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# The higher Morita category of $\mathbb{E}_n$–algebras

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We introduce simple models for associative algebras and bimodules in the context of nonsymmetric $\infty$–operads, and use these to construct an $(\infty,2)$–category of associative algebras, bimodules and bimodule homomorphisms in a monoidal $\infty$–category. By working with $\infty$–operads over $\mathbf{\Delta}^{n,\mathrm{op}}$ we iterate these definitions and generalize our construction to get an $(\infty,n+1)$–category of $\mathbb{E}_n$–algebras and iterated bimodules in an $\mathbb{E}_n$–monoidal $\infty$–category. Moreover, we show that if $\mathcal{C}$ is an $\mathbb{E}_{n+k}$–monoidal $\infty$–category then the $(\infty,n+1)$–category of $\mathbb{E}_n$–algebras in $\mathcal{C}$ has a natural $\mathbb{E}_k$–monoidal structure. We also identify the mapping $(\infty,n)$–categories between two $\mathbb{E}_n$–algebras, which allows us to define interesting nonconnective deloopings of the Brauer space of a commutative ring spectrum.

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## 1 Introduction

The goal of this paper is to construct higher categories of $\mathbb{E}_n$–algebras and their iterated bimodules, using a completely algebraic or combinatorial approach to these objects, and establish some of their basic properties. Our construction is motivated by the interesting connections of these higher categories to topological quantum field theories, and a notion of "higher Brauer groups" that can be extracted from them. We will discuss these potential applications, both of which we intend to explore further in future work, after summarizing the main results of the present paper.

### 1.1 Summary of results

If $\boldsymbol{C}$ is a monoidal category, then the associative algebra objects[1] in $\boldsymbol{C}$ and their bimodules form a bicategory $\mathfrak{Alg}_1(\boldsymbol{C})$. More precisely, this bicategory has

- associative algebras in $\boldsymbol{C}$ as objects,
- $A$–$B$–bimodules in $\boldsymbol{C}$ as 1–morphisms from $A$ to $B$,
- bimodule homomorphisms as 2–morphisms,

[1] Also commonly called associative monoids, but we will reserve the term monoid for the case when the tensor product in $\boldsymbol{C}$ is the cartesian product.

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with composition of 1–morphisms given by taking tensor products: if $M$ is an $A$–$B$–bimodule and $N$ is a $B$–$C$–bimodule then their composite is $M \otimes_B N$ with its natural $A$–$C$–bimodule structure. Moreover, if $\boldsymbol{C}$ is a symmetric monoidal category, such as $\mathrm{Mod}_R$ for $R$ a commutative ring, then $\mathfrak{Alg}_1(\boldsymbol{C})$ inherits a symmetric monoidal structure.[2] When $R$ is a commutative ring, this symmetric monoidal bicategory $\mathfrak{Alg}_1(\mathrm{Mod}_R)$ organizes a wealth of interesting algebraic information — for example, two $R$–algebras are equivalent in $\mathfrak{Alg}_1(\mathrm{Mod}_R)$ precisely when they are Morita equivalent, ie have equivalent categories of modules.

Since all the concepts involved have derived analogues, it is reasonable to expect that there is a derived or higher-categorical version of the bicategory $\mathfrak{Alg}_1(\mathrm{Mod}_R)$, based on chain complexes of $R$–modules up to quasi-isomorphism. More generally, it should be possible to allow $R$ to be a differential graded algebra — or even a ring spectrum, with chain complexes replaced by $R$–modules in spectra up to stable weak equivalence.

In this paper we will indeed construct such generalizations of the bicategory of algebras and bimodules. However, the coherence issues that must be solved to define these seem intractable from the point of view of classical (enriched) category theory. To avoid this problem, we instead work in the setting of $\infty$–categories.

Roughly speaking, an *$\infty$–category* (or *$(\infty,1)$–category*) is a structure that has objects and morphisms like a category, but also "homotopies" (or invertible 2–morphisms) between morphisms, "homotopies between homotopies" (or invertible 3–morphisms), and so on. The morphisms can be composed, but the composition is not strictly associative, only associative up to a coherent choice of (higher) homotopies. Using homotopy theory there are a number of ways of making this idea precise in such a way that one can actually work with the resulting structures; we will make use of the theory of *quasicategories* as developed by Joyal and Lurie [28], which is by far the best-developed variant.

Similarly, one can consider *$(\infty,n)$–categories* for $n > 1$; these have $i$–morphisms for all $i$ that are required to be invertible when $i > n$, and are thus the "$\infty$–version" of $n$–categories. We will encounter them in the guise of Barwick's *$n$–fold Segal spaces* [9], which we will review below in Section 3.3.

In this higher-categorical setting there is a natural notion of a *monoidal* $\infty$–category, ie an $\infty$–category equipped with a tensor product that is associative up to coherent

[2] Although it is intuitively clear that the tensor product on $\boldsymbol{C}$ induces such a symmetric monoidal structure, this seems to have been completely defined only quite recently by Shulman [39], following a construction of a braided monoidal structure by Garner and Gurski [15]. Considering the difficulty of even defining symmetric monoidal bicategories in full generality, this is perhaps not entirely unsurprising — see Schommer-Pries [37, Section 2.1] for a discussion of the history of such definitions.

homotopy. Our first main result, which we will prove in Section 4, is a construction of an $(\infty,2)$–category $\mathfrak{Alg}_1(\mathcal{C})$ of algebras, bimodules, and bimodule homomorphisms in any monoidal $\infty$–category $\mathcal{C}$ that satisfies some mild technical assumptions.

In the $\infty$–categorical setting it is also natural to ask how this structure extends to $\mathbb{E}_n$–*algebras*. In the context of ordinary categories, an object equipped with two compatible associative multiplications is a commutative algebra. When we pass to higher categories, however, this is no longer true. The most familiar example of this phenomenon is iterated algebras in the 2–category of categories — if we consider associative algebras in the appropriate 2–categorical sense, these are monoidal categories; categories with two compatible monoidal structures are then braided monoidal categories, and ones with three or more monoidal structures are symmetric monoidal categories. In general, objects with $k$ compatible associative algebra structures in an $n$–category are commutative algebras for $k > n$ — this is a form of the Baez–Dolan stabilization hypothesis;[3] in other words, in an $n$–category compatible associative algebra structures give $n+1$ different algebraic structures. For an $\infty$–category, then, objects equipped with multiple compatible multiplications give an infinite sequence of algebraic structures lying between associative and commutative algebras, namely the $\mathbb{E}_n$–algebras for $n = 1, 2, \ldots$.[4] In particular, we can consider $\mathbb{E}_n$–algebras in the $\infty$–category $\mathrm{Cat}_\infty$ of $\infty$–categories, which gives the notion of $\mathbb{E}_n$–*monoidal* $\infty$–*categories*, ie $\infty$–categories equipped with $n$ compatible tensor products.

The general version of our first main result, which we will prove in Section 5.3, is then a construction of $(\infty,n+1)$–categories of $\mathbb{E}_n$–algebras in any nice $\mathbb{E}_n$–monoidal $\infty$–category:

**Theorem 1.1** *Let $\mathcal{C}$ be a nice $\mathbb{E}_n$–monoidal $\infty$–category. Then there exists an $(\infty,n+1)$–category $\mathfrak{Alg}_n(\mathcal{C})$ whose objects are $\mathbb{E}_n$–algebras in $\mathcal{C}$, with 1–morphisms given by $\mathbb{E}_{n-1}$–algebras in bimodules in $\mathcal{C}$, 2–morphisms by $\mathbb{E}_{n-2}$–algebras in bimodules in bimodules in $\mathcal{C}$, and so forth.*

Here the precise meaning of "nice" amounts to the existence of well-behaved relative tensor products over algebras in $\mathcal{C}$, which is needed to have well-defined compositions in these higher categories. For example, we can take $\mathcal{C}$ to be the (symmetric monoidal) $\infty$–category $\mathrm{Mod}_R$ of modules over a commutative ring spectrum $R$ or the "derived $\infty$–category" $\mathcal{D}_\infty(R)$ of modules over an associative ring $R$, obtained by inverting the quasi-isomorphisms in the category of chain complexes of $R$–modules. (More

[3] See Lurie [31, Corollary 5.1.1.7] for a proof of this statement.

[4] The Dunn–Lurie additivity theorem [31, Theorem 5.1.2.2] says that this iterative definition agrees with the classical definition in terms of configuration spaces of little discs in $\mathbb{R}^n$.

generally, we can consider the analogous localization of the category of dg-modules over a dg-algebra $R$.)

If $\mathcal{C}$ is a symmetric monoidal $\infty$–category, we will also show that $\mathfrak{Alg}_n(\mathcal{C})$ inherits a symmetric monoidal structure. More precisely, our second main result (proved in Section 5.4) is as follows:

**Theorem 1.2** *If $\mathcal{C}$ is a nice $\mathbb{E}_{m+n}$–monoidal $\infty$–category, then the $(\infty, n+1)$–category $\mathfrak{Alg}_n(\mathcal{C})$ inherits a natural $\mathbb{E}_m$–monoidal structure.*

Finally, our third main result, which we prove in Section 5.5, explains how the $(\infty, n+1)$–categories $\mathfrak{Alg}_n(\mathcal{C})$ are related for different $n$:

**Theorem 1.3** *Suppose $\mathcal{C}$ is a nice $\mathbb{E}_n$–monoidal $\infty$–category. Then for any $\mathbb{E}_n$–algebras $A$ and $B$ in $\mathcal{C}$, the $(\infty, n)$–category $\mathfrak{Alg}_n(\mathcal{C})(A, B)$ of maps from $A$ to $B$ is equivalent to $\mathfrak{Alg}_{n-1}(\mathrm{Bimod}_{A,B}(\mathcal{C}))$, where $\mathrm{Bimod}_{A,B}(\mathcal{C})$ is the $\infty$–category of $A$–$B$–bimodules in $\mathcal{C}$ equipped with a natural $\mathbb{E}_{n-1}$–monoidal structure. In particular, if $I$ is the unit of the monoidal structure then $\mathfrak{Alg}_n(\mathcal{C})(I, I) \simeq \mathfrak{Alg}_{n-1}(\mathcal{C})$.*

## 1.2 Higher Brauer groups

If $\boldsymbol{C}$ is a symmetric monoidal category, we say that an object $X \in \boldsymbol{C}$ is *invertible* if there exists another object $X^{-1}$ such that $X \otimes X^{-1}$ is isomorphic to the identity; by considering the homotopy 1–category this gives a notion of invertible objects in any symmetric monoidal $(\infty, n)$–category.

In particular, if $R$ is a commutative ring then the invertible objects of $\mathfrak{Alg}_1(\mathrm{Mod}_R)$ are those associative $R$–algebras $A$ that have an inverse $A^{-1}$ in the sense that $A \otimes_R A^{-1}$ is Morita equivalent to $R$ — these are precisely the *Azumaya algebras* over $R$. By considering these invertible objects and the invertible 1– and 2–morphisms between them we obtain a symmetric monoidal 2–groupoid $\mathfrak{Br}_1(R)$ with very interesting homotopy groups:

- $\pi_0\mathfrak{Br}_1(R)$, ie the set of isomorphism classes of objects in $\mathfrak{Br}_1(R)$, is the classical Brauer group of Azumaya $R$–algebras.
- $\pi_1\mathfrak{Br}_1(R)$ is the Picard group of invertible $R$–modules.
- $\pi_2\mathfrak{Br}_1(R)$ is the group $R^\times$ of multiplicative units in $R$.

Moreover, the "loop space" $\Omega\mathfrak{Br}_1(R) = \mathfrak{Br}_1(R)(R, R)$ is the *Picard groupoid* of invertible $R$–modules and isomorphisms.

Using the results of this paper, we can also consider the invertible objects in $\mathfrak{Alg}_n(\mathcal{C})$ for any suitable symmetric monoidal $\infty$–category $\mathcal{C}$. Restricting to the invertible $i$–morphisms between these for all $i$, we get a symmetric monoidal $\infty$–groupoid $\mathfrak{Br}_n(\mathcal{C})$,

or equivalently an $\mathbb{E}_\infty$–space; we will call this the *$n$–Brauer space* of $\mathcal{C}$. It is evident from the definition of the invertible objects that this $\mathbb{E}_\infty$–space is *grouplike*, ie the induced multiplication on $\pi_0\mathfrak{Br}_n(\mathcal{C})$ makes this monoid a group, and so it corresponds to a connective spectrum.

It follows immediately from Theorem 1.3 that the loop space $\Omega\mathfrak{Br}_n(\mathcal{C})$ is equivalent to $\mathfrak{Br}_{n-1}(\mathcal{C})$. Thus the $n$–Brauer spaces $\mathfrak{Br}_n(\mathcal{C})$ are a sequence of deloopings, and so we can combine these spaces into a nonconnective "Brauer spectrum" $\mathfrak{BR}(\mathcal{C})$ with

$$\pi_{-k}\mathfrak{BR}(\mathcal{C}) = \pi_{n-k}\mathfrak{Br}_n(\mathcal{C}) \quad \text{for } n \geq k.$$

If $R$ is a commutative ring spectrum, the "$n$–Brauer groups"

$$\mathrm{Br}_n(R) := \pi_{-n}\mathfrak{BR}(\mathrm{Mod}_R) = \pi_0\mathfrak{Br}_n(\mathrm{Mod}_R)$$

can be thought of as consisting of the $\mathbb{E}_n$–analogues of (derived) Azumaya algebras, considered up to an $\mathbb{E}_n$–variant of Morita equivalence. In particular:

- For $n = 1$ we recover the Brauer groups of commutative ring spectra and the derived Brauer groups of commutative rings, as studied by Toën [41], Szymik [40], Baker, Richter and Szymik [8], Antieau and Gepner [2] and others.
- For $n = 0$ we recover the Picard group of invertible $R$–modules, as studied by Hopkins, Mahowald and Sadofsky [20], May [33], Mathew and Stojanoska [32] and others.

The "negative Brauer groups" (ie the positive homotopy groups of $\mathfrak{BR}(\mathrm{Mod}_R)$) are also easy to describe: for $* < 0$ we get the homotopy groups of the units of $R$, ie $\mathrm{Br}_*(R) = \pi_{1-*}(\Omega^\infty R^\times)$, where $\Omega^\infty R^\times$ denotes the components of $\Omega^\infty R$ lying over the units in $\pi_0 R$; for $* < -1$ we thus have $\mathrm{Br}_*(R) = \pi_{1-*}(R)$.

A fascinating question for future research is whether the spaces $\mathfrak{Br}_n(R)$ for $R$ a (connective) commutative ring spectrum satisfy étale descent in the same way as the Brauer spaces $\mathfrak{Br}_1(R)$ (as proved by Toën [41] and Antieau and Gepner [2]).

If the étale-local triviality results of the same authors for $\mathfrak{Br}_1(R)$ also extend to $n > 1$, it should be possible to use the resulting descent spectral sequence to compute the higher Brauer groups in some simple cases. In fact, this would imply that the higher Brauer groups are closely related to étale cohomology; generalizing the known results for $n = 1$ and 0 one might optimistically conjecture that in general

$$\mathrm{Br}_n(R) \cong H^n_{\text{ét}}(R;\mathbb{Z}) \times H^{n+1}_{\text{ét}}(R;\mathbb{G}_m),$$

where the first factor occurs since we are considering nonconnective $R$–modules (or chain complexes of $R$–modules that are not required to be 0 in negative degrees).

## 1.3 Topological quantum field theories

*Topological quantum field theories* (or *TQFTs*) were introduced by Atiyah [3] as a way of formalizing mathematically some particularly simple examples of quantum field theories constructed by Witten. The original definition is quite easy to state:

**Definition 1.4** Let $\mathrm{Bord}(n)$ be the category with objects closed $(n-1)$–manifolds and morphisms (diffeomorphism classes of) $n$–dimensional cobordisms between these (thus a morphism from $M$ to $N$ is an $(n+1)$–manifold with boundary $B$, with an identification of $\partial B$ with $M \amalg N$). The disjoint union of manifolds gives a symmetric monoidal structure on $\mathrm{Bord}(n)$, and an *$n$–dimensional topological quantum field theory* valued in a symmetric monoidal category $\boldsymbol{C}$ is a symmetric monoidal functor $\mathrm{Bord}(n) \to \boldsymbol{C}$.

Requiring the manifolds and cobordisms to be equipped with various structures, such as orientations or framings, gives different variants of the category $\mathrm{Bord}(n)$. We get various flavours of TQFTs, such as oriented or framed TQFTs, by considering these different versions of $\mathrm{Bord}(n)$. In examples the category $\boldsymbol{C}$ is usually the category $\mathrm{Vect}_{\mathbb{C}}$ of complex vector spaces.

One reason mathematicians became interested in TQFTs is that they lead to interesting invariants of manifolds: if $\mathcal{Z}\colon \mathrm{Bord}(n) \to \mathrm{Vect}_{\mathbb{C}}$ is an $n$–dimensional TQFT, then $\mathcal{Z}$ assigns a complex number to any closed $n$–manifold $M$ — we can consider $M$ as a cobordism from the empty set to the empty set, and since this is the unit of the monoidal structure on $\mathrm{Bord}(n)$, $\mathcal{Z}(M)$ is a linear map $\mathbb{C} \to \mathbb{C}$, which is given by multiplication with a complex number.

To compute the number $\mathcal{Z}(M)$ we can cut $M$ along suitable submanifolds of codimension 1 and use the functoriality of $\mathcal{Z}$. This is enough to compute these invariants in very low dimensions ($n \leq 2$). In higher dimensions, however, we would like to be able to cut our manifolds in more flexible ways, for example by choosing a triangulation of $M$, to make the invariants more computable. This led mathematicians to consider the notion of *extended* topological quantum field theories; this was formalized by Baez and Dolan [7] in the language of *$n$–categories* (building on earlier work by Freed [14] and Lawrence [26], among others).

