such that $\pi_\Phi^{-1}(z)$ is a symplectic submanifold of U for all $z \in \mathbb{D}(\epsilon) - \{0\}$. Also, the restriction of θ to $\pi_\Phi^{-1}(\{|z| = \epsilon'\})$ is a contact form for all $0 < \epsilon' \leq \epsilon$.

Proof Since U is relatively compact, it is sufficient for us to show that for every $x \in U \cap \big(\bigcup_{i \in S} V_i\big)$, there is a small open set $U_x \subset X$ containing x such that $\pi_\Phi|_{U_x \cap \pi_\Phi^{-1}(\mathbb{C} - 0)}$ has symplectic fibers and such that the restriction of θ to $\pi_\Phi^{-1}(\{|z| = \epsilon'\}) \cap U_x$ is a contact form. We will first show that the fibers are symplectic. Suppose that $I \subset S$ is the largest set satisfying $x \in V_I$. Let a_R be the function defined in Definition 5.9 and let $\Pi_{(m_i)_{i \in S}, I}(v)$ be as in (5-13). Near x, we have that $a_R(\rho_i) = \rho_i$. Therefore,

$$\pi_\Phi(y) = \Phi_2\big(\Pi_{(m_i)_{i \in S}, I}(\Psi_I^{-1}(y))\big)$$

for all $y \in X$ near to x. Since Ψ_I is a regularization, it is sufficient for us to show that the fibers of $\Phi_2 \circ \Pi_{(m_i)_{i \in S}, I}$ restricted to a small neighborhood of x inside $\mathcal{N}_X V_I$ are symplectic with respect to

$$\omega_{(\rho_{I;i}, \nabla^{(I;i)})_{i \in I}} \overset{(5\text{-}4)}{=} \pi^*_{\mathcal{N}_X V_I}(\omega|_{V_I}) + \frac{1}{2} \bigoplus_{i \in I} \mathrm{pr}^*_{I;i}(d(\rho_{I;i} \alpha_{I;i})), \tag{5-34}$$

where $\mathrm{pr}_{I;i}$ is the natural projection map from (5-32). Let $W_x \subset V_I$ be a small open neighborhood of x that is contractible and choose unitary trivializations

$$T_i \colon \mathcal{N}_X V_i|_{W_x} \to W_x \times \mathbb{C}$$

for all $i \in I$. Let $z_i\colon \mathcal{N}_X V_I|_{W_x} \to \mathbb{C}$ be the composition of $T_i \circ \mathrm{pr}_{I;i}$ with the projection map to $\mathbb{C}$. Hence, along W_x, (5-34) becomes

$$\omega_{(\rho_{I;i},\nabla^{(I;i)})_{i\in I}}|_{W_x} = \pi^*_{\mathcal{N}_X V_I}(\omega|_{V_I}) + \beta + \frac{i}{2}\bigoplus_{i\in I} dz_i \wedge d\bar{z}_i, \tag{5-35}$$

where $\beta \in \Omega^2(\mathcal{N}_X V_I|_{W_x})$ is a closed 2–form whose restriction to the fibers of $\pi_{\mathcal{N}_X V_I}|_{W_x}$ is zero and whose restriction to the zero section is also zero. This means that near x we have that $\omega_{(\rho_{I;i},\nabla^{(I;i)})_{i\in I}}|_{W_x}$ is C^0 close to

$$\check{\omega} \equiv \pi^*_{\mathcal{N}_X V_I}(\omega|_{V_I}) + \frac{i}{2}\bigoplus_{i\in I} dz_i \wedge d\bar{z}_i.$$

We can choose our trivializations T_i so that $\Phi_2 \circ \Pi_{(m_i)_{i\in S},I}$ is equal to $\prod_{i\in I} z_i^{m_i}$ inside $\pi_{\mathcal{N}_X V_I}|_{W_x}$. Since the fibers of $\prod_{i\in I} z_i^{m_i}$ are symplectic with respect to $\check{\omega}$ near $W_x \cap V_I$ and since ω is equal to $\check{\omega}$ at the point x, there is a small neighborhood $\widetilde{U}_x \subset \mathcal{N}_X V_I$ containing x such that the fibers of $\prod_{i\in I} z_i|_{\widetilde{U}_x}$ are symplectic with respect to $\omega_{(\rho_{I;i},\nabla^{(I;i)})_{i\in I}}|_{W_x}$. Hence, the fibers of $\pi_\Phi|_{U_x}$ are symplectic, where $U_x \equiv \Psi_I(\widetilde{U}_x)$.

We now wish to show that θ restricted to $Y_r \equiv \pi_\Phi^{-1}(\{|z| = r\}) \cap U_x$ is a contact form for all $r > 0$. Since the restriction of ω to the fibers of $\pi_\Phi|_{U_x}$ are symplectic, it is sufficient for us to show that θ restricted to the kernel of $\omega|_{Y_r}$ is nonzero at every point for all $r > 0$. By (5-34) and the fact that $\Phi_2 \circ \Pi_{(m_i)_{i\in S},I}$ is equal to $\prod_{i\in I} z_i^{m_i}$ inside $\mathcal{N}_X V_I|_{W_x}$, we get that the kernel of $\Psi_I^*\omega|_{\Psi_I^{-1}(Y_r)}$ is tangent to the fibers of $\pi_{\mathcal{N}_X V_I}$ inside $\mathrm{Dom}(\Psi_I)$. Therefore, since the restriction of the 1–form (5-33) to the fibers of $\pi_{\mathcal{N}_X V_I}$ inside $\mathrm{Dom}(\Psi_I)$ is zero, $\Psi_I^*\theta$ restricted to the kernel of $\Psi_I^*\omega|_{\widetilde{U}_x}$ is nonzero so long as $\widetilde{U}_x \subset \mathrm{Dom}(\Psi_I)$. Hence, θ restricted to Y_r is a contact form for all $r > 0$ so long as $U_x \subset \mathrm{Im}(\Psi_I)$. □

Definition 5.33 Let $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ be a model resolution admitting a radius R regularization

$$\mathcal{R} \equiv ((\rho_i)_{i\in I}, (\Psi_I)_{I\subset S}) \tag{5-36}$$

along a relatively compact open set $U \subset X$ as in Definition 5.29, where U contains $K \equiv \bigcup_{i\in S-\star_S} V_i$. Let $T_{r,I}$ be the radius r tube of V_I as in (5-14). Let $\Phi_2\colon X \to \mathbb{C}$ be the composition of Φ with the natural projection map $X \times \mathbb{C} \to \mathbb{C}$. Choose $\epsilon > 0$ small enough that

$$(\Phi_2 \circ s_{(m_i)_{i\in S}})^{-1}(\mathbb{D}_\epsilon) \cap U \subset \bigcup_{i\in S} T_{R,i} \tag{5-37}$$

and the fibers $(\Phi_2 \circ s_{(m_i)_{i\in S}})^{-1}(z) \cap U$ are symplectic for $z \in \mathbb{D}_\epsilon - 0$ by Lemma 5.32. Let $\check{T}$ be a smoothing of the compact manifold with corners $\bigcup_{i\in S-\star_S} T_{R,i}$ such that

$$\partial \check{T} \cap T_{R,\star_S} = \Psi_{I^*}(\pi^{-1}_{\mathcal{N}_X V_{\star_S}}(V_{I^*} \cap \partial \check{T})) \cap T_{R,\star_S}, \tag{5-38}$$

X_θ points outwards along $\partial \check{T} \cap T_{R,\star_S}$ and such that $\check{T} \subset \bigcup_{i\in S-\star_S} T_{R,i}$. We also assume that the smoothing is small enough that $\bigcup_{i\in S-\star_S} T_{3R/4,i}$ is contained in the interior of $\check{T}$.

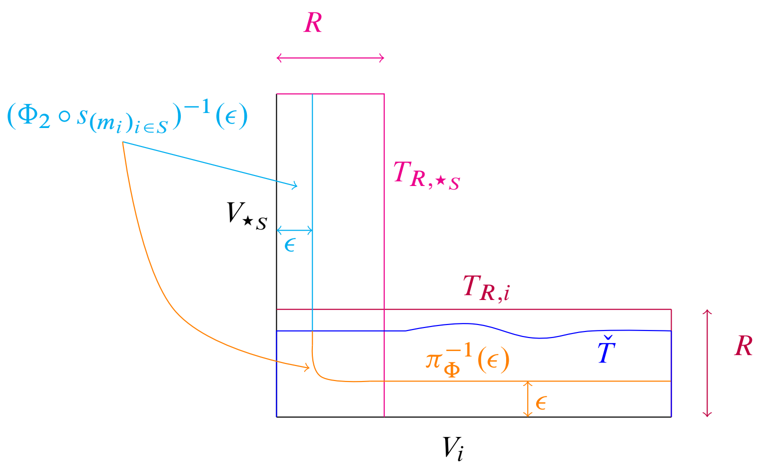
Define

$$\pi_\Phi \colon \check{T} \to \mathbb{C}, \qquad \pi_\Phi \equiv \Phi_2 \circ s_{(m_i)_{i\in S}}|_{\check{T}}.$$

The *Milnor fiber* of $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ is the pair

$$(M, \theta_M) \equiv (\pi_\Phi^{-1}(\epsilon), \theta|_{\pi_\Phi^{-1}(\epsilon)}).$$

This is a Liouville domain for $\epsilon > 0$ small enough since X_θ is tangent to $V_{\star_S} - K$ and X_θ is transverse to $\partial \check{T}$ and pointing outwards:



Because

- the 1–form (5-33) restricted to each fiber of $\pi_{\mathcal{N}_X V_I}|_{\Psi_I^{-1}(X-K)}$ is 0 for every $I \subset S$,
- (5-38) holds,
- Φ is radius R compatible with $\mathcal{R}$ along U, and
- $\bigcup_{i\in S-\star_S} T_{3R/4,i}$ is contained in the interior of $\check{T}$,

we get that the monodromy map $\phi \colon M \to M$ of π_Φ around the loop

$$[0,1] \to \partial\mathbb{D}(\epsilon), \qquad t \to \epsilon e^{2\pi i t},$$

with respect to the symplectic connection associated to ω exists and has compact support. In addition, since $\omega|_{\pi_\Phi^{-1}(\partial\mathbb{D}(\epsilon))} = d\theta|_{\pi_\Phi^{-1}(\partial\mathbb{D}(\epsilon))}$, ϕ is an exact symplectomorphism with compact support. We call (M, θ_M, ϕ) the *abstract contact open book associated to* $\bigl(\mathcal{O}_X\bigl(\sum_{i\in S} m_i V_i\bigr), \Phi, \theta\bigr)$.

Now suppose that our model resolution has a choice of grading. Since $\pi_\Phi^{-1}(\partial\mathbb{D}(\epsilon))$ is a contact submanifold of $(X-K,\omega)$ with contact form given by restricting θ by Lemma 5.32 after possibly shrinking ϵ, we get an induced grading on this contact submanifold by Lemma 5.20. Since the contact distribution is isotopic to $Q \equiv \ker(D\pi_\Phi|_{\pi_\Phi^{-1}(\partial(\epsilon))})$ through hyperplane distributions Q_t for $t \in [0,1]$, where $\omega|_{Q_t}$ is nondegenerate for all t, we get a grading

$$\iota\colon \widetilde{\mathrm{Fr}}(Q) \times_{\widetilde{\mathrm{Sp}}(2n)} \mathrm{Sp}(2n) \cong \mathrm{Fr}(Q)$$

on Q and hence on $(M, d\theta_M)$. Since the parallel transport maps of π_Φ along $\partial\mathbb{D}(\epsilon)$ have lifts to $\widetilde{\mathrm{Fr}}(Q)$, ϕ has an induced grading and hence (M,θ_M,ϕ) is a graded abstract contact open book. We will call this the *induced grading* on (M,θ_M,ϕ).

Lemma 5.34 *Let $(B_t \subset C, \xi_t, \Phi_t)$ for $t \in [0,1]$ be a smooth family of contact pairs. Then there is a smooth family of contactomorphisms $\Psi_t\colon C \to C$ between $(B_t \subset C, \xi_0, \Phi_0)$ and $(B_t \subset C, \xi_t, \Phi_t)$* (as in Definition 3.7) *for all $t \in [0,1]$ such that $\Psi_0 = \mathrm{id}$.*

Proof By Gray's stability theorem, there is a smooth family of contactomorphisms $\check{\Phi}_t\colon C \to C$ starting from the identity map such that $\check{\Phi}_t$ is a contactomorphism between (C,ξ_0) and (C,ξ_t). Therefore, by pulling everything back by $\check{\Phi}_t$, we can assume that $\xi_t = \xi_0$ for all $t \in [0,1]$. Also by Gray's stability theorem, there is a smooth family of embeddings $\iota_t\colon B_0 \to C$ mapping B_0 to B_t such that

- $\iota_0|_{B_0}\colon B_0 \to B_0$ is the identity map, and
- $\iota_t|_{B_t}\colon B_0 \to B_t$ is a contactomorphism.

Again by Gray's stability theorem, there is a neighborhood N of B and a smooth family of contact embeddings $\tilde{\iota}_t\colon (N, \xi_0|_N) \to (C, \xi_C)$ whose restriction to B_0 is ι_t and where $\tilde{\iota}_0|_N\colon N \to N$ is the identity map. Let $H_t\colon \tilde{\iota}_t(N) \to \mathbb{R}$ be a smooth family of functions *generating the contact isotopy* $\tilde{\iota}_t$. By definition this means that there is a contact form α compatible with ξ_0 such that

$$i_{d\tilde{\iota}_t(x)/dt}\alpha = -H_t, \quad i_{d\tilde{\iota}_t(x)/dt} d\alpha = dH_t - (i_R dH_t)\alpha \quad \text{for all } x \in \iota_t(N),\ t \in [0,1],$$

where R is the Reeb vector field of α (see [24, Lemma 3.49]). Choose a smooth family of functions K_t for $t \in [0,1]$ equal to H_t near $\iota_t(B_t)$. Then K_t generates a smooth family of contactomorphisms Ψ_t satisfying the properties we want. □

Lemma 5.35 *Let $\left(\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right), \Phi, \theta\right)$ be a model resolution admitting a regularization*

$$\mathcal{R} \equiv ((\rho_i)_{i\in S}, (\Psi_I)_{I\subset S})$$

of radius $R < 1$ along an open set $U \subset X$ containing $K \equiv \bigcup_{i\in S-\star_S} V_i$ as in Definition 5.29. Since $m_{\star_S} = 1$, we let

$$\Phi_\star \equiv \Phi|_{V_{\star_S}-K} \colon \mathcal{N}_X(V_{\star_S} - K) \to (V_{\star_S} - K) \times \mathbb{C}$$

be the induced trivialization of the normal bundle

$$\mathcal{N}_X(V_{\star_S} - K) = \mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right)\Bigg|_{V_{\star_S}-K}$$

induced by Φ. Let $C \subset \bigcup_{i\in S-\star_S} T_{R,i} - K$ be a closed hypersurface transverse to X_θ and $V_{\star_S}$ and define $B \equiv C \cap V_{\star_S}$. Let Φ_B be a trivialization of the normal bundle of the contact submanifold $B \subset C$ induced by $\Phi_\star|_B$.

Then the contact pair $(B \subset C, \ker(\theta|_C), \Phi_B)$ is contactomorphic to the link of the model resolution $\left(\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right), \Phi, \theta\right)$. If our model resolution is graded, then both of these contact pairs have induced gradings by Lemma 5.20 and the above contactomorphism becomes a graded contactomorphism with respect to these gradings.

Proof Choose $\check{R} > R$ smaller than the tube radius of our model resolution along U so that $\check{R} < 1$. Let $\alpha\colon [0,\check{R}] \to [0,1]$ be a smooth function such that $\alpha' \geq 0$, $\alpha(x) = x$ for all $x \leq R$ and $\alpha(x) = 1$ near $\check{R}$. Define $\alpha_{\rho_i}\colon X - K \to \mathbb{R}$ to be equal to $\alpha(\rho_i)$ inside $T_{\check{R},i} - K$ and 1 otherwise. Define

$$f\colon X - K \to \mathbb{R}, \quad f \equiv \sum_{i\in S-\star_S} \log(\alpha_{\rho_i}).$$

Then f is compatible with $(V_i)_{i\in S-\star_S}$ as in Definition 5.18. Let $c \ll -1$ and define

$$\check{C} \equiv f^{-1}(c), \quad \check{B} \equiv \check{C} \cap V_{\star_S}, \quad \xi_{\check{C}} \equiv \ker(\theta)|_{\check{C}}.$$

The normal bundle of $\check{B}$ inside $\check{C}$ has a natural trivialization $\Phi_{\check{B}}$ induced by the trivialization $\Phi_\star$. Since $df(X_\theta) > 0$ near K and $c \ll -1$, we get that $(\check{B} \subset \check{C}, \xi_{\check{C}}, \Phi_{\check{B}})$ is the link of our model resolution $\left(\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right), \Phi, \theta\right)$ by Definition 5.25.

Since $df(X_\theta) > 0$ inside $\bigcup_{i\in S-\star_S} T_{R,i} - K$ and $\bigcup_i V_i$ is connected, we can choose a smooth family of hypersurfaces $(C_t)_{t\in[0,1]}$ joining C and $\check{C}$ so that C_t is transverse to X_θ and $V_{\star_S}$ for all $t\in[0,1]$. Define $B_t \equiv C_t \cap V_{\star_S}$ and $\xi_t \equiv \ker(\theta|_{C_t})$. Also, let Φ_{B_t} be the trivialization of the normal bundle of B_t inside C_t induced by $\Phi_\star$ such that $\Phi_{B_0} = \Phi_B$ and $\Phi_{B_1} = \Phi_{\check{B}}$. Then $(B_t \subset C_t, \xi_t, \Phi_{B_t})$ is a smooth family of contact pairs joining $(B\subset C, \ker(\theta|_C), \Phi_B)$ and $(\check{B}\subset\check{C}, \xi_{\check{C}}, \Phi_{\check{B}})$. Therefore, they are isomorphic by Lemma 5.34. Also, if $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ is graded then they are graded isomorphic since all of our contact pairs have induced gradings from our model resolution by Lemma 5.20. □

Lemma 5.36 *The link of a (graded) model resolution $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ supports a (graded) contact open book which is contactomorphic to $\mathrm{OBD}(M, \theta_M, \phi)$, where (M, θ_M, ϕ) is the (graded) abstract contact open book associated to this model resolution as in Definition 5.33.*

Proof In this proof we will use the same notation as in Definition 5.33. We will introduce it again here for the sake of clarity. By Lemma 5.31 we can isotope our model resolution so that it admits a regularization

$$\mathcal{R} \equiv ((\rho_i)_{i\in I}, (\Psi_I)_{I\subset S})$$

of radius R along U for some relatively compact open U containing $K \equiv \bigcup_{i\in S-\star_S} V_i$. By Lemma 5.26, the link does not change after this isotopy. Let $T_{r,I}$ be the radius r tube of V_I as in (5-14). Let $\check{T}$ be a smoothing of the manifold with corners $\bigcup_{i\in S-\star_S} T_{R,i}$ as in Definition 5.33. In other words, $\check{T}$ satisfies (5-38), X_θ points outwards along $\partial\check{T}$ and $\check{T}\subset\bigcup_{i\in S-\star_S} T_{R,i}$. Also, we require that $\check{T}$ contains $\bigcup_{i\in S-\star_S} T_{3R/4,i}$. Define

$$\pi_\Phi\colon \check{T}\to\mathbb{C}, \quad \pi_\Phi \equiv \Phi_2 \circ s_{(m_i)_{i\in S}}|_{\check{T}},$$

where $\Phi_2\colon \mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big)\to\mathbb{C}$ is the composition of Φ with the natural projection map $X\times\mathbb{C}\to\mathbb{C}$. Then we can assume that

$$(M, \theta_M) \equiv (\pi_\Phi^{-1}(\epsilon), \theta|_{\pi_\Phi^{-1}(\epsilon)})$$

for $\epsilon > 0$ small enough that (5-37) is satisfied. Let ω be the symplectic form associated to our model resolution. Here $\phi\colon M\to M$ is the monodromy map around the loop

$$[0,1]\to\partial\mathbb{D}(\epsilon), \quad s\to\epsilon e^{2\pi i s}, \tag{5-39}$$

with respect to the symplectic connection associated to ω. Then (M, θ, ϕ) is the abstract open book associated to our model resolution so long as $\epsilon > 0$ is sufficiently small.

