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Equivariant concentration in topological groups

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We prove that, if G is a second-countable topological group with a compatible right-invariant metric d and $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of compactly supported Borel probability measures on G converging to invariance with respect to the mass transportation distance over d and such that $(\operatorname{spt}\mu_n, d\restriction_{\operatorname{spt}\mu_n}, \mu_n\restriction_{\operatorname{spt}\mu_n})_{n\in\mathbb{N}}$ concentrates to a fully supported, compact mm–space (X, d_X, μ_X), then X is homeomorphic to a G–invariant subspace of the Samuel compactification of G. In particular, this confirms a conjecture by Pestov and generalizes a well-known result by Gromov and Milman on the extreme amenability of topological groups. Furthermore, we exhibit a connection between the average orbit diameter of a metrizable flow of an arbitrary amenable topological group and the limit of Gromov's observable diameters along any net of Borel probability measures UEB–converging to invariance over the group.

54H11, 54H20, 22A10, 53C23

1 Introduction

Over the past few decades, the study of the measure concentration phenomenon has become a central theme in topological dynamics, in particular in the context of infinite-dimensional transformation groups. In fact, measure concentration ranges among the two most prominent pathways to extreme amenability, next to Ramsey-type phenomena; see Kechris, Pestov and Todorcevic [14] and Pestov [19; 20]. The origin of this development is marked by the groundbreaking work of Gromov and Milman [13], who showed that every *Lévy group*, ie topological group containing a Lévy family of compact subgroups with dense union, is extremely amenable, and who, moreover, exhibited a number of striking examples of such groups, eg the unitary group of the infinite-dimensional separable Hilbert space equipped with the strong operator topology. Their ideas were followed by numerous other examples; see eg Carderi and Thom [5], Giordano and Pestov [9] and Pestov [22].

In his seminal work on metric measure geometry [12, Chapter $3\frac{1}{2}$], Gromov offered a far-reaching extension of the measure concentration phenomenon: he introduced the *observable distance*, a metric on the set of isomorphism classes of *mm–spaces*,

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ie separable complete metric spaces equipped with a Borel probability measure. The topology generated by this metric, the *concentration topology*, captures the (classical) measure concentration phenomenon in a very natural manner: a sequence of mm–spaces constitutes a Lévy family if and only if it *concentrates* (converges in Gromov's observable distance) to a singleton space. But, of course, the concentration topology allows for nontrivial limit objects, and recent years' growing interest in Polish groups with metrizable universal minimal flow (see for instance Ben Yaacov, Melleray and Tsankov [3], Kechris, Pestov and Todorcevic [14] and Melleray, Nguyen Van Thé and Tsankov [16]) suggests studying manifestations of concentration to nontrivial spaces for topological groups. We attempt to advance this idea, which originated with Giordano and Pestov [9; 21], with our main result.

Theorem 1.1 *Let G be a second-countable topological group equipped with a right-invariant compatible metric d. Suppose that there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of Borel probability measures on G with compact supports $K_n := \operatorname{spt}\mu_n$ $(n \in \mathbb{N})$ such that*

(A) $(\mu_n)_{n\in\mathbb{N}}$ *converges to invariance in the mass transportation distance over d,*

(B) $(K_n, d\restriction_{K_n}, \mu_n\restriction_{K_n})_{n\in\mathbb{N}}$ *concentrates to a fully supported, compact mm–space (X, d_X, μ_X).*

Then there exists a topological embedding $\psi\colon X \to \mathrm{S}(G)$ such that the push-forward measure $\psi_*(\mu_X)$ is G–invariant. In particular, $\psi(X)$ is a G–invariant subspace of $\mathrm{S}(G)$.

Due to recent work of the author and Thom [27, Theorem 3.2], every amenable second-countable topological group in fact admits a sequence of finitely supported probability measures converging to invariance with respect to the mass transportation distance over any right-invariant compatible metric. Furthermore, any Borel probability measure on a second-countable topological space assigns value 1 to its support, and thus the restriction of the measure to the Borel σ–algebra of its support will indeed be a probability measure. In particular, this applies to the measures considered in condition (B) of Theorem 1.1.

Since any minimal invariant closed subspace of the Samuel compactification $\mathrm{S}(G)$ of a topological group G is a — and, up to isomorphism, the — universal minimal flow of G (see Auslander [2, Chapter 8], and also Ellis [6], Uspenskij [30] and de Vries [33]), the theorem above may be used to compute universal minimal flows of topological groups, or at least prove their metrizability. As universal minimal flows of noncompact locally compact groups are always nonmetrizable [14, Theorem A2.2], no such group can possibly satisfy the hypothesis of Theorem 1.1.

In particular, Theorem 1.1 confirms a 2006 conjecture by Pestov [21, Conjecture 7.4.26]: if G is a metrizable topological group, equipped with a compatible right-invariant metric d, and $(K_n)_{n\in\mathbb{N}}$ is an increasing sequence of compact subgroups such that

- the union $\bigcup_{n\in\mathbb{N}} K_n$ is everywhere dense in G, and
- $(K_n, d\restriction_{K_n}, \mu_n)_{n\in\mathbb{N}}$ concentrates to a fully supported, compact mm–space (X, d_X, μ_X), where μ_n denotes the normalized Haar measure on K_n,

then the topological space X supports the structure of a G–flow, with respect to which it admits a morphism to every G–flow. Indeed, it is easily seen that, by density of the increasing union of compact subgroups, the corresponding Haar measures converge to invariance in the mass transportation distance over d, whence Theorem 1.1 asserts that X is homeomorphic to a G–invariant subspace of $\mathrm{S}(G)$, which in turn gives rise to a G–flow on X with the desired property; cf [21, Corollary 3.1.12].

For a discussion of examples of the above kind of nontrivial concentration phenomenon, we refer to [9, Section 7; 21, Chapter 7.4]. In this connection, there is another intriguing question by Pestov [21, Problem 7.4.27]: given a left-invariant metric d on the full symmetric group $\mathrm{Sym}(\mathbb{N})$ compatible with the topology of pointwise convergence, do the subgroups $(\mathrm{Sym}(n))_{n\in\mathbb{N}}$, equipped with their normalized counting measures and the restrictions of d, concentrate to the closed subspace $\mathrm{LO}(\mathbb{N}) \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ of linear orders on $\mathbb{N}$, endowed with the unique $\mathrm{Sym}(\mathbb{N})$–invariant Borel probability measure (see Glasner and Weiss [11]) and a suitable compatible metric? This question has been answered in the negative recently in Schneider [26]: in fact, the considered sequence of finite mm–spaces does not even admit a subsequence that is Cauchy with respect to Gromov's observable distance.

In addition to Theorem 1.1, we will unveil another link between concentration phenomena and topological dynamics: roughly speaking, the average orbit diameter of an arbitrary flow of an amenable topological group, equipped with a continuous pseudometric, is bounded from above by the limit inferior of Gromov's observable diameters [12] (see Definition 5.1) computed for any net of Borel probability measures on the acting group UEB–converging to invariance, with respect to the induced pseudometric. More precisely, if G is a topological group and X is a *G–flow*, ie a nonempty compact Hausdorff space together with a continuous action of G on it, then for any continuous pseudometric d on X and any point $x \in X$ there is a right-uniformly continuous pseudometric $d_{G,x}$ on G defined by

$$d_{G,x}(g,h) := d(gx,hx) \quad \text{for } g,h \in G,$$

which is bounded from above by the bounded continuous right-invariant pseudometric

$$d_{G,X} := \sup_{y \in X} d_{G,y} \colon G \times G \to \mathbb{R}.$$

Theorem 1.2 *Let G be a topological group and let $(\mu_i)_{i \in I}$ be a net of Borel probability measures on G UEB–converging to invariance over G. If d is a continuous pseudometric on a G–flow X and ν is a G–invariant regular Borel probability measure on X, then*

$$\int \sup_{g \in E} d(x,gx)\, d\nu(x) \leq \sup_{\alpha > 0} \liminf_{i \to I} \sup_{x \in X} \operatorname{ObsDiam}(G, d_{G,x}, \mu_i; -\alpha)$$

for every finite subset $E \subseteq G$.

Let us add some remarks about Theorem 1.2. If G is a topological group acting continuously on a compact Hausdorff space X and d is a continuous pseudometric on X, then

$$\sup_{x \in X} \operatorname{ObsDiam}(G, d_{G,x}, \mu; -\alpha) \leq \operatorname{ObsDiam}(G, d_{G,X}, \mu; -\alpha)$$

for every Borel probability measure μ on G and any $\alpha > 0$ (see Remark 5.2), which means that Theorem 1.2 immediately provides a corresponding estimate in terms of $d_{G,X}$. The same is true for Corollary 1.3 below. Of course, if G is separable, then Theorem 1.2 asserts that

$$\int \sup_{g \in G} d(x,gx)\, d\nu(x) \leq \sup_{\alpha > 0} \liminf_{i \to I} \sup_{x \in X} \operatorname{ObsDiam}(G, d_{G,x}, \mu_i; -\alpha)$$

for any net $(\mu_i)_{i \in I}$ of Borel probability measures on G UEB–converging to invariance over G. Moreover, Theorem 1.2 readily implies [24, Theorem 3.9], which is our Corollary 5.7, an extension of the result for Polish groups by Pestov [23, Theorem 5.7], thus entailing earlier work of Glasner, Tsirelson and Weiss [10, Theorem 1.1], who showed that every spatial action of a Lévy group must be trivial. We refer to the end of Section 5 for a brief discussion on this. Furthermore, let us highlight another consequence of Theorem 1.2.

Corollary 1.3 *Let G be a topological group and let $(\mu_i)_{i\in I}$ be a net of Borel probability measures on G UEB–converging to invariance over G. If d is a continuous pseudometric on a G–flow X, then there exists an $x_0 \in X$ such that*

$$\sup_{g\in G} d(x_0, gx_0) \leq \sup_{\alpha>0} \liminf_{i\to I} \sup_{x\in X} \operatorname{ObsDiam}(G, d_{G,x}, \mu_i; -\alpha).$$

The first estimates for orbit diameters, concerning Hölder actions on compact metric spaces, in terms of the isoperimetric behavior of the acting group and covering properties of the phase space, belong to Milman [17]. For generalizations of Milman's results as well as corresponding estimates for actions of Lévy groups on a certain class of noncompact metric spaces, we refer to Funano's work [7].

Let us briefly outline the structure of the present article. In Section 2 we recollect some elementary facts and concepts concerning metrics and measures, leading up to the definition of Gromov's observable distance (Definition 2.1). In Section 3 we provide the background on UEB–convergence to invariance in topological groups necessary for the proof of Theorem 1.1, which is given in Section 4. Finally, Section 5 is devoted to proving Theorem 1.2 and Corollary 1.3, as well as discussing some consequences.