**Remark 1.5** For the definition of Baez and Dolan we consider an $n$–category $\mathrm{Bord}_n$ whose objects are compact 0–manifolds, with morphisms given by 1–dimensional cobordisms between 0–manifolds, and in general $i$–morphisms for $i = 1, \dots, n$ given by $i$–dimensional cobordisms between manifolds with corners. (For the $n$–morphisms we take diffeomorphism classes of these.) The disjoint union should equip this with

a symmetric monoidal structure, but giving a precise definition of this symmetric monoidal $n$–category becomes increasingly intractable as $n$ increases. A complete definition has been given by Schommer-Pries [37] in the case $n = 2$, but for larger $n$ it seems that an appropriate notion of symmetric monoidal $n$–category has not even been defined.

**Definition 1.6** Given such a symmetric monoidal $n$–category $\mathrm{Bord}_n$, an *$n$–dimensional extended TQFT* valued in a symmetric monoidal $n$–category $\mathcal{C}$ is a symmetric monoidal functor $\mathrm{Bord}_n \to \mathcal{C}$. As before, considering various structures on the manifolds in $\mathrm{Bord}_n$ gives different flavours of field theories, such as framed, oriented and unoriented.

Baez and Dolan also conjectured that there is a simple classification of *framed* extended topological quantum field theories:

**Conjecture 1.7** (Cobordism Hypothesis) A framed extended TQFT $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$ is classified by the object $\mathcal{Z}(*) \in \mathcal{C}$. Moreover, the objects of $\mathcal{C}$ that correspond to framed TQFTs admit a simple algebraic description: they are precisely the *$n$–dualizable objects*. (We refer to Lurie [30, Section 2.3] for a precise definition of $n$–dualizable objects.)

At the time, however, the foundations for higher category theory required to realize their ideas did not yet exist. The necessary foundations have only been developed during the past decade, with the work of Barwick, Bergner, Joyal, Lurie, Rezk and many others. The resulting theory of *$(\infty, n)$–categories* is often easier to work with than the more restricted notion of $n$–category — in particular, it is not hard to give a good definition of symmetric monoidal $(\infty, n)$–categories for arbitrary $n$.

We can then consider an $(\infty, n)$–category $\mathrm{Bord}_{(\infty,n)}$ of cobordisms, where we take diffeomorphisms as our $(n+1)$–morphisms, smooth homotopies as the $(n+2)$–morphisms, and so on. This also turns out to be much easier to define than the analogous $n$–category; a sketch of a definition is given in [30], and the full details of the construction have recently been worked out by Calaque and Scheimbauer [11].

It is then natural to define extended TQFTs valued in a symmetric monoidal $(\infty, n)$–category as symmetric monoidal functors from $\mathrm{Bord}_{(\infty,n)}$. In this more general setting, Lurie was able to prove the cobordism hypothesis (although so far only a detailed sketch [30] of the proof has appeared). In fact, Lurie also proves classification theorems for other flavours of TQFTs, such as oriented or unoriented ones, in terms of the homotopy fixed points for an action of the orthogonal group $O(n)$ on the space of $n$–dualizable objects in any symmetric monoidal $(\infty, n)$–category.

The cobordism hypothesis works for an arbitrary symmetric monoidal $(\infty,n)$–category, and so leaves open the question of what the appropriate target is for the interesting field theories that arise in physics and geometry. Motivation from physics (see Freed [14] and Kapustin [25]) suggests that in general a TQFT should assign an $(n-k-1)$–category enriched in vector spaces, or more generally in chain complexes of vector spaces, to a closed $k$–manifold.[5]

The higher category of $\mathbb{E}_n$–algebras and iterated bimodules we will construct here can be considered as a special case of this general target: $\mathbb{E}_n$–algebras in some $\infty$–category $\mathcal{C}$ are the same thing as $(\infty,n)$–categories enriched in $\mathcal{C}$ that have one object, one 1–morphism, … and one $(n-1)$–morphism. In fact, it is possible to extend the definitions we consider here to get definitions of enriched $(\infty,n)$–categories and iterated bimodules between them; I hope to use these to construct an $(\infty,n+1)$–category of enriched $(\infty,n)$–categories in a sequel to this paper.

Although not completely general, the TQFTs valued in the symmetric monoidal $(\infty,n+1)$–category of $\mathbb{E}_n$–algebras are still very interesting. This situation is discussed in [30, Section 4.1], where the following results are stated without proof:

**Conjecture 1.8** (i) All $\mathbb{E}_n$–algebras in $\mathcal{C}$ are $n$–dualizable in $\mathfrak{Alg}_n(\mathcal{C})$, and so give rise to framed $n$–dimensional extended TQFTs. (More precisely, all objects of $\mathfrak{Alg}_n(\mathcal{C})$ are dualizable, and all $i$–morphisms have adjoints for $i=1,\dots,n-1$.)

(ii) The framed $n$–dimensional extended TQFT associated to an $\mathbb{E}_n$–algebra $A$ is given by the *factorization homology* or *topological chiral homology* of $A$. (These invariants were first introduced by Lurie [31, Section 5.5] and also independently by Andrade [1], and have since been extensively developed by a number of other authors, in particular Francis and collaborators; see for example Ayala, Francis and Rozenblyum [4], Ayala, Francis and Tanaka [5] and Francis [13]. An overview can also be found in Ginot's lecture notes [17].)

(iii) An $\mathbb{E}_n$–algebra $A$ is $(n+1)$–dualizable if and only if it is dualizable as a module over its $S^k$–factorization homology for all $k=-1,0,1,\dots,n-1$. (For $n=1$, this is equivalent to $A$ being *smooth* and *proper* — see [31, Section 4.6.4].)

Scheimbauer [36] has constructed factorization homology as an extended TQFT valued in a geometric variant of $\mathfrak{Alg}_n(\mathcal{C})$ (defined using locally constant factorization algebras on certain stratifications of $\mathbb{R}^n$), which confirms the first two parts of this conjecture. It follows from Theorem 1.3 that (i) is equivalent to the 1–morphisms in $\mathfrak{Alg}_n(\mathcal{C})$ having adjoints for all $n\geq 2$, and we hope to use this to give algebraic proofs of (i) and (iii).

[5] To nonclosed manifolds it should assign a higher-categorical generalization of the notion of a bimodule or profunctor between enriched categories, which is somewhat complicated to define.

## 1.4 Related work

As already mentioned, a geometric construction of $(\infty, n+1)$–categories closely related to $\mathfrak{Alg}_n(\mathcal{C})$ has been worked out by Scheimbauer [36]. However, the natural definition of bimodules in the factorization algebra setting is not quite the same as ours: the bimodules that arise from factorization algebras are *pointed*. If $\mathfrak{Alg}_n^{\mathrm{FA}}(\mathcal{C})$ denotes Scheimbauer's $(\infty, n+1)$–category of $\mathbb{E}_n$–algebras in $\mathcal{C}$, we therefore expect the relation to our work to be as follows:

**Conjecture 1.9** Let $\mathcal{C}$ be a nice $\mathbb{E}_n$–monoidal $\infty$–category. Then $\mathfrak{Alg}_n^{\mathrm{FA}}(\mathcal{C})$ is equivalent to $\mathfrak{Alg}_n(\mathcal{C}_{I/})$.

In order to carry out such a comparison, one would need to know that the iterated bimodules we consider can equivalently be described as algebras for $\infty$–operads of "little discs" on certain stratifications of $\mathbb{R}^n$ — this would be a generalization of the Dunn–Lurie additivity theorem for $\mathbb{E}_n$–algebras. Such a result appears to follow from forthcoming work of Ayala and Hepworth (extending their [6]); we hope to use this to compare the algebraic version of $\mathfrak{Alg}_n(\mathcal{C})$ we construct here to the factorization-algebra-based version of Scheimbauer in a sequel to this paper.

An alternative geometric construction of $\mathfrak{Alg}_n(\mathcal{C})$ is also part of unpublished work of Ayala, Francis and Rozenblyum, related to the construction sketched in the work of Morrison and Walker on the blob complex [34].

In the case $n = 1$, an alternative construction of the double $\infty$–categories $\mathfrak{Alg}_1(\mathcal{C})$ using symmetric $\infty$–operads can be extracted from [31, Section 4.4]. Indeed, many of the results in Section 4 are simply nonsymmetric variants of Lurie's — the main advantage of our setup is that our results generalize easily to $n > 1$.

A bicategory of dg-algebras and dg-bimodules, considered up to quasi-isomorphism, is discussed by Johnson [22]. This should be the homotopy bicategory of our $(\infty, 2)$–category of algebras and bimodules in the corresponding "derived $\infty$–category" of chain complexes.

Finally, an extension of our construction has been obtained by Johnson-Freyd and Scheimbauer: in [21] they show that given an $\mathbb{E}_k$–monoidal $(\infty, n)$–category $\mathcal{C}$, our construction (as well as that of Scheimbauer) can be used to obtain an $(\infty, n+k)$–category of $\mathbb{E}_k$–algebras in $\mathcal{C}$.

## 1.5 Overview

We begin by introducing our models for associative algebras, bimodules and their tensor products in Section 2; we discuss them here only in the context of cartesian monoidal

$\infty$–categories, ie ones where the monoidal structure is the cartesian product, as this allows us to clarify their underlying meaning without introducing the machinery of $\infty$–operads. Next, in Section 3 we discuss how iterating these definitions give models for $\mathbb{E}_n$–algebras and iterated bimodules, again in the cartesian setting. In Section 4 we then construct the $(\infty,2)$–categories $\mathfrak{Alg}_1(\mathcal{C})$ for $\mathcal{C}$ a general monoidal $\infty$–category, using nonsymmetric $\infty$–operads. By working with a notion of $\infty$–*operads over* $\mathbf{\Delta}^{n,\mathrm{op}}$ the technical results we prove for associative algebras turn out to extend to the setting of $\mathbb{E}_n$–algebras for $n>2$, and so in Section 5 we construct the $(\infty,n+1)$–categories $\mathfrak{Alg}_n(\mathcal{C})$ without much more work; we also consider the functoriality of these $(\infty,n+1)$–categories and their natural monoidal structures, and finish by identifying their mapping $(\infty,n)$–categories. Finally, in the appendix we discuss the technical results we need about $\mathbf{\Delta}^n$–$\infty$–operads; these are mostly straightforward variants of results from [31].

## 1.6 Notation and terminology

This paper is written in the language of $\infty$–categories, as developed in the guise of quasicategories in the work of Joyal [23] and Lurie [28; 31]. This means that terms such as "colimit", "Kan extension" and "commutative diagram" are used (unless otherwise specified) in their $\infty$–categorical (or "fully weak") senses — for example, a commutative diagram of shape $\mathcal{I}$ in an $\infty$–category $\mathcal{C}$ means a functor of $\infty$–categories $\mathcal{I}\to\mathcal{C}$, and thus means a diagram that commutes up to a coherent choice of (higher) homotopies that is specified by this diagram. In general, we reuse the notation and terminology used by Lurie [28; 31]; here are some exceptions and reminders:

- $\mathbf{\Delta}$ is the simplicial indexing category, with objects the nonempty finite totally ordered sets $[n]:=\{0,1,\dots,n\}$ and morphisms order-preserving functions between them. Similarly, $\mathbf{\Delta}_+$ denotes the augmented simplicial indexing category, which also includes the empty set $[-1]=\varnothing$.

- To avoid clutter, we write $\mathbf{\Delta}^n$ for the product $\mathbf{\Delta}^{\times n}$, and use $\mathbf{\Delta}^{n,\mathrm{op}}_{/I}$ to mean $((\mathbf{\Delta}^{\times n})_{/I})^{\mathrm{op}}$ for any $I\in\mathbf{\Delta}^n$.

- $\mathbf{\Gamma}^{\mathrm{op}}$ is the category of pointed finite sets.

- Generic categories are generally denoted by single capital boldface letters ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$) and generic $\infty$–categories by single calligraphic letters ($\mathcal{A},\mathcal{B},\mathcal{C}$). Specific categories and $\infty$–categories both get names in the normal text font.

- $\mathrm{Set}_\Delta$ is the category of simplicial sets, ie the category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathrm{Set})$ of set-valued presheaves on $\mathbf{\Delta}$.

- $\mathcal{S}$ is the $\infty$–category of spaces; this can be defined as the coherent nerve $N\mathrm{Set}_\Delta^\circ$ of the full subcategory $\mathrm{Set}_\Delta^\circ$ of the category $\mathrm{Set}_\Delta$ spanned by the Kan complexes, regarded as a simplicial category via the internal Hom.

- We make use of the theory of *Grothendieck universes* to allow us to define ($\infty$–)categories without being limited by set-theoretical size issues; specifically, we fix three nested universes, and refer to sets contained in them as *small*, *large* and *very large*. When $\mathcal{C}$ is an $\infty$–category of small objects of a certain type, we generally refer to the corresponding $\infty$–category of large objects as $\widehat{\mathcal{C}}$, without explicitly defining this object. For example, $\mathrm{Cat}_\infty$ is the (large) $\infty$–category of small $\infty$–categories, and $\widehat{\mathrm{Cat}}_\infty$ is the (very large) $\infty$–category of large $\infty$–categories.

- If $\mathcal{C}$ is an $\infty$–category, we write $\iota\mathcal{C}$ for the *interior* or *underlying space* of $\mathcal{C}$, ie the largest subspace of $\mathcal{C}$ that is a Kan complex.

- If a functor $f\colon \mathcal{C} \to \mathcal{D}$ (of $\infty$–categories) is left adjoint to a functor $g\colon \mathcal{D} \to \mathcal{C}$, we will refer to the adjunction as $f \dashv g$.

- We will say that a functor $f\colon \mathcal{C} \to \mathcal{D}$ of $\infty$–categories is *coinitial* if the opposite functor $f^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is cofinal in the sense of Lurie [28, Section 4.1.1].

- If $K$ is a simplicial set, we denote the cone points of the simplicial sets $K^{\triangleright}$ and $K^{\triangleleft}$, obtained by freely adjoining a final and an initial object to $K$, by $\infty$ and $-\infty$, respectively.

- We say an $\infty$–category (or more generally any simplicial set) $\mathcal{C}$ is *weakly contractible* if the map $\mathcal{C} \to \Delta^0$ is a weak equivalence in the Kan–Quillen model structure (as opposed to the Joyal model structure). This is equivalent to the $\infty$–groupoid obtained by inverting all the morphisms in $\mathcal{C}$ being trivial, and to the geometric realization of the simplicial set $\mathcal{C}$ being a contractible topological space.

## 1.7 Some key concepts

As an aid to readers who are not intimately familiar with [28], in this subsection we briefly introduce some key concepts that we will make use of throughout this paper, namely *cocartesian fibrations*, *cofinal functors*, and *relative (co)limits*.

**Definition 1.10** If $f\colon \mathcal{E} \to \mathcal{B}$ is a functor of $\infty$–categories, a morphism $\epsilon\colon e \to e'$ in $\mathcal{E}$ lying over $\beta\colon b \to b'$ in $\mathcal{B}$ is *$p$–cocartesian* if for every $x \in \mathcal{E}$ the commutative[6]

[6]Recall that this means commutative in the $\infty$–categorical sense, so the square really includes the data of a homotopy between the two composites, which we do not explicitly indicate.

square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(e', x) & \xrightarrow{\epsilon^*} & \mathrm{Map}_{\mathcal{E}}(e, x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{B}}(b', f(x)) & \xrightarrow[\beta^*]{} & \mathrm{Map}_{\mathcal{B}}(b, f(x)) \end{array}$$

is cartesian, ie it is a pullback[7] square.

This is equivalent to the induced map on fibres

$$\mathrm{Map}_{\mathcal{E}}(e', x)_f \to \mathrm{Map}_{\mathcal{E}}(e, x)_{\phi \circ \beta}$$

being an equivalence for all maps $\phi\colon b' \to f(x)$, so this definition gives a natural $\infty$–categorical generalization of cocartesian morphisms in ordinary category theory.

**Definition 1.11** We say that a functor of $\infty$–categories $f\colon \mathcal{E} \to \mathcal{B}$ is a *cocartesian fibration* if for every $e \in \mathcal{E}$ and $\beta\colon f(e) \to b$ there exists an $f$–cocartesian morphism $e \to \beta_! e$ over $\beta$; cocartesian fibrations are thus the natural $\infty$–categorical version of *Grothendieck opfibrations*.

If we think of $f$ as a map of simplicial sets, and assume (as we are free to do up to equivalence) that it is an inner fibration, then this definition can be reformulated more concretely in terms of the existence of liftings for certain horns, which is the definition given in [28, Section 2.4.2].

If $f\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration, then [28, Corollary 3.2.2.1] implies that the induced functor $\mathrm{Fun}(K, \mathcal{E}) \to \mathrm{Fun}(K, \mathcal{B})$ is also a cocartesian fibration for any $K$. Given diagrams $p\colon K \to \mathcal{E}$ and $\overline{q}\colon K^{\triangleright} \to \mathcal{B}$ with $f \circ p = q := \overline{q}|_K$ we can therefore define a *cocartesian pushforward* of $p$ to a diagram $p'\colon K \to \mathcal{E}_{\overline{q}(\infty)}$ lying in the fibre over $\overline{q}(\infty)$, by regarding $\overline{q}$ as a morphism in $\mathrm{Fun}(K, \mathcal{B})$ from $q$ to the constant functor at $\overline{q}(\infty)$ and choosing a cocartesian morphism over this with source $p$.