Define

$$L_r \equiv \bigcup_{i \in S - \star_S} T_{r,i}.$$

Let

$$\Phi_\star \equiv \Phi|_{V_{\star_S}-K}\colon \mathcal{N}_X(V_{\star_S}-K) \to (V_{\star_S}-K)\times\mathbb{C}$$

be the trivialization of the normal bundle $\mathcal{N}_X(V_{\star_S}-K) = \mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big)\big|_{V_{\star_S}-K}$ induced by Φ as defined in the statement of Lemma 5.35 and let $\Phi_{\star,2}$ be the composition of $\Phi_\star$ with the natural projection map $V_{\star_S - K}\times\mathbb{C}\to\mathbb{C}$. Let

$$P_{\star_S}\colon \mathrm{Im}(\Psi_{\star_S}) \to V_{\star_S}, \quad P_{\star_S} \equiv \pi_{\mathcal{N}_X V_{\star_S}} \circ \Psi_{\star_S}^{-1},$$

be the natural projection map and (r,ϑ) polar coordinates on $\mathbb{C}$. Let $W\colon L_R \cap V_{\star_S} \to [0,1]$ be a smooth function equal to 0 inside $L_{4R/5}\cap V_{\star_S}$ and equal to 1 inside $(L_R - L_{5R/6})\cap V_{\star_S}$ and define

$$\widetilde{W}\colon T_{R,\star_S}\cap L_R \to \mathbb{R}, \quad \widetilde{W} \equiv W \circ P_{\star_S}.$$

We now define $\theta_t \in \Omega^1\big(((T_{R,\star_S}\cap L_R)\cup L_{4R/5}) - K\big)$ for $t\in[0,1]$ to be θ inside $L_{4R/5}-K$ and equal to

$$(1-t)\theta + t\Big((1-\widetilde{W})\theta + \widetilde{W}\big(P_{\star_S}^*(\theta|_{V_{\star_S}-K}) + \tfrac{1}{2}\rho_{\star_S}(\Psi_{\star_S}^{-1})^*\Phi_{\star,2}^*(d\vartheta)\big)\Big) \tag{5-40}$$

inside $T_{R,\star_S}\cap L_R - K$. For $R_1>0$ small enough with respect to R, we get that $d\theta_t$ is a symplectic form inside $L \equiv L_{4R/5}\cup(L_R\cap T_{R_1,\star_s})$ and $d\theta_t$ restricted to $\pi_\Phi^{-1}(x)\cap L$ is a symplectic form for all $x\in\mathbb{C}-0$ and $t\in[0,1]$.

Let $\kappa\colon V_{\star_S}\to\mathbb{R}$ be a smooth function which is negative in the interior of $\check{T}\cap V_{\star_S}$ and positive outside $\check{T}\cap V_{\star_S}$ and such that $\kappa^{-1}(0) = \partial\check{T}\cap V_{\star_S}$ is a regular level set. We can assume that our perturbation $\check{T}$ from Definition 5.33 is small enough that $\partial\check{T}\subset L_R - L_{5R/6}$. Choose a constant $\check{\delta}>0$ small enough that $\kappa^{-1}(-\check{\delta},0]\subset L_R - L_{5R/6}$ and $X_{\theta_t}^{d\theta_t}$ is transverse to $\widetilde{\kappa}^{-1}(s)\cap L$ for all $s\in(-\check{\delta},\check{\delta})$. Define

$$\widetilde{\kappa}\colon \mathrm{Im}(\Psi_{\star_S})\to\mathbb{R}, \quad \widetilde{\kappa}\equiv\kappa\circ P_{\star_S}.$$

Define $\delta \equiv 1 - e^{-\check{\delta}}$. Let

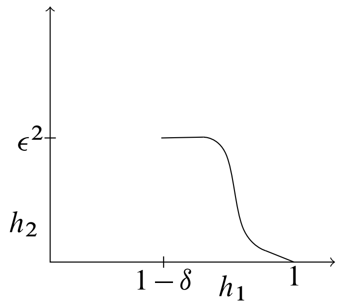
$$h_1, h_2\colon [0,\delta)\to\mathbb{R}$$

be smooth functions satisfying

(1) $h_1'(r) < 0$ and $h_2'(r)\geq 0$ for all $r>0$,

(2) $h_1(r) = 1 - r^2$ and $h_2(r) = \frac{1}{2}r^2$ for r near 0, and

(3) $h_1(r) = 1 - r$ and $h_2(r) = \epsilon^2$ for r in $\left[\frac{1}{2}\delta, \delta\right)$:



Now define

$$C \equiv \left(\pi_\Phi^{-1}(\partial \mathbb{D}_\epsilon) - \widetilde{\kappa}^{-1}((-\check{\delta}, 0])\right) \cup \bigcup_{s \in [0,\delta]} \left(\rho_{\star S}^{-1}(h_2(s)) \cap \widetilde{\kappa}^{-1}\left(\log(h_1(s))\right)\right). \tag{5-41}$$

This is a smooth hypersurface in X since Φ is radius R compatible with $\mathcal{R}$ along U and since $\epsilon > 0$ can be made small enough that $\epsilon < \frac{3}{4}R$. We can also ensure that $\epsilon > 0$ is small enough that $\pi_\Phi^{-1}(\mathbb{D}_\epsilon) \subset L$. This ensures that $d\theta_t$ is a symplectic form near $\pi_\Phi^{-1}(\mathbb{D}_\epsilon)$ and that $d\theta_t$ restricted to the fibers of $\pi_\Phi|_{\pi_\Phi^{-1}(\partial \mathbb{D}_\epsilon)}$ is a symplectic form for all t.

Define $B \equiv C \cap V_{\star S}$. This is also equal to $\check{T} \cap V_{\star S} = \kappa^{-1}(0)$. For R small enough, we have that $(C, \ker(\theta_t)|_C)$ is a smooth family of contact submanifolds of X. The trivialization $\Phi_\star$ gives us a trivialization $\Phi_{B,t}$ of the normal bundle of the contact submanifold B inside $(C, \ker(\theta_t)|_C)$ since C is transverse to $V_{\star S}$. Hence, we get a smooth family of contact pairs

$$P_t \equiv (B \subset C, \ker(\theta_t)|_C, \Phi_{B,t})$$

which are all contactomorphic by Lemma 5.34. Also, by Lemma 5.35, the contact pair P_0 is contactomorphic to the link of $\left(\mathcal{O}_X\left(\sum_{i \in S} m_i V_i\right), \Phi, \theta\right)$ for R small enough and hence P_1 is contactomorphic to the link of $\left(\mathcal{O}_X\left(\sum_{i \in S} m_i V_i\right), \Phi, \theta\right)$. Therefore, to complete this lemma, it is sufficient to show that the contact pair P_1 is contactomorphic to the contact pair associated to $\mathrm{OBD}(M, \theta_M, \phi)$. In fact, since $(M, \theta_t|_M)$ is a smooth family of Liouville domains and since the monodromy map of π_Φ around the path (5-39) with respect to the fiberwise symplectic 2–form $d\theta_t|_{\pi_\Phi^{-1}(\partial \mathbb{D}(\epsilon)}$ is equal to ϕ for all t, it is sufficient for us to show that the contact pair P_1 is contactomorphic to the contact pair associated to $\mathrm{OBD}(M, \theta_1|_M, \phi)$. Note that there is a resemblance between the construction of C and the construction of $\mathrm{OBD}(M, \theta_1|_M, \phi)$ from Definition 3.14. We will now make this precise.

The contact pair P_1 can be constructed as follows: Define

$$V \equiv \pi_\Phi^{-1}(\partial\mathbb{D}_\epsilon) - \widetilde{\kappa}^{-1}\big((\log(1-\tfrac{1}{2}\delta), 0]\big).$$

Let $T_\phi = M \times [0,1]/\sim$ be the mapping torus of ϕ. We have a diffeomorphism $\Phi\colon T_\phi \to \pi_\Phi^{-1}(\partial\mathbb{D}_\epsilon)$ sending (x,s) to the parallel transport of $x \in M$ along $\partial\mathbb{D}(\epsilon)$ in the anticlockwise direction from $\epsilon \in \partial\mathbb{D}(\epsilon)$ to $\epsilon e^{is} \in \partial\mathbb{D}(\epsilon)$ with respect to the 2–form $d\theta_1|_{\pi_\Phi^{-1}(\partial\mathbb{D}(\epsilon))}$. Hence, we will assume that $T_\phi = \pi_\Phi^{-1}(\partial\mathbb{D}_\epsilon)$ under the identification Φ and that V is naturally a subset of T_ϕ. Since ϕ has compact support inside M, we have the standard collar neighborhood

$$(1-\delta, 1] \times \partial M \times (\mathbb{R}/\mathbb{Z}) \subset T_\phi \tag{5-42}$$

as in Definition 3.13 (here $\delta > 0$ is the same small constant defined above, which might have to be made smaller). We can choose κ so that $e^{\widetilde{\kappa}}|_{T_\phi}$ is the natural projection to $(1-\delta, 1]$ in the neighborhood (5-42). This means that θ_1 restricted to the region (5-42) is equal to $e^{\widetilde{\kappa}}\alpha_M + \pi\epsilon^2 dt$, where t parametrizes $\mathbb{R}/\mathbb{Z}$ and where $\alpha_M = \theta_1|_{\partial M}$ by (5-40).

Using the diffeomorphism Φ and definition (5-41) of C, we have that C is naturally diffeomorphic to

$$\check{C} \equiv (\partial M \times \mathbb{D}(\delta)) \sqcup V/\sim,$$

where $\sim$ identifies $(x,z) \in \partial M \times \big(\mathbb{D}(\delta) - \mathbb{D}(\tfrac{1}{2}\delta)\big)$ with

$$\big(1-|z|, x, \tfrac{1}{2\pi}\arg(z)\big) \in \big(1-\delta, 1-\tfrac{1}{2}\delta\big] \times \partial M \times (\mathbb{R}/\mathbb{Z}) \subset V.$$

Because θ_1 restricted to $T_{R,\star_S} \cap (L_R - L_{5R/6})$ is equal to

$$P_{\star_S}^*(\theta|_{V_{\star_S}-K}) + \tfrac{1}{2}\rho_{\star_S}(\Psi_{\star_S}^{-1})^*\Phi_\star^*(d\vartheta)$$

by (5-40) and because

$$P_{\star_S}^*(\theta|_{V_{\star_S}-K})|_{M\cap\widetilde{\kappa}^{-1}(-\check{\delta},0]} = e^{\widetilde{\kappa}}\alpha_M$$

inside the cylinder $(1-\delta, 1] \times \partial M \subset M$, we have that the contact form $\theta_1|_C$ inside $\check{C}$ under the above identification is equal to

$$\alpha_1 \equiv \begin{cases} h_1(r)\alpha_M + \tfrac{1}{2}h_2(r)\,d\vartheta & \text{inside } \partial M \times \mathbb{D}(\tfrac{1}{2}\delta), \\ \theta_1|_{T_\phi} & \text{inside } V. \end{cases} \tag{5-43}$$

Notice that this description of P_1 resembles the construction of the open book associated to the abstract contact open book $(M, \theta_1|_M, \phi)$ as in Definition 3.14. All we need to do is deform the above construction until it is actually equal to $\mathrm{OBD}(M, \theta_1|_M, \phi)$.

We will now do this explicitly. From now on we let $t\colon V \to \mathbb{R}/\mathbb{Z}$ be the coordinate $\frac{1}{2\pi}\pi_\Phi^*(\vartheta)$. Since the monodromy map ϕ has compact support, there is a smooth function $F_\phi\colon M \to \mathbb{R}$ such that $\phi^*(\theta_1|_M) = \theta_1|_M + dF_\phi$. Let $\rho\colon [0,1] \to [0,1]$ be a smooth function equal to 0 near 0 and 1 near 1. Since $T_\phi = M \times [0,1]/\sim$, where $\sim$ identifies $(x,1)$ with $(\phi(x),0)$, we have a well-defined 1–form $\theta_1|_M + d(\rho(t)F_\phi)$ on T_ϕ. For $s \in [0,1]$, define

$$\alpha_s \in \Omega^1(T_\phi), \quad \alpha_s \equiv (1-s)\theta_1|_{T_\phi} + s(\theta_1|_M + d(\rho(t)F_\phi)) + c_s\,dt,$$

where $(c_s)_{s\in[0,1]}$ is a smooth family of constants where $c_0 = 1$ and c_t is sufficiently large that α_s is a contact form for all $s \in [0,1]$. Then (T_ϕ, α_1) is the mapping torus of $(M, \theta_1|_M, \phi)$ as in Definition 3.13.

Choose a smooth family of functions

$$h_1^s, h_2^s\colon [0,\delta) \to [0,\infty), \quad s \in [0,1],$$

satisfying

(1) $(h_1^s)'(r) < 0$ and $(h_2^s)'(r) \geq 0$ for all $r > 0$,

(2) $h_1^s(r) = 1 - r^2$ and $h_2^s(r) = \frac{1}{2}r^2$ for r near 0,

(3) $h_1^s(r) = 1 - r$ and $h_2^s(r) = (1-s)\epsilon^2 + c_s$ for r in $\left[\frac{1}{2}\delta, \delta\right)$,

(4) $h_1^0(r) = h_1(r)$ and $h_2^0(r) = h_2(r)$ for all $r \in [0,\delta)$.

Define

$$\alpha_1^s \equiv \begin{cases} h_1^s(r)\alpha_M + \frac{1}{2}h_2^s(r)\,d\vartheta & \text{inside } \partial M \times \mathbb{D}\left(\frac{1}{2}\delta\right), \\ \alpha_s & \text{inside } V \subset T_\phi, \end{cases} \tag{5-44}$$

for all $s \in [0,1]$. Then $(C, \ker(\alpha_1^s))_{s\in[0,1]}$ is a smooth family of contact manifolds such that $B \subset C$ is a contact submanifold. Also, we have a smooth family of trivializations Φ_1^s of the normal bundle of B inside $(C, \ker(\alpha_1^s))$ such that $\Phi_1^0 = \Phi_{B,1}$. Therefore,

$$\check{P}_t \equiv (B \subset C, \ker(\alpha_1^s), \Phi_1^s)$$

is a smooth family of contact pairs and so, by Lemma 5.34, they are all contactomorphic. By construction, $\check{P}_1$ is equal to $\mathrm{OBD}(M, \theta_M, \phi)$. Since $\check{P}_1$ is contactomorphic to $\check{P}_0 = P_1$ and P_1 is contactomorphic to P_0, which in turn is contactomorphic to the link of our model resolution, we get that $\mathrm{OBD}(M, \theta_M, \phi)$ is contactomorphic to the link of our model resolution. □

5.5 Dynamics of abstract contact open books associated to model resolutions

In this subsection we show that the fixed points of a positive slope perturbation of the symplectomorphism associated to the graded abstract contact open book associated to a model resolution form a union of specific codimension 0 families of fixed points. We also compute the indices of these fixed points.

Definition 5.37 Let $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ be a graded model resolution with associated symplectic form ω on X, where $n+1 = \frac{1}{2}\dim(X)$. Let J be an ω–compatible almost complex structure on X. By Definition A.6, the grading on $\check{X} \equiv X - \bigcup_{i\in S-\star_S} V_i$ corresponds to a trivialization $\Phi\colon \kappa_J|_{\check{X}} \to \check{X}\times\mathbb{C}$ of the canonical bundle. Let U be a small neighborhood of $\bigcup_{i\in S-\star_S} V_i$ which deformation retracts onto V_i. Choose a smooth section s of κ_J which is transverse to 0 and such that $\Phi\circ s|_{\check{X}-U}$ is a nonzero constant section of $(\check{X}-U)\times\mathbb{C}$. By a Mayer–Vietoris argument, the homology group $H_{2n}\big(\bigcup_{i\in S-\star_S} V_i;\mathbb{Z}\big) = H_{2n}(U;\mathbb{Z})$ is freely generated by the fundamental classes $[V_i]$ of V_i. Let $[s^{-1}(0)] \in H_{2n}(U)$ be the homology class represented by the zero set. Then $[s^{-1}(0)] = \sum_{i\in S-\star_S} a_i[V_i]$ for unique numbers $a_i\in\mathbb{Z}$ for $i\in S-\star_S$. The *discrepancy* of V_i is defined to be a_i for all $i\in S-\star_S$.

In the case of Example 5.14, the discrepancy and multiplicity of E_i as defined in Definition 5.37 is identical to the discrepancy and multiplicity of f along E_i as in Definition 2.1. Similarly, we have a notion of multiplicity m separating resolution as in Definition 2.2 for model resolutions which coincide in the case of Example 5.14:

Definition 5.38 A model resolution $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ is called a *multiplicity m separating resolution* if $m_i + m_j > m$ for all $i, j \in S$ satisfying $V_i\cap V_j \neq \varnothing$.

Definition 5.39 Let $\big(\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big), \Phi, \theta\big)$ be a model resolution. Let $i\in S-\star_S$. Define $V_i^o \equiv V_i - \bigcup_{j\in S-i} V_j$ and $X_i \equiv X - \bigcup_{j\in S-i} V_j$. Let U_i be an open neighborhood of V_i^o inside X_i which deformation retracts onto V_i^o and let $\iota_i\colon U_i - V_i^o \to U_i$ be the natural inclusion map. Let $s_{(m_i)_{i\in S}}$ be the canonical section of $\mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big)$ as in (5-12). Let $\Phi_2\colon \mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big) \to \mathbb{C}$ is the composition of Φ with the natural projection map to $\mathbb{C}$. Define

$$Q_i\colon U_i - V_i^o \to \mathbb{C}^*, \quad Q_i(x) := \Phi_2\circ s_{(m_i)_{i\in S}}(x).$$

Then the *natural m_i–fold covering of V_i^o* is the m_i–fold covering of V_i^o given by a disjoint union of covers diffeomorphic to the cover corresponding to the normal subgroup

$$G_i := (\iota_i)_*\bigl(\ker((Q_i)_*)\bigr) \subset \pi_1(U_i) = \pi_1(V_i^o)$$

and the number of such covers is m_i divided by the index of G_i in $\pi_1(V_i^o)$ (see Lemma 5.40 below). Such a cover does not depend on the choice of neighborhood U_i. In fact, it is an invariant of the model resolution up to isotopy.

Lemma 5.40 *The index of G_i divides m_i.*

The proof of this lemma also gives us a geometric interpretation of $\widetilde{V}_i^o$.