2 Metrics, measures, and concentration

We start off by clarifying some notation. Given a set X, we denote by $\ell^\infty(X)$ the unital Banach algebra of all bounded real-valued functions on X equipped with the supremum norm

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in X\} \quad \text{for } f \in \ell^\infty(X).$$

Let X be a topological space. If the topology of X is generated by a metric d, then we call d a *compatible* metric on X. We will denote by $\mathrm{C}(X)$ the set of all continuous real-valued functions on X and we let $\mathrm{CB}(X) := \mathrm{C}(X) \cap \ell^\infty(X)$. Moreover, let us denote by $\mathcal{B}(X)$ the Borel σ–algebra of X and by $\mathrm{P}(X)$ the set of all Borel probability measures on X. The *weak topology* on $\mathrm{P}(X)$ is defined to be the initial topology on $\mathrm{P}(X)$ generated by the maps of the form $\mathrm{P}(X) \to \mathbb{R}$, $\mu \mapsto \int f\, d\mu$, where $f \in \mathrm{CB}(X)$. If X is metrizable (or just perfectly normal), then the weak topology turns $\mathrm{P}(X)$ into a Tychonoff space. The *support* of a measure $\mu \in \mathrm{P}(X)$ is defined as

$$\operatorname{spt}\mu := \{x \in X \mid \text{for all } U \subseteq X \text{ open, } x \in U \Longrightarrow \mu(U) > 0\},$$

which is easily seen to form a closed subset of X. Given $\mu \in \mathrm{P}(X)$ and a Borel subset $B \subseteq X$ with $\mu(B)=1$, we let $\mu\restriction_B := \mu|_{\mathcal{B}(B)} \in \mathrm{P}(B)$. The *push-forward* of a measure $\mu \in \mathrm{P}(X)$ along a Borel map $f\colon X \to Y$ into another topological space Y is defined to be

$$f_*(\mu)\colon \mathcal{B}(Y) \to [0,1], \quad B \mapsto \mu(f^{-1}(B)).$$

Furthermore, let us note that each $\mu \in \mathrm{P}(X)$ gives rise to a pseudometric me_μ on the set of all Borel measurable real-valued functions on X, defined by

$$\mathrm{me}_\mu(f,g) := \inf\bigl\{\varepsilon > 0 \bigm| \mu\bigl(\{x \in X \mid |f(x)-g(x)| > \varepsilon\}\bigr) \le \varepsilon\bigr\}$$

for any two Borel functions $f, g\colon X \to \mathbb{R}$.

Let (X,d) be a pseudometric space. Given a subset $A \subseteq X$, abbreviate $d\restriction_A := d|_{A\times A}$ and define $\operatorname{diam}(A,d) := \sup\{d(x,y) \mid x,y \in A\}$. For $x \in A \subseteq X$ and $\varepsilon > 0$, we let

$$B_d(x,\varepsilon) := \{y \in X \mid d(x,y) < \varepsilon\}, \quad B_d(A,\varepsilon) := \{y \in X \mid d(a,y) < \varepsilon \text{ for some } a \in A\}.$$

Then the *Hausdorff distance* between any two subsets $A, B \subseteq X$ is given by

$$d_{\mathrm{H}}(A,B) := \inf\{\varepsilon > 0 \mid B \subseteq B_d(A,\varepsilon),\ A \subseteq B_d(B,\varepsilon)\}.$$

For $\ell, r \ge 0$, we denote by $\mathrm{Lip}_\ell(X,d)$ the set of all ℓ–Lipschitz real-valued functions on (X,d), and we define

$$\mathrm{Lip}_\ell^\infty(X,d) := \mathrm{Lip}_\ell(X,d) \cap \ell^\infty(X), \quad \mathrm{Lip}_\ell^r(X,d) := \{f \in \mathrm{Lip}_\ell(X,d) \mid \|f\|_\infty \le r\}.$$

Moreover, we let

$$\mathrm{Lip}(X,d) := \bigcup_{\ell \ge 0} \mathrm{Lip}_\ell(X,d) \quad \text{and} \quad \mathrm{Lip}^\infty(X,d) := \mathrm{Lip}(X,d) \cap \ell^\infty(X).$$

The *mass transportation distance*[1] d_{MT} over d is the pseudometric on $\mathrm{P}(X)$ defined by

$$d_{\mathrm{MT}}(\mu,\nu) := \sup_{f \in \mathrm{Lip}_1^1(X,d)} \left| \int f\, d\mu - \int f\, d\nu \right| \quad \text{for } \mu,\nu \in \mathrm{P}(X).$$

Furthermore, the *Prokhorov distance* d_{P} over d is the pseudometric on $\mathrm{P}(X)$ given by

$$\begin{aligned} d_{\mathrm{P}}(\mu,\nu) &:= \inf\{\varepsilon > 0 \mid \mu(B) \le \nu(B_d(B,\varepsilon)) + \varepsilon \text{ for all } B \in \mathcal{B}(X)\} \\ &= \inf\{\varepsilon > 0 \mid \nu(B) \le \mu(B_d(B,\varepsilon)) + \varepsilon \text{ for all } B \in \mathcal{B}(X)\} \end{aligned}$$

[1] Different names appearing in the literature include *Monge–Kontorovich distance*, *bounded Lipschitz distance*, *Wasserstein distance*, and *Fortet–Mourier distance*; see [25; 8; 32].

for $\mu, \nu \in \mathrm{P}(X)$. In the case that (X,d) is a separable metric space, both d_{P} and d_{MT} are metrics compatible with the weak topology on $\mathrm{P}(X)$. For a more comprehensive account on these and other probability metrics, the reader is referred to [25; 8; 32].

Finally in this section, we will recall the basics concerning Gromov's concentration topology [12, Chapter $3\,{}^1\!/_2$.H], following the presentation of Shioya [28, Chapter 5]. This type of convergence refers to mm–spaces. An *mm–space* is a triple (X,d,μ) where (X,d) is a separable complete metric space and μ is a Borel probability measure on X. Moreover, an mm–space (X,d,μ) is called *compact* if (X,d) is compact, and *fully supported* if $\operatorname{spt}\mu = X$. Henceforth, we will denote by λ the Lebesgue measure on $[0,1)$. A *parametrization* of an mm–space (X,d,μ) is a Borel measurable map $\varphi\colon [0,1) \to X$ such that $\varphi_*(\lambda) = \mu$. It is well known that any mm–space admits a parametrization; see eg [28, Lemma 4.2].

Definition 2.1 The *observable distance* between two mm–spaces X and Y is defined to be

$$d_{\mathrm{conc}}(X,Y) := \inf\bigl\{(\mathrm{me}_\lambda)_{\mathrm{H}}(\mathrm{Lip}_1(X)\circ\varphi, \mathrm{Lip}_1(Y)\circ\psi) \mid \varphi \text{ a parametrization of } X,\ \psi \text{ a parametrization of } Y\bigr\}.$$

A sequence of mm–spaces $(X_n)_{n\in\mathbb{N}}$ is said to *concentrate to* an mm–space X if

$$\lim_{n\to\infty} d_{\mathrm{conc}}(X_n, X) = 0.$$

It is known that the observable distance induces a metric on the set of isomorphism classes of mm–spaces; see [28, Theorem 5.16]. In particular, two mm–spaces X and Y are *isomorphic*, ie there exists an mm–space isomorphism between X and Y, if and only if $d_{\mathrm{conc}}(X,Y) = 0$. By an *isomorphism* between mm–spaces (X,d_X,μ_X) and (Y,d_Y,μ_Y) we mean an isometry

$$f\colon (\operatorname{spt}\mu_X, d_X\restriction_{\operatorname{spt}\mu_X}) \to (\operatorname{spt}\mu_Y, d_Y\restriction_{\operatorname{spt}\mu_Y})$$

such that $f_*(\mu_X\restriction_{\operatorname{spt}\mu_X}) = \mu_Y\restriction_{\operatorname{spt}\mu_Y}$. For our purposes, ie the proof of our Theorem 1.1, the following characterization of concentration will be useful:

Theorem 2.2 [28, Corollary 5.35] *A sequence of mm–spaces $(X_n,d_n,\mu_n)_{n\in\mathbb{N}}$ concentrates to an mm–space (X,d,μ) if and only if there is a sequence of Borel maps $p_n\colon X_n \to X$, $n\in\mathbb{N}$, such that*

1. $(p_n)_*(\mu_n) \to \mu$ *in the weak topology as* $n\to\infty$,
2. $(\mathrm{me}_{\mu_n})_{\mathrm{H}}\bigl(\mathrm{Lip}_1(X,d)\circ p_n, \mathrm{Lip}_1(X_n,d_n)\bigr) \to 0$ *as* $n\to\infty$.

3 Topological groups and convergence to invariance

In this section we briefly recollect some results from [27] about UEB–convergence to invariance over topological groups. Throughout the present note, by a *topological group* we will always mean a *Hausdorff* topological group.

Let X be a uniform space. Consider the commutative unital real Banach algebra $\mathrm{UCB}(X)$ of all bounded uniformly continuous real-valued functions on X endowed with the supremum norm. The set $\mathrm{M}(X)$ of all *means* on $\mathrm{UCB}(X)$, ie (necessarily continuous) positive unital linear maps from $\mathrm{UCB}(X)$ to $\mathbb{R}$, equipped with the weak-$*$ topology, ie the initial topology generated by the maps of the form $\mathrm{M}(X) \to \mathbb{R}$, $\mu \mapsto \mu(f)$, where $f \in \mathrm{UCB}(X)$, constitutes a compact Hausdorff space. The set $\mathrm{S}(X)$ of all (necessarily positive and linear) unital ring homomorphisms from $\mathrm{UCB}(X)$ into $\mathbb{R}$ forms a closed subspace of $\mathrm{M}(X)$, which is called the *Samuel compactification* of X. The map $\eta_X\colon X \to \mathrm{S}(X)$ given by

$$\eta_X(x)(f) := f(x) \quad \text{for } x \in X,\ f \in \mathrm{UCB}(X)$$

is uniformly continuous and has dense range in $\mathrm{S}(X)$, and the mapping

$$\mathrm{C}(\mathrm{S}(X)) \to \mathrm{UCB}(X), \quad f \mapsto f \circ \eta_X,$$

is an isometric isomorphism of unital Banach algebras. Furthermore, a subset $H \subseteq \mathrm{UCB}(X)$ is called *UEB* (short for *uniformly equicontinuous bounded*) if H is norm-bounded and *uniformly equicontinuous*, ie for every $\varepsilon > 0$ there exists an entourage U of X such that

$$|f(x) - f(y)| \leq \varepsilon \quad \text{for all } f \in H,\ (x, y) \in U.$$

The collection $\mathrm{UEB}(X)$ of all UEB subsets of $\mathrm{UCB}(X)$ constitutes a convex vector bornology on the vector space $\mathrm{UCB}(X)$. It is easily seen that a subset $H \subseteq \mathrm{UCB}(X)$ belongs to $\mathrm{UEB}(X)$ if and only if H is norm-bounded and there is a uniformly continuous pseudometric d on X such that $H \subseteq \mathrm{Lip}_1(X, d)$. The *UEB topology* on the continuous dual $\mathrm{UCB}(X)^*$ is defined as the topology of uniform convergence on UEB subsets of $\mathrm{UCB}(X)$. This is a locally convex linear topology on the vector space $\mathrm{UCB}(X)^*$ containing the weak-$*$ topology, ie the initial topology generated by the maps $\mathrm{UCB}(X)^* \to \mathbb{R}$, $\mu \mapsto \mu(f)$, where $f \in \mathrm{UCB}(X)$. For more details on the UEB topology, the reader is referred to [18].