Grothendieck proved [18] that Grothendieck opfibrations over a category $\boldsymbol{C}$ correspond to (pseudo)functors from $\boldsymbol{C}$ to the category of categories. Lurie's *straightening equivalence* from [28, Section 3.2] establishes an analogous equivalence between cocartesian fibrations over an $\infty$–category $\mathcal{C}$ and functors from $\mathcal{C}$ to the $\infty$–category $\mathrm{Cat}_\infty$ of $\infty$–categories. For more details on cocartesian fibrations, and the dual concept of *cartesian* fibrations, see [28, Sections 2.4 and 3.2], especially Subsections 2.4.1–2.4.4.

[7] Note that this means that it is a pullback in the $\infty$–categorical sense — if we choose some concrete model for these mapping spaces as simplicial sets, this is equivalent to the corresponding diagram of simplicial sets being a *homotopy* pullback.

**Definition 1.12** A functor $F\colon \mathcal{A} \to \mathcal{B}$ of $\infty$–categories is *cofinal* if for every diagram $p\colon \mathcal{B} \to \mathcal{C}$, the induced functor $\mathcal{C}_{p/} \to \mathcal{C}_{p\circ F/}$ is an equivalence. Dually, $F$ is *coinitial* if $F^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is cofinal, ie the functor $\mathcal{C}_{/p} \to \mathcal{C}_{/p\circ F}$ is an equivalence for every $p$.

Since a colimit of $p$ is the same thing as a final object in $\mathcal{C}_{p/}$, we see that if $F$ is cofinal then $p$ has a colimit if and only if $p\circ F$ has a colimit, and these colimits are necessarily given by the same object in $\mathcal{C}$. The key criterion for cofinality is [28, Theorem 4.1.3.1]: $F\colon \mathcal{A} \to \mathcal{B}$ is cofinal if and only if for every $b \in \mathcal{B}$ the slice $\infty$–category $\mathcal{A}_{b/} := \mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{b/}$ is weakly contractible. For more details, see [28, Section 4.1], especially Subsection 4.1.1.

**Definition 1.13** Given a functor of $\infty$–categories $f\colon \mathcal{E} \to \mathcal{B}$ we say a diagram $\overline{p}\colon K^{\triangleright} \to \mathcal{E}$ is a *colimit relative to* $f$ (or an *$f$–colimit*) of $p := \overline{p}|_K$ if the commutative square of $\infty$–categories

$$\begin{array}{ccc} \mathcal{E}_{\overline{p}/} & \longrightarrow & \mathcal{B}_{f\overline{p}/} \\ \downarrow & & \downarrow \\ \mathcal{E}_{p/} & \longrightarrow & \mathcal{B}_{fp/} \end{array}$$

is cartesian, ie the induced functor

$$\mathcal{E}_{\overline{p}/} \to \mathcal{E}_{p/} \times_{\mathcal{B}_{fp/}} \mathcal{B}_{f\overline{p}/}$$

is an equivalence.

Ordinary colimits in $\mathcal{E}$ are the same thing as colimits relative to the functor $\mathcal{E} \to *$ to the terminal $\infty$–category. Notice also that if $\overline{p}\colon K^{\triangleright} \to \mathcal{E}$ is a diagram such that $f\overline{p}$ is a colimit in $\mathcal{B}$, then $\overline{p}$ is an $f$–colimit if and only if it is a colimit in $\mathcal{E}$.

We can also reformulate the definition in terms of mapping spaces: $\overline{p}\colon K^{\triangleright} \to \mathcal{E}$ is an $f$–colimit if and only if for every $e \in \mathcal{E}$, the commutative square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(\overline{p}(\infty), e) & \longrightarrow & \lim_{k\in K} \mathrm{Map}_{\mathcal{E}}(p(k), e) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{B}}(f\overline{p}(\infty), f(e)) & \longrightarrow & \lim_{k\in K} \mathrm{Map}_{\mathcal{E}}(fp(k), f(e)) \end{array}$$

is cartesian, or equivalently (since limits commute) if and only if for every map $\phi\colon f\overline{p}(\infty) \to f(e)$ the map on fibres

$$\mathrm{Map}_{\mathcal{E}}(\overline{p}(\infty), e)_{\phi} \to \lim_{k\in K} \mathrm{Map}_{\mathcal{E}}(p(k), e)_{\phi\circ(fp(k)\to f\overline{p}(\infty))}$$

is an equivalence.

If $f$ is a cocartesian fibration, then it follows from [28, Propositions 4.3.1.9 and 4.3.1.10] that a diagram $\overline{p}\colon K^{\triangleright} \to \mathcal{E}$ with $x = f\,\overline{p}(\infty)$ is an $f$–colimit if and only if the cocartesian pushforward of $\overline{p}$ to the fibre over $x$ is a colimit in $\mathcal{E}_x$, and for every morphism $\phi\colon x \to y$ in $\mathcal{B}$ the functor $\phi_!\colon \mathcal{E}_x \to \mathcal{E}_y$ induced by the cocartesian morphisms over $\phi$ preserves this colimit. If $f\,\overline{p}$ is a colimit diagram in $\mathcal{B}$, then this gives a useful criterion for relating colimits in $\mathcal{E}$ to colimits in the fibres of $f$. For more on relative colimits (and the dual concept of relative limits), see [28, Section 4.3.1].

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## 2 Algebras and bimodules in the cartesian setting

Our goal in this section is to introduce the models for algebras and bimodules we will use in this paper, and to motivate our approach to defining an $(\infty, 2)$–category of these. Here we will only consider the case where the monoidal $\infty$–category these take values in has the cartesian product as its tensor product — to consider general monoidal $\infty$–categories we must work in the context of (nonsymmetric) $\infty$–operads, and this extra layer of formalism can potentially obscure the simple underlying meaning of our definitions. In Section 2.1 we recall how associative monoids can be modelled as certain simplicial objects, and in Section 2.2 we will see that bimodules can similarly be described as certain presheaves on the slice category $\mathbf{\Delta}_{/[1]}$. Next, in Section 2.3 we discuss how relative tensor products of bimodules can be described in this context, using presheaves on $\mathbf{\Delta}_{/[2]}$. In Section 2.4 we recall that a more general class of simplicial objects can be used to model internal categories in an $\infty$–category — in particular, we review Rezk's *Segal spaces*, which are a model for $\infty$–categories. We then indicate in Section 2.5 how, by considering certain presheaves on $\mathbf{\Delta}_{/[n]}$ for arbitrary $n$, we can construct a Segal space that describes an $\infty$–category of algebras or bimodules — or more generally a double $\infty$–category of these, from which the desired $(\infty, 2)$–category can be extracted.

### 2.1 $\mathbf{\Delta}$ and associative algebras

The observation that simplicial spaces satisfying a certain "Segal condition" give a model for $A_\infty$–spaces, ie spaces equipped with a homotopy-coherently associative

multiplication, goes back to unpublished work of Segal. Formulated in the language of $\infty$–categories, Segal's definition of a homotopy-coherently associative monoid, which in the $\infty$–categorical setting is the only meaningful notion of an associative monoid, is the following:

**Definition 2.1** Let $\mathcal{C}$ be an $\infty$–category with finite products. An *associative monoid* in $\mathcal{C}$ is a simplicial object $A_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that for every $[n]$ in $\mathbf{\Delta}$ the natural map

$$A_n \to A_1 \times \cdots \times A_1,$$

induced by the maps $\rho_i\colon [1] \to [n]$ in $\mathbf{\Delta}$ that send 0 to $i-1$ and 1 to $i$, is an equivalence.

To see that this definition makes sense, observe that the inner face map $d_1\colon [1] \to [2]$ induces a multiplication

$$A_1 \times A_1 \xleftarrow{\sim} A_2 \xrightarrow{d_1} A_1,$$

and the degeneracy $s_0\colon [1] \to [0]$ induces a unit

$$* \xleftarrow{\sim} A_0 \xrightarrow{s_0} A_1.$$

To see that the multiplication is associative, observe that the commutative square

$$\begin{array}{ccc} A_3 & \xrightarrow{d_1} & A_2 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_1} \\ A_2 & \xrightarrow[d_1]{} & A_1 \end{array}$$

exhibits a homotopy between the two possible multiplications $A_1^{\times 3} \to A_1$. Similarly, the higher-dimensional cubes giving compatibilities between the different composites of face maps $[1] \to [n]$ exhibit the higher coherence homotopies for the associative monoid.

## 2.2 $\mathbf{\Delta}_{/[1]}$ and bimodules

We will now see that, just as simplicial objects give a natural notion of associative monoids, presheaves on the slice category $\mathbf{\Delta}_{/[1]}$ give a model for bimodules between associative monoids:

**Definition 2.2** Let $\mathcal{C}$ be an $\infty$–category with finite products. A *bimodule* in $\mathcal{C}$ is a functor

$$M\colon \mathbf{\Delta}^{\mathrm{op}}_{/[1]} \to \mathcal{C}$$

such that, for every object $\phi\colon [n] \to [1]$ in $\mathbf{\Delta}_{/[1]}$, the natural map

$$M(\phi) \to M(\phi\rho_1) \times \cdots \times M(\phi\rho_n),$$

induced by composition with the maps $\rho_i\colon [1] \to [n]$, is an equivalence.

To see that such objects can indeed be interpreted as bimodules, observe that the category $\mathbf{\Delta}_{/[1]}$ can be described as having objects sequences $(i_0, \dots, i_n)$, where $0 \le i_k \le i_{k+1} \le 1$, with a unique morphism $(i_{\phi(0)}, \dots, i_{\phi(n)}) \to (i_0, \dots, i_m)$ for every $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$. In terms of this description a functor $M\colon \mathbf{\Delta}^{\mathrm{op}}_{/[1]} \to \mathcal{C}$ is a bimodule if and only if the object $M(i_0, \dots, i_n)$ decomposes as $M(i_0, i_1) \times \cdots \times M(i_{n-1}, i_n)$. Thus every object decomposes as a product of $M(0,0)$, $M(0,1)$ and $M(1,1)$.

The two maps $[0] \to [1]$ induce functors $\mathbf{\Delta} \to \mathbf{\Delta}_{/[1]}$ — these are the inclusions of the full subcategories of $\mathbf{\Delta}_{/[1]}$ with objects of the form $(0, \dots, 0)$ and $(1, \dots, 1)$. Restricting along these we see that $M(0,0)$ and $M(1,1)$ are associative monoids. The maps $(0,1) \to (0,0,1)$ and $(0,1) \to (0,1,1)$ in $\mathbf{\Delta}_{/[1]}$ give multiplications

$$M(0,0) \times M(0,1) \xleftarrow{\sim} M(0,0,1) \to M(0,1),$$
$$M(0,1) \times M(1,1) \xleftarrow{\sim} M(0,1,1) \to M(0,1),$$

which exhibit $M(0,1)$ as a left $M(0,0)$–module and a right $M(1,1)$–module. Moreover, the commutative square

$$\begin{array}{ccc} M(0,0,1,1) & \longrightarrow & M(0,1,1) \\ \downarrow & & \downarrow \\ M(0,0,1) & \longrightarrow & M(0,1) \end{array}$$

implies that these module structures are compatible. The remaining data given by $M$ shows that these actions are homotopy-coherently associative and compatible with the multiplications in $M(0,0)$ and $M(1,1)$.

## 2.3 $\mathbf{\Delta}_{/[2]}$ and tensor products of bimodules

We can similarly define $\mathbf{\Delta}_{/[2]}$–*monoids* as certain presheaves on $\mathbf{\Delta}_{/[2]}$. If we think of $\mathbf{\Delta}_{/[2]}$ as having as objects sequences $(i_0, \dots, i_m)$ with $0 \le i_k \le i_{k+1} \le 2$, then we can phrase the definition as follows:

**Definition 2.3** Let $\mathcal{C}$ be an $\infty$–category with finite products. Then a $\mathbf{\Delta}_{/[2]}$–*monoid* in $\mathcal{C}$ is a functor $M\colon \mathbf{\Delta}^{\mathrm{op}}_{/[2]} \to \mathcal{C}$ such that, for every object $(i_0, \dots, i_m)$, the natural

map

$$M(i_0,\dots,i_m)\to M(i_0,i_1)\times\cdots\times M(i_{n-1},i_n),$$

induced by composition with the maps $\rho_i$, is an equivalence.

Unravelling this definition, we see that a $\mathbf{\Delta}_{/[2]}$–monoid $M$ in $\mathcal{C}$ is given by the data of

- three associative monoids $M_0=M(0,0)$, $M_1=M(1,1)$ and $M_2=M(2,2)$, given by the restrictions of $M$ along the three natural inclusions $\mathbf{\Delta}^{\mathrm{op}}\to\mathbf{\Delta}^{\mathrm{op}}_{/[2]}$;
- three bimodules: an $M_0$–$M_1$–bimodule $M(0,1)$, an $M_1$–$M_2$–bimodule $M(1,2)$ and an $M_0$–$M_2$–bimodule $M(0,2)$, given by the restrictions of $M$ along the three natural inclusions $\mathbf{\Delta}^{\mathrm{op}}_{/[1]}\to\mathbf{\Delta}^{\mathrm{op}}_{/[2]}$;
- an $M_1$–balanced map $M(0,1)\times M(1,2)\simeq M(0,1,2)\to M(0,2)$, which we can think of as the restriction of $M$ along the inclusion $j\colon\mathbf{\Delta}^{\mathrm{op}}_{+}\to\mathbf{\Delta}^{\mathrm{op}}_{/[2]}$ that sends $[n]$ to $(0,1,\dots,1,2)$ (with $n+1$ 1's for $n=-1,\,0,\dots$).

We would like to understand what it means for the bimodule $M(0,2)$ to be the tensor product $M(0,1)\otimes_{M_1}M(1,2)$ in terms of this data. In classical algebra, if $A$ is an associative algebra and $M$ is a right and $N$ a left $A$–module, the tensor product $M\otimes_A N$ can be defined as the reflexive coequalizer of the two multiplication maps $M\times A\times N\to M\times N$. As usual, in the $\infty$–categorical setting this coequalizer must be replaced by its "derived" version, namely the colimit of a simplicial diagram, commonly known as the "bar construction": specifically, this is the diagram $B(M,A,N)_\bullet:=M\times A^{\times\bullet}\times N$ with face maps given by multiplications and degeneracies determined by the unit of $A$.

For a $\mathbf{\Delta}_{/[2]}$–monoid $M$, this diagram is precisely the restriction of the augmented simplicial diagram $j$ to $\mathbf{\Delta}^{\mathrm{op}}$. Thus, the bimodule $M(0,2)$ is a tensor product precisely when $j$ is a colimit diagram, which leads us to make the following definition:

**Definition 2.4** We say a $\mathbf{\Delta}^{\mathrm{op}}_{/[2]}$–monoid $M$ in $\mathcal{C}$ is *composite* if the map

$$\mathbf{\Delta}^{\mathrm{op}}_{+}\xrightarrow{\ j\ }\mathbf{\Delta}^{\mathrm{op}}_{/[2]}\xrightarrow{\ M\ }\mathcal{C}$$

is a colimit diagram.

## 2.4 $\mathbf{\Delta}$ and $\infty$–categories

As originally observed by Rezk [35], a generalization of Segal's definition of associative monoids gives a model for $\infty$–categories, namely *Segal spaces*. In the $\infty$–categorical

context, these are a special case of the natural definition of an internal category or category object:

**Definition 2.5** Let $\mathcal{C}$ be an $\infty$–category with finite limits. A *category object* in $\mathcal{C}$ is a simplicial object $X_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that for all $[n] \in \mathbf{\Delta}$ the natural map

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

induced by the maps $\rho_i\colon [1] \to [n]$ and the maps $[0] \to [n]$, is an equivalence. We write $\mathrm{Cat}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$ spanned by the category objects.

A *Segal space* is a category object in the $\infty$–category $\mathcal{S}$ of spaces. We can think of a Segal space $X_\bullet$ as having a space $X_0$ of "objects" and a space $X_1$ of "morphisms"; the face maps $X_1 \rightrightarrows X_0$ assign the source and target object to each morphism, and the degeneracy $s_0\colon X_0 \to X_1$ assigns an identity morphism to every object. Then $X_n \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is the space of composable sequences of $n$ morphisms, and the face map $d_1\colon [1] \to [2]$ gives a composition

$$X_1 \times_{X_0} X_1 \xleftarrow{\simeq} X_2 \xrightarrow{d_1} X_1.$$

The remaining data in $X_\bullet$ gives the homotopy-coherent associativity data for this composition and its compatibility with the identity maps.

**Remark 2.6** We can regard the $\infty$–category $\mathrm{Cat}_\infty$ of $\infty$–categories as the localization of the $\infty$–category of Segal spaces at the fully faithful and essentially surjective functors (in the appropriate homotopically correct sense). The main theorem of [35] is that this localization is given by the full subcategory $\mathrm{CSS}(\mathcal{S})$ of $\mathrm{Cat}(\mathcal{S})$ spanned by the *complete* Segal spaces. It was proved by Joyal and Tierney [24] that the model category of complete Segal spaces is Quillen equivalent to Joyal's model category of quasicategories, and so the $\infty$–category $\mathrm{Cat}_\infty$, defined using quasicategories, is equivalent to $\mathrm{CSS}(\mathcal{S})$.

We will also make use of category objects in $\mathrm{Cat}_\infty$. These give a notion of *double $\infty$–categories*, just as double categories can be thought of as internal categories in Cat. We will see below in Section 3.3 that, just as a double category has two underlying bicategories, a double $\infty$–category has two underlying $(\infty, 2)$–categories.