Proof After an isotopy, we can assume that our model resolution admits a regularization

$$\mathcal{R} \equiv ((\rho_j)_{j\in I}, (\Psi_I)_{I\subset S})$$

of radius R along U for some relatively compact open U containing $\bigcup_{j\in S-\star_S} V_i$. Let $T_{r,i}$ be the radius r tube of V_i as in (5-14) for some $r < \frac{3}{4}R$. We can assume that the open neighborhood U_i from Definition 5.39 is equal to $T_{r,i} - \bigcup_{j\in S-i} V_j$. We have that our map Q_i is equal to

$$Q_i\colon U_i - V_i^o \to \mathbb{C}^*, \quad Q_i(x) \equiv \Phi_2 \circ s_{(m_i)_{i\in S}}(x).$$

Define $\mathbb{D}(\epsilon)^* \equiv \mathbb{D}(\epsilon) - 0$, where $\mathbb{D}(\epsilon) \subset \mathbb{C}$ is the ϵ–disk. Then Q_i restricted to $Q_i^{-1}(\mathbb{D}(\epsilon)^*)$ for $\epsilon > 0$ small enough is a fibration whose fibers are smooth manifolds with corners. Combining this with the fact that $\pi_2(\mathbb{D}(\epsilon)^*) = 0$, we get that the map

$$\pi_1(Q_i^{-1}(\epsilon)) \to \ker((Q_i|_{Q_i^{-1}(\mathbb{D}(\epsilon)^*)})_*) = \ker((Q_i)_*)$$

is an isomorphism by a fibration long exact sequence argument. Therefore, the natural map

$$\pi_1(Q_i^{-1}(\epsilon)) \to \pi_1(U_i)$$

has image G_i. Also, for $0 < \epsilon \ll r \ll 1$, the map

$$P\colon Q_i^{-1}(\epsilon) \to V_i^o, \quad P(x) \equiv \pi_{\mathcal{N}_X} \circ \Psi_i^{-1}|_{Q_i^{-1}(\epsilon)}(x),$$

is a covering map of order m_i over $\mathrm{Im}(P)$ and V_i^o is homotopic to the image $\mathrm{Im}(P)$. Hence, the index of G_i divides m_i. □

Theorem 5.41 *Let $m \in \mathbb{N}_{>0}$ and let $\left(\mathcal{O}_X\left(\sum_{i\in S} m_i V_i\right), \Phi, \theta\right)$ be a graded model resolution that is also a multiplicity m separating resolution and define $V_i^o \equiv V_i - \bigcup_{j\in S-i} V_j$ for all $i \in S$ and define*

$$S_m \equiv \{i \in S - \star_S : m_i \text{ divides } m\}.$$

Let a_i be the discrepancy of V_i for each $i \in S - \star_S$. Then there is a graded abstract contact open book (M, θ_M, ϕ) such that the contact pair associated to it is graded contactomorphic to the link of our model resolution. Also, there is small positive slope perturbation $\check{\phi}$ of ϕ^m such that the fixed-point set of $\check{\phi}$ is a disjoint union of codimension 0 families of fixed points $(B_i)_{i\in S_m}$ satisfying

1. *$H^*(B_i; \mathbb{Z}) = H^*(\widetilde{V}_i^o; \mathbb{Z})$, where $\widetilde{V}_i^o$ is the natural m_i–fold covering of V_i^o as in Definition 5.39,*
2. *the action of B_i is equal to $-m_i w_i - \pi(m_i - m)\epsilon^2$, where w_i is the wrapping number of θ around V_i, and*
3. *$\mathrm{CZ}(\check{\phi}, B_i) = 2k_i(a_i + 1)$, where $k_i \equiv m/m_i$,*

for all $i \in S - \star_S$.

Proof of Theorem 5.41 We will use the same notation as in Definition 5.33. We will introduce it again here for the sake of clarity. After an isotopy, we can assume that our model resolution admits a regularization

$$\mathcal{R} \equiv ((\rho_i)_{i\in I}, (\Psi_I)_{I\subset S})$$

of radius R along U for some relatively compact open U containing $K \equiv \bigcup_{i\in S-\star_S} V_i$ since, by Lemma 5.26, the link does not change after this isotopy. Our abstract contact open book (M, θ_M, ϕ) will be the graded abstract contact open book associated to this model resolution as in Definition 5.33. By Lemma 5.36, the link of $\mathrm{OBD}(M, \theta_M, \phi)$ is contactomorphic to the link of our model resolution.

We now wish to show that ϕ satisfies properties (1)–(3) listed in the statement of this theorem. To do this, we need to recall the construction of (M, θ_M, ϕ). Let $T_{r,I}$ be the radius r tube of V_I as in (5-14) and let $T_{r,I}^o$ be the interior of $T_{r,I}$. Let $\check{T}$ be a smoothing of the manifold with corners $\bigcup_{i\in S-\star_S} T_{R,i}$ as in Definition 5.33. In other words, $\check{T}$ satisfies (5-38), X_θ points outwards along $\partial\check{T}$, $\check{T} \subset \bigcup_{i\in S-\star_S} T_{R,i}$ and $\bigcup_{i\in S-\star_S} T_{3R/4,i}$ is contained in the interior of $\check{T}$. Define

$$\pi_\Phi \colon \check{T} \to \mathbb{C}, \quad \pi_\Phi \equiv \Phi_2 \circ s_{(m_i)_{i\in S}}|_{\check{T}},$$

where $\Phi_2\colon \mathcal{O}_X\big(\sum_{i\in S} m_i V_i\big) \to \mathbb{C}$ is the composition of Φ with the natural projection map $X \times \mathbb{C} \to \mathbb{C}$. Then we can assume that

$$(M, \theta_M) \equiv (\pi_\Phi^{-1}(\epsilon), \theta|_{\pi_\Phi^{-1}(\epsilon)})$$

for some small $\epsilon > 0$. We will assume that $\epsilon > 0$ is small enough that $M \subset \bigcup_{i\in S} T_{R/4,i}$ and the fibers of $\pi_\Phi|_{\pi_\Phi^{-1}(\mathbb{D}(\epsilon))}$ are symplectic by Lemma 5.32. Let ω be the associated symplectic form of our model resolution. Define $\phi\colon M \to M$ to be the monodromy map around the loop

$$[0,1] \to \partial D(\epsilon), \quad t \to e^{2\pi i t},$$

with respect to the symplectic connection associated to ω.

First of all, we will compute the fixed points of the map ϕ^m. To do this, we will show that they correspond to certain periodic orbits of the flow of a Hamiltonian on $\check{T}$. Define

$$H\colon \mathrm{Dom}(\pi_\Phi) = \check{T} \to \mathbb{R}, \qquad H(x) = |F(x)| \quad \text{for all } x \in \check{T}.$$

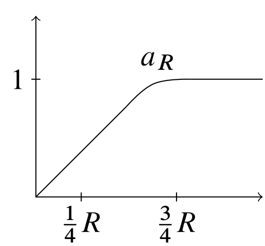
It is sufficient for us to find the periodic orbits of X_H starting inside M which map under π_Φ to loops in $\mathbb{C}^*$ which wrap around 0 exactly m times in the anticlockwise direction. This is because there is a one-to-one correspondence between fixed points of ϕ^m and such orbits. This correspondence sends a fixed point p of ϕ^m to the unique flowline of X_H starting and ending at p whose image under π_Φ wraps around 0 exactly m times in the anticlockwise direction.

Define $\check{T}_{R,I} \equiv T^o_{3R/4,I} - \bigcup_{i\in S-I} T^o_{3R/4,i}$ for each $I \subset S$. Since $M \subset \bigcup_{I\subset S} \check{T}_{R,I}$, it is sufficient for us to calculate the fixed points of ϕ^m inside $M \cap \check{T}_{R,I}$ for each $I \subset S$. Therefore, we will now compute the periodic orbits of X_H starting inside $M \cap \check{T}_{R,I}$ for all $I \subset S$ which project to loops in $\mathbb{C}^*$ wrapping m times around 0 in an anticlockwise direction. Let $a_R\colon [0,\infty) \to [0,\infty)$ be the smooth function defined in Definition 5.9. In other words, a_R satisfies

(1) $a_R'(x) > 0$ for $x \in \big[0, \frac{3}{4}R\big)$,

(2) $a_R(x) = x$ for $x \le \frac{1}{4}R$,

(3) $a_R(x) = 1$ for $x \ge \frac{3}{4}R$:



Define

$$b_R\colon [0,\infty) \to [0,\infty), \quad b_R(x) \equiv \sqrt{a_R(x)}.$$

Let

$$p_I\colon T_{R,I} \to V_I, \quad p_I(x) \equiv \pi_{\mathcal{N}_X V_I}(\Psi_I^{-1}(x)),$$

be the natural projection map. Inside $\check{T}_{R,I}$ we have that

$$H(x) = \prod_{i\in I} \bigl(b_R(\rho_i(x))\bigr)^{m_i} \quad \text{for all } x \in \check{T}_{R,I}$$

since the bundle trivialization Φ is radius R compatible with our regularization $\mathcal{R}$ along $T_{R,I}$. Hence,

$$X_H|_x = \sum_{i\in I} \biggl(m_i b_R'(\rho_i(x)) b_R(\rho_i(x))^{m_i} \prod_{j\in I-i} b_R(\rho_j(x))^{m_j} \biggr) X_{\rho_i}|_x \quad \text{for all } x \in \check{T}_{R,I}. \tag{5-45}$$

This means that all the periodic orbits of X_H starting inside $\check{T}_{R,I}$ are contained inside the fibers of p_I since the vector fields X_{ρ_i} are tangent to these fibers. Also, since $b_R(\rho_i(x)) > 0$ and $b_R'(\rho_i(x)) > 0$ for all $x \in \check{T}_{R,I}$ and all $i \in I$, we have that any disk contained inside a fiber of p_I bounding any such orbit must intersect V_i positively for all $i \in I$. This implies that the projection of this orbit to $\mathbb{C}^*$ wraps around 0 more than m times if $|I| > 1$ since our model resolution is a multiplicity m separating resolution. This means that if the set of periodic orbits of X_H starting inside $M \cap \check{T}_{R,I}$ whose image in $\mathbb{C}^*$ wraps m times around 0 is nonempty then $|I| = 1$. Hence, all fixed points of ϕ^m are contained inside $\bigcup_{i\in S} M \cap \check{T}_{R,i}$. Similar reasoning ensures that $i \in S_m \cup \{\star_S\}$ and that the set of fixed points of ϕ^m inside $T_{R,i}$ is $B_i \equiv M \cap \check{T}_{R,i}$ for all such i.

By Lemma 5.42 below with $W = T_{R,i}$, $h = \pi\rho_i|_{T_{R,i}}$, $B_1 = h^{-1}(\pi\sqrt{\epsilon})$, $B_2 = T_{R,i} \cap \{|\pi_\Phi| = \epsilon\}$, and $f_j \equiv \frac{1}{2\pi}\arg(\pi_\Phi)|_{B_j}$ for $j = 1,2$, we have that B_i is a codimension 0 family of fixed points of ϕ^m for all $i \in S_m$. Since B_i is homotopic to $\pi_\Phi^{-1}(\epsilon) \cap T_{R,i}$, which in turn is homotopic to the fiber $Q_i^{-1}(\epsilon)$ constructed in the proof of Lemma 5.40, we have $H^*(B_i;\mathbb{Z}) = H^*(\widetilde{V}_i^o;\mathbb{Z})$ for all $i \in S - \star_S$.

We now need to construct a small positive slope perturbation $\check{\phi}$ of ϕ^m without creating any extra fixed points such that $B_{\star_S}$ disappears and such that $\phi = \check{\phi}$ near $\bigcup_{i\in S_m} B_i$. Since $B_{\star_S}$ is a codimension 0 family of fixed points of ϕ, there is a neighborhood $N_{\star_S}$ of $B_{\star_S}$ and a Hamiltonian $H_{\star_S}\colon N_{\star_S} \to (-\infty, 0]$ such that ϕ^m is the time 1 flow of $H_{\star_S}$ inside $N_{\star_S}$ and such that $B_{\star_S} = H_{\star_S}^{-1}(0)$. Choose $\delta_\star > 0$ small enough that $H_{\star_S}$ has no q–periodic orbits inside $H_{\star_S}^{-1}(-\delta_\star, 0)$ for all $q \in [0,2]$. Since the vector field (5-45) is tangent to the fibers of p_I inside $T_{R,I} \cap T_{3R/4,\star_S}$ for all $I \subset S$ and since Ψ_I is a regularization, we have that $H_{\star_S}$ must be a function of the variables $(\rho_i)_{i \in I}$ inside $T_{R,I} \cap T_{3R/4,\star_S}$ only. This implies that we can construct a smooth function $\check{b}\colon N_{\star_S} \to \mathbb{R}$ for $\delta_\star > 0$ small enough that

- $\check{b} = F \circ H_{\star_S}$ for some smooth function $F\colon \mathbb{R} \to \mathbb{R}$ inside $H_{\star_S}^{-1}\big([-\delta_{\star_S}, -\frac{1}{3}\delta_{\star_S}]\big)$, where $F \circ H_{\star_S} = H_{\star_S}$ near $H_{\star_S}^{-1}(-\delta_{\star_S})$,
- $\check{b}$ is C^2 small inside $H_{\star_S}^{-1}\big([-\frac{1}{3}\delta_{\star_S}, 0]\big)$
- $\check{b} = \delta r_M$ near ∂M, where r_M is the radial coordinate on M, and
- $\check{b}$ has no critical points.

This implies that the time 1 flow of $\check{b}$ has no fixed points inside $N_{\star_S}$ and is equal to $H_{\star_S}$ outside a compact subset of $N_{\star_S}$. Define $\check{\phi}$ to be equal to ϕ^m outside $N_{\star_S}$ and the time 1 flow of $\check{b}$ inside $N_{\star_S}$. This is a positive slope perturbation of ϕ^m such that the set of fixed points of $\check{\phi}$ is $\bigcup_{i \in S - \star_S} B_i$ and $\check{\phi} = \phi^m$ in a neighborhood of these fixed points.

Next we need to compute the action of B_i for each $i \in S - \star_S$. Let $p \in B_i$ and let $\gamma\colon \mathbb{R}/\mathbb{Z} \to \check{T}$ be the unique loop starting at $p \in M$ which is symplectically orthogonal to the fibers of π_Φ and satisfying $\pi_\Phi \circ \gamma(t) = e^{2i\pi m t}$ for all $t \in \mathbb{R}$. Then the action of p is equal to $-\int_0^1 \gamma^*\theta + \pi m \epsilon^2 = -m_i w_i - \pi(m_i - m)\epsilon^2$.

We now need to compute the Conley–Zehnder index of B_i for each $i \in S_m$. Fix $i \in S_m$ and let $p \in B_i \subset M$. Let $\gamma\colon \mathbb{R}/\mathbb{Z} \to \check{T}$ be the unique loop starting at $p \in M$ which is symplectically orthogonal to the fibers of π_Φ and satisfying $\pi_\Phi \circ \gamma(t) = e^{2i\pi m t}$ for all $t \in \mathbb{R}$. Let J be an ω–compatible almost complex structure on X such that π_Φ becomes J–holomorphic. Let

$$T^{\mathrm{ver}}\check{T} \equiv \ker(D\pi_\Phi)|_{\check{T}-K} \subset T(\check{T} - K)$$

be the vertical tangent bundle. Let $(T^{\mathrm{ver}}\check{T})^\perp \subset T(\check{T} - K)$ be the set of vectors which are ω–orthogonal to the vertical tangent bundle. This is a J–holomorphic subbundle of $T(\check{T} - K)$. Let $\tau_{\mathbb{C}^*}\colon T\mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}$ be the holomorphic trivialization which sends $\partial/\partial\vartheta$ to the constant section 1 and let $\tau_{\mathbb{C}^*,2}\colon T\mathbb{C}^* \to \mathbb{C}$ be the composition of $\tau_{\mathbb{C}^*}$ with the natural projection map $\mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$. We then have a trivialization

$$\tau^\perp\colon (T^{\mathrm{ver}}\check{T})^\perp \to (\check{T} - K) \times \mathbb{C},$$
$$\tau_{T^\perp}(Y) \equiv \big(x, \tau_{\mathbb{C}^*,2}(D\pi_\Phi(Y))\big) \quad \text{for all } Y \in (T^{\mathrm{ver}}\check{T})^\perp|_x,\ x \in \check{T} - K.$$

Let

$$(\tau^{\perp})^*\colon ((T^{\mathrm{ver}}\check{T})^{\perp})^* \to (\check{T} \cap \check{X}) \times \mathbb{C}$$

be the corresponding trivialization of the dual bundle.

Let $\kappa_{J,\phi}$ be the canonical bundle of $T^{\mathrm{ver}}\check{T}$. Then we have a canonical isomorphism

$$\kappa_J|_{\check{T}-K} \cong \kappa_{J,\phi} \otimes ((T^{\mathrm{ver}}\check{T})^{\perp})^*. \tag{5-46}$$

Since $(X-K,\omega)$ is a graded symplectic manifold, we get a natural choice of trivialization $\tau\colon \kappa_J|_{X-K} \to (X-K)\times\mathbb{C}$ by Definition A.7. The trivializations τ and $(\tau^{\perp})^*$ give us a trivialization $\tau^{\mathrm{ver}}\colon \kappa_{J,\phi} \to (\check{T}-K)\times\mathbb{C}$ of $\kappa_{J,\phi}$ by the identity (5-46).

Let s_ϕ be a section of $\kappa_{J,\phi}$ that is equal to the constant section 1 with respect to our trivialization τ^{ver}. Let s be a section of κ_J such that $s^{-1}(0)$ is transverse to 0 and contained inside a small neighborhood N of $\bigcup_{i\in S-\star_S} V_i$ which deformation retracts onto $\bigcup_{i\in S-\star_S} V_i$ and such that $\tau\circ s|_{X-N}$ is the constant section 1. Then by definition $[s^{-1}(0)]$ is a homology class homologous to $\sum_{i\in S} a_i[V_i]$. Now define $\check{T}_\epsilon \equiv \pi_{\Phi}^{-1}(\partial\mathbb{D}(\epsilon))$. By construction, $\tau^{\mathrm{ver}}|_{\check{T}_\epsilon}$ is homotopic to the induced trivialization from Lemma 5.20 (after identifying the contact hyperplane distribution in $\check{T}_\epsilon$ with $T^{\mathrm{ver}}\check{T}$ using an isotopy between these symplectic subbundles).