Now let G be a topological group. Denote by $\mathcal{U}(G)$ the neighborhood filter of the neutral element in G and endow G with its *right uniformity* defined by the basic entourages

$$\{(x,y) \in G \times G \mid yx^{-1} \in U\} \quad \text{for } U \in \mathcal{U}(G).$$

Referring to the right uniformity, we denote by $\mathrm{RUCB}(G)$ the set of all bounded uniformly continuous real-valued functions on G and by $\mathrm{RUEB}(G)$ the set of all UEB subsets of $\mathrm{RUCB}(G)$. It is easily seen that a subset $H \subseteq \mathrm{RUCB}(G)$ belongs to $\mathrm{RUEB}(G)$ if and only if H is norm-bounded and there is a continuous right-invariant pseudometric d on G with $H \subseteq \mathrm{Lip}_1(G,d)$. Furthermore, for $g \in G$, we define $\lambda_g \colon G \to G$ by $x \mapsto gx$ and $\rho_g \colon G \to G$ by $x \mapsto xg$. We note that G acts continuously on $\mathrm{M}(G)$ by

$$(g\mu)(f) := \mu(f \circ \lambda_g) \quad \text{for } g \in G,\ \mu \in \mathrm{M}(G),\ f \in \mathrm{RUCB}(G),$$

and that $\mathrm{S}(G)$ constitutes a G–invariant subspace of $\mathrm{M}(G)$. Let us recall that G is *amenable* (resp. *extremely amenable*) if $\mathrm{M}(G)$ (resp. $\mathrm{S}(G)$) admits a G–fixed point. It is well known that G is amenable (resp. extremely amenable) if and only if every G–flow admits a G–invariant regular Borel probability measure (resp. a G–fixed point). For a more comprehensive account on (extreme) amenability of general topological groups, we refer to [21].

We will need a characterization of amenability in terms of almost invariant finitely supported probability measures from [27].

Definition 3.1 Let G be a topological group. A net $(\mu_i)_{i \in I}$ of Borel probability measures on G is said to *UEB–converge to invariance* (*over* G) if for all $g \in G$ and all $H \in \mathrm{RUEB}(G)$,

$$\sup_{f \in H} \left| \int f \circ \lambda_g \, d\mu_i - \int f \, d\mu_i \right| \to 0 \quad \text{as } i \to I.$$

Theorem 3.2 [27, Theorem 3.2] *A topological group is amenable if and only if it admits a net of (finitely supported regular) Borel probability measures UEB–converging to invariance.*

We note some elementary properties of UEB–convergence to invariance.

Lemma 3.3 *Let G be a topological group. Let $(\mu_i)_{i\in I}$ be a net of Borel probability measures UEB–converging to invariance over G. The following hold:*

(1) *For any $(g_i)_{i\in I} \in G^I$, the net $((\rho_{g_i})_*(\mu_i))_{i\in I}$ UEB–converges to invariance over G.*

(2) *If $\varphi\colon G \to H$ is a continuous homomorphism with dense range in a topological group H, then $(\varphi_*(\mu_i))_{i\in I}$ UEB–converges to invariance over H.*

(3) *For each $i \in I$, let C_i be a Borel subset of G such that $\mu_i(C_i) = 1$. Then $\bigcup_{i\in I} C_iC_i^{-1}$ is dense in G.*

Proof (1) Let $(g_i)_{i\in I} \in G^I$. Consider any $F \in \mathrm{RUEB}(G)$. Then it is straightforward to check that $\{f \circ \rho_{g_j} \mid f \in F,\ j \in I\} \in \mathrm{RUEB}(G)$. Hence, for every $g \in G$,

$$\sup_{(f,j)\in F\times I}\left|\int f\circ\rho_{g_j}\,d\mu_i - \int f\circ\rho_{g_j}\circ\lambda_g\,d\mu_i\right| \to 0 \quad \text{as } i \to I,$$

and, in particular,

$$\begin{aligned}\sup_{f\in F}\left|\int f\,d(\rho_{g_i})_*(\mu_i) - \int f\circ\lambda_g\,d(\rho_{g_i})_*(\mu_i)\right| &= \sup_{f\in F}\left|\int f\circ\rho_{g_i}\,d\mu_i - \int f\circ\lambda_g\circ\rho_{g_i}\,d\mu_i\right| \\ &= \sup_{f\in F}\left|\int f\circ\rho_{g_i}\,d\mu_i - \int f\circ\rho_{g_i}\circ\lambda_g\,d\mu_i\right| \to 0 \quad \text{as } i \to I.\end{aligned}$$

(2) Let $h \in H$ and $F \in \mathrm{RUEB}(H)$. We wish to show that

$$\sup_{f\in F}\left|\int f\,d\varphi_*(\mu_i) - \int f\circ\lambda_h\,d\varphi_*(\mu_i)\right| \to 0 \quad \text{as } i \to I.$$

Let $\varepsilon > 0$. Since F belongs to $\mathrm{RUEB}(H)$, there exists $U \in \mathcal{U}(H)$ such that

$$\|f - (f\circ\lambda_u)\|_\infty \le \tfrac{\varepsilon}{2}$$

for all $f \in F$ and $u \in U$. Due to $\varphi(G)$ being dense in H, there exists $g \in G$ with $\varphi(g) \in Uh$. As $\varphi\colon G \to H$ is uniformly continuous with regard to the respective right uniformities, it follows that $F \circ \varphi \in \mathrm{RUEB}(G)$. Hence, we find $i_0 \in I$ such that

$$\sup_{f\in F}\left|\int f\circ\varphi\,d\mu_i - \int f\circ\varphi\circ\lambda_g\,d\mu_i\right| \le \tfrac{\varepsilon}{2} \quad \text{for all } i \in I,\ i \ge i_0.$$

For every $i \in I$ with $i \geq i_0$, we conclude that

$$\begin{aligned}
&\left|\int f \, d\varphi_*(\mu_i) - \int f \circ \lambda_h \, d\varphi_*(\mu_i)\right| \\
&\quad = \left|\int f \circ \varphi \, d\mu_i - \int f \circ \lambda_h \, d\varphi_*(\mu_i)\right| \\
&\quad \leq \left|\int f \circ \varphi \, d\mu_i - \int f \circ \varphi \circ \lambda_g \, d\mu_i\right| + \left|\int f \circ \lambda_{\varphi(g)} \, d\varphi_*(\mu_i) - \int f \circ \lambda_h \, d\varphi_*(\mu_i)\right| \\
&\quad \leq \tfrac{\varepsilon}{2} + \|(f \circ \lambda_{\varphi(g)}) - (f \circ \lambda_h)\|_\infty \\
&\quad = \tfrac{\varepsilon}{2} + \|f - (f \circ \lambda_{\varphi(g)h^{-1}})\|_\infty \\
&\quad \leq \varepsilon
\end{aligned}$$

for all $f \in F$; ie $\sup_{f \in H} \left|\int f \, d\varphi_*(\mu_i) - \int f \circ \lambda_h \, d\varphi_*(\mu_i)\right| \leq \varepsilon$, as desired.

(3) Let $C := \bigcup_{i \in I} C_i C_i^{-1}$. Consider any $g \in G$ and $U \in \mathcal{U}(G)$. We are going to show that $gU \cap C \neq \varnothing$. By Urysohn's lemma for uniform spaces, there exists a right-uniformly continuous function $f\colon G \to [0,1]$ such that $f(e) = 1$ and $f(x) = 0$ whenever $x \in G \setminus U$. For every subset $S \subseteq G$, define $f_S\colon G \to [0,1]$ by

$$f_S(x) := \sup_{s \in S} f(xs^{-1}) \quad \text{for } x \in G.$$

It is straightforward to check that the set $\{f_S \mid S \subseteq G\}$ belongs to $\mathrm{RUEB}(G)$. Since $(\mu_i)_{i \in I}$ UEB–converges to invariance over G, there exists $i_0 \in I$ such that

$$\sup_{S \subseteq G} \left|\int f_S \, d\mu_i - \int f_S \circ \lambda_{g^{-1}} \, d\mu_i\right| \leq \tfrac{1}{2} \quad \text{for all } i \in I \text{ with } i \geq i_0.$$

Let $i \in I$. We observe that $\int f_{C_i} \, d\mu_i = 1$, as $\mu_i(C_i) = 1$ and $C_i \subseteq f_{C_i}^{-1}(1)$. Hence, if $i \geq i_0$, then $\int f_{C_i} \circ \lambda_{g^{-1}} \, d\mu_i \geq \frac{1}{2}$ and so $(f_{C_i} \circ \lambda_{g^{-1}})|_{C_i} \neq \mathbf{0}$, thus $gUC_i \cap C_i \neq \varnothing$. This entails that $gU \cap C \neq \varnothing$. Consequently, C is dense in G. $\square$

Let us note the following consequence of Lemma 3.3(3):

Corollary 3.4 *If a metrizable topological group G admits a sequence $(\mu_n)_{n \in \mathbb{N}}$ of Borel probability measures UEB–converging to invariance such that, for each $n \in \mathbb{N}$, there is a compact subset $C_n \subseteq G$ with $\mu_n(C_n) = 1$, then G is separable.*

For metrizable topological groups, one may reformulate UEB–convergence to invariance in terms of mass transportation distances over compatible right-invariant metrics; see Corollary 3.6. This will be a consequence of the following fact about metrizable uniformities:

Lemma 3.5 *Let (X,d) be a metric space and $H \in \mathrm{UEB}(X,d)$. For every $\varepsilon > 0$, there exists an $\ell \geq 1$ such that for all $f \in H$, there exists an $f' \in \mathrm{Lip}_\ell^\ell(X,d)$ such that*

$$\|f - f'\|_\infty \leq \varepsilon.$$

Proof Upon translating H by a suitable constant function, we may and will assume that $f \geq 0$ for all $f \in H$. Put $s := \sup_{f \in H} \|f\|_\infty$. Let $\varepsilon \in (0,1]$. Since H is uniformly equicontinuous, we find $\delta > 0$ such that for all $f \in H$ and all $x, y \in X$,

$$d(x,y) < \delta \implies |f(x) - f(y)| \leq \varepsilon.$$

Let $k := (s+\varepsilon)/\delta$ and $\ell := \max\{k, s+1\}$. For each $f \in H$, define $f_k \colon X \to \mathbb{R}$ by

$$f_k(x) := \inf_{y \in X} f(y) + kd(x,y) \quad \text{for } x \in X.$$

Note that $f_k \colon (X,d) \to \mathbb{R}$ is k–Lipschitz for every $f \in H$. Let us now prove that $\|f - f_k\|_\infty \leq \varepsilon$ for all $f \in H$. For this purpose, let $f \in H$. Since $f_k \leq f$, it suffices to show that $f_k \geq f - \varepsilon$. To this end, let $x \in X$. For each $y \in X$, either $f(y) \geq f(x) - \varepsilon$ and therefore

$$f(y) + kd(x,y) \geq f(y) \geq f(x) - \varepsilon,$$

or $f(y) < f(x) - \varepsilon$ and thus $d(x,y) \geq \delta$, which entails that

$$f(y) + kd(x,y) \geq k\delta = s + \varepsilon \geq f(x) + \varepsilon.$$

In any case, $f(y) + kd(x,y) \geq f(x) - \varepsilon$ for all $y \in X$. Consequently, $f_k(x) \geq f(x) - \varepsilon$, as desired. In turn, $\|f_k\|_\infty \leq \|f\|_\infty + \varepsilon \leq s+1$ and hence, $f_k \in \mathrm{Lip}_\ell^\ell(X,d)$. □

Corollary 3.6 *Let G be a topological group and let d be a compatible right-invariant metric on G. A net $(\mu_i)_{i \in I}$ of Borel probability measures UEB–converges to invariance over G if and only if $(\mu_i)_{i \in I}$ converges to invariance in the mass transportation distance over d, ie for all $g \in G$,*

$$d_{\mathrm{MT}}((\lambda_g)_*(\mu_i), \mu_i) \to 0 \quad \text{as } i \to I.$$

Proof Since d is continuous and right-invariant, $\mathrm{Lip}_1^1(G,d)$ belongs to $\mathrm{RUEB}(G)$, whence the former implies the latter. To prove the converse, let $(\mu_i)_{i \in I}$ be a net of Borel probability measures converging to invariance in the mass transportation distance over d. Given that d is right-invariant and generates the topology of G, it is easily seen that the right uniformity of G coincides with the uniformity induced by d. Let