## 2.5 $\mathbf{\Delta}_{/[n]}$ and the $(\infty, 2)$–category of algebras and bimodules

As a preliminary to discussing the $(\infty, 2)$–category of algebras and bimodules in an $\infty$–category $\mathcal{C}$ with finite products, let us consider the underlying $\infty$–category $\mathfrak{Alg}_1(\mathcal{C})$ of

algebras and bimodules as a Segal space. From our discussion so far, we have an obvious choice for the space $\mathfrak{alg}_1(\mathcal{C})_0$ of objects, namely the space of associative monoids in $\mathcal{C}$, and for the space $\mathfrak{alg}_1(\mathcal{C})_1$ of morphisms, namely the space of $\mathbf{\Delta}_{/[1]}$–monoids in $\mathcal{C}$. These spaces are simply the appropriate collections of connected components in the spaces $\mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$ and $\mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}}_{/[1]}, \mathcal{C})$, respectively. The source and target maps are induced by composition with $d_1$ and $d_0\colon [0] \to [1]$, and composition with $s_0\colon [1] \to [0]$ sends a monoid $A$ to $A$ considered as an $A$–$A$–bimodule, giving the correct identity morphisms.

In order to construct a Segal space, the spaces $\mathfrak{alg}_1(\mathcal{C})_2$ and $\mathfrak{alg}_1(\mathcal{C})_1 \times_{\mathfrak{alg}_1(\mathcal{C})_0} \mathfrak{alg}_1(\mathcal{C})_1$ must be equivalent. On the other hand, the composition in $\mathfrak{alg}_1(\mathcal{C})$ should be given by relative tensor products of bimodules, which we saw above corresponds to taking a composite $\mathbf{\Delta}_{/[2]}$–monoid and composing with the middle face map $d_1\colon [1] \to [2]$; this suggests that the space $\mathfrak{alg}_1(\mathcal{C})_2$ should be the space of composite $\mathbf{\Delta}_{/[2]}$–monoids. Luckily, it will turn out that the space of composite $\mathbf{\Delta}_{/[2]}$–monoids is indeed equivalent to $\mathfrak{alg}_1(\mathcal{C})_1 \times_{\mathfrak{alg}_1(\mathcal{C})_0} \mathfrak{alg}_1(\mathcal{C})_1$ via the appropriate forgetful maps, so this does actually make sense.

To define the spaces $\mathfrak{alg}_1(\mathcal{C})_n$ for general $n$, we similarly consider *composite* $\mathbf{\Delta}_{/[n]}$–*monoids* for arbitrary $n$. If we think of $\mathbf{\Delta}_{/[n]}$ as having objects sequences $(i_0, \dots, i_m)$ with $0 \le i_k \le i_{k+1} \le n$, we have the following definition:

**Definition 2.7** Let $\mathcal{C}$ be an $\infty$–category with finite products. Then a $\mathbf{\Delta}_{/[n]}$–*monoid* in $\mathcal{C}$ is a functor $M\colon (\mathbf{\Delta}_{/[n]})^{\mathrm{op}} \to \mathcal{C}$ such that, for every object $(i_0, \dots, i_m)$, the natural map

$$M(i_0, \dots, i_m) \to M(i_0, i_1) \times \cdots \times M(i_{n-1}, i_n),$$

induced by composition with the maps $\rho_i$, is an equivalence.

A $\mathbf{\Delta}_{/[n]}$–monoid in $\mathcal{C}$ describes

- $n+1$ associative monoids $M_0 = M(0,0), M_1 = M(1,1), \dots, M_n = M(n,n)$;
- an $M_i$–$M_j$–bimodule $M(i,j)$ for each pair $(i,j)$ with $0 \le i < j \le n$;
- an $M_j$–balanced map $M(i,j) \times M(j,k) \to M(i,k)$ for each triple $(i,j,k)$ with $0 \le i < j < k \le n$, compatible with the actions of $M_i$ and $M_k$;

such that these bilinear maps are compatible, eg if $0 \le i < j < k < l \le n$ then the diagram

$$\begin{array}{ccc} M(i,j) \times M(j,k) \times M(k,l) & \longrightarrow & M(i,j) \times M(j,l) \\ \downarrow & & \downarrow \\ M(i,k) \times M(k,l) & \longrightarrow & M(i,l) \end{array}$$

commutes. Composition with the maps $\phi_*\colon \mathbf{\Delta}_{/[n]} \to \mathbf{\Delta}_{/[m]}$ given by composition with a map $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$ takes $\mathbf{\Delta}_{/[m]}$–monoids in $\mathcal{C}$ to $\mathbf{\Delta}_{/[n]}$–monoids.

We say that a $\mathbf{\Delta}_{/[n]}$–monoid $M$ is *composite* if these maps exhibit the bimodule $M(i,j)$ as the iterated tensor product

$$M(i,i+1) \otimes_{M_{i+1}} M(i+1,i+2) \otimes_{M_{i+2}} \cdots \otimes_{M_{j-1}} M(j-1,j).$$

As in the case $n=2$, this condition can be formulated precisely in terms of certain (multi)simplicial diagrams being colimits — we will discuss this in more detail below in Section 4.2.

If $\mathfrak{alg}_1(\mathcal{C})_n$ denotes the space of composite $\mathbf{\Delta}_{/[n]}$–monoids, then the main results of Section 4 will tell us:

- The composite monoids are preserved under composition, with the maps $\mathbf{\Delta}_{/[n]} \to \mathbf{\Delta}_{/[m]}$ coming from maps in $\mathbf{\Delta}$. Thus the spaces $\mathfrak{alg}_1(\mathcal{C})_\bullet$ fit together into a simplicial space.
- The spaces $\mathfrak{alg}_1(\mathcal{C})_\bullet$ satisfy the Segal condition, ie the map

  $$\mathfrak{alg}_1(\mathcal{C})_n \to \mathfrak{alg}_1(\mathcal{C})_1 \times_{\mathfrak{alg}_1(\mathcal{C})_0} \cdots \times_{\mathfrak{alg}_1(\mathcal{C})_0} \mathfrak{alg}_1(\mathcal{C})_1$$

  is an equivalence for all $n$.

In other words, $\mathfrak{alg}_1(\mathcal{C})_\bullet$ is a Segal space. This (or more precisely its completion) is our $\infty$–category of algebras and bimodules.

We can just as easily consider the $\infty$–*categories* $\mathfrak{ALG}_1(\mathcal{C})_n$ of composite $\mathbf{\Delta}_{/[n]}$–monoids, ie the appropriate full subcategories of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{/[n]}, \mathcal{C})$. We'll show that these form a category object $\mathfrak{ALG}_1(\mathcal{C})$ in $\mathrm{Cat}_\infty$, ie a double $\infty$–category — this has associative monoids as objects, algebra homomorphisms as vertical morphisms, bimodules as horizontal morphisms and bimodule homomorphisms as commutative squares. As we will see below in Section 3.3, from this double $\infty$–category we can then extract an $(\infty,2)$–category $\mathfrak{Alg}_1(\mathcal{C})$ of algebras, bimodules and bimodule homomorphisms.

# 3 $\mathbb{E}_n$–algebras and iterated bimodules in the cartesian setting

The definitions we considered in Section 2 can be iterated, and in this section we will discuss how this leads to an $(\infty, n+1)$–category of $\mathbb{E}_n$–algebras, again in the cartesian case. In Section 3.1 we consider iterated $\mathbf{\Delta}$–monoids, which gives a model for

$\mathbb{E}_n$–algebras. Then in Section 3.2 we see that, similarly, iterating the notion of category object gives *$n$–uple* $\infty$–categories, in the form of $n$–uple Segal spaces. This leads to a notion of $(\infty, n)$–categories in the form of Barwick's *iterated Segal spaces*, which we review in Section 3.3; this is the model of $(\infty, n)$–categories we will use below in Section 5. Finally, in Section 3.4 we indicate how the definition of the double $\infty$–category of algebras and bimodules can be iterated to get $(n+1)$–uple $\infty$–categories of $\mathbb{E}_n$–algebras in a cartesian monoidal $\infty$–category.

## 3.1 $\Delta^n$ and $\mathbb{E}_n$–algebras

The Dunn–Lurie additivity theorem [31, Theorem 5.1.2.2] implies that, in the $\infty$–categorical setting, $\mathbb{E}_n$–algebras in some $\infty$–category $\mathcal{C}$ are equivalent to associative algebras in $\mathbb{E}_{n-1}$–algebras in $\mathcal{C}$. In the cartesian case we would thus expect that associative monoids in associative monoids in … in $\mathcal{C}$ give a model for $\mathbb{E}_n$–algebras in $\mathcal{C}$ — we will prove a precise version of this claim below in Section A.3. Unwinding the definition, we see that these objects can be described as certain multisimplicial objects in $\mathcal{C}$:

**Definition 3.1** Let $\mathcal{C}$ be an $\infty$–category with finite products. A $\boldsymbol{\Delta^n}$*–monoid* in $\mathcal{C}$ is a multisimplicial object

$$A_{\bullet,\dots,\bullet}\colon \boldsymbol{\Delta}^{n,\mathrm{op}} \to \mathcal{C}$$

such that, for every object $([i_1],\dots,[i_n]) \in \boldsymbol{\Delta}^n$, the natural map

$$A_{i_1,\dots,i_n} \to \prod_{j_1=1}^{i_1} \cdots \prod_{j_n=1}^{i_n} A_{1,\dots,1},$$

induced by the maps $(\rho_{j_1},\dots,\rho_{j_n})$, is an equivalence.

**Remark 3.2** It is convenient to introduce some notation to simplify this definition: let $C_n$ denote the object $([1],\dots,[1])$ in $\boldsymbol{\Delta}^{n,\mathrm{op}}$, and for $I \in \boldsymbol{\Delta}^{n,\mathrm{op}}$ let $|I|$ denote the set of (levelwise) inert maps $C_n \to I$, ie the maps $(\rho_{i_1},\dots,\rho_{i_n})$. Then the Segal condition for a $\boldsymbol{\Delta}^n$–monoid $A$ can be stated as: for every $I \in \boldsymbol{\Delta}^{n,\mathrm{op}}$, the natural map $A_I \to A_{C_n}^{\times|I|}$ induced by the maps in $|I|$ is an equivalence.

## 3.2 $\Delta^n$ and $n$–uple $\infty$–categories

Just as we can iterate the notion of associative monoid to get a definition of $\mathbb{E}_n$–algebras in the cartesian setting, we can iterate the definition of a category object to get a definition of *$n$–uple internal categories*. To state this definition more explicitly, it is useful to first introduce some notation:

**Definition 3.3** A morphism $f\colon [n] \to [m]$ in $\mathbf{\Delta}$ is *inert* if it is the inclusion of a subinterval of $[m]$, ie $f(i) = f(0) + i$ for all $i$, and *active* if it preserves the extremal elements, ie $f(0) = 0$ and $f(n) = m$. More generally, we say a morphism $(f_1, \dots, f_n)$ in $\mathbf{\Delta}^n$ is *inert* or *active* if each $f_i$ is inert or active. We write $\mathbf{\Delta}^n_{\mathrm{act}}$ and $\mathbf{\Delta}^n_{\mathrm{int}}$ for the subcategories of $\mathbf{\Delta}^n$ with active and inert morphisms, respectively.

**Lemma 3.4** *The active and inert morphisms form a factorization system on* $\mathbf{\Delta}^n$.

**Proof** This is a special case of [10, Lemma 8.3]; it is also easy to check by hand. □

**Remark 3.5** Since the objects of $\mathbf{\Delta}^n$ have no nontrivial automorphisms, the factorizations into active and inert morphisms are actually *strictly* unique, rather than just unique up to isomorphism.

**Definition 3.6** Let $S$ be a subset of $\{1, \dots, n\}$. We write $C_S := ([i_1], \dots, [i_n])$, where $i_j$ is 1 for $j \in S$ and 0 otherwise. We refer to the objects $C_S$ as *cells* and write $\mathrm{Cell}^n$ for the full subcategory of $\mathbf{\Delta}^n_{\mathrm{int}}$ spanned by the objects $C_S$ for all $S \subseteq \{1, \dots, n\}$. Note that we have $C_n = C_{\{1,\dots,n\}}$.

**Remark 3.7** The category $\mathrm{Cell}^n$ is equivalent to the product $(\mathrm{Cell}^1)^{\times n}$, where $\mathrm{Cell}^1$ is the category with objects $[0]$ and $[1]$ and the two inclusions $[0] \to [1]$ as its only nonidentity morphisms.

**Definition 3.8** For $I \in \mathbf{\Delta}^n$, we write $\mathrm{Cell}^n_{/I}$ for the category $(\mathbf{\Delta}^n_{\mathrm{int}})_{/I} \times_{\mathbf{\Delta}^n_{\mathrm{int}}} \mathrm{Cell}^n$ of inert morphisms from cells to $I$.

**Definition 3.9** Let $\mathcal{C}$ be an $\infty$–category with finite limits. An *$n$–uple category object* in $\mathcal{C}$ is a multisimplicial object $X_{\bullet,\dots,\bullet}\colon \mathbf{\Delta}^{n,\mathrm{op}} \to \mathcal{C}$ such that, for all $I = ([i_1], \dots, [i_n]) \in \mathbf{\Delta}$, the natural map

$$X_I \to \lim_{C \to I \in \mathrm{Cell}^{n,\mathrm{op}}_{/I}} X_C$$

is an equivalence. We write $\mathrm{Cat}^n(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{n,\mathrm{op}}, \mathcal{C})$ spanned by the $n$–uple category objects.

**Remark 3.10** To see that this is equivalent to iterating the definition of a category object in $\mathcal{C}$, observe that for $I = ([i_1], \dots, [i_n])$ in $\mathbf{\Delta}^n$, the category $\mathrm{Cell}^n_{/I}$ is simply the product $\mathrm{Cell}^1_{/[i_1]} \times \cdots \times \mathrm{Cell}^1_{/[i_n]}$, and so decomposing the limit we see that $X_{\bullet,\dots,\bullet}$ is an $n$–uple category object if and only if $X_{i,\bullet,\dots,\bullet}$ is an $(n-1)$–uple category object for all $i$, and $X_\bullet$ is a category object in $(n-1)$–simplicial objects in $\mathcal{C}$.

If $\mathcal{C}$ is the $\infty$–category $\mathcal{S}$ of spaces, an $n$–uple category object $X_{\bullet,\dots,\bullet}$ can be thought of as consisting of

- a space $X_{0,\dots,0}$ of objects;
- spaces $X_{1,0,\dots,0}, \dots, X_{0,\dots,0,1}$ of $n$ different kinds of 1–morphism, each with a source and target in $X_{0,\dots,0}$;
- spaces $X_{1,1,0,\dots,0}$, etc, of "commutative squares" between any two kinds of 1–morphism;
- spaces $X_{1,1,1,0,\dots,0}$, etc, of "commutative cubes" between any three kinds of 1–morphism;

$\vdots$

- a space $X_{1,1,\dots,1}$ of "commutative $n$–cubes";

together with units and coherently homotopy-associative composition laws for all these different types of morphisms. In other words, an $n$–uple category object in $\mathcal{S}$ can be regarded as an *$n$–uple $\infty$–category*.

**Remark 3.11** Since $\infty$–categories can be thought of as (complete) Segal spaces, ie category objects in $\mathcal{S}$, we can think of $n$–uple category objects in $\mathrm{Cat}_\infty$ as *$(n+1)$–uple $\infty$–categories*. More precisely, regarding $\mathrm{Cat}_\infty$ as the $\infty$–category of complete Segal spaces we have an inclusion $\mathrm{Cat}_\infty \hookrightarrow \mathrm{Cat}(\mathcal{S})$, and this induces an inclusion $\mathrm{Cat}^n(\mathrm{Cat}_\infty) \hookrightarrow \mathrm{Cat}^{n+1}(\mathcal{S})$.

## 3.3 $\Delta^n$ and $(\infty, n)$–categories

We can view $(\infty, n)$–categories as given by the same kind of data as an $n$–uple $\infty$–category, except that there is only one type of 1–morphism, etc, so to define $(\infty, n)$–categories as a special kind of $n$–uple $\infty$–category we want to require certain spaces to be "trivial". This leads to Barwick's definition of an $n$–fold Segal object in an $\infty$–category:

**Definition 3.12** Suppose $\mathcal{C}$ is an $\infty$–category with finite limits. A *1–fold Segal object* in $\mathcal{C}$ is just a category object in $\mathcal{C}$. For $n > 1$ we inductively define an *$n$–fold Segal object* in $\mathcal{C}$ to be an $n$–uple category object $X$ such that

(i) the $(n-1)$–uple category object $X_{0,\bullet,\dots,\bullet}$ is constant;

(ii) the $(n-1)$–uple category object $X_{k,\bullet,\dots,\bullet}$ is an $(n-1)$–fold Segal object for all $k$.

We write $\mathrm{Seg}_n(\mathcal{C})$ for the full subcategory of $\mathrm{Cat}^n(\mathcal{C})$ spanned by the $n$–fold Segal objects. When $\mathcal{C}$ is the $\infty$–category $\mathcal{S}$ of spaces, we refer to $n$–fold Segal objects in $\mathcal{S}$ as *$n$–fold Segal spaces*.

**Remark 3.13** Unwinding the definition, an $n$–fold Segal space $X$ consists of

- a space $X_{0,\dots,0}$ of objects,
- a space $X_{1,0,\dots,0}$ of 1–morphisms,
- a space $X_{1,1,0,\dots,0}$ of 2–morphisms,
- $\vdots$
- a space $X_{1,\dots,1}$ of $n$–morphisms,

together with units and coherently homotopy-associative composition laws for these morphisms.

Given a double category object $X\colon \mathbf{\Delta}^{2,\mathrm{op}} \to \mathcal{C}$, there is a canonical way to extract a 2–fold Segal object $X'$:

- We take $X'_{0,\bullet}$ to be the constant simplicial object at $X_{0,0}$.
- For $n > 0$ we define $X'_{n,\bullet}$ to be the pullback

$$\begin{array}{ccc} X'_{n,\bullet} & \longrightarrow & X_{n,\bullet} \\ \downarrow & & \downarrow \\ X'_{0,\bullet} & \longrightarrow & X_{0,\bullet} \end{array}$$

  where the bottom horizontal map is induced by the degeneracies. This amounts to forgetting the objects of $X_{0,1}$ that are not in the image of the degeneracy map $X_{0,0} \to X_{0,1}$ — ie we are forgetting all the nontrivial 1–morphisms of one kind.