We can choose J so that, near the image γ, the Hamiltonian flow $\phi_t^{\rho_i/2}\colon T_{R,i}\to T_{R,i}$ of $\frac{1}{2}\rho_i|_{T_{R,i}}$ is J–holomorphic. Hence, on some neighborhood N_γ of γ invariant under the flow of $X_{\rho_i/2}$, we have that $\phi_t^{\rho_i/2}$ lifts to a map

$$\widetilde{\phi}_t\colon \kappa_J|_x \to \kappa_J|_{\phi_t^{\rho_i/2}(x)}$$

for all $x\in \check{T}_\epsilon\cap N_\gamma$ given by the highest wedge power of the J–holomorphic bundle map $((D\phi_t^{\rho_i/2})^{-1})^*$. Also, since $D\phi_t^{\rho_i/2}(v)\in T^{\mathrm{ver}}\check{T}_\epsilon$ for all $v\in T^{\mathrm{ver}}\check{T}_\epsilon|_{N_\gamma\cap\check{T}_\epsilon}$, we get an induced map

$$\widetilde{\phi}_t^{\mathrm{ver}}\colon \kappa_{J,\phi}|_{\check{T}_\epsilon\cap N_\gamma} \to \kappa_{J,\phi}|_{\check{T}_\epsilon\cap N_\gamma}.$$

Let $\tau_2\colon \kappa_J|_{X-K}\to\mathbb{C}$ and $\tau_2^{\mathrm{ver}}\colon \kappa_{J,\phi}\to\mathbb{C}$ be the compositions of τ and τ^{ver}, respectively, with the natural projection map to $\mathbb{C}$. The winding number of the map

$$w_\phi\colon \mathbb{R}/2\pi m\mathbb{Z}\to\mathbb{C}^*\cong \mathrm{Aut}(\mathbb{C},\mathbb{C}),\quad w_\phi(t)=\tau_2^{\mathrm{ver}}\circ(\widetilde{\phi}_t^{\mathrm{ver}}|_p)\circ(\tau_2^{\mathrm{ver}}|_{\kappa_{J,\phi}|_p})^{-1},$$

is equal to the winding number of the map

$$w_\tau\colon \mathbb{R}/2\pi m\mathbb{Z}\to\mathbb{C}^*\cong \mathrm{Aut}(\mathbb{C},\mathbb{C}),\quad w_\tau(t)=\tau_2\circ(\widetilde{\phi}_t|_p)\circ(\tau_2|_{\kappa_J|_p})^{-1}.$$

Since $[s^{-1}(0)]$ is homologous to $\sum_{i\in S-\star_S} a_i[V_i]$ and $\gamma(t)=\phi_t^{\rho_i/2}(p)$ for all t, we have that the winding number of w_τ is equal to

$$k_i(-1-a_i).$$

Hence, by Lemma A.8 and the fact that the winding number of w_ϕ is the winding number of w_τ, we have that the Conley–Zehnder index of the fixed point p of ϕ^m is -2 times the winding number of w_τ and so $\mathrm{CZ}(\phi^m, B_i)=2k_i(a_i+1)$. □

Here is a technical lemma that was used in the proof of Theorem 5.41 above:

Lemma 5.42 *Let (W,ω) be a symplectic manifold admitting a free Hamiltonian S^1–action generated by a Hamiltonian $h\colon W\to\mathbb{R}$* (ie $\phi_1^h=\mathrm{id}_W$ and $\phi_t^h(x)\neq x$ for all $x\in W$ and $t\in(0,1)$). *Let $B_1,B_2\subset W$ be two real hypersurfaces inside W with maps*

$$f_1\colon B_1\to\mathbb{R}/\mathbb{Z},\quad f_2\colon B_2\to\mathbb{R}/\mathbb{Z}$$

such that

1. *the fibers of f_i are symplectic submanifolds of W and the corresponding monodromy map $\phi_i\colon f_i^{-1}(0)\to f_i^{-1}(0)$ of f_i around $\mathbb{R}/\mathbb{Z}$ is well defined* (ie no points parallel transport off to infinity in finite time) *for $i=1,2$,*
2. *$\phi_1=\mathrm{id}_{B_1}$, $B_1=h^{-1}(C)$ and $B_2\subset h^{-1}([C,\infty))$ for some $C>0$,*
3. *B_i is invariant under our Hamiltonian S^1–action for $i=1,2$ and, for all $t\in S^1=\mathbb{R}/\mathbb{Z}$ and $x\in B_1$, we have $f_1(t\cdot x)=f_1(x)+t$, and*
4. *$f_1^{-1}(0)\cap f_2^{-1}(0)$ is equal to a compact codimension 0 submanifold of $f_2^{-1}(0)$ with boundary and corners and $f_1|_{B_1\cap B_2}=f_2|_{B_1\cap B_2}$.*

Then $f_1^{-1}(0)\cap f_2^{-1}(0)$ is a codimension 0 family of fixed points of ϕ_2^m for all $m>0$.

Proof Let $Q\subset B_1$ be an S^1–invariant relatively compact open set containing $B_1\cap B_2$, let $V\equiv Q\cap f_1^{-1}(0)$ and $\omega_V\equiv\omega|_V$. For all $r_1,r_2>0$ let $A_{r_1,r_2}\subset\mathbb{C}$ be the open annulus whose inner radius is r_1 and whose outer radius is r_2 with the standard symplectic form. Let $r\colon\mathbb{C}\to[0,\infty)$ be the radial function $z\to|z|$ and $\theta\colon\mathbb{C}-0\to\mathbb{R}/2\pi\mathbb{Z}$ the angle coordinate. Define $\check{C}\equiv\sqrt{C/\pi}$. Let $S_{\check{C}}\subset\mathbb{C}$ be the circle of radius $\check{C}$. After shrinking Q slightly we can, by an equivariant Moser theorem (see [18]), find an S^1–equivariant open set $U\subset W$ symplectomorphic to

$$\left(V\times A_{\check{C}-\delta,\check{C}+\delta},\omega_V+\tfrac{1}{2}\,d(r^2)\wedge d\vartheta\right)$$

such that $Q=V\times S_{\check{C}}$ and $h|_U=\pi r^2$.

If we smoothly deform f_2 inside B_2 through fibrations whose fibers are always transverse to the line field given by $\ker(\omega|_{B_2})$ then the symplectic form on the fibers and the monodromy map do not change. This is because such a deformation can be realized by a flow along a vector field tangent to the line field $\ker(\omega|_{B_2})$. In particular, we can assume in some small S^1–invariant neighborhood $\check{U} \subset U$ of $B_1 \cap B_2$ that

$$f_2|_{\check{U}\cap B_2} = \left(\frac{\theta}{2\pi}\right)\Big|_{\check{U}\cap B_2}. \tag{5-47}$$

Let $\mathrm{pr}_1\colon V \times A_{\check{C}-\delta,\check{C}+\delta} \to V$ be the natural projection map. Define $V_2 \equiv f_2^{-1}(0) \cap \check{U}$ and $\omega_{V_2} \equiv \omega|_{V_2}$. We can assume that $\check{U}$ is small enough that $\mathrm{pr}_1|_{V_2}\colon V_2 \to V$ is a diffeomorphism onto its image. This map is also a symplectic embedding by (5-47). Define

$$H\colon V_2 \to \mathbb{R}, \quad H(x) \equiv \pi r(x)^2 - 2\pi\check{C} = h|_{V_2} - 2\pi\check{C}.$$

Since $\omega_V + \frac{1}{2} d(r^2) \wedge d\vartheta = \omega_V + dh \wedge d\left(\frac{1}{2\pi}\vartheta\right)$ inside $\check{U}$ and $\mathrm{pr}_1|_{V_2}$ is a symplectic embedding, we get that the vector field

$$-X_H^{\omega_{V_2}} + 2\pi\frac{\partial}{\partial\vartheta}$$

is tangent to $\check{U} \cap B_2$ and lies in the kernel of $\omega|_{B_2\cap\check{U}}$. Then for all $m > 0$, ϕ_2^m is equal to the time 1 flow of $-mH$ near $B_1 \cap f_2^{-1}(0)$ inside the symplectic manifold (V_2, ω_{V_2}). Hence, $B_1 \cap f_2^{-1}(0)$ is a codimension 0 family of fixed points of ϕ_2. □

6 Proof of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2 Let $L \equiv (L_f \subset S_\epsilon, \xi_{S_\epsilon}, \Phi_f)$ be the contact pair associated to f with the standard grading as in Example 3.8. Let $\left(\mathcal{O}_X\left(\sum_{i\in S} m_i E_i\right), \Phi, \theta\right)$ be a graded model resolution coming from the log resolution $\pi\colon Y \to \mathbb{C}^{n+1}$ as in Example 5.14. The wrapping number of θ around E_j is w_j for all $j \in S - \star_S$. By using the function $|z|^2$ on $\mathbb{C}^{n+1}$ combined with Lemma 5.34, one can show that the link of this model resolution is contactomorphic to L. Hence, Theorem 1.2 follows from Theorem 5.41, (HF2) and (HF3). □

Lemma 6.1 *Suppose we have a cohomological spectral sequence converging to a $\mathbb{Z}$–graded abelian group G^* with E^1 page $(E^1_{p,q})_{p\in\mathbb{Z},q\in\mathbb{Z}}$. Define*

$$m \equiv \sup\{p+q : E_1^{p,q} \neq 0\},$$
$$k_p \equiv \sup\{p+q : q \in \mathbb{Z},\ E_1^{p,q} \neq 0\} \quad \textit{for all } p \in \mathbb{Z}.$$

Suppose that m is finite and $k_p \neq m-1$ for all $p \in \mathbb{Z}$. Then $G^m \neq 0$ and $G^k = 0$ for all $k > m$.

Proof Let

$$p_m \equiv \inf\{p \in \mathbb{Z} : p+q = m \text{ and } E_1^{p,q} \neq 0 \text{ for some } q \in \mathbb{Z}\}.$$

We will show that each element of $E_j^{p_m,m-p_m}$ can never kill or be killed by the spectral sequence differential for each $j \geq 1$. Since $k_{p_m-j} \neq m-1$ for all j, we get that $k_{p_m-j} < m-1$ for all $j \geq 1$. Therefore, $E_j^{p_m-j,m-p_m+j-1} = 0$ for all $j \geq 1$. Hence, the differential

$$d_j^{p_m-j,m-p_m+j-1} \colon E_j^{p_m-j,m-p_m+j-1} \to E_j^{p_m,m-p_m}$$

is zero for all j. Also, since $(p_m+j)+(m-p_m-j+1) = m+1 > m$, we get that $E_j^{p_m+j,m-p_m-j+1} = 0$ for all j. Therefore, the differential

$$d_j^{p_m,m-p_m} \colon E_j^{p_m,m-p_m} \to E_j^{p_m+j,m-p_m-j+1}$$

is zero for all j. Hence, $G^m \neq 0$. Also, $G^k = 0$ for all $k > m$ since $E_1^{p,q} = 0$ for all $p,q \in \mathbb{Z}$ satisfying $p+q=k$. □

Proof of Corollary 1.3 The numbers μ_m do not depend on the choice of log resolution for all $m > 0$ by Lemmas 2.3 and 2.6. Hence, Corollary 1.3 follows immediately from Theorem 1.2 combined with Lemmas 2.4 and 6.1. □

Appendix A Gradings and canonical bundles

In this section we will develop tools so that we can construct gradings (see Definition 3.2) and relate them to other kinds of topological information. In this paper we will only need to study gradings up to isotopy, which will be defined now. We will first give a definition of a grading for any principal G bundle and then relate it to gradings of (E,Ω). Throughout this section, G will be a Lie group, $\widetilde{G}$ its universal cover and $p \colon W \to B$ will be a principal G bundle. Also, $\pi \colon E \to V$ will be a symplectic vector bundle with symplectic form Ω whose fibers have dimension $2n$.

Definition A.1 A *grading* of p consists of a principal $\widetilde{G}$ bundle $\widetilde{p} \colon \widetilde{W} \to B$ along with a G bundle isomorphism

$$\iota \colon \widetilde{W} \times_{\widetilde{G}} G \cong W.$$

Note that a grading of (E,Ω) is equivalent to a grading of the principal $\mathrm{Sp}(2n)$ bundle $\mathrm{Fr}(E)$. Let

$$\iota_j\colon \widetilde{W}_j \times_{\widetilde{G}} G \cong W, \quad j = 0,1,$$

be gradings of p. An *isotopy* between these two gradings consists of a $\widetilde{G}$–bundle isomorphism

$$\Psi\colon \widetilde{W}_0 \to \widetilde{W}_1$$

together with a smooth family of G bundle isomorphisms

$$\check{\iota}_t\colon \widetilde{W}_0 \times_{\widetilde{G}} G \cong W$$

joining ι_0 and $\iota_1 \circ \check{\Psi}$, where $\check{\Psi}\colon \widetilde{W}_0 \times_{\widetilde{G}} G \to \widetilde{W}_1 \times_{\widetilde{G}} G$ is the natural isomorphism induced by Ψ. An *isotopy* between two gradings of (E,Ω) is an isotopy between the corresponding gradings on the principal $\mathrm{Sp}(2n)$ bundle $\mathrm{Fr}(E)$. We can define isotopies of gradings of symplectic manifolds and contact manifolds in a similar way.

Definition A.2 Let

$$\iota\colon \widetilde{W} \times_{\widetilde{G}} G \cong W$$

be a grading of p. The *associated covering map* of this grading is the natural map

$$\widetilde{W} \to \widetilde{W} \times_{\widetilde{G}} G \xrightarrow{\iota} W.$$

The following lemma gives a topological characterization of gradings. For simplicity we will assume that the base B is connected. Let $\star \in W$ be a choice of basepoint.

Lemma A.3 *Let N_W be the set of normal subgroups $A \triangleleft \pi_1(W,\star)$ such that*

$$p_*\colon \pi_1(W,\star) \to \pi_1(B, p(\star))$$

restricted to A is an isomorphism. Let Gr_W be the set of isotopy classes of gradings of W. Then the map $Q_W\colon \mathrm{Gr}_W \to N_W$ sending a grading to

$$\mathrm{Im}(P_*) \subset \pi_1(W,\star) \tag{A-1}$$

is an isomorphism, where P is the associated covering map of this grading.

Proof We will first show that the map Q_W is well defined by showing that the image (A-1) is a normal subgroup. Let

$$\iota\colon \widetilde{W} \times_{\widetilde{G}} G \cong W$$

be a grading of p. The covering map P of such a grading is isomorphic (using the map ι) to the natural map

$$\widetilde{W} \to \widetilde{W} \times_{\widetilde{G}} G.$$

The deck transformations of this map are equal to $\ker(\widetilde{G} \to G)$ and these act transitively. Hence, the image (A-1) is a normal subgroup. Combining this with the fact that the fibers of the natural fibration $\widetilde{W} \to B$ are simply connected, the image (A-1) is contained in N_W and hence the map Q_W is well defined.

We will now construct an inverse to Q_W. Let $N \triangleleft \pi_1(W, \star)$ be an element of N_W. Let

$$P\colon \widetilde{W} \to W$$

be a covering map with a choice of basepoint $\widetilde{\star} \in \widetilde{W}$ mapping to $\star$ such that the map

$$P_*\colon \pi_1(\widetilde{W}, \widetilde{\star}) \to \pi_1(W, \star)$$

has image equal to N. Let F be a fiber of p. Let $\iota_F\colon F \hookrightarrow W$ be the natural inclusion map. Since $P_*|_{P_*^{-1}(N)}$ is injective, we get that $(\iota_F)_*^{-1}(P_*^{-1}(N)) = \{\mathrm{id}\}$ and hence $P|_{P^{-1}(F)}\colon P^{-1}(F) \to F$ is the universal covering map. This implies that each fiber has a natural $\widetilde{G}$ action and hence

$$p \circ P\colon \widetilde{W} \to B$$

is a $\widetilde{G}$ bundle with a natural isomorphism

$$\widetilde{W} \times_{\widetilde{G}} W \cong W.$$

We define $Q_W^{-1}(N)$ to be the above grading. This is an inverse to Q_W. □

We have the following immediate corollary of Lemma A.3:

Corollary A.4 *Suppose that $p_j\colon W_j \to B$ is a principal G_j bundle for some Lie group G_j for $j = 1, 2$. Let $\Phi\colon W_1 \to W_2$ be a map of fiber bundles such that the induced map on the fibers is a fundamental group isomorphism. Then the map $\Phi_*\colon N_{W_1} \to N_{W_2}$ induces a natural bijection between isotopy classes of gradings on W_1 and isotopy classes of gradings on W_2.*

Here the sets N_{W_1} and N_{W_2} in this corollary are defined as in Lemma A.3.

Lemma A.5 *Let $\pi_K\colon K \to B$ be a principal $U(1)$ bundle. Then there is a natural one-to-one correspondence between homotopy classes of trivializations of π_K and isotopy classes of gradings of π_K.*

Proof There is a one-to-one correspondence between trivializations $\Phi\colon K \to B \times U(1)$ of K up to homotopy and sections of π_K up to homotopy given by the map sending Φ to the section whose image is $\Phi^{-1}(1)$. Hence, all we need to do is construct a natural one-to-one correspondence between sections up to homotopy and isotopy classes of gradings. Let S be the set of sections up to homotopy and Gr_K the set of isotopy classes of gradings. By Lemma A.3, it is sufficient for us to construct a bijection between S and N_K. We define

$$\Psi\colon S \to N_K, \quad \Psi(s) = \mathrm{Im}\big(s_*\colon \pi_1(B,\star) \to \pi_1(K, s(\star))\big).$$

The inverse of this map is constructed as follows: We start with a normal subgroup $N \in N_K$. This gives us a grading

$$\iota\colon \widetilde{K} \times_{\mathbb{R}} U(1) \cong K$$

by Lemma A.3 since $\widetilde{U(1)} = \mathbb{R}$. Since the fibers of $\widetilde{K}$ are contractible, there is a smooth section $\tilde{s}\colon B \to \widetilde{K}$ by [32, Theorem 9] combined with the Steenrod approximation theorem [31, Section 6.7, Main Theorem]. The composition $s \equiv P_K \circ \tilde{s}$, where $P_K\colon \widetilde{K} \to K$ is the associated covering map, is then a smooth section of K. We then define $\Psi^{-1}(N) \equiv s$. This is the inverse of Ψ. □

We will now focus on the principal $\mathrm{Sp}(2n)$ bundle $\mathrm{Fr}(E)$.

Definition A.6 Let J be a complex structure on E compatible with Ω. The *frame bundle* $\mathrm{Fr}(E,\Omega,J)$ of the unitary bundle (E,Ω,J) is the principal $U(n)$–bundle whose fiber over $v \in V$ is the space of unitary bases $e_1,\dots,e_n$ of $(E,\Omega,J)|_v$. The *anticanonical bundle* κ_J^* of (E,J) is the highest exterior power of the complex vector bundle (E,J). The associated $U(1)$–bundle $\kappa_{\Omega,J}^* \subset \kappa_J^*$ of κ_J^* has a fiber at $v \in V$ equal to the subset of elements $e_1 \wedge \dots \wedge e_n$, where $e_1,\dots,e_n$ is a unitary basis for $(E,\Omega,J)|_v$. Therefore, we have a natural map $\det_{\Omega,J}\colon \mathrm{Fr}(E,\Omega,J) \to \kappa_{\Omega,J}$.

The *canonical bundle* κ_J of (E,J) is the dual of κ_J^* (or equivalently the anticanonical bundle of the dual bundle of (E,J)). In a similar way, we can define the *(anti)canonical bundle* of a symplectic manifold with a choice of compatible almost complex structure, or of a contact manifold (C,ξ_C) with a choice of compatible contact form α and a $d\alpha|_{\xi_C}$–compatible almost complex structure J on ξ_C.

Definition A.7 Let J be an Ω–compatible complex structure on E. Let

$$\iota_J\colon \mathrm{Fr}(E,\Omega,J) \to \mathrm{Fr}(E,\Omega)$$

be the natural inclusion map. By [25, Propositions 2.22 and 2.23], the natural maps $\det_{\Omega,J}$ and ι_J above are bundle maps whose restriction to each fiber is a fundamental group isomorphism. Hence, by Corollary A.4 and Lemma A.5 there is a natural one-to-one correspondence between isotopy classes of gradings of (E,Ω) and homotopy classes of trivializations of $\kappa^*_{\Omega,J}$. Combining this with the fact that there is a natural one-to-one correspondence between homotopy classes of trivializations of $\kappa^*_{\Omega,J}$ and homotopy classes of trivializations of κ^*_J and hence of trivializations of the canonical bundle κ_J, we get a natural one-to-one correspondence

$$\text{(A-2)} \qquad \mathrm{Gr}\colon \{\text{trivializations of } \kappa_J\}/\text{homotopy} \xrightarrow{1\text{–}1} \{\text{gradings of } (E,\Omega)\}/\text{isotopy}.$$

Given a trivialization Φ of κ_J we will call the grading $\mathrm{Gr}(\Phi)$ the *grading associated to* Φ. Given a grading g of (E,Ω), we will call $\mathrm{Gr}^{-1}(g)$ the *trivialization associated to this grading*.