$H \in \mathrm{RUEB}(G) = \mathrm{UEB}(X, d)$ and $g \in G$. Consider any $\varepsilon > 0$. Thanks to Lemma 3.5, there exists $\ell \geq 1$ with

$$H \subseteq B_{\|\cdot\|_\infty}\big(\mathrm{Lip}_\ell^\ell(G, d), \tfrac{\varepsilon}{3}\big).$$

By assumption, we find $i_0 \in I$ such that for all $i \in I$ with $i \geq i_0$,

$$\sup_{f \in \mathrm{Lip}_1^1(G,d)} \left| \int f \circ \lambda_g \, d\mu_i - \int f \, d\mu_i \right| \leq \frac{\varepsilon}{3\ell}.$$

Let $i \in I$ with $i \geq i_0$. For each $f \in H$, there exists $f' \in \mathrm{Lip}_\ell^\ell(G, d) = \ell \cdot \mathrm{Lip}_1^1(G, d)$ with $\|f - f'\|_\infty \leq \frac{\varepsilon}{3}$, and thus

$$\begin{aligned}
&\left| \int f \circ \lambda_g \, d\mu_i - \int f \, d\mu_i \right| \\
&\qquad \leq \|(f \circ \lambda_g) - (f' \circ \lambda_g)\|_\infty + \left| \int f' \circ \lambda_g \, d\mu_i - \int f' \, d\mu_i \right| + \|f' - f\|_\infty \\
&\qquad \leq \frac{\varepsilon}{3} + \ell \frac{\varepsilon}{3\ell} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

ie $\sup_{f \in H} \left| \int f \circ \lambda_g \, d\mu_i - \int f \, d\mu_i \right| \leq \varepsilon$. So $(\mu_i)_{i \in I}$ UEB–converges to invariance over G. □

4 Equivariant concentration

Our proof of Theorem 1.1 will make a distinction between the precompact and the nonprecompact case. Whereas the former may be settled by a very simple and straightforward argument, the treatment of the latter is somewhat more complicated. Recall that a topological group G is said to be *precompact* if, for every $U \in \mathcal{U}(G)$, there exists a finite subset $F \subseteq G$ such that $G = UF$. It is well known that a topological group is precompact if and only if it embeds into a compact group; see [1, Corollary 3.7.17]. The following characterization of precompact groups was obtained independently by Uspenskij (unpublished, cf a footnote in [31]) and Solecki [29]. A short proof may be found in [4, Proposition 4.3].

Lemma 4.1 *Let G be a topological group. If for every $U \in \mathcal{U}(G)$ there exists a finite subset $F \subseteq G$ with $G = FUF$, then G is precompact.*

We will need the above in the form of Corollary 4.3.

Corollary 4.2 *Let G be a topological group. If for every $U \in \mathcal{U}(G)$ there exists a compact subset $K \subseteq G$ with $G = KUK$, then G is precompact.*

Proof We apply Lemma 4.1. Let $U \in \mathcal{U}(G)$. Pick $V \in \mathcal{U}(G)$ with $V^3 \subseteq U$. By assumption, we find a compact subset $K \subseteq G$ such that $G = KVK$. Since K is compact, there is a finite set $F \subseteq G$ with $K \subseteq VF$ and $K \subseteq FV$. It follows that $G = KVK \subseteq FV^3F \subseteq FUF$. □

Corollary 4.3 *Let G be a topological group. If G is not precompact, then there is a $U \in \mathcal{U}(G)$ such that, for every sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G, there is a sequence $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ with*

$$UK_m g_m \cap UK_n g_n = \varnothing \quad \text{for all } m, n \in \mathbb{N} \text{ with } m \neq n.$$

Proof Since G is not precompact, Corollary 4.2 above asserts the existence of some $V \in \mathcal{U}(G)$ such that $G \neq KVK$ for any compact set $K \subseteq G$. Let $U \in \mathcal{U}(G)$ with $U^{-1}U \subseteq V$. We claim that U has the desired property. To see this, let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of G. We select $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ recursively as follows: we let $g_0 := e$, and if $g_0, \dots, g_{n-1} \in G$ are chosen appropriately, then we pick

$$g_n \in G \setminus (K_n^{-1} V(K_0 g_0 \cup \dots \cup K_{n-1} g_{n-1}))$$

and note that $K_n g_n \cap V(K_0 g_0 \cup \dots \cup K_{n-1} g_{n-1}) = \varnothing$, whence

$$UK_n g_n \cap U(K_0 g_0 \cup \dots \cup K_{n-1} g_{n-1}) = \varnothing.$$

Evidently, the sequence $(g_n)_{n\in\mathbb{N}}$ is as desired. □

Before moving on to the proof of Theorem 1.1, let us note two basic preliminary observations (Lemmas 4.5 and 4.6). The first one, concerning the metrizability of topological groups, will be deduced from the following more general fact.

Lemma 4.4 *Let X be a dense subset of a Hausdorff uniform space Y. Suppose that X admits a metric d generating the subspace uniformity inherited from Y. Then there exists a unique continuous map $D\colon Y \times Y \to \mathbb{R}$ with $D|_{X\times X} = d$, and furthermore D is a metric generating the uniformity of Y.*

Proof Uniqueness of the desired map is an immediate consequence of X being dense in Y. Let us prove the existence. Since $X \times X$ is a dense subspace of $Y \times Y$ and the metric $d\colon X \times X \to \mathbb{R}$ is uniformly continuous, there exists a unique uniformly continuous map $D\colon Y \times Y \to \mathbb{R}$ with $D|_{X\times X} = d$. Due to D being continuous, $S := \{(x,y) \in Y^2 \mid D(x,y) \geq 0,\ D(x,y) = D(y,x)\}$ is closed in Y^2 and $T := \{(x,y,z) \in Y^3 \mid D(x,z) \leq D(x,y) + D(y,z)\}$ is closed in Y^3. Since $D|_{X\times X} = d$ is a metric, $X^2 \subseteq S$ and $X^3 \subseteq T$, and therefore $Y^2 = S$ and $Y^3 = T$ by density of X in Y. Hence, D is a uniformly continuous pseudometric on Y. It remains to prove that the uniformity of Y is contained in the one generated by D, which will then imply that D is a metric. To this end, let U be an arbitrary entourage of Y. Choose a symmetric entourage V of Y such that $V \circ V \circ V \subseteq U$. As d generates the uniformity of X, there exists $\varepsilon > 0$ such that $\{(x,y) \in X^2 \mid d(x,y) < \varepsilon\} \subseteq V$. We will show that U contains $W := \left\{(x,y) \in Y^2 \mid D(x,y) < \frac{\varepsilon}{3}\right\}$. Let $(x,y) \in W$. As D is continuous and X is dense in Y, we find $x' \in X \cap B_D\left(x, \frac{\varepsilon}{3}\right)$ and $y' \in X \cap B_D\left(y, \frac{\varepsilon}{3}\right)$ with $(x,x') \in V$ and $(y,y') \in V$. Then

$$d(x',y') = D(x',y') \leq D(x',x) + D(x,y) + D(y,y') < \varepsilon,$$

thus $(x',y') \in V$ and hence $(x,y) \in V \circ V \circ V \subseteq U$. Therefore, $W \subseteq U$, as desired. □

Lemma 4.5 *Let G be a dense subgroup of a topological group H. Suppose that d is a right-invariant compatible metric on G. Then there exists a unique continuous map $D\colon H \times H \to \mathbb{R}$ with $D|_{G\times G} = d$, and furthermore D is a right-invariant compatible metric on H.*

Proof As d is a right-invariant compatible metric on G, it generates the right uniformity of G, which coincides with the restriction of the right uniformity of H to G. Thus, by Lemma 4.4, there exists a unique continuous map $D\colon H \times H \to \mathbb{R}$ with $D|_{G\times G} = d$, and moreover D is a compatible metric on H. It remains to show that D is right-invariant. Indeed, the continuity of D implies that

$$T := \{(x,y,z) \in H^3 \mid D(xz,yz) = D(x,y)\}$$

is closed in H^3. Since $G^3 \subseteq T$ by right invariance of $d = D|_{G\times G}$ and G is dense in H, it follows that $H^3 = T$, as desired. □

Our second preliminary note is the following variation on the well-known fact that quotients of Banach spaces by closed linear subspaces are again Banach spaces. We include a proof for the sake of convenience.

Lemma 4.6 *Let X and Y be Banach spaces, and let Y_0 be any dense linear subspace of Y. If $T\colon X \to Y$ is a bounded linear operator such that*

$$\|y\|_Y = \inf\{\|x\|_X \mid x \in T^{-1}(y)\} \quad \text{for all } y \in Y_0,$$

then $T(X) = Y$.

Proof Let $y \in Y$. As Y_0 is dense in Y, there is a sequence $(z_n)_{n\in\mathbb{N}}$ in Y_0 with $\|y - z_n\|_Y \le 2^{-n}$ for each $n \in \mathbb{N}$. By assumption, there exists $x_0 \in T^{-1}(z_0)$ such that $\|x_0\|_X \le \|z_0\|_Y + 1$. Likewise, our hypothesis asserts that, for each $n \in \mathbb{N}$, we find

$$x_{n+1} \in T^{-1}(z_{n+1} - z_n)$$

with $\|x_{n+1}\|_X \le \|z_{n+1} - z_n\|_Y + 2^{-(n+1)}$. For each $n \in \mathbb{N}$, consider the element $x_n^* := \sum_{i\le n} x_i \in X$ and note that $T(x_n^*) = \sum_{i\le n} T(x_i) = z_n$. Furthermore, for all $m, n \in \mathbb{N}$ where $m > n$,

$$\|x_m^* - x_n^*\|_X \le \sum_{i=n+1}^{m} \|x_i\|_X \le \sum_{i=n+1}^{m} \|z_i - z_{i-1}\|_Y + 2^{-i} \le 3 \sum_{i=n}^{m-1} 2^{-i}.$$

Hence, $(x_n^*)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, thus convergent to some point $x^* \in X$. Since T is continuous, it follows that $T(x^*) = y$, as desired. □

Now everything is in place for the proof of Theorem 1.1.

Proof of Theorem 1.1 Let us start off with a remark about the last assertion of Theorem 1.1: since $X = \operatorname{spt} \mu_X$ is compact, $\psi(X) = \psi(\operatorname{spt} \mu_X) = \operatorname{spt}(\psi_* \mu_X)$ for any continuous mapping $\psi\colon X \to \mathrm{S}(G)$, whence the G–invariance of $\psi_*(\mu_X)$ would imply the G–invariance of $\psi(X)$. We will establish the existence of the desired embedding by case analysis.