This construction can be iterated to extract an $n$–fold Segal object from an $n$–uple category object — in fact, by permuting the $n$ coordinates we can extract $n$ different Segal objects. More formally, we have:

**Proposition 3.14** [19, Proposition 4.12] *Let $\mathcal{C}$ be an $\infty$–category with finite limits. The inclusion $\mathrm{Seg}_n(\mathcal{C}) \hookrightarrow \mathrm{Cat}^n(\mathcal{C})$ has a right adjoint $U_{\mathrm{Seg}}\colon \mathrm{Cat}^n(\mathcal{C}) \to \mathrm{Seg}_n(\mathcal{C})$.* □

Although $n$–fold Segal spaces describe $(\infty, n)$–categories, the $\infty$–category $\mathrm{Seg}_n(\mathcal{S})$ is not the correct homotopy theory of $(\infty, n)$–categories, as we have not inverted the appropriate class of fully faithful and essentially surjective maps. This localization can be obtained by restricting to the full subcategory $\mathrm{CSS}_n(\mathcal{S})$ of *complete* $n$–fold Segal spaces, as proved by Barwick [9]; we denote the localization $\mathrm{Seg}_n(\mathcal{S}) \to \mathrm{CSS}_n(\mathcal{S})$ by $L_n$, but we will not need the details of the definition in this paper.

**Remark 3.15** There is a canonical way to extract an $(\infty,n)$–category from an $n$–uple $\infty$–category $\mathcal{C}$, namely the completion $L_nU_{\mathrm{Seg}}\mathcal{C}$ of the underlying $n$–fold Segal space of $\mathcal{C}$. Moreover, the functor $L_nU_{\mathrm{Seg}}\colon \mathrm{Cat}^n(\mathcal{S}) \to \mathrm{Cat}_{(\infty,n)}$ is symmetric monoidal with respect to the cartesian product — since $U_{\mathrm{Seg}}$ is a right adjoint it preserves products, and $L_n$ preserves products by [19, Lemma 7.10]. In particular, if $\mathcal{C}$ is an $\mathbb{E}_m$–monoidal $n$–uple $\infty$–category, then $L_nU_{\mathrm{Seg}}\mathcal{C}$ is an $\mathbb{E}_m$–monoidal $(\infty,n)$–category. Similarly, we can extract an underlying $(\infty,n+1)$–category from an $n$–uple category object $\mathcal{C}$ in $\mathrm{Cat}_\infty$ as $L_{n+1}U_{\mathrm{Seg}}i\,\mathcal{C}$ where $i$ denotes the inclusion $\mathrm{Cat}^n(\mathrm{Cat}_\infty) \hookrightarrow \mathrm{Cat}^{n+1}(\mathcal{S})$. The functor $L_{n+1}U_{\mathrm{Seg}}i\colon \mathrm{Cat}^n(\mathrm{Cat}_\infty) \to \mathrm{Cat}_{(\infty,n+1)}$ also preserves products, since $i$ is another right adjoint.

## 3.4 $\boldsymbol{\Delta^n_{/I}}$ and iterated bimodules

We will now consider how to extend the definition of the double $\infty$–category $\mathfrak{ALG}_1(\mathcal{C})$ of algebras, algebra homomorphisms, and bimodules in $\mathcal{C}$ we outlined above to get an $(n+1)$–uple $\infty$–category $\mathfrak{ALG}_n(\mathcal{C})$ of $\mathbb{E}_n$–algebras. We take the $\infty$–category $\mathfrak{ALG}_1(\mathcal{C})_{0,\dots,0}$ of objects to be the $\infty$–category of $\boldsymbol{\Delta}^n$–monoids in $\mathcal{C}$ — a full subcategory of $\mathrm{Fun}(\boldsymbol{\Delta}^{n,\mathrm{op}},\mathcal{C})$. To define the remaining structure, we first observe that we can iterate the definition of $\boldsymbol{\Delta}_{/[i]}$–monoids to get a notion of $\boldsymbol{\Delta}^n_{/I}$–monoids for all $I \in \boldsymbol{\Delta}^n$:

**Definition 3.16** Let $\mathcal{C}$ be an $\infty$–category with products, and suppose $I \in \boldsymbol{\Delta}^n$. A $\boldsymbol{\Delta}^n_{/I}$*–monoid* in $\mathcal{C}$ is a functor $X\colon \boldsymbol{\Delta}^{n,\mathrm{op}}_{/I} \to \mathcal{C}$ such that, for every object $\phi\colon J \to I$, the natural map

$$X(\phi) \to \prod_{\alpha \in |J|} X(\phi \circ \alpha)$$

is an equivalence.

Just as in the case $n = 1$, however, we do not want $\mathfrak{ALG}_n(\mathcal{C})_I$ to contain *all* the $\boldsymbol{\Delta}^n_{/I}$–monoids, only those that are "composite" in the sense that they decompose appropriately as tensor products. We will define this notion precisely below in Section 5.2. The main result of this paper, restricted to the cartesian case, is then that this does indeed give an $(n+1)$–uple $\infty$–category. More precisely, if for every $I \in \boldsymbol{\Delta}^n$, we let $\mathfrak{ALG}_n(\mathcal{C})_I$ denote the $\infty$–category of composite $\boldsymbol{\Delta}^n_{/I}$–monoids (a full subcategory of $\mathrm{Fun}(\boldsymbol{\Delta}^{n,\mathrm{op}}_{/I},\mathcal{C})$), then:

- The composite monoids are preserved under composition with the maps $\boldsymbol{\Delta}^n_{/I} \to \boldsymbol{\Delta}^n_{/J}$ coming from maps $I \to J$ in $\boldsymbol{\Delta}^n$. Thus the objects $\mathfrak{ALG}_n(\mathcal{C})_{\bullet,\dots,\bullet}$ define a multisimplicial $\infty$–category.

- The $\infty$–categories $\mathfrak{AlgBm}_n(\mathcal{C})_{\bullet,\dots,\bullet}$ satisfy the Segal condition, ie the map

$$\mathfrak{AlgBm}_n(\mathcal{C})_I \to \lim_{C \to I \in \mathrm{Cell}_{/I}^{n,\mathrm{op}}} \mathfrak{AlgBm}_n(\mathcal{C})_C$$

  is an equivalence for all $n$.

In other words, $\mathfrak{AlgBm}_n(\mathcal{C})$ is an $n$–uple category object in $\mathrm{Cat}_\infty$. From this we can then extract an $(\infty, n+1)$–category $\mathfrak{Alg}_n(\mathcal{C})$ as the underlying complete $(n+1)$–fold Segal space $L_{n+1}U_{\mathrm{Seg}}i\mathfrak{AlgBm}_n(\mathcal{C})$, as discussed above.

# 4 Algebras and bimodules

In Section 2 we sketched our approach to constructing a double $\infty$–category of algebras and bimodules in the cartesian case, ie when the algebras are defined with respect to the monoidal structure given by the cartesian product. However, although this case is certainly not without interest, many key examples of symmetric monoidal $\infty$–categories where we want to consider algebras and bimodules have noncartesian tensor products — for example spectra, modules over a ring spectrum, or the "derived $\infty$–category" of chain complexes in an abelian category with quasi-isomorphisms inverted. To extend our definitions to apply also to such noncartesian examples, we will work with the theory of $\infty$–*operads*. Specifically, in this section we will make use of the theory of *nonsymmetric* $\infty$–*operads* to construct a double $\infty$–category $\mathfrak{AlgBm}_1(\mathcal{C})$ of associative algebras in any nice monoidal $\infty$–category $\mathcal{C}$, with algebra homomorphisms and bimodules as the two kinds of 1–morphisms.

In Section 4.1 we recall the basics of nonsymmetric $\infty$–operads, and then in Section 4.2 we observe that using these the definition of bimodules we discussed above in Section 2 has a natural extension to the noncartesian setting, which lets us define the $\infty$–categories $\mathfrak{AlgBm}_1(\mathcal{C})_k$ that will make up the simplicial $\infty$–category $\mathfrak{AlgBm}_1(\mathcal{C})$. In Section 4.3 we check that these $\infty$–categories satisfy the Segal condition, and in Section 4.4 we show that they are functorial and so do indeed form a simplicial object in $\mathrm{Cat}_\infty$. Finally, in Section 4.5 we study the forgetful functor from bimodules to pairs of algebras in more detail — the results we prove here will be used below in Section 5.5.

## 4.1 Nonsymmetric $\infty$–operads

In this subsection we will review some basic notions from the theory of nonsymmetric $\infty$–operads. For more motivation for these definitions, we refer the reader to the extensive discussion in [16, Sections 2.1–2.2].

In ordinary category theory a monoidal category can be viewed as being precisely an associative monoid in the 2–category of categories, provided we interpret "associative

monoid" in an appropriately 2–categorical sense. Similarly, we can define a *monoidal $\infty$–category* to be an associative monoid in the $\infty$–category $\mathrm{Cat}_\infty$ of $\infty$–categories. As we saw in Section 2.1, we can take this to mean a simplicial object in $\mathrm{Cat}_\infty$ satisfying a "Segal condition". Using Lurie's straightening equivalence, we get an equivalent definition of monoidal $\infty$–categories as certain cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$:

**Definition 4.1** A *monoidal $\infty$–category* is a cocartesian fibration $\mathcal{C}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$ such that for each $[n]$ the map $\mathcal{C}^{\otimes}_{[n]} \to (\mathcal{C}^{\otimes}_{[1]})^{\times n}$, induced by the cocartesian morphisms over the maps $\rho_i$ in $\mathbf{\Delta}^{\mathrm{op}}$, is an equivalence.

One advantage of this definition is that it can be weakened to give a definition of *nonsymmetric $\infty$–operads*:

**Definition 4.2** A *nonsymmetric $\infty$–operad* is a functor of $\infty$–categories $\pi\colon \mathcal{O} \to \mathbf{\Delta}^{\mathrm{op}}$ such that:

(i) For every inert morphism $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and every $X \in \mathcal{O}_{[n]}$ there exists a $\pi$–cocartesian morphism $X \to \phi_! X$ over $\phi$.

(ii) For every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ the functor

$$\mathcal{O}_{[n]} \to (\mathcal{O}_{[1]})^{\times n}$$

induced by the cocartesian morphisms over the inert maps $\rho_i$ for $i = 1, \dots, n$ is an equivalence of $\infty$–categories.

(iii) For every morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$, $X \in \mathcal{O}_{[n]}$ and $Y \in \mathcal{O}_{[m]}$, composition with the cocartesian morphisms $Y \to Y_i$ over the inert morphisms $\rho_i$ gives an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{O}}(X, Y) \xrightarrow{\simeq} \prod_i \mathrm{Map}^{\rho_i \circ \phi}_{\mathcal{O}}(X, Y_i),$$

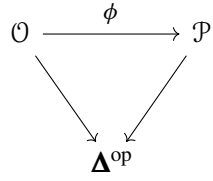
where $\mathrm{Map}^{\phi}_{\mathcal{O}}(X, Y)$ denotes the subspace of $\mathrm{Map}_{\mathcal{O}}(X, Y)$ of morphisms that map to $\phi$ in $\mathbf{\Delta}^{\mathrm{op}}$. (Equivalently, $Y$ is a *$\pi$–limit* of the $Y_i$ in the sense of [28, Section 4.3.1].)

**Remark 4.3** To see how this definition is related to the usual notion of nonsymmetric (coloured) operad (or multicategory), recall that to any nonsymmetric (coloured) operad in sets we can associate its *category of operators*, which is a category over $\mathbf{\Delta}^{\mathrm{op}}$. These categories of operators are characterized precisely by the 1–categorical analogues of conditions (i)–(iii) above — for more details see [16, Section 2.2].

**Remark 4.4** This definition is a special case of Barwick's notion of an $\infty$–operad over an operator category [10], namely the case where the operator category is the category of finite ordered sets.

**Remark 4.5** Since $\mathbf{\Delta}^{\mathrm{op}}$ is an ordinary category, a map $\mathcal{O} \to \mathbf{\Delta}^{\mathrm{op}}$ where $\mathcal{O}$ is an $\infty$–category is automatically an inner fibration by [28, Proposition 2.3.1.5].

**Definition 4.6** If $\mathcal{O}$ and $\mathcal{P}$ are nonsymmetric $\infty$–operads, a *morphism of nonsymmetric $\infty$–operads* from $\mathcal{O}$ to $\mathcal{P}$ is a commutative diagram




such that $\phi$ carries cocartesian morphisms in $\mathcal{O}$ that map to inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$ to cocartesian morphisms in $\mathcal{P}$. We will also refer to a morphism of nonsymmetric $\infty$–operads $\mathcal{O} \to \mathcal{P}$ as an $\mathcal{O}$–*algebra* in $\mathcal{P}$. We write $\mathrm{Alg}^{1}_{\mathcal{O}}(\mathcal{P})$ for the $\infty$–category of $\mathcal{O}$–algebras in $\mathcal{P}$, defined as a full subcategory of the $\infty$–category of functors from $\mathcal{O}$ to $\mathcal{P}$ over $\mathbf{\Delta}^{\mathrm{op}}$.

We will actually need to work with a somewhat more general notion than that of nonsymmetric $\infty$–operad. To introduce this, recall from Section 2.4 that a *double $\infty$–category* can be defined as a simplicial object in $\mathrm{Cat}_{\infty}$ that satisfies a more general variant of the Segal condition that defines monoids. Reformulating this in terms of cocartesian fibrations, we get the following analogue of our definition of a monoidal $\infty$–category above:

**Definition 4.7** A *double $\infty$–category* is a cocartesian fibration $\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$ such that for each $[n]$ the map

$$\mathcal{M}_{[n]} \to \mathcal{M}_{[1]} \times_{\mathcal{M}_{[0]}} \cdots \times_{\mathcal{M}_{[0]}} \mathcal{M}_{[1]},$$

induced by the cocartesian morphisms over the maps $\rho_i$ and the maps $[n] \to [0]$ in $\mathbf{\Delta}^{\mathrm{op}}$, is an equivalence.

Now we can contemplate the analogous variant of the definition of a nonsymmetric $\infty$–operad:

**Definition 4.8** A *generalized nonsymmetric $\infty$–operad* is a functor of $\infty$–categories $\pi\colon \mathcal{O} \to \mathbf{\Delta}^{\mathrm{op}}$ such that:

(i) For every inert morphism $\phi\colon [m]\to[n]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and every $X\in\mathcal{O}_{[n]}$ there exists a $\pi$–cocartesian morphism $X\to\phi_! X$ over $\phi$.

(ii) For every $[n]\in\mathbf{\Delta}^{\mathrm{op}}$ the functor

$$\mathcal{O}_{[n]}\to\mathcal{O}_{[1]}\times_{\mathcal{O}_{[0]}}\cdots\times_{\mathcal{O}_{[0]}}\mathcal{O}_{[1]}$$

induced by the cocartesian arrows over the inert maps $\rho_i$ $(i=1,\dots,n)$ and the maps $[n]\to[0]$ is an equivalence of $\infty$–categories.

(iii) Given $Y\in\mathcal{O}_{[m]}$, choose a cocartesian lift of the diagram of inert morphisms from $[m]$ to $[1]$ and $[0]$: let $Y\to Y_{(i-1)i}$ be a cocartesian morphism over the map $\rho_i\colon[m]\to[1]$ $(i=1,\dots,m)$ and let $Y\to Y_i$ $(i=0,\dots,m)$ be a cocartesian morphism over the map $\sigma_i\colon[m]\to[0]$ corresponding to the inclusion of $\{i\}$ in $[m]$. Then for any map $\phi\colon[n]\to[m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and $X\in\mathcal{O}_{[n]}$, composition with these cocartesian morphisms induces an equivalence

$$\begin{aligned}\mathrm{Map}^{\phi}_{\mathcal{O}}(X,Y)\xrightarrow{\sim}\mathrm{Map}^{\rho_1\circ\phi}_{\mathcal{O}}(X,Y_{01})\times_{\mathrm{Map}^{\sigma_1\circ\phi}_{\mathcal{O}}(X,Y_1)}\\ \cdots\times_{\mathrm{Map}^{\sigma_{m-1}\circ\phi}_{\mathcal{O}}(X,Y_{m-1})}\mathrm{Map}^{\rho_1\circ\phi}_{\mathcal{O}}(X,Y_{(m-1)m}).\end{aligned}$$

(Equivalently, any cocartesian lift of the diagram of inert maps from $[m]$ to $[1]$ and $[0]$ is a $\pi$–limit diagram in $\mathcal{O}$.)

**Remark 4.9** As discussed in [16, Sections 2.3–2.4], generalized nonsymmetric $\infty$–operads are an $\infty$–categorical analogue of the ***fc*-multicategories** of Leinster [27] (also called *virtual double categories* in [12]), which are a common generalization of double categories and multicategories.

We can define morphisms of generalized nonsymmetric $\infty$–operads in the same way as we define morphisms of nonsymmetric $\infty$–operads, ie as maps over $\mathbf{\Delta}^{\mathrm{op}}$ that preserve cocartesian morphisms over inert morphisms. Again, we will refer to a morphism $\mathcal{M}\to\mathcal{N}$ of generalized nonsymmetric $\infty$–operads as an $\mathcal{M}$–*algebra* in $\mathcal{N}$, and define an $\infty$–category $\mathrm{Alg}^{1}_{\mathcal{M}}(\mathcal{N})$ of these as a full subcategory of the $\infty$–category of functors from $\mathcal{M}$ to $\mathcal{N}$ over $\mathbf{\Delta}^{\mathrm{op}}$.