The above discussion enables us to compute the Conley–Zehnder index of a fixed point of a graded symplectomorphism in some nice cases. Let $\phi\colon M \to M$ be a graded exact symplectomorphism of a Liouville domain (M,θ_M). Let V be the unique vector field on the mapping torus T_ϕ from Definition 3.13 given by the lift of the vector field d/dt on $\mathbb{R}/\mathbb{Z}$ satisfying $\iota_V d\alpha_{T_\phi} = 0$. Let ϕ^V_t be the time t flow of V. Let x be a fixed point of ϕ. Suppose that there is a compatible complex structure J on the vertical tangent bundle $(T^{\mathrm{ver}}T_\phi \equiv \ker(D\pi_{T_\phi}), d\alpha_{T_\phi})$ such that $D\phi^V_t$ restricted to $T^{\mathrm{ver}}T_\phi|_x$ is J–holomorphic for all $t \in [0,1]$. Then ϕ^V_t lifts to a map

$$\widetilde{\phi}_t\colon \kappa_J|_x \to \kappa_J|_{\phi^V_t(x)}, \quad \widetilde{\phi}_t(\wedge_{i=1}^n e_i^*) = \wedge_{i=1}^n (\phi_t^*)^{-1} e_i^* \quad \text{for all } e_1^*,\dots,e_n^* \in T^*_x M.$$

Since ϕ is graded, we see by Definition 3.13 that there is a natural grading on the vertical tangent bundle. Therefore, by Definition A.7, there is a natural trivialization $\Phi\colon \kappa_J \to T_\phi \times \mathbb{C}$ of κ_J associated to this grading. Let $\Phi_2\colon \kappa_J \to \mathbb{C}$ be the composition of Φ with the natural projection map to $\mathbb{C}$.

Lemma A.8 *Let $x \in M = \pi^{-1}_{T_\phi}(0)$ be a fixed point of ϕ and suppose that*

$$D\phi|_x\colon T_xM \to T_xM$$

is the identity map. Then $\mathrm{CZ}(\phi,x)$ is equal to -2 times the winding number of the map

$$w\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}^* = \mathrm{Aut}(\mathbb{C},\mathbb{C}), \quad t \to \Phi_2 \circ \widetilde{\phi}_t \circ (\Phi_2|_{\kappa_J|_x})^{-1}.$$

Proof Let $\gamma\colon \mathbb{R}/m\mathbb{Z} \to T_\phi$ be the m–periodic orbit of V whose initial point is x. Then there is a unique (up to homotopy) unitary trivialization T of $\gamma^* T^{\mathrm{ver}} T_\phi$ such that Φ is equal to the highest wedge power of T. Because of the correspondence (A-2), we can also ensure that T maps the grading on $\gamma^* T^{\mathrm{ver}} T_\phi$ (given by pulling back the grading on $T^{\mathrm{ver}} T_\phi$ via γ) to the trivial grading on $(\mathbb{R}/m\mathbb{Z}) \times \mathbb{C}^{n+1}$ (maybe after changing the grading to an isotopic one).

Under this trivialization, the flow of V corresponds to a smooth family of Hermitian matrices $(A_t)_{t\in[0,1]}$ and the degree of w is -1 times the winding number of $t \to \det_{\mathbb{C}}(A_t)$. Using the correspondence (A-2) and the trivialization T, such a family of matrices corresponds to a point in the universal cover $\widetilde{\mathrm{Fr}}(TM)|_x$ of $\mathrm{Fr}(TM)|_x$. Hence, the Conley–Zehnder index of $(A_t)_{t\in[0,1]}$ is equal to $\mathrm{CZ}(\phi, x)$. Since A_t are unitary matrices, we get that $\mathrm{CZ}((A_t)_{t\in[0,1]})$ is equal to twice the winding number of $t \to \det_{\mathbb{C}}(A_t)$. Hence, $\mathrm{CZ}(\phi, x)$ is -2 times the winding number of w. □

Appendix B Contactomorphisms of mapping tori and Floer cohomology

The aim of this section is to show that property (HF2) holds. Here is a statement of this property:

Suppose that $(M_1, \theta_{M_1}, \phi_1)$ and $(M_2, \theta_{M_2}, \phi_2)$ are graded abstract contact open books such that the graded contact pairs associated to them are graded contactomorphic. Then $\mathrm{HF}^*(\phi_0, +) = \mathrm{HF}^*(\phi_1, +)$.

We will prove this by using an intermediate Floer cohomology group called S^1–equivariant Hamiltonian Floer cohomology on a certain mapping cylinder of our symplectomorphism.

Definition B.1 Let (M, θ_M, ϕ) be an abstract contact open book. Let $\check{\phi}$ be a small positive slope perturbation of ϕ. The *mapping cylinder of* $\check{\phi}$ is a triple $(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$ where

1. $W_{\check{\phi}} \equiv (\mathbb{R} \times \mathbb{R} \times M)/\mathbb{Z}$, where the $\mathbb{Z}$ action on $(\mathbb{R} \times \mathbb{R} \times M)$ has the property that $1 \in \mathbb{Z}$ sends (s, t, x) to $(s, t-1, \check{\phi}(x))$,
2. $\pi_{\check{\phi}}\colon W_{\check{\phi}} \to \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$ sends (s, t, x) to $(s, t) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$, and

(3) $\theta_{\check{\phi}} = s\,dt + \kappa\theta_M + \kappa\, d(\rho(t) F_{\check{\phi}})$, where

- $F_{\check{\phi}}\colon M \to \mathbb{R}$ is a smooth function with support in the interior of M that satisfies $(\check{\phi})^*\theta_M = \theta_M + dF_{\check{\phi}}$,
- $\rho\colon [0,1] \to [0,1]$ is a smooth function equal to 0 near 0 and 1 near 1, and
- $\kappa > 0$ is a constant small enough to ensure that $d\theta_{\check{\phi}}$ is symplectic.

Let $r_M\colon (0,1] \times \partial M \to (0,1]$ be the cylindrical coordinate on M. Let $\delta_{\check{\phi}} > 0$ be small enough that the symplectomorphism $\check{\phi}$ is equal to the time 1 flow of δr_M inside $(1-\delta_{\check{\phi}}, 1] \times \partial M$ for some $\delta > 0$. Let $\phi_t^{\delta r_M}\colon (1-\delta_{\check{\phi}}, 1] \times \partial M \to (1-\delta_{\check{\phi}}, 1] \times \partial M$ be the time t flow of δr_M. Then we have a natural embedding

$$\iota_{\check{\phi}}\colon C_{\check{\phi}} \equiv (\mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \times (1-\delta_{\check{\phi}}, 1] \times \partial M) \hookrightarrow W_{\check{\phi}}, \quad \iota_{\check{\phi}}(s,t,r_M,y) \equiv (s,t,\phi_{-t}^{\delta r_M}(r_M,y)),$$

called the *vertical cylindrical end of* $W_{\check{\phi}}$. The coordinate

$$r_{\check{\phi}}\colon C_{\check{\phi}} \to (1-\delta_{\check{\phi}}, 1], \quad r_{\check{\phi}}(s,t,x) \equiv r_M(x), \tag{B-1}$$

is called the *vertical cylindrical coordinate*. A *grading* on a mapping cylinder

$$(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$$

is a grading on the symplectic manifold $(W_{\check{\phi}}, d\theta_{\check{\phi}})$.

Two mapping cylinders $(W_{\check{\phi}_1}, \pi_{\check{\phi}_1}, \theta_{\check{\phi}_1})$, $(W_{\check{\phi}_2}, \pi_{\check{\phi}_2}, \theta_{\check{\phi}_2})$ are *isomorphic* if there is a diffeomorphism $\Phi\colon W_{\check{\phi}_1} \to W_{\check{\phi}_2}$ and a constant $\delta > 0$ such that

- $\Phi^*\theta_{\check{\phi}_2} = \theta_{\check{\phi}_1} + dq$ for some $q\colon W_{\check{\phi}_1} \to \mathbb{R}$, where q has support in the set $W_{\check{\phi}_1} - \{r_{\check{\phi}_1} \geq 1-\delta\}$, and
- $\pi_{\check{\phi}_1}|_{\{r_{\check{\phi}_1} \geq 1-\delta\}} = (\pi_{\check{\phi}_2} \circ \Phi)|_{\{r_{\check{\phi}_1} \geq 1-\delta\}}$.

They are *graded isomorphic* if, in addition, Φ is a graded symplectomorphism from $(W_{\check{\phi}_1}, d\theta_{\check{\phi}_1})$ to $(W_{\check{\phi}_2}, d\theta_{\check{\phi}_2})$.

Note that the definition above has many similarities with the definition of the mapping torus from Definition 3.13. Also, if we define the mapping torus

$$\pi_{T_{\check{\phi}}}\colon T_{\check{\phi}} \to \mathbb{R}/\mathbb{Z} \quad \text{and} \quad \alpha_{T_{\check{\phi}}}$$

of our positive slope perturbation $\check{\phi}$ in exactly the same way as in Definition 3.13, then $W_{\check{\phi}} = \mathbb{R} \times T_{\check{\phi}}$, $\theta_{\check{\phi}} = (s-\kappa)\,dt + (\kappa/C)\alpha_{\check{\phi}}$ for some $C > 0$ and $\pi_{\check{\phi}} = \mathrm{id}_{\mathbb{R}} \times \pi_{T_{\check{\phi}}}$. The following calculation will be useful later on. If we have a Hamiltonian H equal to $\pi_{\check{\phi}}^* K$ for some $K\colon \mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \to \mathbb{R}$ then X_H is equal to the horizontal lift of $X_K^{ds \wedge dt}$ with respect to the symplectic connection associated to $d\theta_{\check{\phi}}$.

Lemma B.2 *Let*

$$(B_1 \subset C_1, \xi_1, \tau_1), \quad (B_2 \subset C_2, \xi_2, \tau_2)$$

be the (graded) contact pairs associated to the (graded) abstract contact open books $(M_1, \theta_{M_1}, \phi_1)$ *and* $(M_2, \theta_{M_2}, \phi_2)$*, respectively. Let* $\check{\phi}_1$ *be a small positive slope perturbation of* ϕ_1*. If the above contact pairs are (graded) contactomorphic then* $(M_2, \theta_{M_2}, \phi_2)$ *is (graded) isotopic to an abstract contact open book* $(M_3, \theta_{M_3}, \phi_{M_3})$ *such that the mapping cylinders*

$$(W_{\check{\phi}_1}, \pi_{\check{\phi}_1}, \theta_{\check{\phi}_1}), \quad (W_{\check{\phi}_3}, \pi_{\check{\phi}_3}, \theta_{\check{\phi}_3})$$

are (graded) isomorphic, where $\check{\phi}_3$ *is a small positive slope perturbation of* ϕ_3*.*

Proof Since the corresponding open books are contactomorphic, and the boundary ∂M_j is contactomorphic to the binding of $\mathrm{OBD}(M_j, \theta_{M_j}, \phi_j)$ for $j = 1, 2$, we get that ∂M_1 is contactomorphic to ∂M_2. Hence, there is a diffeomorphism $\Psi\colon \partial M_2 \to \partial M_1$ such that $\Psi^*\alpha_1 = f\alpha_2$, where $\alpha_j = \theta_{M_j}|_{\partial M_j}$ for $j = 1, 2$ and $f\colon \partial M_2 \to (0, \infty)$. After multiplying θ_2 by a positive constant, we can assume that $f > 1$. Choose $\delta > 0$ small enough that the subset of the cylindrical end $(1-\delta, 1] \times \partial M_2 \subset M_2$ is disjoint from the support of ϕ_2 and let $r_{M_2}\colon (1-\delta, 1] \times \partial M_2 \to (1-\delta, 1]$ be the associated cylindrical coordinate. Let $\rho\colon (1-\delta, 1] \to [0, 1]$ be a smooth function with nonnegative derivative that is equal to 0 near $1-\delta$ and 1 near 1. Let $F_t\colon M_2 \to \mathbb{R}$, $t \in [0, 1]$ be a smooth family of functions equal to $1 + t\rho(r_{M_2})(f-1)$ inside $(1-\delta] \times \partial M_2$ and 1 otherwise. Then $(M_2, F_t\theta_{M_2}, \phi_2)$ is an isotopy of abstract contact open books. Define $(M_3, \theta_{M_3}, \phi_3) \equiv (M_2, F_1\theta_{M_2}, \phi_2)$. Hence, there is a diffeomorphism $\check{\Psi}\colon \partial M_3 \to \partial M_1$ such that $(\check{\Psi})^*\alpha_{M_1} = \alpha_{M_3}$, where $\alpha_{M_3} \equiv \theta_{M_3}|_{\partial M_3}$.

Choose a small positive slope perturbation $\check{\phi}_3$ of ϕ_3 so that $\check{\phi}_3|_{(1-\check{\delta},1]\times\partial M_3}$ is equal to $(\mathrm{id}_{(1-\check{\delta},1]} \times \check{\Psi})^*\check{\phi}_1|_{(1-\check{\delta},1]\times\partial M_1}$ for some $\check{\delta} > 0$ smaller than δ. Let $(T_{\phi_j}, \pi_{T_{\phi_j}}, \alpha_{\phi_j})$ be the mapping torus of ϕ_j and let $(T_{\check{\phi}_j}, \pi_{T_{\check{\phi}_j}}, \alpha_{\check{\phi}_j})$ be the mapping torus of $\check{\phi}_j$ for $j = 1, 3$. Let $(B_3 \subset C_3, \xi_3, \tau_3)$ be the contact pair associated to $(M_3, \theta_{M_3}, \phi_3)$. By Lemma 5.34, we get that the contact pair associated to $(M_3, \theta_{M_3}, \phi_3)$ is contactomorphic to the contact pair associated to $(M_2, \theta_{M_3}, \phi_2)$ which in turn is contactomorphic to the contact pair associated to $(M_1, \theta_{M_1}, \phi_1)$. There is a contactomorphism $Q\colon T_{\phi_3} \to T_{\phi_1}$ such that $Q|_{B_3}\colon B_3 \to B_1$ is equal to $\check{\Psi}$ under the identification $B_i = M_i$ for $i = 1, 3$ and such that $\pi_{T_{\phi_3}} = \pi_{T_{\phi_1}} \circ Q$ near ∂T_{ϕ_3}. Hence, we can find a contactomorphism $\check{Q}\colon T_{\check{\phi}_3} \to T_{\check{\phi}_1}$ satisfying $(\check{Q})^*\alpha_{\check{\phi}_1} = \alpha_{\check{\phi}_3}$ near $\partial T_{\check{\phi}_3}$ and such that $\pi_{T_{\check{\phi}_3}} = \pi_{T_{\check{\phi}_1}} \circ \check{Q}$ near $\partial T_{\check{\phi}_3}$.

Since $W_{\check{\phi}_j}$ is naturally diffeomorphic to $\mathbb{R} \times T_{\check{\phi}_j}$ for $j = 1, 3$, we can define

$$W\colon W_{\check{\phi}_3} \to W_{\check{\phi}_1}, \quad W \equiv (\mathrm{id}_{\mathbb{R}}, \check{Q}).$$

Now $W^*\theta_{\check{\phi}_3} = \theta_{\check{\phi}_1}$ outside a subset $K \subset W_{\check{\phi}_3}$ whose intersection with each fiber of $\pi_{\check{\phi}_3}$ is compact. Since $\theta_{\check{\phi}_3}$ and $W^*\theta_{\check{\phi}_1}$ scale at most linearly in C^1 norm as we translate in the s coordinate direction, we can use a Moser argument applied to $\tau\theta_{\check{\phi}_3} + (1-\tau)W^*\theta_{\check{\phi}_1}$ for $\tau \in [0,1]$, giving us our isomorphism. □

Lemma B.3 *Suppose that $(M_1, \theta_{M_1}, \phi_1)$ and $(M_2, \theta_{M_2}, \phi_2)$ are isotopic abstract contact open books. Then $\mathrm{HF}^*(\phi_1, +) = \mathrm{HF}^*(\phi_2, +)$.*

Proof Since the above abstract contact open books are isotopic, we can assume that $M_1 = M_2$ and that there is a smooth family of Liouville forms $(\theta_s)_{s \in [0,1]}$ such that $\theta_0 = \theta_{M_1}$ and $\theta_1 = \theta_{M_2}$. Also, there is a smooth family of exact symplectomorphisms $\psi_s\colon M_1 \to M_1$ for $s \in [0,1]$ with respect to θ_s with support in a fixed compact set joining ϕ_1 and ϕ_2. Let r_s be the cylindrical coordinate for (M_1, θ_s) and choose $\delta > 0$ small enough that $\{r_s \geq 1 - \delta\}$ is disjoint from the support of ψ_s for all $s \in [0,1]$. By pulling back r_s and θ_s and ψ_s by a smooth family of diffeomorphisms starting at the identity and parametrized by $s \in [0,1]$, we can assume that $r_s = r_0$ inside the region $\{r_0 \geq 1 - \delta\} = (1-\delta, 1] \times \partial M_1$. By Gray's stability theorem, we can also assume that $\theta_s = r_0 f_s \alpha$ inside $\{r_0 \geq 1 - \delta\} = (1-\delta, 1] \times \partial M_1$ for some contact form α on ∂M_1 and some smooth family of functions $f_s\colon \partial M_1 \to (0, \infty)$ for $s \in [0,1]$.

Now choose a smooth family of functions $g_s\colon (1-\delta, 1] \times \partial M_1 \to (0, \infty)$ for $s \in [0,1]$ so that $\partial g_s / \partial r_0 > 0$, g_s is equal to f_s inside $\bigl(1-\delta, 1-\frac{1}{2}\delta\bigr] \times \partial M_1$, and $g_s = C f_0$ inside $\bigl(1 - \frac{1}{4}\delta, 1\bigr] \times \partial M_1$ for some large constant $C > 0$ and for all $t \in [0,1]$. Define $\check{\theta}_s$ to be equal to θ_s outside $(1-\delta, 1] \times \partial M_1$ and equal to $r_0 g_s \alpha$ inside this region. Then $(M_1, \check{\theta}_s)$ for $s \in [0,1]$ is a smooth family of Liouville domains such that $\check{\theta}_s = \theta_s$ inside $\bigl(1-\delta, 1-\frac{1}{2}\delta\bigr] \times \partial M_1$ and $\check{\theta}_s$ is independent of s near ∂M_1.

Let $(K_{s,t})_{(s,t) \in [0,1]^2}$ be a smooth family of almost complex structures compatible with $d\theta_s$ such that $K_{s,t}$ is cylindrical inside $(1-\delta] \times M_1$ with respect to θ_s for all $(s,t) \in [0,1]^2$. Choose a smooth family of almost complex structures $(J_{s,t})_{(s,t) \in [0,1]^2}$ compatible with $d\check{\theta}_s$ equal to $K_{s,t}$ outside $\bigl(1 - \frac{1}{2}\delta, 1\bigr] \times \partial M_1$ and equal to $K_{0,t}$ inside $\bigl(1 - \frac{1}{4}\delta, 1\bigr] \times \partial M_1$. Let $\check{\psi}_s$ be a smooth family of exact symplectomorphisms with respect to $\check{\theta}_s$ which are small positive slope perturbations of ψ_s such that $\check{\psi}_s$ is the time 1 flow of ηr_s inside $(1-\delta, 1] \times \partial M_1$ for some very small $\eta > 0$ (so that there are no fixed points of this symplectomorphism in this region). Let $\widehat{\psi}_s$ be a smooth family of positive slope perturbations of ψ_s with respect to θ_s which are equal to the time 1 flow of ηr_s inside $(1-\delta,1]\times\partial M_1$ with respect to the symplectic structure $d\theta_s$. We assume that η is small enough that $\widehat{\psi}_s$ has no fixed points inside $(1-\delta,1]\times\partial M_1$ for all $s\in[0,1]$.