We first treat the precompact case. Assuming that G is precompact, we find an embedding $h\colon G \to K$ into a compact group K such that $K = \overline{h(G)}$ (cf [1, Corollary 3.7.17]). By Lemma 4.5, there is a unique continuous metric $d_K\colon K \times K \to \mathbb{R}$ such that $h\colon (G, d) \to (K, d_K)$ is isometric, and furthermore d_K is a compatible right-invariant metric on K. Let us denote by ν_K the normalized Haar measure on K. We prove that the sequence $(K_n, d\restriction_{K_n}, \mu_n\restriction_{K_n})_{n\in\mathbb{N}}$ concentrates to (K, d_K, ν_K). For each $n \in \mathbb{N}$, the map $p_n := h|_{K_n}\colon (K_n, d\restriction_{K_n}) \to (K, d_K)$ is an isometric embedding, whence $\operatorname{Lip}_1(K_n, d\restriction_{K_n}) = \operatorname{Lip}_1(K, d_K) \circ p_n$. According to Theorem 2.2, it now remains to show that $(p_n)_*(\mu_n\restriction_{K_n}) = h_*(\mu_n) \to \nu_K$ weakly as $n \to \infty$. To this end, let $f \in \mathrm{C}(K) = \mathrm{RUCB}(K)$. By Corollary 3.6 and Lemma 3.3(2), the sequence $(h_*(\mu_n))_{n\in\mathbb{N}}$ UEB–converges to invariance over K; in particular, for all $x \in K$,

$$\left|\int f(xy)\, dh_*(\mu_n)(y) - \int f(y)\, dh_*(\mu_n)(y)\right| \to 0 \quad \text{as } n \to \infty.$$

Due to Lebesgue's dominated convergence theorem, it follows that

$$\int\left|\int f(xy)\, dh_*(\mu_n)(y) - \int f(y)\, dh_*(\mu_n)(y)\right| d\nu_K(x) \to 0 \quad \text{as } n \to \infty.$$

Thanks to the right invariance of ν_K along with Fubini's theorem, we also have

$$\begin{aligned}\left|\int f\, d\nu_K - \int f\, dh_*(\mu_n)\right| &= \left|\iint f(xy)\, d\nu_K(x)\, dh_*(\mu_n)(y) - \int f(y)\, dh_*(\mu_n)(y)\right| \\ &= \left|\iint f(xy)\, dh_*(\mu_n)(y)\, d\nu_K(x) - \int f(y)\, dh_*(\mu_n)(y)\right| \\ &\leq \int\left|\int f(xy)\, dh_*(\mu_n)(y) - \int f(y)\, dh_*(\mu_n)(y)\right| d\nu_K(x)\end{aligned}$$

for all $n \in \mathbb{N}$, which by the above implies that $\int f\, dh_*(\mu_n) \to \int f\, d\nu_K$ as $n \to \infty$. This shows that $(p_n)_*(\mu_n\restriction_{K_n}) = h_*(\mu_n) \to \nu_K$ weakly as $n \to \infty$, from which it follows that $(K_n, d\restriction_{K_n}, \mu_n\restriction_{K_n})_{n\in\mathbb{N}}$ indeed concentrates to (K, d_K, ν_K). In view of (B), this necessitates that $d_{\mathrm{conc}}((K, d_K), (X, d_X)) = 0$, wherefore the mm–spaces (K, d_K, ν_K) and (X, d_X, μ_X) are isomorphic [28, Theorem 5.16]. Since both ν_K and μ_X have full support, this entails the existence of an isometric bijection $\zeta\colon (X, d_X) \to (K, d_K)$ with $\zeta_*(\mu_X) = \nu_K$. Also, given that h embeds the topological group G densely into K, we obtain a G–equivariant homeomorphism $\chi\colon \mathrm{S}(G) \to \mathrm{S}(K)$ by setting

$$\chi(\xi)(f) := \xi(f \circ h) \quad \text{for } \xi \in \mathrm{S}(G),\ f \in \mathrm{C}(K),$$

where $\mathrm{C}(K) = \mathrm{RUCB}(K)$ by compactness of K. As the K–equivariant map $\eta_K\colon K \to \mathrm{S}(K)$ is a homeomorphism due to Gelfand duality, $\varphi := \chi^{-1} \circ \eta_K\colon K \to \mathrm{S}(G)$ is a G–equivariant homeomorphism. Therefore, we conclude that $\psi := \varphi \circ \zeta\colon X \to \mathrm{S}(G)$ is a homeomorphism and that $\psi_*(\mu_X) = \varphi_*(\nu_K)$ is G–invariant. This settles the precompact case.

For the rest of the proof, let us assume that G is not precompact. Let $U \in \mathcal{U}(G)$ be as in Corollary 4.3. Since $K_n := \operatorname{spt} \mu_n$ is compact for every $n \in \mathbb{N}$, there exists $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ with $UK_m g_m \cap UK_n g_n = \varnothing$ for any two distinct $m, n \in \mathbb{N}$. For each $n \in \mathbb{N}$, consider the push-forward Borel probability measure $\nu_n := (\rho_{g_n})_*(\mu_n)$ on G, and note that $S_n := \operatorname{spt} \nu_n = K_n g_n$. We conclude that

(i) $$US_m \cap US_n = \varnothing \quad \text{for all } m, n \in \mathbb{N} \text{ with } m \neq n.$$

Due to Corollary 3.6 and Lemma 3.3(1), the sequence $(\nu_n)_{n\in\mathbb{N}}$ UEB–converges to invariance over G. As the metric d is right-invariant, the map

$$(K_n, d\restriction_{K_n}, \mu_n\restriction_{K_n}) \to (S_n, d\restriction_{S_n}, \nu_n\restriction_{S_n}), \quad x \mapsto xg_n,$$

is an mm–space isomorphism for every $n \in \mathbb{N}$. Thus, since $(K_n, d\restriction_{K_n}, \mu_n\restriction_{K_n})_{n\in\mathbb{N}}$ concentrates to (X, d_X, μ_X), so does $(S_n, d\restriction_{S_n}, \nu_n\restriction_{S_n})_{n\in\mathbb{N}}$. Consider the Prokhorov distance $(d_X)_{\mathrm{P}}$ on $\mathrm{P}(X)$ associated with the metric d_X (see Section 2). Due to Theorem 2.2, there exists a sequence of Borel maps $p_n\colon S_n \to X$ ($n \in \mathbb{N}$) such that

(1) $(d_X)_{\mathrm{P}}((p_n)_*(\overline{\nu}_n), \mu_X) \to 0$ as $n \to 0$,
(2) $(\mathrm{me}_{\overline{\nu}_n})_{\mathrm{H}}(\mathrm{Lip}_1(X, d_X) \circ p_n, \mathrm{Lip}_1(S_n, d_n)) \to 0$ as $n \to \infty$,

where $\overline{\nu}_n := \nu_n\restriction_{S_n}$ and $d_n := d\restriction_{S_n}$ for $n \in \mathbb{N}$. We show that for every $f \in \mathrm{Lip}^\infty(G, d)$,

$$T(f) := \left\{(f_n)_{n\in\mathbb{N}} \in \mathrm{Lip}_\ell^\ell(X, d_X)^{\mathbb{N}} \;\middle|\; \ell \geq 0,\ \lim_{n\to\infty} \mathrm{me}_{\overline{\nu}_n}(f_n \circ p_n, f|_{S_n}) = 0\right\}$$

is a nonempty set. For this purpose, let $f \in \mathrm{Lip}_\ell^\ell(G, d)$ for some $\ell \geq 1$. By (2), there exists a sequence $(f_n)_{n\in\mathbb{N}} \in \mathrm{Lip}_1(X, d_X)^{\mathbb{N}}$ such that $\mathrm{me}_{\overline{\nu}_n}(f_n \circ p_n, \ell^{-1} f|_{S_n}) \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, it follows that

$$f_n' := ((\ell f_n) \wedge \ell) \vee (-\ell) \in \mathrm{Lip}_\ell^\ell(X, d_X),$$

and moreover,

$$\mathrm{me}_{\overline{\nu}_n}(f_n' \circ p_n, f|_{S_n}) \leq \mathrm{me}_{\overline{\nu}_n}((\ell f_n) \circ p_n, f|_{S_n}) \leq \ell \mathrm{me}_{\overline{\nu}_n}(f_n \circ p_n, \ell^{-1} f|_{S_n}),$$

which shows that $(f_n')_{n\in\mathbb{N}} \in T(f)$, as desired.

Next let us prove that for all $f \in \mathrm{Lip}^\infty(G, d)$ and all $(f_n)_{n\in\mathbb{N}}$ with $(f_n')_{n\in\mathbb{N}} \in T(f)$,

(ii) $$\lim_{n\to\infty} \|f_n - f_n'\|_\infty = 0.$$

To this end, let $f \in \mathrm{Lip}^\infty(G, d)$ and $(f_n)_{n\in\mathbb{N}}, (f_n')_{n\in\mathbb{N}} \in T(f)$. Fix $\ell \geq 1$ such that

$$\{f_n \mid n \in \mathbb{N}\} \cup \{f_n' \mid n \in \mathbb{N}\} \subseteq \mathrm{Lip}_\ell^\ell(X, d_X).$$

According to (1), there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive real numbers converging to 0 such that $(d_X)_{\mathrm{P}}((p_n)_*(\overline{\nu}_n), \mu_X) < \delta_n$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Consider $\sigma_n := \mathrm{me}_{\overline{\nu}_n}(f_n \circ p_n, f|_{S_n})$ and $\tau_n := \mathrm{me}_{\overline{\nu}_n}(f_n' \circ p_n, f|_{S_n})$. Note that

$$B_n := \{x \in X \mid |f_n(x) - f_n'(x)| \leq \sigma_n + \tau_n + \ell\delta_n\}$$

is a Borel subset of X containing $B_{d_X}(C_n, \delta_n)$ for

$$C_n := \{x \in X \mid |f_n(x) - f'_n(x)| \leq \sigma_n + \tau_n\}.$$

Considering the Borel sets

$$D_n := \{s \in S_n \mid |f_n(p_n(s)) - f(s)| \leq \sigma_n\},$$
$$D'_n := \{s \in S_n \mid |f'_n(p_n(s)) - f(s)| \leq \tau_n\},$$

we observe that $D_n \cap D'_n \subseteq p_n^{-1}(C_n)$, and therefore

$$\overline{\nu}_n(p_n^{-1}(C_n)) \geq \overline{\nu}_n(D_n \cap D'_n) \geq 1 - \overline{\nu}_n(S_n \setminus D_n) - \overline{\nu}_n(S_n \setminus D'_n) \geq 1 - \sigma_n - \tau_n.$$

It follows that

$$\mu_X(B_n) \geq \mu_X(B_{d_X}(C_n, \delta_n)) \geq \overline{\nu}_n(p_n^{-1}(C_n)) - \delta_n \geq 1 - \sigma_n - \tau_n - \delta_n.$$

This shows that $\mathrm{me}_{\mu_X}(|f_n - f'_n|, \mathbf{0}) \leq \sigma_n + \tau_n + \ell\delta_n$. As this is true for arbitrary $n \in \mathbb{N}$, the definition of $T(f)$ and our choice of $(\delta_n)_{n\in\mathbb{N}}$ imply that $\mathrm{me}_{\mu_X}(|f_n - f'_n|, \mathbf{0}) \to 0$ as $n \to \infty$, ie $|f_n - f'_n| \to \mathbf{0}$ in the measure μ_X as $n \to \infty$. Let $k := 2\ell$. Since $\operatorname{spt} \mu_X = X$, the restriction of me_{μ_X} to $\mathrm{C}(X)$ is a metric, wherefore the induced topology on $\mathrm{C}(X)$, ie the topology of convergence in μ_X, is Hausdorff. By the Arzelà–Ascoli theorem, $\mathrm{Lip}_k^k(X, d_X)$ is compact with respect to the topology of uniform convergence. Given that the latter contains the topology of convergence in μ_X, the two topologies coincide on the set $\mathrm{Lip}_k^k(X, d_X)$. Since

$$\{|f_n - f'_n| \mid n \in \mathbb{N}\} \cup \{\mathbf{0}\} \subseteq \mathrm{Lip}_k^k(X, d_X),$$

we conclude that $|f_n - f'_n| \to \mathbf{0}$ uniformly as $n \to \infty$. This proves (ii).