## 4.2 Bimodules and their tensor products

We now have a natural way to extend the definitions of Section 2 to the noncartesian setting because of the following observation:

**Lemma 4.10** *The projection $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}\to\mathbf{\Delta}^{\mathrm{op}}$ is a double $\infty$–category for all $[n]\in\mathbf{\Delta}$.*

**Proof** This projection is the opfibration associated to the functor

$$\mathrm{Hom}_{\mathbf{\Delta}}(-,[n])\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}.$$

It thus suffices to check that this functor satisfies the Segal condition, which it does since $[k]$ is the iterated pushout $[1] \amalg_{[0]} \cdots \amalg_{[0]} [1]$ in $\mathbf{\Delta}$. □

**Remark 4.11** As a double ($\infty$–)category, $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$ is rather degenerate: it is the double category corresponding to the category (or partially ordered set)

$$0 \to 1 \to \cdots \to n.$$

In particular, it has no nontrivial morphisms in one direction.

**Definition 4.12** Let $\mathcal{C}$ be a monoidal $\infty$–category. An *associative algebra object* in $\mathcal{C}$ is a $\mathbf{\Delta}^{\mathrm{op}}$–algebra, and a *bimodule* in $\mathcal{C}$ is a $\mathbf{\Delta}^{\mathrm{op}}_{/[1]}$–algebra.

Thus, to define the double $\infty$–category $\mathbb{ALG}_1(\mathcal{C})$, natural choices for the $\infty$–categories of objects and morphisms are $\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{C})$ and $\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}_{/[1]}}(\mathcal{C})$, respectively. At the next level, we want to consider a full subcategory of $\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}_{/[2]}}(\mathcal{C})$ that consists of "composite" $\mathbf{\Delta}^{\mathrm{op}}_{/[2]}$–algebras. We want the composition of bimodules in $\mathbb{ALG}_1(\mathcal{C})$ to be given by tensor products, so the composite $\mathbf{\Delta}^{\mathrm{op}}_{/[2]}$–algebras should be those algebras $M$ where $M(0,2)$ is exhibited as the tensor product $M(0,1) \otimes_{M(1,1)} M(1,2)$. As discussed in Section 2.3, this amounts to the diagram $\mathbf{\Delta}^{\mathrm{op}}_{+} \to \mathcal{C}$, obtained by taking the cocartesian pushforward of

$$\mathbf{\Delta}^{\mathrm{op}}_{+} \xrightarrow{j} \mathbf{\Delta}^{\mathrm{op}}_{/[2]} \to \mathcal{C}^{\otimes}$$

to the fibre over $[1]$, being a colimit diagram. To get a more convenient version of this condition, and its generalization to $\mathbf{\Delta}_{/[n]}$–algebras, it will be useful to reformulate it in terms of operadic Kan extensions. In order to do this, we must first introduce some notation:

**Definition 4.13** A morphism $\phi\colon [k] \to [m]$ in $\mathbf{\Delta}$ is *cellular* if $\phi(i+1) \leq \phi(i)+1$ for all $i = 0, \ldots, k$. We write $\mathbf{\Lambda}_{/[n]}$ for the full subcategory of $\mathbf{\Delta}_{/[n]}$ spanned by the cellular maps. (In other words, $\mathbf{\Lambda}_{/[n]}$ is the full subcategory of $\mathbf{\Delta}_{/[n]}$ spanned by the objects $(i_0, \ldots, i_k)$ where $i_{t+1} - i_t \leq 1$.)

**Lemma 4.14** *The projection $\mathbf{\Lambda}^{\mathrm{op}}_{/[n]} \to \mathbf{\Delta}^{\mathrm{op}}$ is a generalized nonsymmetric $\infty$–operad, and the inclusion*

$$\tau_n\colon \mathbf{\Lambda}^{\mathrm{op}}_{/[n]} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}_{/[n]}$$

*is a morphism of generalized nonsymmetric $\infty$–operads.*

This is a special case of the following observation:

**Lemma 4.15** *Suppose $\pi\colon \mathcal{O} \to \mathbf{\Delta}^{\mathrm{op}}$ is a generalized nonsymmetric $\infty$–operad and $\mathcal{C}$ a full subcategory of $\mathcal{O}_{[1]}$. Let $\mathcal{P}$ be the full subcategory of $\mathcal{O}$ spanned by the objects $X$ such that $\rho_{i,!}X$ lies in $\mathcal{C}$ for all inert maps $\rho_i\colon [1] \to \pi(X)$. Then the restricted projection $\mathcal{P} \to \mathbf{\Delta}^{\mathrm{op}}$ is also a generalized nonsymmetric $\infty$–operad, and the inclusion $\mathcal{P} \hookrightarrow \mathcal{O}$ is a morphism of generalized nonsymmetric $\infty$–operads.*

**Proof** If $X \in \mathcal{P}_{[n]}$, $\phi\colon [m] \to [n]$ is an inert map, and $X \to \phi_!X$ is a cocartesian morphism over $\phi$ in $\mathcal{O}$, then $\phi_!X$ is also in $\mathcal{P}$. Hence $\mathcal{P}$ has cocartesian morphisms over inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$, which is condition (i) in Definition 4.8, and the inclusion $\mathcal{P} \hookrightarrow \mathcal{O}$ preserves these. Moreover, for every $[n]$ we have a pullback diagram

$$\begin{array}{ccc} \mathcal{P}_{[n]} & \longrightarrow & \mathcal{C} \times_{\mathcal{O}_{[0]}} \cdots \times_{\mathcal{O}_{[0]}} \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{O}_{[n]} & \longrightarrow & \mathcal{O}_{[1]} \times_{\mathcal{O}_{[0]}} \cdots \times_{\mathcal{O}_{[0]}} \mathcal{O}_{[1]} \end{array}$$

which implies condition (ii) since the bottom horizontal map is an equivalence. Condition (iii) is also satisfied, since $\mathcal{P}$ is a full subcategory. □

**Proof of Lemma 4.14** A map $\phi\colon [m] \to [n]$ is cellular if and only if all its composites $\phi\rho_i\colon [1] \to [n]$ with the inert maps $[1] \to [m]$ is cellular. Thus $\mathbf{\Lambda}^{\mathrm{op}}_{/[n]}$ is the full subcategory of $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$ determined by a full subcategory over $[1]$. It is therefore a generalized nonsymmetric $\infty$–operad by Lemma 4.15. □

The $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$–algebras that are given by tensor products in the appropriate way will turn out to be those that are left operadic Kan extensions along the inclusion $\tau_n\colon \mathbf{\Lambda}^{\mathrm{op}}_{/[n]} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}_{/[n]}$. For this to make sense, we must first check that the map $\tau_n$ is extendable in the sense of Definition A.49, so that we can apply Proposition A.50:

**Proposition 4.16** *The inclusion $\tau_i\colon \mathbf{\Lambda}^{\mathrm{op}}_{/[i]} \to \mathbf{\Delta}^{\mathrm{op}}_{/[i]}$ is extendable for all $i$.*

**Proof** We must show that, for any map $\xi\colon [j] \to [i]$ in $\mathbf{\Delta}$, the map

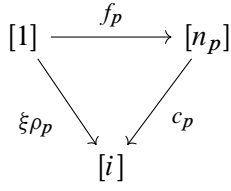
$$(\mathbf{\Lambda}^{\mathrm{op}}_{/[i]})^{\mathrm{act}}_{/\xi} \to \prod_{p=1}^{j} (\mathbf{\Lambda}^{\mathrm{op}}_{/[i]})^{\mathrm{act}}_{/\xi\rho_p}$$

is cofinal, or equivalently that the map

$$(\mathbf{\Lambda}_{/[i]})^{\mathrm{act}}_{\xi/} \to \prod_{p=1}^{j} (\mathbf{\Lambda}_{/[i]})^{\mathrm{act}}_{\xi\rho_p/}$$

is coinitial, where $\rho_p\colon [1]\to[j]$ is the inert map sending 0 to $p-1$ and 1 to $p$. By [28, Theorem 4.1.3.1], to see this it suffices to show that for every $X\in\prod_{p=1}^{j}(\mathbf{\Lambda}_{/[i]})^{\mathrm{act}}_{\xi\rho_p/}$, the $\infty$–category $((\mathbf{\Lambda}_{/[i]})^{\mathrm{act}}_{\xi/})_{/X}$ is weakly contractible.

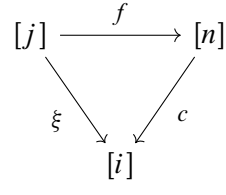
The object $X$ is given by diagrams




for $p=1,\dots,j$, where $f_p$ is active and $c_p$ is cellular. But since the $f_p$ are active we see that

$$c_p(n_p)=c_p f_p(1)=\xi(p)=c_{p+1}f_{p+1}(0)=c_{p+1}(0),$$

so the $c_p$ glue together to a unique map $c\colon[n]\to[i]$ such that $c\eta_p=c_p$, where we let $n=\sum_{p=1}^{j}n_p$ and $\eta_p\colon[n_p]\to[n]$ is the inert map $\eta_p(q)=n_1+\cdots+n_{p-1}+q$. Moreover, $c$ is clearly cellular. The maps $f_p$ then glue to an active map $f\colon[j]\to[n]$ given by $f(p)=n_1+\ldots+n_p$. The resulting object




is then final in $((\mathbf{\Lambda}_{/[i]})^{\mathrm{act}}_{\xi/})_{/X}$, hence this $\infty$–category is indeed weakly contractible. □

The following observation lets us analyze operadic Kan extensions along $\tau_n$:

**Lemma 4.17** *For all $(i,i+k)\in(\mathbf{\Delta}^{\mathrm{op}}_{/[n]})_{[1]}$ (with $k\geq 1$) the functor*

$$\mathbf{\Delta}^{(k-1),\mathrm{op}}\to(\mathbf{\Lambda}^{\mathrm{op}}_{/[n]})^{\mathrm{act}}_{/(i,i+k)}$$

*that sends $([a_1],\dots,[a_{k-1}])$ to $(i,i+1,\dots,i+1,\dots,i+(k-1),\dots,i+(k-1),i+k)$, where there are $a_j+1$ copies of $i+j$, is cofinal. In particular, there is a cofinal map from a product of copies of $\mathbf{\Delta}^{\mathrm{op}}$ to $(\mathbf{\Lambda}^{\mathrm{op}}_{/[n]})^{\mathrm{act}}_{/(i,j)}$ for all $i$ and $j$, and so a cofinal map from $\mathbf{\Delta}^{\mathrm{op}}$ by [28, Lemma 5.5.8.4]; the simplicial set $(\mathbf{\Lambda}^{\mathrm{op}}_{/[n]})^{\mathrm{act}}_{/(i,i+k)}$ is thus sifted.*

**Proof** This follows from [28, Theorem 4.1.3.1], since the category $(\mathbf{\Delta}^{(k-1),\mathrm{op}})_{X/}$ has an initial object for all $X\in(\mathbf{\Lambda}^{\mathrm{op}}_{/[n]})^{\mathrm{act}}_{/(i,i+k)}$. □

**Definition 4.18** We say a monoidal $\infty$–category has *good relative tensor products* if it is $\tau_n$–compatible (in the sense of Definition A.59) for all $n$. Similarly, we say a monoidal functor is *compatible with relative tensor products* if it is $\tau_n$–compatible (in the sense of Definition A.62) for all $n$.

**Lemma 4.19** *Let $\mathcal{C}$ be a monoidal $\infty$–category. Then $\mathcal{C}$ has good relative tensor products if and only if for every algebra $A\colon \mathbf{\Lambda}^{\mathrm{op}}_{/[2]} \to \mathcal{C}^{\otimes}$, the diagram $\mathbf{\Delta} \to (\mathbf{\Lambda}^{\mathrm{op}}_{/[2]})^{\mathrm{act}}_{/(0,2)} \to \mathcal{C}$, obtained from $A$ by cocartesian pushforward to the fibre over $[1]$, has a colimit, and this colimit is preserved by tensoring (on either side) with any object of $\mathcal{C}$. Moreover, a monoidal functor is compatible with relative tensor products if and only if it preserves these colimits.*

**Proof** This follows from Lemma 4.17 and Corollary A.44. □

Applying Corollary A.60, we get:

**Corollary 4.20** *Suppose $\mathcal{C}$ is a monoidal $\infty$–category with good relative tensor products. Then the restriction*

$$\tau_n^*\colon \mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}_{/[n]}}(\mathcal{C}) \to \mathrm{Alg}^1_{\mathbf{\Lambda}^{\mathrm{op}}_{/[n]}}(\mathcal{C})$$

*has a fully faithful left adjoint $\tau_{n,!}$. A $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$–algebra $M$ is in the image of $\tau_{n,!}$ if and only if $M$ exhibits $M(i,j)$ as the tensor product*

$$M(i,i+1)\otimes_{M(i+1,i+1)} M(i+1,i+2)\otimes_{M(i+2,i+2)}\cdots\otimes_{M(j-1,j-1)} M(j-1,j). \quad \square$$

Thus, the following is a good definition of the $\infty$–categories $\mathfrak{ALG}_1(\mathcal{C})_n$ for all $n$:

**Definition 4.21** Let $\mathcal{C}$ be a monoidal $\infty$–category with good relative tensor products. We say that a $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$–algebra $M$ in $\mathcal{C}$ is *composite* if the counit map $\tau_{n,!}\tau_n^* M \to M$ is an equivalence, or equivalently if $M$ is in the essential image of the functor $\tau_{n,!}$. We write $\mathfrak{ALG}_1(\mathcal{C})_n$ for the full subcategory of $\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}_{/[n]}}(\mathcal{C})$ spanned by the composite $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$–algebras.

## 4.3 The Segal condition

Our goal in this subsection is to prove that the $\infty$–categories $\mathfrak{ALG}_1(\mathcal{C})_i$ satisfy the Segal condition, ie that the natural map

$$\mathfrak{ALG}_1(\mathcal{C})_i \to \mathfrak{ALG}_1(\mathcal{C})_1 \times_{\mathfrak{ALG}_1(\mathcal{C})_0} \cdots \times_{\mathfrak{ALG}_1(\mathcal{C})_0} \mathfrak{ALG}_1(\mathcal{C})_1$$

is an equivalence of $\infty$–categories. We will prove this by showing that for every $i$ the generalized nonsymmetric $\infty$–operad $\mathbf{\Lambda}^{\mathrm{op}}_{/[i]}$ is equivalent to the colimit

$$\mathbf{\Delta}^{\mathrm{op}}_{/[1]} \amalg_{\mathbf{\Delta}^{\mathrm{op}}_{/[0]}} \cdots \amalg_{\mathbf{\Delta}^{\mathrm{op}}_{/[0]}} \mathbf{\Delta}^{\mathrm{op}}_{/[1]}$$

in $\mathrm{Opd}_\infty^{\mathbf{\Delta},\mathrm{gen}}$. To do this we use the model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_1^{\mathrm{gen}}}$ defined in Section A.1 and check that $\mathbf{\Lambda}^{\mathrm{op}}_{/[i]}$ is a homotopy colimit; this boils down to checking that a certain map is a trivial cofibration.

We write $\mathbf{\Delta}^{\amalg,\mathrm{op}}_{/[i]}$ for the ordinary colimit $\mathbf{\Delta}^{\mathrm{op}}_{/[1]} \amalg_{\mathbf{\Delta}^{\mathrm{op}}_{/[0]}} \cdots \amalg_{\mathbf{\Delta}^{\mathrm{op}}_{/[0]}} \mathbf{\Delta}^{\mathrm{op}}_{/[1]}$ in (marked) simplicial sets (over $\mathbf{\Delta}^{\mathrm{op}}$). Since this colimit can be written as an iterated pushout along injective maps of simplicial sets, this colimit in simplicial sets is a homotopy colimit corresponding to the $\infty$–categorical colimit we're interested in. Moreover, there is an obvious inclusion $\mathbf{\Delta}^{\amalg,\mathrm{op}}_{/[i]} \hookrightarrow \mathbf{\Lambda}^{\mathrm{op}}_{/[i]}$. Our aim in this subsection is then to prove the following:

**Proposition 4.22** *The inclusion $\mathbf{\Delta}^{\amalg,\mathrm{op}}_{/[i]} \hookrightarrow \mathbf{\Lambda}^{\mathrm{op}}_{/[i]}$ is a trivial cofibration in the model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_1^{\mathrm{gen}}}$.*

Before we turn to the proof, let us first see that this does indeed imply the Segal condition for $\mathfrak{ALG}_n(\mathcal{C})$:

**Corollary 4.23** *Let $\mathcal{M}$ be a generalized nonsymmetric $\infty$–operad. The restriction map*

$$\mathrm{Alg}^1_{\mathbf{\Lambda}^{\mathrm{op}}_{/[n]}}(\mathcal{M}) \to \mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}_{/[1]}}(\mathcal{M}) \times_{\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M})} \cdots \times_{\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M})} \mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}_{/[1]}}(\mathcal{M})$$

*is an equivalence of $\infty$–categories.*

**Proof** Since the model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_1^{\mathrm{gen}}}$ is enriched in marked simplicial sets and the inclusion $\mathbf{\Delta}^{\amalg,\mathrm{op}}_{/[n]} \hookrightarrow \mathbf{\Lambda}^{\mathrm{op}}_{/[n]}$ is a trivial cofibration by Proposition 4.22, for any generalized nonsymmetric $\infty$–operad $\mathcal{M}$ the restriction map $\mathrm{Alg}^1_{\mathbf{\Lambda}^{\mathrm{op}}_{/[n]}}(\mathcal{M}) \to \mathrm{Alg}^1_{\mathbf{\Delta}^{\amalg,\mathrm{op}}_{/[n]}}(\mathcal{M})$ is a trivial Kan fibration, and the map

$$\mathrm{Alg}^1_{\mathbf{\Delta}^{\amalg,\mathrm{op}}_{/[n]}}(\mathcal{M}) \to \mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}_{/[1]}}(\mathcal{M}) \times_{\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M})} \cdots \times_{\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M})} \mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}_{/[1]}}(\mathcal{M})$$

is an equivalence of $\infty$–categories since $\mathbf{\Delta}^{\amalg,\mathrm{op}}_{/[n]}$ is a homotopy colimit. □

**Corollary 4.24** *Let $\mathcal{C}$ be a monoidal $\infty$–category with good relative tensor products. Then the natural restriction map*

$$\mathfrak{ALG}_1(\mathcal{C})_n \to \mathfrak{ALG}_1(\mathcal{C})_1 \times_{\mathfrak{ALG}_1(\mathcal{C})_0} \cdots \times_{\mathfrak{ALG}_1(\mathcal{C})_0} \mathfrak{ALG}_1(\mathcal{C})_1$$

*is an equivalence.*

**Proof** This map factors as a composite of the maps

$$\mathfrak{ALG}_1(\mathcal{C})_n \to \mathrm{Alg}^1_{\mathbf{\Lambda}^{\mathrm{op}}_{/[n]}}(\mathcal{C}) \to \mathfrak{ALG}_1(\mathcal{C})_1 \times_{\mathfrak{ALG}_1(\mathcal{C})_0} \cdots \times_{\mathfrak{ALG}_1(\mathcal{C})_0} \mathfrak{ALG}_1(\mathcal{C})_1,$$

where the first is an equivalence by definition and the second by Corollary 4.23. □

We will deduce Proposition 4.22 from a rather technical result about trivial cofibrations in $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_1^{\mathrm{gen}}}$. To state this, we first need to introduce some terminology for simplices in the nerve of $\mathbf{\Delta}^{\mathrm{op}}$.