Since

- $\check{\psi}_s$ and $\widehat{\psi}_s$ are the time 1 flows of a linear Hamiltonian inside $\bigl(1-\delta,1-\frac{1}{2}\delta\bigr)\times\partial M_1$,
- $J_{s,t}$ and $K_{s,t}$ are cylindrical inside this region,
- $(\check{\psi}_s,(J_{s,t})_{t\in[0,1]})$ and $(\widehat{\psi}_s,(K_{s,t})_{t\in[0,1]})$ are equal outside $(1-\delta,1]\times\partial M_1$, and
- $\check{\psi}_s$ and $\widehat{\psi}_s$ has no fixed points inside $(1-\delta,1]\times\partial M_1$,

a maximum principle [1, Lemma 7.2] tells us that

$$\mathrm{HF}^*(\check{\psi}_s,(J_{s,t})_{t\in[0,1]})=\mathrm{HF}^*(\widehat{\psi}_s,(K_t)_{t\in[0,1]}) \tag{B-2}$$

for all $s\in[0,1]$.

Since $\check{\theta}_s$ is independent of s near ∂M_1, we can assume, by a Moser argument, that $\check{\theta}_s=\theta_0+\beta_s$ for some smooth family of compactly supported closed 1–forms β_s for $s\in[0,1]$. Then, by [41, Theorem 2.34], we get that $\mathrm{HF}^*(\check{\psi}_s,(J_{s,t})_{t\in[0,1]})$ is independent of $s\in[0,1]$. Hence, $\mathrm{HF}^*(\widehat{\psi}_s,(K_s)_{t\in[0,1]})$ is independent of s by (B-2), which implies that $\mathrm{HF}^*(\phi_1,+)=\mathrm{HF}^*(\phi_2,+)$. □

Definition B.4 Let $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ be a mapping cylinder. An almost complex structure J on $W_{\check{\phi}}$ is *strictly compatible with* $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ if

(1) J is compatible with $d\theta_{\check{\phi}}$,
(2) $\pi_{\check{\phi}}\colon W_{\check{\phi}}\to\mathbb{R}\times\mathbb{R}/\mathbb{Z}$ is (J,j)–holomorphic (ie $D\pi_{\check{\phi}}\circ J=j\circ D\pi_{\check{\phi}}$), where j is the complex structure sending $\partial/\partial s$ to $\partial/\partial t$, where (s,t) parametrizes $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$,
(3) the restriction of J to the cylindrical end $C_{\check{\phi}}$ is a product $j\oplus J_M$, where J_M is a fixed cylindrical almost complex structure inside the cylindrical end $(1-\delta_{\check{\phi}},1]\times\partial M$, and
(4) J is invariant under translations in the s coordinate.

We will call J_M the *associated cylindrical almost complex structure on* M. An almost complex structure J on $W_{\check{\phi}}$ is *compatible with* $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ if there is an almost complex structure $\check{J}$ compatible with $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ and a compact subset K in the interior of $W_{\check{\phi}}$ such that $J|_{W_{\check{\phi}}-K}=\check{J}|_{W_{\check{\phi}}-K}$.

Definition B.5 Recall that a 1–periodic orbit of a time-dependent Hamiltonian

$$H_t\colon W_{\check{\phi}} \to \mathbb{R}$$

is a smooth map $\gamma\colon \mathbb{R}/\mathbb{Z} \to M$ satisfying $d\gamma/dt = X_{H_t}^{d\theta_{\check{\phi}}}$. Since there is a natural one-to-one correspondence between 1–periodic orbits and fixed points of the corresponding Hamiltonian symplectomorphism $\phi_1^{H_t}$, we will call γ the *1–periodic orbit associated to the fixed point* $\gamma(0)$. A Hamiltonian is *autonomous* if it does not depend on time. An *S^1–family of 1–periodic orbits* of H is a family $(\gamma_t)_{t\in\mathbb{R}/\mathbb{Z}}$ of 1–periodic orbits where $\gamma_t(\check{t}) = \gamma_0(\check{t}+t)$ for all $t,\check{t} \in \mathbb{R}/\mathbb{Z}$.

Let (H,J) and $(\check{H},\check{J})$ be pairs consisting of autonomous Hamiltonians H and $\check{H}$ and almost complex structures J and $\check{J}$. A smooth family of pair $(H_s,J_s)_{s\in\mathbb{R}}$ *joins* (H,J) *and* $(\check{H},\check{J})$ if $(H_s,J_s) = (H,J)$ for all sufficiently negative s and $(H_s,J_s) = (\check{H},\check{J})$ for all sufficiently positive s.

Let $h\colon \mathbb{R} \to \mathbb{R}$ be a smooth function which is bounded from below with positive derivative satisfying $h''(s) = 0$ whenever s is sufficiently positive and $h'(s) < 1$ for s sufficiently negative. The value of $h'(s)$ for large enough s is called the *slope of h*. A Hamiltonian is *strictly compatible with* $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ if it is equal to $\pi_{\check{\phi}}^* h(s)$ everywhere. A Hamiltonian is *compatible with* $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ if it equals $\pi_{\check{\phi}}^* h(s)$ outside a compact subset of the interior of $W_{\check{\phi}}$. The slope of such a Hamiltonian is defined to be the slope of h.

All the 1–periodic orbits of $h(s)$ on the symplectic manifold $(\mathbb{R}\times\mathbb{R}/\mathbb{Z}, ds\wedge dt)$ wrapping around $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$ once come in S^1 families in the region $h'(s) = 1$ and for the unique s satisfying $h'(s) = 1$ we have 1–periodic orbits

$$\gamma_{s,q}\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}\times\mathbb{R}/\mathbb{Z}, \quad \gamma_q(t) = (s, t+q),$$

for all $q \in [0,1)$. Also, the 1–periodic orbits of $\pi_{\check{\phi}}^* h(s)$ project to 1–periodic orbits of $h(s)$.

A pair (H,J) is *(strictly) compatible with* $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ if H is a Hamiltonian (strictly) compatible with $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ and J is an almost complex structure (strictly) compatible with $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$. A smooth family $(H_s,J_s)_{s\in\mathbb{R}}$ of pairs compatible with $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ has *nonincreasing slope* if the slope of H_s is greater than or equal to the slope of $H_{\check{s}}$ for all $s \le \check{s}$.

Definition B.6 Let $(H_\sigma,J_\sigma)_{\sigma\in\mathbb{R}}$ be a smooth family of pairs of Hamiltonians and almost complex structures compatible with a mapping cylinder $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$. An open subset $V \subset W_{\check{\phi}}$ satisfies the *maximum principal with respect to* $(H_\sigma, J_\sigma)_{\sigma\in\mathbb{R}}$ if for every compact codimension 0 submanifold $\Sigma \subset \mathbb{R}\times\mathbb{R}/\mathbb{Z}$ and every smooth map $u\colon \Sigma \to W_{\check{\phi}}$ satisfying

(1) $u(\partial\Sigma) \subset V$,

(2) $\partial_\sigma u(\sigma,\tau) + J_\sigma \partial_\tau u(\sigma,\tau) = J_\sigma X_{H_\sigma}$

also satisfies $\operatorname{Im}(u) \subset V$.

We say that $(H_\sigma, J_\sigma)_{\sigma\in\mathbb{R}}$ satisfies the *maximum principle* if there is a sequence of relatively compact open sets $(V_i)_{i\in\mathbb{N}}$ whose union is $W_{\check{\phi}}$ such that $(H_\sigma, J_\sigma)_{\sigma\in\mathbb{R}}$ satisfies the maximum principle with respect to V_i for all $i \in \mathbb{N}$.

Lemma B.7 *Let (M, θ_M, ϕ) be an abstract contact open book and let*

$$W \equiv (W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$$

be a mapping cylinder of some positive slope perturbation of ϕ. Let

$$\big(K_\sigma \equiv \pi^*_{\check{\phi}}(k_\sigma(s))\big)_{\sigma\in\mathbb{R}}$$

be a smooth family of Hamiltonians strictly compatible with W such that $dk_\sigma/d\sigma \le 0$, $dk'_\sigma/d\sigma \le 0$ and $dk''_\sigma/d\sigma \le 0$. Let Y be an almost complex structure strictly compatible with W.

Let $\delta, S > 0$ and let $r_{\check{\phi}}$ be the vertical cylindrical coordinate of W. Let $(H_\sigma, J_\sigma)_{\sigma\in\mathbb{R}}$ be a smooth family of pairs compatible with W which are equal to (K_σ, Y) near the boundary of the set

$$V_{\delta,S} \equiv \pi^{-1}_{\check{\phi}}((-S,S)\times(\mathbb{R}/\mathbb{Z})) \subset W_{\check{\phi}}$$

and also in the region $\{r_{\check{\phi}} \ge 1-\delta\}$. Then $V_{\delta,S}$ satisfies the maximum principal with respect to $(H_\sigma, J_\sigma)_{\sigma\in\mathbb{R}}$.

Proof Let $u\colon \Sigma \to W_{\check{\phi}}$ be as in Definition B.6 with V replaced by $V_{\delta,S}$. Let

$$\iota_{\check{\phi}}\colon C_{\check{\phi}} \equiv (\mathbb{R}\times(\mathbb{R}/\mathbb{Z})\times(1-\delta_{\check{\phi}},1]\times\partial M) \hookrightarrow W_{\check{\phi}}, \quad \iota_{\check{\phi}}(s,t,r_M,y) \equiv \big(s,t,(\phi^{\delta r_M}_{-t}(r_M,y))\big),$$

be the vertical cylindrical end of $W_{\check{\phi}}$. Then $r_{\check{\phi}}\colon C_{\check{\phi}} \to (1-\delta_{\check{\phi}},1]$ is the natural projection map. Let $P_M\colon C_{\check{\phi}} \to (1-\delta_{\check{\phi}},1]\times\partial M$ be the natural projection map. Consider the map

$$\hat{u}\colon u^{-1}(C_{\check{\phi}}) \to (1-\delta_{\check{\phi}},1]\times\partial M, \quad \hat{u}(\sigma,\tau)(x) \equiv P_M \circ u(x).$$

Let $J_{M,\sigma,t}$ be the natural almost complex structure on $(1-\delta_{\check{\phi}}) \times \partial M$ induced by $J_\sigma|_{\pi_{\check{\phi}}^{-1}(s,t)}$ for some s. Such an almost complex structure does not depend on s and is cylindrical by definition. Since $J_\sigma X_{H_\sigma}$ is a multiple of $\partial/\partial s$, we get that $\hat{u}$ satisfies

$$\frac{\partial \hat{u}}{\partial \sigma} + J_M \frac{\partial \hat{u}}{\partial \tau} - J_M X_{\delta r_M} = 0,$$

where r_M is the cylindrical coordinate on M and $\delta > 0$ is some constant. Therefore, by applying [1, Lemma 7.2] to $\hat{u}$ we see that such a map cannot intersect the region $(1-\delta, 1] \times \partial M)$ and hence the image of u cannot intersect the region $\{r_{\check{\phi}} \geq 1-\delta\}$.

Therefore, we only need to show that the image of u is contained inside the set $\pi_{\check{\phi}}^{-1}((-S,S)\times \mathbb{R}/\mathbb{Z})$. First of all we can make S very slightly smaller, so that u is transverse to $\pi_{\check{\phi}}^{-1}(\{S,-S\})$ and $u(\partial\Sigma) \subset V_{\delta,S}$. This implies that $\check{\Sigma} \subset \mathbb{R}\times\mathbb{R}/\mathbb{Z}$ is a smooth submanifold with boundary. Suppose for a contradiction that $\check{\Sigma}$ is nonempty. Choose $\delta_1 > 0$ small enough that

$$(H_\sigma, J_\sigma) = (K_\sigma, Y_\sigma)$$

inside $\pi_{\check{\phi}}^{-1}(([-S-\delta_1,-S]\cup[S,S+\delta_1])\times\mathbb{R}/\mathbb{Z})$ for all $\sigma\in\mathbb{R}$ and u intersects $\pi_{\check{\phi}}^{-1}(\{s\}\times\mathbb{R}/\mathbb{Z})$ transversely for all $s\in[-S-\delta_1,-S]\cup[S,S+\delta_1]$. Now let $\beta\colon \mathbb{R}\to\mathbb{R}$ be a smooth function satisfying $\beta' \geq 0$,

$$\beta|_{(-S-\delta_1/2,S+\delta_1/2)} = 0, \quad \beta'|_{(-S-3\delta_1/4,-S-\delta_1/2)\cup(S+\delta_1/2,S+3\delta_1/4)} > 0$$

and β is constant outside $\left(-S-\frac{3}{4}\delta_1, S+\frac{3}{4}\delta_1\right)$. Choose a smooth function $q_\sigma\colon \mathbb{R}\to\mathbb{R}$ such that $q_\sigma|_{(-S-\delta_1/2,S+\delta_1/2)} = 0$ and $q'_\sigma = \beta' k'_\sigma$. Then $dq'_\sigma/d\sigma \leq 0$. Hence,

$$\begin{aligned}
0 &< \int_{\check{\Sigma}} \beta'(s(u))\left|(\pi_{\check{\phi}})_*\left(\frac{\partial u}{\partial\sigma}\right)\right|^2 d\sigma\wedge d\tau \\
&= \int_{\check{\Sigma}} u^* d(\beta(s)dt)\left(\frac{\partial u}{\partial\sigma}, \frac{\partial u}{\partial\tau} - X_{H_\sigma}\right) d\sigma\wedge d\tau \\
&= \int_{\check{\Sigma}} u^* d(\beta(s)dt) - u^*(\beta'(s)dH_\sigma)\wedge d\tau \\
&= \int_{\check{\Sigma}} u^* d(\beta(s)dt) - d\left(u^*(q_\sigma(s))\right)\wedge d\tau + u^*\left(\frac{dq_{\sigma(s)}}{d\sigma}\right) d\sigma\wedge d\tau \\
&\leq \int_{\check{\Sigma}} u^* d(\beta(s)dt) - d\left(q_\sigma(s(u))d\tau\right) \\
&= \int_{\partial\check{\Sigma}} u^*(\beta(s)dt) - q_\sigma(s(u))\,d\tau = 0,
\end{aligned}$$

giving us a contradiction. □

Corollary B.8 *Any smooth family $(H_\sigma, J_\sigma)_{\sigma\in\mathbb{R}}$ compatible with $(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$ with nonincreasing slope satisfies the maximum principle.*

Definition B.9 For any mapping cylinder $W \equiv (W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$, define $\beta_{\check{\phi}} \subset H_1(W_{\check{\phi}})$ to be the set of homology classes represented by loops which project under $\pi_{\check{\phi}}$ to loops homotopic to

$$\mathbb{R}/\mathbb{Z} \to \mathbb{R}\times\mathbb{R}/\mathbb{Z}, \quad t \to (0,t).$$

For each $a, b \in [-\infty,\infty]$ where $a < b$, any nondegenerate Hamiltonian $(H_t)_{t\in[0,1]}$ and smooth family of almost complex structures $(J_t)_{t\in[0,1]}$ compatible with W, we can define $\mathrm{HF}^*_{[a,b],\beta_{\check{\phi}}}(\phi_1^{H_t})$ in the same way as Floer cohomology of $\phi_1^{H_t}$ except that we only consider fixed points of action in $[a,b]$ and whose associated 1–periodic orbits represent an element of $\beta_{\check{\phi}}$. We also define $\mathrm{HF}^*_{[-\infty,\infty],\beta_{\check{\phi}}}(\phi_1^{H_t}) \equiv \mathrm{HF}^*_{\beta_{\check{\phi}}}(\phi_1^{H_t})$.

A *nondegenerate autonomous Hamiltonian* is an autonomous Hamiltonian $H\colon W_{\check{\phi}} \to \mathbb{R}$ such that for every fixed point p of the time 1 flow ϕ_1^H, the eigenspace of $D\phi_1^H(p)$ associated with the eigenvalue 1 is 1–dimensional (this eigenspace is the tangent line to the 1–periodic orbit associated to p). Now suppose that the mapping cylinder $(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$ is graded with grading

$$\iota\colon \widetilde{\mathrm{Fr}}(TW_{\check{\phi}}) \times_{\widetilde{\mathrm{Sp}}(2n)} \mathrm{Sp}(2n) \cong \mathrm{Fr}(TW_{\check{\phi}}).$$

Since any Hamiltonian H_t is isotopic to the identity map $\mathrm{id}_{W_{\check{\phi}}}$, we get that $\phi_1^{H_t}$ is naturally graded since the identity map on $\widetilde{\mathrm{Fr}}(TW_{\check{\phi}})$ makes $\mathrm{id}_{W_{\check{\phi}}}$ into a graded symplectomorphism. We will call this the *standard grading* and from now on we will assume that every graded Hamiltonian symplectomorphism has the standard grading. A *standard perturbation* of a nondegenerate autonomous Hamiltonian H where ϕ_1^H is graded is a time-dependent Hamiltonian $(H_t)_{t\in[0,1]}$ which is C^∞ close to H and equal to H outside a compact set, where

- $(H_t)_{t\in[0,1]}$ is nondegenerate,
- every 1–periodic orbit γ of $(H_t)_{t\in[0,1]}$ is a 1–periodic orbit of H, and
- for every S^1 family of 1–periodic orbits γ of H there are exactly two 1–periodic orbits γ_- and γ_+ in this family which are also 1–periodic orbits of H_t. These orbits satisfy $\mathrm{CZ}(\phi_1^{H_t}, \gamma_\pm) = \mathrm{CZ}(\phi_1^H, \gamma) \pm \frac{1}{2}$.

Such a perturbation exists by [10, Proposition 2.2].

In order to prove our theorem we need another group, called S^1–*equivariant Hamiltonian Floer cohomology*. See [43; 7] for a definition. We will not define this here but we will just state some of the properties that we need. We write these groups as $\mathrm{HF}^*_{S^1,[a,b],\beta_{\check{\phi}}}(H,J)$ for any nondegenerate autonomous Hamiltonian H and almost complex structures J compatible with $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ and any $a,b\in[-\infty,\infty]$ satisfying $a<b$. We also define $\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H,J)\equiv\mathrm{HF}^*_{S^1,[-\infty,\infty],\beta_{\check{\phi}}}(H,J)$.

These groups satisfy the following properties:

(S1HF1) Let S_H be the set of S^1 families of 1–periodic orbits of H with action in $[a,b]$ representing a class in $\beta_{\check{\phi}}$. Let $(H_t)_{t\in[0,1]}$ be a C^∞ small standard perturbation of H. This means that for each $\gamma\in S_H$, there are two 1–periodic orbits γ_- and γ_+ of $(H_t)_{t\in[0,1]}$ which are also orbits in γ satisfying $\mathrm{CZ}(\phi_1^{H_t},\gamma_\pm)=\mathrm{CZ}(\phi_1^H,\gamma)\pm\frac{1}{2}$. Let S_{H_t} be the set of such orbits $\gamma_\pm$.