Fix a nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Appealing to (ii) and the Arzelà–Ascoli theorem again, let us define

$$\Phi\colon \mathrm{Lip}^\infty(G, d) \to \mathrm{Lip}(X, d_X)$$

by setting

$$\Phi(f) := \lim_{n\to\mathcal{F}} f_n \quad \text{for } f \in \mathrm{Lip}^\infty(G, d) \text{ with } (f_n)_{n\in\mathbb{N}} \in T(f)),$$

where the ultrafilter convergence applies to the uniform topology. We will show that Φ is a homomorphism of unital $\mathbb{R}$–algebras. Evidently, $(\boldsymbol{r})_{n\in\mathbb{N}} \in T(\boldsymbol{r})$ and thus $\Phi(\boldsymbol{r}) = \lim_{n\to\mathcal{F}} \boldsymbol{r} = \boldsymbol{r}$ for any $r \in \mathbb{R}$. Let $f, f' \in \mathrm{Lip}^\infty(G, d)$. Pick any $(f_n)_{n\in\mathbb{N}} \in T(f)$ and

$(f'_n)_{n\in\mathbb{N}} \in T(f')$, and choose $\ell \geq 1$ with $\{f_n \mid n \in \mathbb{N}\} \cup \{f'_n \mid n \in \mathbb{N}\} \subseteq \mathrm{Lip}^{\ell}_{\ell}(X, d_X)$ and

$$\max\{\|f\|_\infty,\ \|f'\|_\infty\} \leq \ell.$$

It is easy to check that $f_n + f'_n \in \mathrm{Lip}^{2\ell}_{2\ell}(X, d_X)$ and $f_n f'_n \in \mathrm{Lip}^{\ell^2}_{2\ell^2}(X, d_X)$ for every $n \in \mathbb{N}$. Let $\sigma_n := \mathrm{me}_{\overline{\nu}_n}(f_n \circ p_n, f|_{S_n})$ and $\tau_n := \mathrm{me}_{\overline{\nu}_n}(f'_n \circ p_n, f'|_{S_n})$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$,

$$\begin{aligned}\{s \in S_n \mid |(f_n + f'_n)(p_n(s)) - (f + f')(s)| > \sigma_n + \tau_n\} \\ \subseteq \{s \in S_n \mid |f_n(p_n(s)) - f(s)| > \sigma_n\} \cup \{s \in S_n \mid |f'_n(p_n(s)) - f'(s)| > \tau_n\}\end{aligned}$$

and thus

$$\mathrm{me}_{\overline{\nu}_n}((f_n + f'_n) \circ p_n, (f + f')|_{S_n}) \leq \sigma_n + \tau_n,$$

as well as

$$\begin{aligned}\{s \in S_n \mid |(f_n f'_n)(p_n(s)) - (f f')(s)| > \ell(\sigma_n + \tau_n)\} \\ \subseteq \{s \in S_n \mid |f_n(p_n(s)) - f(s)| > \sigma_n\} \cup \{s \in S_n \mid |f'_n(p_n(s)) - f'(s)| > \tau_n\}\end{aligned}$$

and therefore

$$\mathrm{me}_{\overline{\nu}_n}((f_n f'_n) \circ p_n, (f f')|_{S_n}) \leq \max\{\ell(\sigma_n + \tau_n), \sigma_n + \tau_n\} \leq \ell(\sigma_n + \tau_n).$$

Hence, $(f_n + f'_n)_{n\in\mathbb{N}} \in T(f + f')$ and $(f_n f'_n)_{n\in\mathbb{N}} \in T(f f')$, which readily implies

$$\Phi(f + f') = \lim_{n\to\mathcal{F}} f_n + f'_n = (\lim_{n\to\mathcal{F}} f_n) + (\lim_{n\to\mathcal{F}} f'_n) = \Phi(f) + \Phi(f')$$

and likewise

$$\Phi(f f') = \lim_{n\to\mathcal{F}} f_n f'_n = (\lim_{n\to\mathcal{F}} f_n)(\lim_{n\to\mathcal{F}} f'_n) = \Phi(f)\Phi(f').$$

This shows that Φ is indeed a homomorphism between unital $\mathbb{R}$–algebras.

Next let us prove that

(iii) $\quad \|f\|_\infty = \min\{\|f^*\|_\infty \mid f^* \in \Phi^{-1}(f)\} \quad$ for all $f \in \mathrm{Lip}(X, d_X)$.

Note that this will imply that Φ is surjective. We start our proof of (iii) by observing that Φ is contractive with respect to the supremum norm, ie

$$\|\Phi(f)\|_\infty \leq \|f\|_\infty \quad \text{for all } f \in \mathrm{Lip}^\infty(G, d).$$

Indeed, if $f \in \mathrm{Lip}^\infty(G, d)$ and $(f_n)_{n\in\mathbb{N}} \in T(f)$, then it is easily checked that

$$((f_n \wedge \|f\|_\infty) \vee (-\|f\|_\infty))_{n\in\mathbb{N}} \in T(f),$$

and thus

$$\Phi(f) = \lim_{n\to\mathcal{F}}((f_n \wedge \|f\|_\infty) \vee (-\|f\|_\infty)),$$

whence $\|\Phi(f)\|_\infty \leq \|f\|_\infty$, as desired. Furthermore, since d generates the topology of G, there exists $\varepsilon > 0$ such that $B_d(e,\varepsilon) \subseteq U$. By right invariance of d,

$$B_d(S_m,\varepsilon) \cap B_d(S_n,\varepsilon) \subseteq US_m \cap US_n = \varnothing$$

for any two distinct $m,n \in \mathbb{N}$. To prove (iii), let $\ell \geq 1$ and $f \in \mathrm{Lip}_\ell(X,d_X)$, and put $c := \|f\|_\infty$. According to (2), there exists a sequence of functions $f_n \in \mathrm{Lip}_1(S_n,d_n)$ such that $\mathrm{me}_{\overline{\nu}_n}((\ell^{-1}f)\circ p_n, f_n) \to 0$ as $n \to \infty$. For every $n \in \mathbb{N}$, we have

$$f_n' := ((\ell f_n) \wedge c) \vee (-c) \in \mathrm{Lip}_\ell^c(S_n,d_n),$$

$$\mathrm{me}_{\overline{\nu}_n}(f\circ p_n, f_n') \leq \mathrm{me}_{\overline{\nu}_n}(f\circ p_n, \ell f_n) \leq \ell\,\mathrm{me}_{\overline{\nu}_n}((\ell^{-1}f)\circ p_n, f_n).$$

Consider the set $S := \bigcup_{n\in\mathbb{N}} S_n$ and define $f'\colon S \to \mathbb{R}$ by setting $f'|_{S_n} = f_n'$ for every $n \in \mathbb{N}$. Then $f' \in \mathrm{Lip}_k^c(S, d\restriction_S)$ for $k := \max\{\ell, 2c\varepsilon^{-1}\}$, since

$$B_d(S_m,\varepsilon) \cap B_d(S_n,\varepsilon) = \varnothing$$

for any two distinct $m,n \in \mathbb{N}$. Utilizing a standard construction, we define $f^*\colon G \to \mathbb{R}$ by

$$f^*(g) := \bigl(\bigl(\inf_{s\in S} f'(s) + kd(g,s)\bigr) \wedge c\bigr) \vee (-c) \quad \text{for } g \in G,$$

and observe that $f^* \in \mathrm{Lip}_k^c(G,d)$. Since moreover $f^*|_S = f'$, it follows that

$$\mathrm{me}_{\overline{\nu}_n}(f\circ p_n, f^*|_{S_n}) = \mathrm{me}_{\overline{\nu}_n}(f\circ p_n, f_n') \leq \ell\,\mathrm{me}_{\overline{\nu}_n}((\ell^{-1}f)\circ p_n, f_n)$$

for all $n \in \mathbb{N}$. Hence, $(f)_{n\in\mathbb{N}} \in T(f^*)$ and therefore $\Phi(f^*) = \lim_{n\to\mathcal{F}} f = f$. Finally,

$$\|f\|_\infty = \|\Phi(f^*)\|_\infty \leq \|f^*\|_\infty \leq c = \|f\|_\infty$$

and thus $\|f^*\|_\infty = \|f\|_\infty$, as desired.

Let us consider the unique continuous linear operator $\overline{\Phi}\colon \mathrm{RUCB}(G) \to \mathrm{C}(X)$ which extends Φ. Since Φ is a homomorphism of unital $\mathbb{R}$–algebras, so is $\overline{\Phi}$. Moreover, $\overline{\Phi}$ is surjective by (iii), Lemma 4.6, and the density of $\mathrm{Lip}(X,d_X)$ in $\mathrm{C}(X)$. The map

$$\nu\colon \mathrm{RUCB}(G) \to \mathbb{R}, \quad f \mapsto \lim_{n\to\mathcal{F}} \int f\,d\nu_n,$$

is a left-invariant mean; cf [27, Proof of Theorem 3.2]. We will show that

(iv) $$\int \overline{\Phi}(f)\,d\mu_X = \nu(f) \quad \text{for all } f \in \mathrm{RUCB}(G).$$

Since both ν and the map $\mathrm{RUCB}(G) \to \mathbb{R}$ given by $f \mapsto \int \overline{\Phi}(f)\, d\mu_X$ are continuous linear maps and $\mathrm{Lip}^\infty(G,d)$ is a dense linear subspace of $\mathrm{RUCB}(G)$, it suffices to prove that

$$\int \Phi(f)\, d\mu_X = \nu(f) \quad \text{for all } f \in \mathrm{Lip}^\infty(G,d).$$

Let $f \in \mathrm{Lip}^\infty(G,d)$ and $(f_n)_{n\in\mathbb{N}} \in T(f)$. Then $c := \|f\|_\infty \vee \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. Since $(d_X)_{\mathrm{P}}$ metrizes the weak topology on $\mathrm{P}(X)$, assertion (1) implies that

$$\varepsilon_n := \left| \int \Phi(f)\, d\mu_X - \int \Phi(f)\, d(p_n)_*(\overline{\nu}_n) \right| \to 0 \quad \text{as } n \to \infty.$$

Considering that $\mathcal{F}$ is nonprincipal and, moreover,

$$\begin{aligned} \left| \int \Phi(f)\, d\mu_X - \int f\, d\nu_n \right| &\leq \left| \int \Phi(f)\, d\mu_X - \int \Phi(f)\, d(p_n)_*(\overline{\nu}_n) \right| \\ &\qquad + \left| \int \Phi(f)\, d(p_n)_*(\overline{\nu}_n) - \int f_n\, d(p_n)_*(\overline{\nu}_n) \right| \\ &\qquad + \left| \int f_n \circ p_n\, d\overline{\nu}_n - \int f|_{S_n}\, d\overline{\nu}_n \right| \\ &\leq \varepsilon_n + \|\Phi(f) - f_n\|_\infty + (1+2c)\mathrm{me}_{\overline{\nu}_n}(f_n \circ p_n, f|_{S_n}) \end{aligned}$$

for all $n \in \mathbb{N}$, we conclude that $|\int \Phi(f)\, d\mu_X - \int f\, d\nu_n| \to 0$ as $n \to \mathcal{F}$. This proves (iv).