**Warning 4.25** Throughout the remainder of this section we are really working with *marked* simplicial sets. However, to simplify the notation we will not indicate the marking in any way — thus if eg $\mathcal{O}$ is a generalized nonsymmetric $\infty$–operad we are really thinking of it as the marked simplicial set $(\mathcal{O}, I)$ where $I$ is the collection of inert morphisms. Similarly, all simplicial subsets of generalized nonsymmetric $\infty$–operads are really marked by the inert morphisms that they contain.

**Definition 4.26** Let $\sigma$ be an $n$–simplex in $N\mathbf{\Delta}^{\mathrm{op}}$, ie a diagram

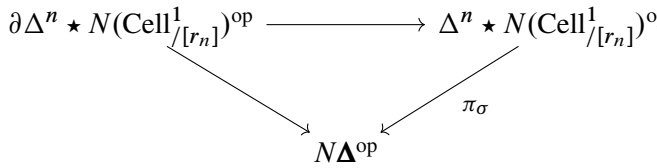
$$\sigma = [r_0] \xrightarrow{f_1} [r_1] \xrightarrow{f_2} \cdots \xrightarrow{f_n} [r_n]$$

in $\mathbf{\Delta}^{\mathrm{op}}$ (where, in terms of the category $\mathbf{\Delta}$, each $f_i$ is a map of ordered sets from $[r_i]$ to $[r_{i-1}]$); for convenience, we will let the symbols $[r_i]$ and $f_i$ denote the objects and morphisms in any such $n$–simplex we encounter from now on. We say that $\sigma$ is *narrow* if $r_n = 1$ and *wide* if $r_n > 1$. If $\sigma$ is wide, we have an induced diagram $\pi_\sigma\colon \Delta^n \star N(\mathrm{Cell}^1_{/[r_n]})^{\mathrm{op}} \to N\mathbf{\Delta}^{\mathrm{op}}$ by adding the inert morphisms from $[r_n]$ to $[1]$ and $[0]$. The *decomposition simplices* of $\sigma$ are the simplices in the image of this diagram.

**Definition 4.27** We say a morphism $\phi$ in $\mathbf{\Delta}^{\mathrm{op}}$ is *neutral* if it is neither active nor inert. If $\sigma$ is an $n$–simplex of $N\mathbf{\Delta}^{\mathrm{op}}$ such that $f_k$ is neutral, we say that $\sigma$ is *$k$–factorizable*. The *$k$–factored $(n+1)$–simplex* of $\sigma$ is then that obtained by taking the inert–active factorization of $f_k$.

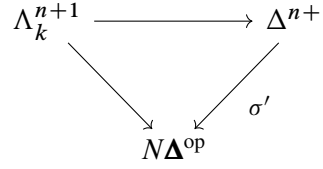
From the definition of the model structure for a categorical pattern $\mathfrak{P}$ in [31, Section B.2] it follows that the $\mathfrak{P}$–anodyne morphisms defined in [31, Definition B.1.1] are trivial cofibrations. In the case $\mathfrak{P} = \mathfrak{O}_1^{\mathrm{gen}}$, we have in particular that:

- If $\sigma$ is a wide $n$–simplex in $N\mathbf{\Delta}^{\mathrm{op}}$ and $\pi_\sigma\colon \Delta^n \star N(\mathrm{Cell}^1_{/[r_n]})^{\mathrm{op}} \to N\mathbf{\Delta}^{\mathrm{op}}$ is the diagram as above, then the inclusion




is a trivial cofibration.

- If $\sigma$ is a $k$–factorizable $n$–simplex and $\sigma'$ is its $k$–factored $(n+1)$–simplex, then the inclusion




is a trivial cofibration.

We will prove Proposition 4.22 by constructing a rather intricate filtration where each inclusion is a pushout of a trivial cofibration of one of these two types. To define this we need some more notation:

**Notation 4.28** We define the following sets of simplices in $N\mathbf{\Delta}^{\mathrm{op}}$:

- For $1 \le r < k \le n$, let $A_n(k,r)$ be the set of nondegenerate narrow $n$–simplices $\sigma$ such that $f_r$ is inert, $f_k$ is neutral and $f_p$ is active for $r < p < k$ and $p > k$.
- For $1 \le r < k \le n$, let $A'_n(k,r)$ be the set of nondegenerate $(n+1)$–simplices $\sigma$ such that $r_n = 1$, $r_{n+1} = 0$, $f_r$ is inert, $f_k$ is neutral and $f_p$ is active for $r < p < k$ and $p > k$.
- For $1 \le k \le n$, let $B_n(k)$ be the set of nondegenerate narrow $n$–simplices $\sigma$ such that $f_k$ is neutral, $f_p$ is active for $p > k$ and $\sigma$ is not contained in $A_n(k,r)$ for any $r$.
- For $1 \le k \le n$, let $B'_n(k)$ be the set of nondegenerate $(n+1)$–simplices $\sigma$ such that $r_n = 1$, $r_{n+1} = 0$, $f_k$ is neutral, $f_p$ is active for $k < p < n+1$ and $\sigma$ is not contained in $A'_n(k,r)$ for any $r$.

Now define $\mathfrak{F}_n \subseteq N\mathbf{\Delta}^{\mathrm{op}}$ to be the simplicial subset containing all the nondegenerate $i$–simplices for $i \le n$ together with:

- For every wide $i$–simplex with $i \le n$, its decomposition simplices.
- The $k$–factored $(i+1)$–simplices of the simplices in $A_i(k,r)$ and $B_i(k)$ for all $k$, $r$ and all $i \le n$.

- The $k$–factored $(i+2)$–simplices of the simplices in $A'_i(k,r)$ and $B'_i(k)$ for all $k$, $r$ and all $i \leq n$.

Then let $\mathfrak{F}_n^+$ denote the simplicial subset containing the simplices in $\mathfrak{F}_n$ together with the narrow *active* $(n+1)$–simplices, meaning those such that all the morphisms $f_i$ are active.

A "prototype" version of our technical result is then: For every $n$, the inclusion $\mathfrak{F}_{n-1}^+ \hookrightarrow \mathfrak{F}_n$ is a trivial cofibration in the generalized nonsymmetric $\infty$–operad model structure. We actually need a slightly more general "relative" version of this, which we are ready to state and prove after introducing a little more notation:

**Notation 4.29** Let $\boldsymbol{O}$ be an ordinary category whose objects have no nontrivial automorphisms, equipped with a map $\boldsymbol{O} \to \boldsymbol{\Delta}^{\mathrm{op}}$ that exhibits $\boldsymbol{O}$ as a generalized nonsymmetric $\infty$–operad. We say a simplex in $N\boldsymbol{O}$ is *narrow*, *wide* or *$k$–factorizable* if this is true of its image in $N\boldsymbol{\Delta}^{\mathrm{op}}$. For such $\boldsymbol{O}$ the inert–active factorizations in $\boldsymbol{O}$ are strictly unique (rather than just unique up to isomorphism), and we can define the decomposition simplices of a wide simplex and the $k$–factored $(n+1)$–simplex of a $k$–factorizable $n$–simplex just as before. If $N\boldsymbol{O}_0$ is a simplicial subset of $N\boldsymbol{O}$ we (slightly abusively) write $\mathfrak{F}_n\boldsymbol{O}$ for the simplicial subset of $N\boldsymbol{O}$ containing the simplices in $N\boldsymbol{O}_0$ together with those lying over the simplices in $\mathfrak{F}_n$; we also define $\mathfrak{F}_n^+\boldsymbol{O}$ similarly.

**Proposition 4.30** *Let $\boldsymbol{O}$ be as above. Suppose $N\boldsymbol{O}_0$ is a simplicial subset of $N\boldsymbol{O}$ such that:*

- *For every wide simplex contained in $N\boldsymbol{O}_0$, its decomposition simplices are also contained in $N\boldsymbol{O}_0$.*
- *For every $n$–simplex in $N\boldsymbol{O}_0$ whose image in $N\boldsymbol{\Delta}^{\mathrm{op}}$ is in $A_n(k,r)$ or $B_n(k)$ for some $k$ and $r$, its $k$–factored $(n+1)$–simplex is also in $N\boldsymbol{O}_0$.*
- *For every $(n+1)$–simplex in $N\boldsymbol{O}_0$ whose image in $N\boldsymbol{\Delta}^{\mathrm{op}}$ is in $A'_n(k,r)$ or $B'_n(k)$ for some $k$ and $r$, its $k$–factored $(n+2)$–simplex is also in $N\boldsymbol{O}_0$.*

*Then the inclusion*

$$\mathfrak{F}_{n-1}^+\boldsymbol{O} \hookrightarrow \mathfrak{F}_n\boldsymbol{O}$$

*is a trivial cofibration in the generalized nonsymmetric $\infty$–operad model structure.*

**Remark 4.31** It is not really necessary to assume that the objects of $\boldsymbol{O}$ have no automorphisms for the proof to go through: it suffices, as in the proof of [31, Theorem 3.1.2.3], to assume that the inert–active factorization system can be refined to a *strict*

factorization system, ie one where the factorizations are defined uniquely, not just up to isomorphism. This slight generalization is not needed for any of our applications, however.

**Proof** The basic idea of the proof is to define a filtration of $\mathfrak{F}_n\boldsymbol{O}$, starting with $\mathfrak{F}^+_{n-1}\boldsymbol{O}$, such that each step in the filtration is a pushout of a trivial cofibration of one of the two types we discussed above.

Let us say that a simplex in $\mathfrak{F}_n\boldsymbol{O}$ is *old* if it is contained in $\mathfrak{F}^+_{n-1}\boldsymbol{O}$, and *new* if it is not. We also write $\bar{A}_n(k,r)$ for the set of new $n$–simplices whose image in $N\boldsymbol{\Delta}^{\mathrm{op}}$ lies in $A_n(k,r)$, and define $\bar{A}'_n(k,r)$, $\bar{B}_n(k)$ and $\bar{B}'_n(k)$ similarly. The filtration is then defined as follows:

- Set $\mathcal{F}_0 := \mathfrak{F}^+_{n-1}\boldsymbol{O}$.
- Let $S_1$ be the set of nondegenerate wide new $n$–simplices such that $f_n$ is inert. We let $\mathcal{F}_1$ be the simplicial subset of $\mathfrak{F}_n\boldsymbol{O}$ containing $\mathcal{F}_0$ together with the $n$–simplices in $S_1$ as well as their decomposition $(n+1)$– and $(n+2)$–simplices.
- Let $S_2(r)$ be the set of nondegenerate wide new $n$–simplices such that $f_r$ is inert and $f_p$ is active for $p > r$. We set $\mathcal{F}_2$ to be the simplicial subset of $\mathfrak{F}_n\boldsymbol{O}$ containing $\mathcal{F}_1$ together with
  - the $n$–simplices in $S_2(r)$ for all $r$ and their decomposition $(n+1)$– and $(n+2)$–simplices;
  - the $n$–simplices in $\bar{A}_n(k,r)$ for all $k$ and $r$ and their $k$–factored $(n+1)$–simplices;
  - the $(n+1)$–simplices in $\bar{A}'_n(k,r)$ for all $k$ and $r$ and their $k$–factored $(n+2)$–simplices.
- Let $\mathcal{F}_3$ be the simplicial subset of $\mathfrak{F}_n\boldsymbol{O}$ containing $\mathcal{F}_2$ together with the $n$–simplices in $\bar{B}_n(k)$ for all $k$ and their $k$–factored $(n+1)$–simplices, as well as the $(n+1)$–simplices in $\bar{B}'_n(k)$ for all $k$ and their $k$–factored $(n+2)$–simplices.
- Let $S_4$ be the set of nondegenerate wide new $n$–simplices that are not contained in $\mathcal{F}_3$. Then $\mathcal{F}_4 := \mathfrak{F}_n\boldsymbol{O}$ consists of the simplices in $\mathcal{F}_3$ together with the $n$–simplices in $S_4$ and their decomposition $(n+1)$– and $(n+2)$–simplices.

We then need to prove that the four inclusions $\mathcal{F}_{m-1} \hookrightarrow \mathcal{F}_m$ are all trivial cofibrations.

$\boldsymbol{m=1}$ If $\sigma$ is an $n$–simplex in $N\boldsymbol{O}$, we write $\pi_\sigma$ for the induced diagram

$$\Delta^n \star N(\mathrm{Cell}^1_{/[r_n]})^{\mathrm{op}} \to N\boldsymbol{O}$$

and $\pi_\sigma^\partial$ for the restriction of this map to $\partial\Delta^n \star N(\mathrm{Cell}^1_{/[r_n]})^{\mathrm{op}}$. For $\sigma$ in $S_1$, observe that since any narrow new $n$–simplex whose final map is inert is contained in $\mathcal{F}_0$, as is any new $(n+1)$–simplex whose final map is $[1]\to[0]$ and whose penultimate map is inert, the map $\pi_\sigma^\partial$ factors through $\mathcal{F}_0$. Thus we have a pushout diagram:

$$\begin{array}{ccc} \coprod_{\sigma\in S_1} \partial\Delta^n \star N(\mathrm{Cell}^1_{/[r_n]})^{\mathrm{op}} & \longrightarrow & \coprod_{\sigma\in S_1} \Delta^n \star N(\mathrm{Cell}^1_{/[r_n]})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 \end{array}$$

Since the upper horizontal map is $\mathfrak{O}_1^{\mathrm{gen}}$–anodyne, so is the lower horizontal map.

***m* = 2** This is the most convoluted step, as we must consider several subsidiary filtrations for the inclusion $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$. We will inductively define a filtration

$$\mathcal{F}_1 = \mathcal{G}_n \subseteq \mathcal{G}'_{n-1} \subseteq \mathcal{G}_{n-1} \subseteq \cdots \subseteq \mathcal{G}'_1 \subseteq \mathcal{G}_1 = \mathcal{F}_2,$$

where $\mathcal{G}'_r$ is itself defined via a filtration

$$\mathcal{G}_{r+1} = \mathcal{I}_{r,r} \subseteq \mathcal{I}'_{r,r+1} \subseteq \mathcal{I}_{r,r+1} \subseteq \cdots \subseteq \mathcal{I}'_{r,n} \subseteq \mathcal{I}_{r,n} = \mathcal{G}'_r.$$

This goes as follows:

- We define $\mathcal{I}'_{r,k}$ to be the simplicial subset of $\mathcal{F}_2$ containing the simplices in $\mathcal{I}_{r,k-1}$ together with the $n$–simplices in $\bar{A}_n(k,r)$ as well as their $k$–factored $(n+1)$–simplices.
- We define $\mathcal{I}_{r,k}$ to be the simplicial subset of $\mathcal{F}_2$ containing the simplices in $\mathcal{I}'_{r,k}$ together with the $(n+1)$–simplices in $\bar{A}'_n(k,r)$ as well as their $k$–factored $(n+2)$–simplices.
- We define $\mathcal{G}_r$ to be the simplicial subset of $\mathcal{F}_2$ containing $\mathcal{G}'_r$ together with the $n$–simplices in $S_2(r)$ as well as their decomposition $(n+1)$– and $(n+2)$–simplices.

Then it suffices to show that the inclusions $f_{r,k}\colon \mathcal{I}_{r,k-1} \hookrightarrow \mathcal{I}'_{r,k}$, $g_{r,k}\colon \mathcal{I}'_{r,k} \hookrightarrow \mathcal{I}_{r,k}$ and $h_r\colon \mathcal{G}'_r \hookrightarrow \mathcal{G}_r$ are all trivial cofibrations.