Then the chain complex $\mathrm{CF}^*_{S^1,[a,b],\beta_{\check{\phi}}}(H)$ defining $\mathrm{HF}^*_{S^1,[a,b],\beta_{\check{\phi}}}(H,J)$ is a free $\mathbb{Z}[u]$–module generated by S_{H_t} and graded by the Conley–Zehnder index taken with negative sign and where the degree of u is -2. Let

$$C^*_{[a,b],\beta_{\check{\phi}}}(H)\subset C^*_{S^1,[a,b],\beta_{\check{\phi}}}(H)$$

be the $\mathbb{Z}$–submodule generated by elements of S_{H_t}. Then the differential ∂ on $\mathrm{CF}^*_{S^1,[a,b],\beta_{\check{\phi}}}(H)$ is equal to $\partial_0+\partial_1$, where $\partial_0(u^iC^*_{[a,b],\beta_{\check{\phi}}}(H))\subset u^iC^*_{[a,b],\beta_{\check{\phi}}}(H)$ and

$$\partial_1(u^iC^*_{[a,b],\beta_{\check{\phi}}}(H))\subset\bigoplus_{j=0}^{i-1}u^jC^*_{[a,b],\beta_{\check{\phi}}}(H)$$

for all i. Here ∂_0 is equal to the differential defining $\mathrm{HF}^*_{[a,b],\beta_{\check{\phi}}}(H,J)$. Also, $\partial(u^i\gamma_-)=u^{i-1}\gamma_+$ plus 1–periodic orbits of higher action for all $i\geq 1$. The differential is $\mathbb{Z}$–linear but not necessarily $\mathbb{Z}[u]$–linear.

(S1HF2) If $(H_\sigma,J_\sigma)_{\sigma\in\mathbb{R}}$ is a smooth family of pairs compatible with $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$ with nonincreasing slope joining (H,J) and $(\check{H},\check{J})$ then there is a group homomorphism

$$\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(\check{H},\check{J})\to\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H,J).$$

If in addition $dH_\sigma/d\sigma\leq 0$, then we have a group homomorphism

$$\mathrm{HF}^*_{S^1,[a,b],\beta_{\check{\phi}}}(\check{H},\check{J})\to\mathrm{HF}^*_{S^1,[a,b],\beta_{\check{\phi}}}(H,J)$$

for all $a<b$. These are called *continuation maps*. They do not depend on the choice of path $(H_\sigma,J_\sigma)_{\sigma\in\mathbb{R}}$ and the composition of two continuation maps is a continuation

map. Also, if (H_σ, J_σ) does not depend on $\sigma \in \mathbb{R}$ then the associated continuation map is the identity map. If $a = -\infty$ and $b = \infty$ and if $H_\sigma = H + f(\sigma)$ for some function $f\colon \mathbb{R} \to \mathbb{R}$ then the corresponding continuation map is also an isomorphism.

(S1HF3) If (H, J) and $(\check{H}, \check{J})$ are compatible with $W \equiv (W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$ and

- satisfy the maximum principle with respect to some $V \subset W_{\check{\phi}}$,
- all the 1–periodic orbits of action of H and $\check{H}$ in $[a,b]$ representing elements of $\beta_{\check{\phi}}$ are contained in V, and
- $(H, J)|_V = (\check{H}, \check{J})|_V$,

then

$$\mathrm{HF}^*_{S^1,[a,b],\beta_{\check{\phi}}}(\check{H}, \check{J}) \cong \mathrm{HF}^*_{S^1,[a,b],\beta_{\check{\phi}}}(H, J).$$

This is due to the fact that their chain complexes are identical. Also, if we have two additional pairs (H', J') and $(\check{H}', \check{J}')$ satisfying the same properties and a smooth nondecreasing family of pairs $(H'_\sigma, J'_\sigma)_{\sigma\in\mathbb{R}}$ and $(\check{H}'_\sigma, \check{J}'_\sigma)_{\sigma\in\mathbb{R}}$ compatible with W joining (H, J) and (H', J') and joining $(\check{H}, \check{J})$ and $(\check{H}', \check{J})'$, respectively, satisfying the maximum principle with respect to V and which are equal inside V for all σ, then the induced continuation maps commute with the above isomorphisms.

Definition B.10 We define

$$\mathrm{SH}^*_{S^1,\beta_{\check{\phi}}}(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}}) \equiv \varinjlim_{(H,J)} \mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H, J),$$

where the direct limit is taken over all pairs (H, J) compatible with $(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$ using the partial ordering $\leq$ on H.

Let $\preceq$ be a partial order on a set S. A *cofinal family* is a subset $S' \subset S$ such that, for all $s \in S$, there exists an $s' \in S'$ such that $s \preceq s'$. In the definition above, it is sufficient to compute $\mathrm{SH}^*_{S^1,\beta_{\check{\phi}}}(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$ by taking the direct limit over some cofinal family of pairs (H, J) as above.

Lemma B.11 *If the slope of a pair (H, J) compatible with $(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$ is greater than 1 then the natural map*

$$\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H, J) \to \mathrm{SH}^*_{S^1,\beta_{\check{\phi}}}(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$$

is an isomorphism.

Proof Let $(\check{H}, \check{J})$ be a pair which is strictly compatible with our mapping cylinder and such that the slope of $\check{H}$ is equal to the slope of H. Then there is a constant $c > 0$ such that $\check{H} + c > H$ and $H + c > \check{H}$. Consider the continuation maps

$$\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H,J) \xrightarrow{\alpha} \mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(\check{H}+c,\check{J}) \to \mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H+2c,J) \to \mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(\check{H}+3c,\check{J}).$$

By (S1HF2), the composition of any two such maps is an isomorphism and hence the continuation map α is an isomorphism. Therefore, it is sufficient for us to assume that (H, J) is strictly compatible with our mapping cylinder. We can also assume that $H = \pi^*_{\check{\phi}} h(s)$ where $h''(s) \geq 0$.

Choose $S > 0$ large enough that $\pi^{-1}_{\check{\phi}}((-S,S)\times\mathbb{R}/\mathbb{Z})$ contains all the 1–periodic orbits of H representing a class in $\beta_{\check{\phi}}$. Let $h_\sigma\colon \mathbb{R}\to\mathbb{R}$ for $\sigma\in[0,\infty)$ be a smooth family of functions such that

- $h_\sigma(s) = h(s)$ for all $s \in (-S, S)$ and $h_0(s) = h(s)$ for all $s \in \mathbb{R}$,
- $h'_\sigma(s), h''_\sigma(s), dh_\sigma(s)/d\sigma \geq 0$,
- $h''_\sigma(s) = 0$ for all large enough s, and
- the slope of h_σ tends to infinity as σ tends to infinity.

By (S1HF3) combined with Lemma B.7, the natural continuation map

$$\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H,J) \to \mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(\pi^*_{\check{\phi}} h_\sigma(s), J)$$

is an isomorphism for all $\sigma \geq 0$. Also, by (S1HF2), the natural continuation map

$$\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H,J) \to \mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(\pi^*_{\check{\phi}} h_\sigma(s)+\sigma, J)$$

is an isomorphism for all $\sigma \geq 0$. Since $(\pi^*_{\check{\phi}} h_\sigma(s)+\sigma, J)$ is a cofinal family of pairs with respect to the ordering $\leq$, we get our result by (S1HF2). □

Lemma B.12 *Fix $q \in \mathbb{R}$. We have*

$$\mathrm{SH}^*_{S^1,\beta_{\check{\phi}}}(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}}) = \varinjlim_{(H,J)} \mathrm{HF}^*_{S^1,(-\infty,0],\beta_{\check{\phi}}}(H,J),$$

where the direct limit is taken over pairs (H, J) compatible with our mapping cylinder satisfying $H|_{\pi^{-1}_{\check{\phi}}((-\infty,q]\times\mathbb{R}/\mathbb{Z})} < 0$.

Proof Since the continuation map

$$\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H,J) \to \mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H+c,J)$$

is an isomorphism by (S1HF2) for every pair (H,J) compatible with our mapping cylinder, we have

$$\mathrm{SH}^*_{S^1,\beta_{\check{\phi}}}(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}}) = \varinjlim_{(H,J)} \mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H,J),$$

where the direct limit is taken over pairs (H,J) compatible with our mapping cylinder satisfying

$$H|_{\pi_{\check{\phi}}^{-1}((-\infty,q]\times\mathbb{R}/\mathbb{Z})} < 0.$$

Let A be a constant smaller than the action of all the 1–periodic orbits of $\check{\phi}$. We say that $h\colon \mathbb{R}\to\mathbb{R}$ for $i\in\mathbb{N}$ is a *compatible function with respect to q* if

- $h', h'' \geq 0$, h is bounded below,
- if $h'(x)=1$ then $h(x) < x+\kappa A$, and
- $h_i|_{(-\infty,q]} < 0$.

Then

$$\mathrm{SH}^*_{S^1,\beta_{\check{\phi}}}(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}}) = \varinjlim_{h} \mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(\pi_{\check{\phi}}^* h(s),J),$$

where the direct limit is taken over compatible functions with respect to q ordered by $\leq$ and where J is an almost complex structure compatible with our mapping cylinder. Since $\pi_{\check{\phi}}^* h(s)$ has no 1–periodic orbits representing $\beta_{\check{\phi}}$ of positive action where h is any compatible function with respect to q we have by, (S1HF3),

$$\varinjlim_{h} \mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(\pi_{\check{\phi}}^* h(s),J) = \varinjlim_{h} \mathrm{HF}^*_{S^1,(-\infty,0],\beta_{\check{\phi}}}(\pi_{\check{\phi}}^* h(s),J),$$

proving our result. □

Lemma B.13 *Let $q\in\mathbb{R}$. Let $(\check{H},\check{J})$ be compatible with $(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}})$; then*

$$\mathrm{SH}^*_{S^1,\beta_{\check{\phi}}}(W_{\check{\phi}},\pi_{\check{\phi}},\theta_{\check{\phi}}) = \varinjlim_{(H,J)} \mathrm{HF}^*_{S^1,(-\infty,0],\beta_{\check{\phi}}}(H,J),$$

where the direct limit is taken over pairs (H,J) compatible with our mapping cylinder satisfying $H|_{\pi_{\check{\phi}}^{-1}((-\infty,q]\times\mathbb{R}/\mathbb{Z})} < 0$ and

$$(H,J)|_{\pi_{\check{\phi}}^{-1}([q+1,\infty)\times\mathbb{R}/\mathbb{Z})} = (\check{H}+C_H,J)|_{\pi_{\check{\phi}}^{-1}([q+1,\infty)\times\mathbb{R}/\mathbb{Z})}$$

for some constant $C_H\in\mathbb{R}$.

Proof Let $(H_i, J_i)_{i\in\mathbb{N}}$ be a cofinal family of pairs with respect to the directed system mentioned in the statement of this lemma. Such a countable family exists since

$$\sup H|_{\pi_{\check{\phi}}^{-1}((-\infty,q]\times(\mathbb{R}/\mathbb{Z}))} < 0$$

for any (H, J) in the directed system above. We have that (H_i, J_i) is compatible with our mapping cylinder and $H_i|_{(\pi_{\check{\phi}}^{-1}((-\infty,q]\times\mathbb{R}/\mathbb{Z}))} < 0$ and

$$(H_i, J_i)|_{\pi_{\check{\phi}}^{-1}([q+1,\infty)\times\mathbb{R}/\mathbb{Z})} = (\check{H} + C_{H_i}, J_i)|_{\pi_{\check{\phi}}^{-1}([q+1,\infty)\times\mathbb{R}/\mathbb{Z}))}$$

for some constant $C_{H_i} \in \mathbb{R}$. We can assume that $H_i < H_{i+1}$ and hence $C_{H_i} < C_{H_{i+1}}$ for all $i \in \mathbb{N}$. After passing to a subsequence, we can assume that $C_{H_i} > i$ for all $i \in \mathbb{N}$.

Let (s, t) be standard coordinates for the base $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$. Let $K\colon W_{\check{\phi}} \to \mathbb{R}$ be a Hamiltonian equal to $\pi_{\check{\phi}}^* k(s)$, where $k(s) = 0$ for $s \le q + 1$, $k'(s) > 0$ for $s > q + 1$, and $k'(s)$ is constant for $s > q + 2$. Since $C_{H_i} > i$ for all i, there is a $\delta > 0$ small enough that the set of 1–periodic orbits of H_i of nonpositive action are equal to the set of 1–periodic orbits of $H_i + \delta i K$ of nonpositive action. Hence, by (S1HF3),

$$\mathrm{HF}^*_{S^1,(-\infty,0],\beta_{\check{\phi}}}(H_i, J_i) = \mathrm{HF}^*_{S^1,(-\infty,0],\beta_{\check{\phi}}}(H_i + \delta i K, J_i).$$

Combining this with Lemma B.12 gives us our result. □

Lemma B.14 *Suppose the mapping cylinders $(W_{\check{\phi}_1}, \pi_{\check{\phi}_1}, \theta_{\check{\phi}_1})$ and $(W_{\check{\phi}_2}, \pi_{\check{\phi}_2}, \theta_{\check{\phi}_2})$ are isomorphic. Then*

$$\mathrm{SH}^*_{S^1,\beta_{\check{\phi}_1}}(W_{\check{\phi}_1}, \pi_{\check{\phi}_1}, \theta_{\check{\phi}_1}) = \mathrm{SH}^*_{S^1,\beta_{\check{\phi}_2}}(W_{\check{\phi}_2}, \pi_{\check{\phi}_2}, \theta_{\check{\phi}_2}).$$

Proof Since these mapping cylinders are isomorphic, we can assume that $W_{\check{\phi}_1} = W_{\check{\phi}_2}$ and $\pi_{\check{\phi}_2} = \pi_{\check{\phi}_2}$ inside $\{r_{\check{\phi}_1} \ge 1 - \delta\}$ for some $\delta > 0$, and $\theta_{\check{\phi}_1} = \theta_{\check{\phi}_2} + k$, where $k\colon W_{\check{\phi}_1} \to \mathbb{R}$ has support disjoint from $\{r_{\check{\phi}_1} \ge 1 - \delta\}$.

Let (H_1, J_1) and (H_2, J_2) be pairs strictly compatible with

$$W_1 \equiv (W_{\check{\phi}_1}, \pi_{\check{\phi}_1}, \theta_{\check{\phi}_1}) \quad \text{and} \quad W_2 \equiv (W_{\check{\phi}_2}, \pi_{\check{\phi}_2}, \theta_{\check{\phi}_2}),$$

respectively, of slope less than some small $\check{\delta} > 0$. If $\check{\delta} > 0$ is small enough then we can construct a pair (H_3, J_3), compatible with W_1, which is equal to (H_2, J_2) in the region $\pi_{\check{\phi}_2}^{-1}((-1, 3)\times\mathbb{R}/\mathbb{Z})$ and equal to (H_1, J_1) inside $\pi_{\check{\phi}_2}^{-1}\big(((-\infty, -2)\cup(4, \infty))\times\mathbb{R}/\mathbb{Z}\big)$, so that H_3 has no 1–periodic orbits.

Let $(\check{H}_i, \check{J}_i)_{i \in \mathbb{N}}$ be a family of pairs strictly compatible with W_2 such that

- $(\check{H}_i, \check{J}_i)$ is equal to $(H_2 + C_{\check{H}_i}, J_2)$ inside $\pi^{-1}_{\check{\phi}_2}([1,\infty) \times \mathbb{R}/\mathbb{Z})$ for some constants $C_{\check{H}_i} \in \mathbb{R}$,
- the restriction $\check{H}_i|_{\pi^{-1}_{\check{\phi}_2}((-\infty,0)\times\mathbb{R}/\mathbb{Z})}$ is negative and uniformly tends to 0 in the C^1 norm as i tends to infinity and $\check{H}_i(x) \to \infty$ as $i \to \infty$ for all x in $\pi^{-1}_{\check{\phi}_2}((0,\infty) \times \mathbb{R}/\mathbb{Z})$.

Let $(\hat{H}_i, \hat{J}_i)_{i \in \mathbb{N}}$ be a family of pairs compatible with W_1 such that

- $(\hat{H}_i, \hat{J}_i)$ is equal to $(H_3 + C_{\check{H}_i}, \check{J}_2)$ inside $\pi^{-1}_{\check{\phi}_1}([1,\infty) \times \mathbb{R}/\mathbb{Z})$,
- $(\hat{H}_i, \hat{J}_i)$ equals $(\check{H}_i, \check{J}_i)$ inside $\pi^{-1}_{\check{\phi}_2}((-1,3) \times \mathbb{R}/\mathbb{Z})$,
- the restriction $\hat{H}_i|_{\pi^{-1}_{\check{\phi}_1}((-\infty,0)\times\mathbb{R}/\mathbb{Z})}$ is negative and uniformly tends to 0 in the C^1 norm as i tends to infinity and $\hat{H}_i(x) \to \infty$ as $i \to \infty$ for all x in $\pi^{-1}_{\check{\phi}_1}((0,\infty) \times \mathbb{R}/\mathbb{Z})$.

Then, by Lemma B.13,

$$\mathrm{SH}^*_{S^1,\beta_{\check{\phi}_1}}(W_{\check{\phi}_2}, \pi_{\check{\phi}_2}, \theta_{\check{\phi}_2}) \equiv \varinjlim_{i \in \mathbb{N}} \mathrm{HF}^*_{S^1,(-\infty,0]}(\check{H}_i, \check{J}_i). \tag{B-3}$$

and

$$\mathrm{SH}^*_{S^1,\beta_{\check{\phi}_1}}(W_{\check{\phi}_1}, \pi_{\check{\phi}_1}, \theta_{\check{\phi}_1}) \equiv \varinjlim_{i \in \mathbb{N}} \mathrm{HF}^*_{S^1,(-\infty,0]}(\hat{H}_i, \hat{J}_i). \tag{B-4}$$

Also, by property (S1HF3),

$$\mathrm{HF}^*_{S^1,(-\infty,0],\beta_{\check{\phi}_1}}(\hat{H}_i, \hat{J}_i) \cong \mathrm{HF}^*_{S^1,(-\infty,0],\beta_{\check{\phi}_1}}(\check{H}_i, \check{J}_i)$$

for all i and the continuation maps between these groups commute with these isomorphisms. Hence,

$$\mathrm{SH}^*_{S^1,\beta_{\check{\phi}_1}}(W_{\check{\phi}_1}, \pi_{\check{\phi}_1}, \theta_{\check{\phi}_1}) \cong \mathrm{SH}^*_{S^1,\beta_{\check{\phi}_1}}(W_{\check{\phi}_2}, \pi_{\check{\phi}_2}, \theta_{\check{\phi}_2})$$

by equations (B-4) and (B-3). □

Lemma B.15 *Let A_- and A_+ be free abelian groups. Define $B_- \equiv A_- \otimes \mathbb{Z}[u]$ and $B_+ \equiv A_+ \otimes \mathbb{Z}[u]$. Let*

$$\partial \equiv \partial_0 + \partial_1\colon B_- \oplus B_+ \to B_- \oplus B_+$$

be a $\mathbb{Z}$–linear differential, where $\partial_0(A_-) \subset A_-$ and $\partial_1(A_-) = 0$. Now suppose that we have a filtration $B_- \oplus B_+ = F_0 \supset F_1 \supset F_2 \supset \cdots$ for the chain complex $(B_- \oplus B_+, \partial)$ such that if $V^i_\pm \equiv (B_\pm \cap F_i)/(B_\pm \cap F_{i+1})$ then $\partial_1(uV^i_-) \subset V^i_+$ and

$$\partial_1|_{uV^i_-} \colon uV^i_- \to V^i_+ \tag{B-5}$$

is an isomorphism for all $i \geq 0$. Then

$$H(B_- \oplus B_+, \partial) = H(A_-, \partial_0).$$

If these groups are graded then the above isomorphism respects this grading.