Finally, let us consider the continuous map $\psi\colon X \to \mathrm{S}(G)$ given by

$$\psi(x)(f) := \overline{\Phi}(f)(x) \quad \text{for } x \in X,\ f \in \mathrm{RUCB}(G).$$

Since $\overline{\Phi}$ is onto, ψ is a topological embedding. To see that $\psi_*(\mu_X)$ is G–invariant, let us note the following: for every $f \in \mathrm{C}(\mathrm{S}(G))$, since $f(\xi) = \xi(f \circ \eta_G)$ for all $\xi \in \mathrm{S}(G)$, we have

$$f(\psi(x)) = \psi(x)(f \circ \eta_G) = \overline{\Phi}(f \circ \eta_G)(x)$$

for all $x \in X$, ie $f \circ \psi = \overline{\Phi}(f \circ \eta_G)$. Also, being a Borel probability measure on a metrizable compact space, μ_X is regular. As ψ is a continuous map between compact Hausdorff spaces, $\psi_*(\mu_X)$ must be regular as well. Therefore, in order to establish the G–invariance of $\psi_*(\mu_X)$, it suffices to observe that, for all $f \in \mathrm{C}(\mathrm{S}(G))$ and $g \in G$,

$$\begin{aligned} \int f \circ \tau_g\, d\psi_*(\mu_X) &= \int f \circ \tau_g \circ \psi\, d\mu_X \\ &= \int \overline{\Phi}(f \circ \tau_g \circ \eta_G)\, d\mu_X = \int \overline{\Phi}(f \circ \eta_G \circ \lambda_g)\, d\mu_X \end{aligned}$$

$$\overset{\text{(iv)}}{=} \nu(f \circ \eta_G \circ \lambda_g) = \nu(f \circ \eta_G) \overset{\text{(iv)}}{=} \int \overline{\Phi}(f \circ \eta_G)\, d\mu_X$$
$$= \int f \circ \psi \, d\mu_X = \int f \, d\psi_*(\mu_X),$$

where $\tau_g \colon \mathrm{S}(G) \to \mathrm{S}(G)$ is given by $\xi \mapsto g\xi$. This completes the proof. □

5 Observable diameters

In this section we will prove Theorem 1.2 and then deduce Corollary 1.3. For a start, let us briefly recall Gromov's concept of observable diameters [12, Chapter $3\frac{1}{2}$]. For further reading, we refer to [12; 15; 28].

Definition 5.1 Let $\alpha > 0$. The *α–partial diameter* of a Borel probability measure ν on $\mathbb{R}$ is

$$\mathrm{PartDiam}(\nu, 1-\alpha) := \inf\{\mathrm{diam}(B, d_{\mathbb{R}}) \mid B \subseteq \mathbb{R} \text{ Borel}, \ \nu(B) \geq 1-\alpha\},$$

where $d_{\mathbb{R}}$ denotes the Euclidean metric on $\mathbb{R}$. Given any Borel probability measure μ on a topological space X and a continuous pseudometric d on X, we refer to the quantity

$$\mathrm{ObsDiam}(X, d, \mu; -\alpha) := \sup\{\mathrm{PartDiam}(f_*(\mu), 1-\alpha) \mid f \in \mathrm{Lip}_1(X, d)\}$$

as the corresponding *α–observable diameter*.

Remark 5.2 Let $\alpha > 0$. If μ is a Borel probability measure on a topological space X and $d_0 \leq d_1$ are continuous pseudometrics on X, then

$$\mathrm{ObsDiam}(X, d_0, \mu; -\alpha) \leq \mathrm{ObsDiam}(X, d_1, \mu; -\alpha).$$

In particular, if G is a topological group acting continuously on a compact Hausdorff space X and d is a continuous pseudometric on X, then

$$\sup_{x \in X} \mathrm{ObsDiam}(G, d_{G,x}, \mu; -\alpha) \leq \mathrm{ObsDiam}(G, d_{G,X}, \mu; -\alpha)$$

for every Borel probability measure μ on G.

Proof of Theorem 1.2 Let X be a G–flow equipped with a G–invariant regular Borel probability measure ν. Consider a continuous pseudometric d on X and let

$$D := \sup_{\alpha > 0} \liminf_{i \to I} \sup_{x \in X} \mathrm{ObsDiam}(G, d_{G,x}, \mu_i; -\alpha).$$

Let $E \subseteq G$ be finite and let $\varepsilon > 0$. We show that

$$\int \sup_{g \in E} d(x, gx)\, d\nu(x) \leq D + \varepsilon.$$

To this end, let $U := B_{d_{G,X}}(e, \frac{\varepsilon}{8})$ and pick a right-uniformly continuous function $p\colon G \to [0,1]$ with $p(e) = 1$ and $p(x) = 0$ for all $x \in G \setminus U$. For any $S \subseteq G$, define $p_S\colon G \to [0,1]$ by

$$p_S(x) := \sup_{s \in S} p(xs^{-1}) \quad \text{for } x \in G.$$

It is easy to see that $\{p_S \mid S \subseteq G\}$ belongs to $\mathrm{RUEB}(G)$. Let $\delta := \mathrm{diam}(X, d) + 1$. Since the net $(\mu_i)_{i \in I}$ UEB–converges to invariance over G, we find $i_0 \in I$ such that for all $i \in I$ with $i \geq i_0$ and all $g \in E$,

$$(*) \qquad \sup_{S \subseteq G} \left| \int p_S\, d\mu_i - \int p_S \circ \lambda_g\, d\mu_i \right| \leq \frac{\varepsilon}{8\delta(|E|+1)}.$$

Since X is compact, there exists a finite nonempty subset $F \subseteq \mathrm{Lip}_1(X, d)$ such that

$$d(x, y) \leq \sup_{f \in F} |f(x) - f(y)| + \tfrac{\varepsilon}{2} \quad \text{for all } x, y \in X;$$

cf [21, Exercise 7.4.13]. For the rest of the proof, fix any $i \in I$ such that $i \geq i_0$ and

$$\sup_{x \in X} \mathrm{ObsDiam}\Bigl(G, d_{G,x}, \mu_i; -\frac{\varepsilon}{8\delta|F|(|E|+1)}\Bigr) \leq D + \tfrac{\varepsilon}{8}.$$

Let us prove that

$$(**) \qquad \sup_{x \in X} \int \sup_{g \in E} \sup_{f \in F} |f(hx) - f(ghx)|\, d\mu_i(h) \leq D + \tfrac{\varepsilon}{2}.$$

Let $x \in X$. Since for each $f \in F$ the map $f_x\colon G \to \mathbb{R}$ given by $h \mapsto f(hx)$ belongs to $\mathrm{Lip}_1(G, d_{G,x})$, our choice of i ensures that

$$\sup_{f \in F} \mathrm{PartDiam}\Bigl((f_x)_*(\mu_i), 1 - \frac{\varepsilon}{8\delta|F|(|E|+1)}\Bigr) \leq D + \tfrac{\varepsilon}{8}.$$

Hence, for each $f \in F$ there exists a Borel set $B_f \subseteq G$ such that

$$\mu_i(B_f) \geq 1 - \frac{\varepsilon}{8\delta|F|(|E|+1)}$$

and $\mathrm{diam}(f_x(B_f), d_{\mathbb{R}}) \leq D + \frac{\varepsilon}{8}$. Considering the Borel set $B := \bigcap_{f \in F} B_f$, we deduce that $\mu_i(B) \geq 1 - \varepsilon/(8\delta(|E|+1))$ and moreover $\mathrm{diam}(f_x(B), d_{\mathbb{R}}) \leq D + \frac{\varepsilon}{8}$ for each $f \in F$. The former implies that $\int p_B\, d\mu_i \geq 1 - \varepsilon/(8\delta(|E|+1))$. Thus, $(*)$ asserts that $\int p_B \circ \lambda_g \, d\mu_i \geq 1 - \varepsilon/(4\delta(|E|+1))$ for each $g \in G$. Since $UB = B_{d_{G,X}}\bigl(B, \frac{\varepsilon}{8}\bigr)$ by right invariance of $d_{G,X}$, it readily follows that

$$\mu_i\bigl(g^{-1} B_{d_{G,X}}\bigl(B, \tfrac{\varepsilon}{8}\bigr)\bigr) \geq 1 - \frac{\varepsilon}{4\delta(|E|+1)}.$$

Considering the Borel set

$$C := B \cap \bigcap_{g \in E} g^{-1} B_{d_{G,X}}\bigl(B, \tfrac{\varepsilon}{8}\bigr),$$

we now conclude that $\mu_i(C) \geq 1 - \varepsilon/(4\delta)$. Furthermore,

$$\sup_{g \in E} \sup_{f \in F} |f(hx) - f(ghx)| \;\leq\; D + \tfrac{\varepsilon}{4}$$

for all $h \in C$. To see this, let $h \in C$. For each $g \in E$, there is $h_g \in B$ with $d_{G,X}(h_g, gh) \leq \frac{\varepsilon}{8}$. Hence,

$$\begin{aligned} |f(hx) - f(ghx)| &\leq |f(hx) - f(h_g x)| + |f(h_g x) - f(ghx)| \\ &\leq \bigl(D + \tfrac{\varepsilon}{8}\bigr) + d_{G,X}(h_g, gh) \leq D + \tfrac{\varepsilon}{4} \end{aligned}$$

for all $g \in E$ and $f \in F$, as desired. Consequently, since F is contained in $\mathrm{Lip}_1(X, d)$,

$$\begin{aligned} &\int \sup_{g \in E} \sup_{f \in F} |f(hx) - f(ghx)| \, d\mu_i(h) \\ &\qquad\qquad = \int_C \sup_{g \in E} \sup_{f \in F} |f(hx) - f(ghx)| \, d\mu_i(h) + \delta \mu_i(G \setminus C) \\ &\qquad\qquad \leq \bigl(D + \tfrac{\varepsilon}{4}\bigr) + \tfrac{\varepsilon}{4} \leq D + \tfrac{\varepsilon}{2}. \end{aligned}$$

This proves $(**)$. By the G–invariance of ν along with Fubini's theorem,

$$\begin{aligned} \int \sup_{g \in E} \sup_{f \in F} |f(x) - f(gx)| \, d\nu(x) &= \iint \sup_{g \in E} \sup_{f \in F} |f(hx) - f(ghx)| \, d\nu(x) \, d\mu_i(h) \\ &= \iint \sup_{g \in E} \sup_{f \in F} |f(hx) - f(ghx)| \, d\mu_i(h) \, d\nu(x) \\ &\overset{(**)}{\leq} D + \tfrac{\varepsilon}{2}, \end{aligned}$$

and therefore

$$\int \sup_{g \in E} d(x, gx) \, d\nu(x) \leq \int \sup_{g \in E} \sup_{f \in F} |f(x) - f(gx)| \, d\nu(x) + \tfrac{\varepsilon}{2} \leq D + \varepsilon. \qquad \square$$

Proof of Corollary 1.3 Let X be a G–flow. Fix a continuous pseudometric d on X and let

$$D := \sup_{\alpha>0} \liminf_{i\to I} \sup_{x\in X} \operatorname{ObsDiam}(G, d_{G,x}, \mu_i; -\alpha).$$

Since G is amenable due to Theorem 3.2, there exists a G–invariant regular Borel probability measure ν on X. Note that, for every finite subset $E \subseteq G$ and every $\varepsilon > 0$, there exists some $x \in X$ with $\sup_{g\in E} d(x, gx) < D + \varepsilon$. Otherwise, there would exist a finite subset $E \subseteq G$ and some $\varepsilon > 0$ such that $\sup_{g\in E} d(x, gx) \geq D + \varepsilon$ for all $x \in X$, whence

$$\int \sup_{g\in E} d(x, gx)\, d\nu(x) \geq D + \varepsilon,$$

contradicting the conclusion of Theorem 1.2. Appealing to the compactness of X, hence we deduce the existence of a point $x \in X$ with $\sup_{g\in G} d(x, gx) \leq D$. □

We conclude this section with some remarks about further consequences of Theorem 1.2. For this purpose, we briefly clarify the connection between observable diameters and the Lévy property in uniform spaces, cf [20, Definition 2.6].