Proof Define $\check{B} \equiv \partial(uB_-)$ and define $\partial_{\check{B}} \colon uB_- \to \check{B}$, $\partial_{\check{B}}(x) = \partial(x)$. Since the map (B-5) is an isomorphism for all $i \geq 0$, we get that $\partial_1|_{uB_-} \colon uB_- \to (B_- \oplus B_+)/(B_- \oplus 0)$ is an isomorphism. Hence, $B_- \oplus B_+ \cong A_- \oplus uB_- \oplus \check{B}$ and the differential with respect to this splitting is the matrix

$$\begin{pmatrix} \partial_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \partial_{\check{B}} & 0 \end{pmatrix}.$$

Computing the homology of this chain complex using the above matrix gives us our result since $\partial_{B'}$ is an isomorphism. □

Lemma B.16 *Let $(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$ be a mapping cylinder. Then*

$$\mathrm{SH}^*_{\beta_{\check{\phi}}}(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}}) = \mathrm{HF}^*(\phi, +).$$

Proof Let $(H = \pi^*_{\check{\phi}} h(s), J)$ be a pair strictly compatible with $(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$, where H has slope 1.5, $h'' \geq 0$ and where $h'|_{(-\infty,0)} < 1$. Let $h_t \colon \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ for $t \in [0,1]$ be a standard perturbation of $h(s)$ viewed as a Hamiltonian on the symplectic manifold $(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, ds \wedge dt)$. Then h_t has exactly two 1–periodic orbits γ_- and γ_+. Also, $H_t \equiv \pi^*_{\check{\phi}} h_t$ is a standard perturbation of H and the 1–periodic orbits of H_t project to γ_- or γ_+. A compactness argument [6] tells us that (H_t, J) satisfies the maximum principle so long as H_t is sufficiently C^∞ close to H and hence we can define $\mathrm{HF}^*_{\beta_{\check{\phi}}}(\phi_1^{H_t})$ in the usual way. We can also define $\mathrm{HF}^*_{S^1, \beta_{\check{\phi}}}(H, J)$ using the standard perturbation H_t by the same compactness argument.

By [27, Theorem 1.3], we have that $\mathrm{HF}^*_{\beta_{\check{\phi}}}(\phi_1^{H_t}, J)$ is isomorphic to

$$\mathrm{HF}^*_{\beta_{\check{\phi}}}(\phi_1^{H_t}, +) \oplus \mathrm{HF}^{*+1}_{\beta_{\check{\phi}}}(\phi_1^{H_t}, +).$$

In fact, in the proof of the above theorem it was shown that if A_- (resp. A_+) is the free abelian group generated by 1–periodic orbits of H_t which project to γ_- (resp. γ_+), then the differential

$$\partial_{H_t,J}\colon A_- \oplus A_+ \to A_- \oplus A_+$$

satisfies $\partial_{H_t,J}(A_\pm) \subset A_\pm$ and the homology of

$$\partial_{H_t,J}|_{A_-}\colon A_- \to A_-$$

is equal to $\mathrm{HF}^*(\phi_1^{H_t}, J)$.

If $\check{\phi}$ is a sufficiently generic positive slope perturbation of ϕ, then we can find a sequence $(\alpha_i)_{i\in\mathbb{N}_{\geq 0}}$ such that there are exactly two 1–periodic orbits $\check{\gamma}_-$ and $\check{\gamma}_+$ of H_t of action in the interval $[\alpha_i, \alpha_{i+1})$ and all 1–periodic orbits are contained in one such interval. Let $B_\pm \equiv A_\pm \otimes \mathbb{Z}[u]$, where u has degree -2. Let $B_- \oplus B_+ = F_0 \supset F_1 \supset \cdots$ be a filtration, where F_i is the $\mathbb{Z}[u]$–submodule generated by orbits of action $\geq \alpha_i$. Define $V^i_\pm \equiv (B_\pm \cap F_i)/(B_\pm \cap F_{i+1})$ for all $i \in \mathbb{N}_{\geq 0}$. By (S1HF1), the differential $\partial\colon B_- \oplus B_+ \to B_- \oplus B_+$ computing $\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H,J)$ is equal to $\partial_0 + \partial_1$, where $\partial_0(A_-) \subset A_-$, $\partial_1(A_-) = 0$, $\partial_0|_{A_-} = \partial_{H_t,J}$, $\partial_1(uV^i_-) \subset V^i_+$ and $\partial_1|_{uV^i_-}\colon uV^i_- \to V^i_+$ is an isomorphism for all $i \in \mathbb{N}_{\geq 0}$.

Therefore, by Lemma B.15 we have that $\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H,J) = H(A_-, \partial_0) = \mathrm{HF}^*(\phi, +)$. Also, $\mathrm{HF}^*_{S^1,\beta_{\check{\phi}}}(H,J) = \mathrm{SH}^*_{S^1,\beta_{\check{\phi}}}(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}})$ by Lemma B.11 and hence

$$\mathrm{SH}^*_{S^1,\beta_{\check{\phi}}}(W_{\check{\phi}}, \pi_{\check{\phi}}, \theta_{\check{\phi}}) = \mathrm{HF}^*(\phi, +). \qquad \square$$

Proof of (HF2) This follows from Lemmas B.2, B.3, B.14 and B.16. □

Now the only issue is if we have two polynomials with embedded contactomorphic links. Then we need to show that the associated contact pairs are isomorphic. In other words, we need to show that the normal bundles coincide up to homotopy. This is contained in the proof of the following lemma:

Lemma B.17 *Let $f, g\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be polynomials with isolated singularities at 0 with embedded contactomorphic links, with $n \geq 1$. Then $\mathrm{HF}^*(\phi^m, +) = \mathrm{HF}^*(\psi^m, +)$, where ϕ (resp. ψ) is the monodromy map of the Milnor open book associated to f (resp. g) as in Example 3.11.*

Proof Let $(L_f \subset S_\epsilon, \xi_{S_\epsilon}, \Phi_f)$ and $(L_g \subset S_\epsilon, \xi_{S_\epsilon}, \Phi_g)$ be the contact pairs associated to f and g, respectively, as in Example 3.8. Let $\Psi\colon S_\epsilon \to S_\epsilon$ be the contactomorphism sending L_f to L_g. We need to show that Ψ is in fact a contactomorphism of graded contact pairs by (HF2). Since $H_1(S_\epsilon;\mathbb{Q}) = 0$, we get that Ψ is a graded contactomorphism by (A-2) in Definition A.7. Therefore, we just need to show that the composition

$$\mathcal{N}_{S_\epsilon} L_f \xrightarrow{d\Psi|_{L_f}} \mathcal{N}_{S_\epsilon} L_g \xrightarrow{\Phi_g} L_g \times \mathbb{C} \xrightarrow{\Psi^{-1}\times \mathrm{id}_{\mathbb{C}}} L_f \times \mathbb{C}$$

is homotopic to Φ_f.

This is true since the trivialization Φ_f (and similarly Φ_g) is uniquely determined by the following topological property: Let $\Psi_f\colon \check{\mathcal{N}}_{S_\epsilon} L_f \to S_\epsilon$ be a regularization of L_f as in Definition 5.2. Then the trivialization Φ_f gives us a section s of $\check{\mathcal{N}}_{S_\epsilon} L_f$ whose image under the trivialization Φ_f is a constant section. Then s is the unique section up to homotopy with the property that the image of $H_1(L_f;\mathbb{Q}) \xrightarrow{\Psi_f \circ s} H_1(S_\epsilon - L_f;\mathbb{Q})$ is zero. This trivialization could be thought of as a generalization of the Seifert framing of links. □

Appendix C A Morse–Bott spectral sequence

In this section, we will show that property (HF3) holds. Here is a statement of this property:

Let (M, θ_M, ϕ) be a graded abstract contact open book, where $\dim(M) = 2n$. Suppose that the set of fixed points of a small positive slope perturbation $\check{\phi}$ of ϕ is a disjoint union of codimension 0 families of fixed points $B_1, \dots, B_l$ and let $\iota\colon \{1, \dots, l\} \to \mathbb{N}$ be a function where

- *$\iota(i) = \iota(j)$ if and only if the action of B_i equals the action of B_j, and*
- *$\iota(i) < \iota(j)$ if the action of B_i is less than the action of B_j.*

Then there is a cohomological spectral sequence converging to $\mathrm{HF}^*(\phi, +)$ whose E_1 page is equal to

$$E_1^{p,q} = \bigoplus_{\{i \in \{1,\dots,l\} : \iota(i) = p\}} H_{n-(p+q)-\mathrm{CZ}(\phi, B_j)}(B_p;\mathbb{Z}). \tag{C-1}$$

The spectral sequence above is an example of a *Morse–Bott spectral sequence*. Before we prove this statement we need some preliminary definitions and lemmas.

Definition C.1 Let (M, θ_M, ϕ) be a graded abstract contact open book. Let $\check{\phi}\colon M \to M$ be a small positive slope perturbation of ϕ and $(J_t)_{t\in[0,1]}$ a C^∞ generic family of $d\theta_M$–compatible almost complex structures. Let $a, b \in \mathbb{R}$ be real numbers with the property that no fixed point of $\check{\phi}$ has action equal to a or b.

We define $\mathrm{HF}^*_{[a,b]}(\check{\phi}, (J_t)_{t\in[0,1]})$ in the following way: Let $\check{\phi}'$ be a C^∞ small generic perturbation of $\check{\phi}$ inside a compact set such that all of the fixed points of $\check{\phi}'$ are nondegenerate. Then $\mathrm{HF}^*_{[a,b]}(\check{\phi}, (J_t)_{t\in[0,1]})$ is defined in the same way as $\mathrm{HF}^*(\check{\phi}', (J_t)_{t\in[0,1]})$ except that we only consider orbits inside the action window $[a, b]$. This group does not depend on the choice of perturbation $\check{\phi}'$ so long as no fixed point of $\check{\phi}$ has action equal to a or b.

We can define this group in the following equivalent way: Let $\mathrm{CF}^*(\check{\phi}')$ be the chain complex for $\check{\phi}'$. Then the subspace $\mathrm{CF}^*_{[a,\infty]}(\check{\phi}')$ consisting of fixed points of action $\geq a$ is a subcomplex. We define

$$\mathrm{HF}^*_{[a,b]}(\check{\phi}', (J_t)_{t\in[0,1]})$$

to be the homology of the quotient complex $\mathrm{CF}^*_{[a,\infty]}(\check{\phi}')/\mathrm{CF}^*_{[b,\infty]}(\check{\phi}')$.

Suppose that B is the set of fixed points of $\check{\phi}$ of action c and suppose that there is some $a < c < b$ such that there are no fixed points of action in $[a, b] - c$. We define

$$\mathrm{HF}^*(\check{\phi}, B) \equiv \mathrm{HF}^*_{[a,b]}(\check{\phi}, (J_t)_{t\in[0,1]}).$$

This does not depend on the choice of a, b or $(J_t)_{t\in[0,1]}$.

Lemma C.2 *Let (M, θ_M, ϕ) be a graded abstract contact open book. Let $\check{\phi}\colon M \to M$ be the composition of ϕ with a C^∞ small Hamiltonian such that $\check{\phi}$ has small positive slope. Let $B \subset M$ be an isolated family of fixed points of $\check{\phi}$.*

Let $(J_t)_{t\in[0,1]}$ be a smooth family of almost complex structures cylindrical near ∂M. Then there is an neighborhood $N_B \subset M$ of B such that for any sufficiently small C^∞ perturbation $\check{\phi}'$, any Floer trajectory of $(\check{\phi}', (J_t)_{t\in[0,1]})$ connecting nondegenerate fixed points $p, \check{p} \in N_B$ of $\check{\phi}'$ is contained inside N_B.

Proof We choose a relatively compact open neighborhood N_B of B such that any fixed point of $\check{\phi}$ inside $\overline{N}_B$ is actually contained inside B. Let $\check{N}_B \subset M$ be an open neighborhood of B whose closure is contained in N_B.

Let $(\phi_k)_{k\in\mathbb{N}}$ be a sequence of symplectomorphisms of M which C^∞ converges to $\check{\phi}$. Suppose (for a contradiction) that ϕ_k has a fixed point $p_k \in N_B - \check{N}_B$ for all k. Then, after passing to a subsequence, we have that p_i converges to some $p \in \overline{N}_B - \check{N}_B$.

Since p is a fixed point of ϕ, we get that $p \in B$, which is impossible. Therefore, ϕ_k has no fixed points inside $N_B - \check{N}_B$ for all sufficiently large k.

Now suppose that p_k and $\check{p}_k$ are fixed points of ϕ_k and suppose that we have a sequence of Floer trajectories

$$u_k \colon \mathbb{R} \times [0,1] \to M$$

of $(\check{\phi}', (J_t)_{t\in[0,1]})$ joining p_k and $\check{p}_k$. Define $W \equiv \mathbb{R} \times [0,1] \times M$ with a symplectic form $\omega_W \equiv ds \wedge dt + d\theta_M$, where s and t are the standard coordinates on $\mathbb{R} \times [0,1]$. Let i be the standard complex structure on $\mathbb{R} \times [0,1] \subset \mathbb{C}$, where (s,t) is identified with $s + it$. Define $J^W|_{(s,t,x)} \equiv i|_{(s,t)} \oplus J_t|_{(s,t)}$. Define

$$u_k^W \colon \mathbb{R} \times [0,1] \to W, \quad u_k^W(s,t) \equiv (s,t,u_k(s,t)),$$

for all $k \in \mathbb{N}$. This is a sequence of J^W–holomorphic maps.

Now suppose (for a contradiction) that the image of u_k is not contained inside N_B for all k. Then, after passing to a subsequence, there is a sequence of points $(s_k, t_k) \in \mathbb{R} \times [0,1]$ such that $u_k(s_k,t_k) \in N_B - \check{N}_B$ and $u_k(s_k,t_k)$ converges to some point $q \in \overline{N}_B - \check{N}_B$. After reparametrizing the domain by translations in the s direction, we can assume that $s_k = 0$ for all k. Also, after passing to a subsequence we can assume that $t_k \to \check{t} \in [0,1]$ for some $\check{t}$. Define $w_k \equiv u_k^W|_{[-1,1]\times[0,1]}$. Then, by the main result in [16], we get that w_k C^0 converges to a continuous map $v \colon [-1,1] \times [0,1] \to W$ which is smooth and J^W–holomorphic on a dense open subset of its domain.

Let $\pi_M \colon W \to M$ be the natural projection map. Since p_k and $\check{p}_k$ converge to points in B, their difference in action converges to zero, which implies that

$$\int_{[-1,1]\times\mathbb{R}/\mathbb{Z}} v^* \, d\theta_M = 0.$$

Hence, $\pi_M \circ v$ is constant. Since $\check{\phi}\big(\pi_M(v(0,1))\big) = \pi_M\big(v((0,0))\big)$ and $\pi_M \circ v$ is constant, we get that the image of $\pi_M \circ v$ is a fixed point of $\check{\phi}$ inside $N_B - \check{N}_B$. But this is impossible since $v(0,t) \in N_B - \check{N}_B$. □

As a result of the above lemma, we have the following definition:

Definition C.3 Let $B \subset M$ be an isolated family of fixed points of some positive slope perturbation $\check{\phi}$ of ϕ and let N_B be a neighborhood of B as in Lemma C.2. Let $\check{\phi}'$ be a C^∞ small perturbation such that all the fixed points of $\check{\phi}'$ inside N_B are nondegenerate. Since all Floer trajectories of $(\check{\phi}', (J_t)_{t\in[0,1]})$ are contained inside N_B, we can define the Floer cohomology group $\mathrm{HF}^*(\check{\phi}, B)$ in the usual way, where we only consider fixed points inside N_B. Such a group is called the *local Floer cohomology of B*. Again it does not depend on the choice of perturbation $\check{\phi}'$ or $(J_t)_{t\in[0,1]}$ although we will not need this fact here.

Note that if B is the only set of fixed points of $\check{\phi}$ of action in the interval $[a,b]$, then the above definition coincides with the definition of $\mathrm{HF}^*(\check{\phi}, B)$ from Definition C.1. More generally, if B is a union of isolated families of fixed points $B_1, \dots, B_l$ all of the same action then $\mathrm{HF}^*(\check{\phi}, B) = \bigoplus_{i=1}^l \mathrm{HF}^*(\check{\phi}, B_i)$.

Lemma C.4 *Let (M, θ_M, ϕ) be a graded abstract contact open book. Let $\check{\phi}\colon M \to M$ be a small positive slope perturbation of ϕ. Suppose that the set of all the fixed points of $\check{\phi}$ of action in $[a,b]$ is equal to $B = \bigsqcup_{i=1}^l B_i$, where $B_1, \dots, B_l$ are codimension 0 families of fixed points, all of the same action. Then*

$$\mathrm{HF}^*(\check{\phi}, B) \equiv \bigoplus_{i=1}^{l} H_{n-*-\mathrm{CZ}(\phi,B_i)}(B_i;\mathbb{Z}). \tag{C-2}$$

Proof Let $N_{B_i} \subset M$ be a small neighborhood of B_i with the property that $\check{\phi}$ is the time 1 flow of a Hamiltonian $H_{B_i}\colon N_{B_i} \to (-\infty, 0]$ satisfying $B_i = H_{B_i}^{-1}(0)$ for each i. Let $\phi_1^{H_{B_i}}\colon N_{B_i} \to N_{B_i}$ be the time 1 flow of H_B for each i. After possibly shrinking each neighborhood N_{B_i}, we can assume that $N_{B_1}, \dots, N_{B_l}$ are all disjoint. Let $(J_t)_{t\in[0,1]}$ be a generic smooth family of almost complex structures cylindrical near ∂M. By Lemma C.2, any sufficiently small C^∞ perturbation $\check{\phi}'$ of $\check{\phi}$ has the property that any Floer trajectory connecting fixed points inside $\bigcup_{i=1}^l N_{B_i}$ is actually contained inside N_{B_j} for some j. Therefore,

$$\mathrm{HF}^*(B) = \bigoplus_{i=1}^{l} \mathrm{HF}^*(\phi_1^{H_{B_i}}, B_i).$$

Hence, by [35, Theorem 7.1] combined with (CZ4), we have that (C-2) holds. □

Proof of (HF3) Let $\check{\phi}'$ be a C^∞ small perturbation of $\check{\phi}$ and let $(J_t)_{t\in[0,1]}$ be a C^∞ generic smooth family of almost complex structures cylindrical near ∂M. Let α_i be the action of B_i for each $i \in \{1, \dots, l\}$. For each $p \in \mathbb{N}$, choose $\beta_p \in \mathbb{R}$ so that $\alpha_i \neq \beta_p$ for all $i \in \{1, \dots, l\}$ and $\alpha_i > \beta_p$ if and only if $\iota(i) \geq p$. Let F_p be the subgroup of the chain complex $\mathrm{CF}^*(\check{\phi}')$ generated by fixed points of action greater than β_p. Then

$(F_p)_{i\in p}$ is a filtration on this chain complex. By Lemma C.4,

$$H^*(F_p/F_{p-1}) = \mathrm{HF}^*_{[\beta_{p-1},\beta_p]}(\check{\phi}, B_p) = \bigoplus_{\{i\in\{1,\dots,l\}:\iota(i)=p\}} H_{n-*-\mathrm{CZ}(\phi,B_j)}(B_p;\mathbb{Z})$$

for all $p = 1,\dots,l$ and $H^*(F_p/F_{p-1}) = 0$ if $p \in \mathbb{N} - \{1,\dots,l\}$. Therefore, the spectral sequence associated to the filtration $(F_p)_{p\in\mathbb{N}}$ is (C-1). □

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