Definition 5.3 Let X be a uniform space. For an entourage U of X, let

$$U[A] := \{y \in X \mid \text{there exists an } x \in A \text{ with } (x, y) \in U\} \quad \text{for } A \subseteq X.$$

A net $(\mu_i)_{i\in I}$ of Borel probability measures on X is called a *Lévy net* in X if, for every family $(B_i)_{i\in I}$ of Borel subsets of X and any open entourage U of X,

$$\liminf_{i\to I} \mu_i(B_i) > 0 \implies \lim_{i\to I} \mu_i(U[B_i]) = 1.$$

Let us recall that every measurable real-valued function $f\colon X \to \mathbb{R}$ on a probability measure space $(X, \mathscr{B}, \mu)$ admits a (not necessarily unique) *median*, ie a real number $m \in \mathbb{R}$ with

$$\mu(\{x \in X \mid f(x) \geq m\}) \geq \tfrac{1}{2} \leq \mu(\{x \in X \mid f(x) \leq m\}).$$

We will need the following well-known fact.

Lemma 5.4 [13, Definition 2.5] *Let X be a uniform space, let d be a uniformly continuous pseudometric on X, and let $(\mu_i)_{i\in I}$ be a Lévy net of Borel probability* *measures on X. For each pair $(i, f) \in I \times \mathrm{Lip}_1(X, d)$, let $m_i(f)$ be a median of f with respect to μ_i. For every $\varepsilon > 0$,*

$$\sup_{f \in \mathrm{Lip}_1(X,d)} \mu_i(\{x \in X \mid |f(x) - m_i(f)| > \varepsilon\}) \to 0 \quad \text{as } i \to I.$$

Proof We include the proof for the sake of convenience. Let $\varepsilon > 0$. Since d is uniformly continuous, there is a symmetric open entourage U of X such that $d(x, y) \le \varepsilon$ for all $(x, y) \in U$. For all $(i, f) \in I \times \mathrm{Lip}_1(X, d)$, we conclude that $U[A_i(f)] \subseteq B_i(f)$ and $U[A'_i(f)] \subseteq B'_i(f)$, where

$$A_i(f) := \{x \in X \mid f(x) \le m_i(f)\}, \quad B_i(f) := \{x \in X \mid f(x) \le m_i(f) + \varepsilon\},$$
$$A'_i(f) := \{x \in X \mid f(x) \ge m_i(f)\}, \quad B'_i(f) := \{x \in X \mid f(x) \ge m_i(f) - \varepsilon\}.$$

Hence, $U[A_i(f)] \cap U[A'_i(f)] \subseteq B_i(f) \cap B'_i(f)$ for all $(i, f) \in i \times \mathrm{Lip}_1(X, d)$. We will show that

$$\sup_{f \in \mathrm{Lip}_1(X,d)} \mu_i\big(X \setminus (B_i(f) \cap B'_i(f))\big) \to 0 \quad \text{as } i \to I.$$

Let $\delta > 0$. For each pair $(i, f) \in I \times \mathrm{Lip}_1(X, d)$, our hypothesis on $m_i(f)$ asserts that

$$\min\{\mu_i(A_i(f)), \mu_i(A'_i(f))\} \ge \tfrac{1}{2}.$$

Since $(\mu_i)_{i \in I}$ is a Lévy net in X, there exists $i_0 \in I$ such that for all $i \in I$ with $i \ge i_0$ and for all $f \in \mathrm{Lip}_1(X, d)$,

$$\mu_i(U[A_i(f)]) \ge 1 - \tfrac{\delta}{2}.$$

Otherwise, the subset

$$\big\{i \in I \mid \text{there exists an } f \in \mathrm{Lip}_1(X, d) \text{ such that } \mu_i(U[A_i(f)]) < 1 - \tfrac{\delta}{2}\big\}$$

would be cofinal in I, which is easily seen to contradict the Lévy property. Likewise, there exists some $i_1 \in I$ with $i_1 \ge i_0$ such that for all $i \in I$ with $i \ge i_1$ and for all $f \in \mathrm{Lip}_1(X, d)$,

$$\mu_i(U[A'_i(f)]) \ge 1 - \tfrac{\delta}{2}.$$

Consequently, if $i \in I$ with $i \ge i_1$, then

$$\mu_i(B_i(f) \cap B'_i(f)) \ge \mu_i(U[A_i(f)] \cap U[A'_i(f)]) \ge 1 - \delta$$

for every $f \in \mathrm{Lip}_1(X, d)$, which means that

$$\sup_{f \in \mathrm{Lip}_1(X,d)} \mu_i\big(X \setminus (B_i(f) \cap B'_i(f))\big) \le \delta. \qquad \square$$

The following is a fairly well-known fact about mm–spaces (see eg [28, Proposition 5.7]) adapted to uniform spaces in a straightforward manner.

Proposition 5.5 *Let $(\mu_i)_{i\in I}$ be a net of Borel probability measures on a uniform space X. The following are equivalent:*

(1) *$(\mu_i)_{i\in I}$ is a Lévy net in X.*

(2) *For every uniformly continuous pseudometric d on X and every $\alpha > 0$,*

$$\lim_{i\to I} \operatorname{ObsDiam}(X, d, \mu_i; -\alpha) = 0.$$

(3) *For every bounded uniformly continuous pseudometric d on X and every $\alpha > 0$,*

$$\lim_{i\to I} \operatorname{ObsDiam}(X, d, \mu_i; -\alpha) = 0.$$

Proof (1) $\Longrightarrow$ (2) Consider a uniformly continuous pseudometric d on X. For every $i \in I$ and $f \in \operatorname{Lip}_1(X, d)$, let $m_i(f)$ be a median of f with respect to μ_i. Let $\alpha > 0$. We show that

$$\lim_{i\to I} \operatorname{ObsDiam}(X, d, \mu_i; -\alpha) = 0.$$

Let $\varepsilon > 0$. By (1) and Lemma 5.4, there exists $i_0 \in I$ such that for all $i \in I$ with $i \geq i_0$,

$$(*) \qquad \sup_{f\in\operatorname{Lip}_1(X,d)} \mu_i\big(\big\{x \in X \;\big|\; |f(x) - m_i(f)| \geq \tfrac{\varepsilon}{2}\big\}\big) \leq \alpha.$$

We argue that for all $i \in I$ with $i \geq i_0$,

$$\operatorname{ObsDiam}(X, d, \mu_i; -\alpha) \leq \varepsilon.$$

Let $i \in I$ where $i \geq i_0$. If $f \in \operatorname{Lip}_1(X, d)$, then $B_i(f) := B_{d_{\mathbb{R}}}\big(m_i(f), \frac{\varepsilon}{2}\big)$ is a Borel subset of $\mathbb{R}$ with $\operatorname{diam}(B_i(f), d_{\mathbb{R}}) \leq \varepsilon$ and

$$f_*(\mu_i)(B_i(f)) = \mu_i\big(\big\{x \in X \;\big|\; |f(x) - m_i(f)| < \tfrac{\varepsilon}{2}\big\}\big) \geq 1 - \alpha$$

thanks to $(*)$, wherefore $\operatorname{PartDiam}(f_*(\mu_i), 1 - \alpha) \leq \varepsilon$. This completes the argument.

(2) $\Longrightarrow$ (3) Trivial.

(3) $\Longrightarrow$ (1) Let U be an open entourage of X. According to classical work of Weil [34], there exist $\delta > 0$ and a uniformly continuous pseudometric d on X with $\operatorname{diam}(X, d) \leq 1$ such that $\{(x, y) \in X \times X \mid d(x, y) < \delta\} \subseteq U$. Consider a family $(B_i)_{i\in I}$ of Borel subsets of X such that $\sigma := \liminf_{i\to I} \mu_i(B_i) > 0$. In order to verify

that $\lim_{i\to I}\mu_i(U[B_i])=1$, let $\varepsilon>0$. Fix any $i_0\in I$ with $\inf\{\mu_i(B_i)\mid i\in I,\ i\geq i_0\}>\frac{\sigma}{2}$. By (3), there is an $i_1\in I$ with $i_1\geq i_0$ such that for all $i\in I$ with $i\geq i_1$,

$$(**)\qquad \operatorname{ObsDiam}\big(X,d,\mu_i;-\min\{\varepsilon,\tfrac{\sigma}{2}\}\big)<\delta.$$

We argue that for all $i\in I$ with $i\geq i_1$,

$$\mu_i(U[B_i])\geq 1-\varepsilon.$$

To this end, let $i\in I$ with $i\geq i_1$. Note that $f_i\colon X\to\mathbb{R},\ x\mapsto\inf\{d(x,y)\mid y\in B_i\}$ belongs to $\mathrm{Lip}_1(X,d)$. Hence, by $(**)$, there exists a Borel subset $C_i\subseteq X$ with

$$\mu_i(C_i)\geq 1-\min\{\varepsilon,\tfrac{\sigma}{2}\}\quad\text{and}\quad\operatorname{diam}(f_i(C_i),d_{\mathbb{R}})<\delta.$$

The former implies that $\mu_i(B_i\cap C_i)>0$ and thus $B_i\cap C_i\neq\varnothing$, wherefore $0\in f_i(C_i)$. Since $\operatorname{diam}(f_i(C_i),d_{\mathbb{R}})<\delta$, we now conclude that $f_i(C_i)\subseteq(-\delta,\delta)$ and hence $C_i\subseteq B_d(B_i,\delta)\subseteq U[B_i]$. It follows that $\mu_i(U[B_i])\geq\mu_i(C_i)\geq 1-\varepsilon$, as desired. □

As any bounded right-uniformly continuous pseudometric on a topological group is bounded from above by a bounded continuous right-invariant pseudometric, we arrive at the following characterization of the Lévy property on topological groups (with their right uniformity).

Corollary 5.6 *Let $(\mu_i)_{i\in I}$ be a net of Borel probability measures on a topological group G. The following are equivalent:*

(1) *$(\mu_i)_{i\in I}$ is a Lévy net in G.*

(2) *For every continuous right-invariant pseudometric d on G and every $\alpha>0$,*

$$\lim_{i\to I}\operatorname{ObsDiam}(G,d,\mu_i;-\alpha)=0.$$

(3) *For every bounded continuous right-invariant pseudometric d on G and every $\alpha>0$,*

$$\lim_{i\to I}\operatorname{ObsDiam}(G,d,\mu_i;-\alpha)=0.$$

In view of Corollary 5.6, let us point out two consequences of our results. For one thing, Corollary 1.3 yields a quantitative version of [13, Theorem 7.1], ie the extreme amenability of Lévy groups. And for another thing, Theorem 1.2 readily implies [24, Theorem 3.9], an extension of the result for Polish groups by Pestov [23, Theorem 5.7] generalizing earlier work of Glasner, Tsirelson and Weiss [10, Theorem 1.1].

Corollary 5.7 [24, Theorem 3.9] *If a topological group G admits a Lévy net of Borel probability measures UEB–converging to invariance over G, then G is whirly amenable, ie*

- *G is amenable, and*
- *every invariant regular Borel probability measure on a G–flow is supported on the set of fixed points.*

Proof The amenability of G is due to Theorem 3.2, while the second assertion follows immediately from Theorem 1.2 combined with Corollary 5.6 and Remark 5.2. □

Let us finish with an open problem.

Remark 5.8 Since the Lévy property can be stated in the more general framework of uniform spaces, it would be very interesting to know if Gromov's concentration topology admits an equally natural extension in that context. If so, then one may hope to generalize Theorem 1.1 to the case of topological groups with nonmetrizable universal minimal flow.

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