Observe that for $\sigma$ in $\bar{A}_n(k,r)$ with $k$–factored $(n+1)$–simplex $\tau$, the faces $d_j\tau$ with $j\neq k$ are contained in $\mathcal{I}_{r,k-1}$. Thus we get a pushout diagram

$$\begin{array}{ccc} \coprod_{\sigma\in\bar{A}_n(k,r)} \Lambda^{n+1}_k & \longrightarrow & \coprod_{\sigma\in\bar{A}_n(k,r)} \Delta^{n+1} \\ \downarrow & & \downarrow \\ \mathcal{I}_{r,k-1} & \longrightarrow & \mathcal{I}'_{r,k} \end{array}$$

and so $f_{r,k}$ is a trivial cofibration. Similarly, for $\sigma$ in $\bar{A}'_n(k,r)$ with $k$–factored $(n+2)$–simplex $\tau$ the faces $d_j\tau$ with $j \neq k$ are contained in $\mathcal{I}'_{r,k}$. We therefore have another pushout diagram

$$\begin{array}{ccc} \coprod_{\sigma\in\bar{A}'_n(k,r)} \Lambda^{n+2}_k & \longrightarrow & \coprod_{\sigma\in\bar{A}'_n(k,r)} \Delta^{n+2} \\ \downarrow & & \downarrow \\ \mathcal{I}'_{k,r} & \longrightarrow & \mathcal{I}_{k,r} \end{array}$$

hence $g_{r,k}$ is also a trivial cofibration.

Now for $\sigma \in S_2(r)$ the map $\pi^{\partial}_{\sigma}$ factors through $\mathcal{G}'_r$, so we have a pushout square

$$\begin{array}{ccc} \coprod_{\sigma\in S_2(k)} \partial\Delta^n \star N(\mathrm{Cell}^1_{/[r_n]})^{\mathrm{op}} & \longrightarrow & \coprod_{\sigma\in S_2(k)} \Delta^n \star N(\mathrm{Cell}^1_{/[r_n]})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathcal{G}'_r & \longrightarrow & \mathcal{G}_r \end{array}$$

which implies that $h_r$ is a trivial cofibration.

***m* = 3** We again need to define a subsidiary filtration

$$\mathcal{F}_2 = \mathcal{H}_0 \subseteq \mathcal{H}'_1 \subseteq \mathcal{H}_1 \subseteq \cdots \subseteq \mathcal{H}'_n \subseteq \mathcal{H}_n = \mathcal{F}_3.$$

Here we inductively define $\mathcal{H}'_k$ to be the subset of $\mathcal{F}_3$ containing $\mathcal{H}_{k-1}$ together with the $n$–simplices in $\bar{B}_n(k)$ as well as their $k$–factored $(n+1)$–simplices, and then define $\mathcal{H}_k$ to be that containing $\mathcal{H}'_k$ together with the $(n+1)$–simplices in $\bar{B}'_n(k)$ as well as their $k$–factored $(n+2)$–simplices. It then suffices to prove that the inclusions $\mathcal{H}_{k-1} \hookrightarrow \mathcal{H}'_k$ and $\mathcal{H}'_k \hookrightarrow \mathcal{H}_k$ are trivial cofibrations. If $\sigma \in \bar{B}_n(k)$ and $\tau$ is its $k$–factored simplex, then $d_j\tau$ lies in $\mathcal{H}_{k-1}$ for $j \neq k$, so we have a pushout square

$$\begin{array}{ccc} \coprod_{\sigma\in\bar{B}_n(k)} \Lambda^{n+1}_k & \longrightarrow & \coprod_{\sigma\in\bar{B}_n(k)} \Delta^{n+1} \\ \downarrow & & \downarrow \\ \mathcal{H}_{k-1} & \longrightarrow & \mathcal{H}'_k \end{array}$$

and hence the inclusion $\mathcal{H}_{k-1} \hookrightarrow \mathcal{H}'_k$ is a trivial cofibration. Similarly, we have a pushout square

$$\begin{array}{ccc} \coprod_{\sigma\in\bar{B}'_n(k)} \Lambda^{n+2}_k & \longrightarrow & \coprod_{\sigma\in\bar{B}'_n(k)} \Delta^{n+2} \\ \downarrow & & \downarrow \\ \mathcal{H}'_k & \longrightarrow & \mathcal{H}_k \end{array}$$

so the inclusion $\mathcal{H}'_k \hookrightarrow \mathcal{H}_k$ is also a trivial cofibration.

**$m = 4$** Observe that for $\sigma$ in $S_4$ the diagram $\pi^{\partial}_{\sigma}$ factors through $\mathcal{F}_3$, and so we have a pushout diagram:

$$\begin{array}{ccc} \coprod_{\sigma\in S_4} \partial\Delta^n \star N(\mathrm{Cell}^1_{/[r_n]})^{\mathrm{op}} & \longrightarrow & \coprod_{\sigma\in S_4} \Delta^n \star N(\mathrm{Cell}^1_{/[r_n]})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathcal{F}_3 & \longrightarrow & \mathcal{F}_4 \end{array}$$

The inclusion $\mathcal{F}_3 \to \mathcal{F}_4$ is therefore also a trivial cofibration, which completes the proof. □

**Corollary 4.32** *Let $\boldsymbol{O}$ be an ordinary category whose objects have no nontrivial automorphisms, equipped with a map $\boldsymbol{O} \to \boldsymbol{\Delta}^{\mathrm{op}}$ that exhibits $\boldsymbol{O}$ as a generalized nonsymmetric $\infty$–operad. Suppose $N\boldsymbol{O}_0$ is a simplicial subset of $N\boldsymbol{O}$ such that*

- *every narrow active simplex in $N\boldsymbol{O}$ is contained in $N\boldsymbol{O}_0$;*
- *for every wide simplex contained in $N\boldsymbol{O}_0$, its decomposition simplices are also contained in $N\boldsymbol{O}_0$;*
- *for every $n$–simplex in $N\boldsymbol{O}_0$ whose image in $N\boldsymbol{\Delta}^{\mathrm{op}}$ is in $A_n(k,r)$ and $B_n(k)$ for some $k$ and $r$, its $k$–factored $(n+1)$–simplex is also in $N\boldsymbol{O}_0$;*
- *for every $(n+1)$–simplex in $N\boldsymbol{O}_0$ whose image in $N\boldsymbol{\Delta}^{\mathrm{op}}$ is in $A'_n(k,r)$ and $B'_n(k)$ for some $k$ and $r$, its $k$–factored $(n+2)$–simplex is also in $N\boldsymbol{O}_0$.*

*Then the inclusion $N\boldsymbol{O}_0 \hookrightarrow N\boldsymbol{O}$ is a trivial cofibration of generalized nonsymmetric $\infty$–operads.*

**Proof** Since $N\boldsymbol{O}_0$ contains all narrow active simplices in $N\boldsymbol{O}$, the simplicial subsets $\mathfrak{F}_n\boldsymbol{O}$ and $\mathfrak{F}^+_n\boldsymbol{O}$ of $N\boldsymbol{O}$ coincide. The inclusion $N\boldsymbol{O}_0 \hookrightarrow N\boldsymbol{O}$ is therefore the composite of the inclusions $\mathfrak{F}_{n-1}\boldsymbol{O} = \mathfrak{F}^+_{n-1}\boldsymbol{O} \hookrightarrow \mathfrak{F}_n\boldsymbol{O}$, which are all trivial cofibrations by Proposition 4.30. □

**Proof of Proposition 4.22** We apply Corollary 4.32 to the inclusion $\boldsymbol{\Delta}^{\amalg,\mathrm{op}}_{/[i]} \hookrightarrow \boldsymbol{\Lambda}^{\mathrm{op}}_{/[i]}$. The required hypotheses hold since a simplex of $\boldsymbol{\Lambda}^{\mathrm{op}}_{/[i]}$ lies in $\boldsymbol{\Delta}^{\amalg,\mathrm{op}}_{/[i]}$ if and only if its source is of the form $(i_0,\dots,i_n)$ with $i_n - i_0 \le 1$. □

## 4.4 The double $\infty$–category of algebras

Our goal in this subsection is to prove that the $\infty$–categories $\mathfrak{ALG}_1(\mathcal{C})_n$ fit together into a simplicial $\infty$–category. We will do this by checking that composite $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$–algebras map to composite $\mathbf{\Delta}^{\mathrm{op}}_{/[m]}$–algebras under composition with the map

$$\phi_*\colon \mathbf{\Delta}^{\mathrm{op}}_{/[m]} \to \mathbf{\Delta}^{\mathrm{op}}_{/[n]}$$

induced by a map $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$.

**Definition 4.33** Suppose $\mathcal{C}$ is a monoidal $\infty$–category. Let $\overline{\mathfrak{ALG}}_1(\mathcal{C}) \to \mathbf{\Delta}^{\mathrm{op}}$ denote a cocartesian fibration associated to the functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends $[n]$ to $\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}_{/[n]}}(\mathcal{C})$. Write $\mathfrak{ALG}_1(\mathcal{C})$ for the full subcategory of $\overline{\mathfrak{ALG}}_1(\mathcal{C})$ spanned by the objects of $\mathfrak{ALG}_1(\mathcal{C})_n$ for all $n$, ie the composite $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$–algebras for all $n$.

We wish to prove that the restricted projection $\mathfrak{ALG}_1(\mathcal{C}) \to \mathbf{\Delta}^{\mathrm{op}}$ is a cocartesian fibration, with the cocartesian morphisms inherited from $\overline{\mathfrak{ALG}}_1(\mathcal{C})$. The key step in the proof is showing that a certain functor is cofinal; to state the required result we first need the following technical generalization of cellular maps:

**Definition 4.34** Suppose $\phi\colon [m] \to [n]$ is an injective morphism in $\mathbf{\Delta}$. We say a morphism $\alpha\colon [k] \to [n]$ is *$\phi$–cellular* if

(i) for $\alpha(i) < \phi(0)$ we have $\alpha(i+1) \leq \alpha(i) + 1$,

(ii) for $\phi(j) \leq \alpha(i) < \phi(j+1)$ we have $\alpha(i+1) \leq \phi(j+1)$,

(iii) for $\alpha(i) \geq \phi(m)$ we have $\alpha(i+1) \leq \alpha(i) + 1$.

**Remark 4.35** We recover the previous notion of *cellular* maps to $[n]$ as the $\phi$–cellular maps with $\phi = \mathrm{id}_{[n]}$.

**Definition 4.36** For $[n] \in \mathbf{\Delta}$ and $\phi\colon [m] \to [n]$ any injective morphism in $\mathbf{\Delta}$, we write $\mathbf{\Lambda}_{/[n]}[\phi]$ for the full subcategory of $\mathbf{\Delta}_{/[n]}$ spanned by the $\phi$–cellular maps to $[n]$.

**Proposition 4.37** (i) *If $\phi\colon [m] \to [n]$ is an injective morphism in $\mathbf{\Delta}$, then for any $\gamma\colon [k] \to [m]$ the map*

$$\phi_*\colon (\mathbf{\Lambda}_{/[m]})^{\mathrm{act}}_{\gamma/} \to (\mathbf{\Lambda}_{/[n]}[\phi])^{\mathrm{act}}_{\phi\gamma/}$$

*given by composition with $\phi$ is coinitial.*

(ii) *If $\phi\colon [m] \to [n]$ is a surjective morphism in $\mathbf{\Delta}$, then for any $\gamma\colon [k] \to [m]$ the map*

$$\phi_*\colon (\mathbf{\Lambda}_{/[m]})^{\mathrm{act}}_{\gamma/} \to (\mathbf{\Lambda}_{/[n]})^{\mathrm{act}}_{\phi\gamma/}$$

*given by composition with $\phi$ is coinitial.*

**Proof** We first prove (i), ie we consider an injective map $\phi$. To show that $\phi_*$ is coinitial, recall that by [28, Theorem 4.1.3.1] it suffices to prove that for each $X \in (\mathbf{\Lambda}_{/[n]}[\phi])^{\mathrm{act}}_{\phi\gamma/}$, the category $((\mathbf{\Lambda}_{/[l]})^{\mathrm{act}}_{\gamma/})_{/X}$ is weakly contractible. The object $X$ is a diagram

$$\begin{array}{ccc} [k] & \xrightarrow{\alpha} & [p] \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\xi} \\ [l] & \xrightarrow[\phi]{} & [n] \end{array}$$

where $\xi$ is a $\phi$–cellular map and $\alpha$ is active, and an object $\overline{X} \in ((\mathbf{\Lambda}_{/[l]})^{\mathrm{act}}_{\gamma/})_{/X}$ is a diagram

$$\begin{array}{ccccc} & & \overset{\alpha}{\frown} & & \\ [k] & \xrightarrow[\pi]{} & [q] & \xrightarrow[\lambda]{} & [p] \\ {\scriptstyle\gamma}\downarrow & \swarrow{\scriptstyle\theta} & & & \downarrow{\scriptstyle\xi} \\ [l] & & \xrightarrow[\phi]{} & & [n] \end{array}$$

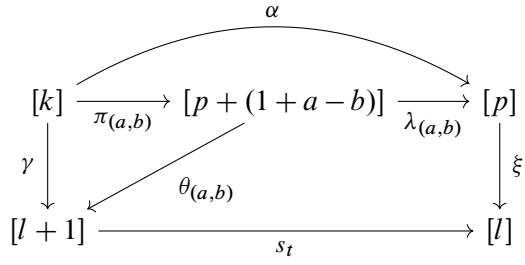
where $\theta$ is a cellular map and $\pi$ and $\lambda$ are active.

Since $\phi$ is injective, this category has a final object, given as follows: Let $[q] = \xi^{-1}([l])$, let $\lambda$ be the inclusion $[q] \to [p]$ and let $\theta$ be the induced projection $[q] \to [l]$ — since $\xi$ is $\phi$–cellular, $\theta$ hits everything in $[l]$ and so is cellular. Moreover, $\alpha$ factors through a map $\pi\colon [k] \to [q]$, since $\xi\alpha = \phi\gamma$ and so the image of $\alpha$ in $[p]$ maps to the image of $\phi$ in $[n]$. But then, since $\alpha$ is active, the maps $\pi$ and $\lambda$ must also be active, so we have defined an object of $((\mathbf{\Lambda}_{/[l]})^{\mathrm{act}}_{\gamma/})_{/X}$. Any other object of the category has a unique map to this, ie this is a final object. This implies that the category $((\mathbf{\Lambda}_{/[l]})^{\mathrm{act}}_{\gamma/})_{/X}$ is weakly contractible.

We now consider (ii), the surjective case. We can write $\phi$ as a composite of elementary degeneracies, and so it suffices to consider the case where $\phi$ is an elementary degeneracy $s_t\colon [l+1] \to [l]$. We again wish to apply [28, Theorem 4.1.3.1] and show that for each $X \in (\mathbf{\Lambda}_{/[l]})^{\mathrm{act}}_{s_t\gamma/}$ the category $((\mathbf{\Lambda}_{/[l+1]})^{\mathrm{act}}_{\gamma/})_{/X}$ is weakly contractible. Let $X$ be as above, and let $\Lambda_X$ denote the partially ordered set of pairs $(a, b)$ where

- $a, b \in [p]$,
- $\xi(a) = \xi(b) = t$,
- $a \leq b$,
- if $i \in [k]$ satisfies $\gamma(i) = t$ then $\alpha(i) \leq a$,
- if $i \in [k]$ satisfies $\gamma(i) = t+1$ then $\alpha(i) \geq b$,

where $(a,b) \leq (a',b')$ if $a \leq a' \leq b' \leq b$. Define a functor $G_X\colon \Lambda_X \to ((\mathbf{\Lambda}_{/[l+1]})^{\mathrm{act}}_{\gamma/})_{/X}$ by sending $(a,b)$ to the diagram




where

$$\theta_{(a,b)}(i) = \begin{cases} \xi(i), & i \leq a, \\ \xi(i)+1, & i > a, \end{cases}$$

$$\lambda_{(a,b)}(i) = \begin{cases} i, & i \leq a \\ i-(1+a-b), & i > a, \end{cases}$$

$$\pi_{(a,b)}(i) = \begin{cases} \alpha(i), & i \leq a, \\ \alpha(i)+(1+a-b), & i > a. \end{cases}$$

Here $\theta_{(a,b)}$ is cellular, the maps $\lambda_{(a,b)}$ and $\pi_{(a,b)}$ are active, and the diagram commutes. The maps from $(a,b)$ to $(a,b-1)$ and $(a+1,b)$ are sent by $G_X$ to the obvious transformations of diagrams including the face maps $d_b$, $d_a\colon [p+(1+a-b)] \to [p+(2+a-b)]$, respectively.

Now observe that $G_X$ has a left adjoint $F_X\colon ((\mathbf{\Lambda}^{\mathrm{act}}_{/[l]})_{\gamma/})_{/X} \to \Lambda_X$. This sends a diagram as above to $(a,b)$ where $a$ is maximal such that there exists $i \in [q]$ with $\theta(i) = t$ and $\lambda(i) = a$, and $b$ is minimal such that there exists $i$ with $\theta(i) = t+1$ and $\lambda(i) = b$. We have $F_X G_X = \mathrm{id}$, and the unit map $\mathrm{id} \to G_X F_X$ is given by the natural diagram containing the map $\overline{\lambda}\colon [q] \to [p+(1+a-b)]$ defined by

$$\overline{\lambda}(i) = \begin{cases} \lambda(i), & i \leq a, \\ \lambda(i)+(1+a-b), & i > a. \end{cases}$$

Since adjunctions of $\infty$–categories are in particular weak homotopy equivalences of simplicial sets, it follows that $((\mathbf{\Delta}^{\mathrm{cell,act}}_{/[l]})_{\gamma/})_{/X}$ is weakly contractible if and only if $\Lambda_X$ is. But $\Lambda_X$ has an initial object, namely $(A,B)$ where $A$ is minimal such that $\xi(A) = t$ and $A \geq \alpha(i)$ for any $i \in [k]$ such that $\gamma(i) = t$, and $B$ is maximal such that $\xi(B) = t+1$ and $B \leq \alpha(i)$ for any $i \in [k]$ such that $\gamma(i) = t+1$. This implies that $\Lambda_X$ is indeed weakly contractible, which completes the proof. □

**Corollary 4.38** *Suppose $\mathcal{C}$ is a monoidal $\infty$–category with good relative tensor products. Then the projection $\mathfrak{ALG}_1(\mathcal{C}) \to \mathbf{\Delta}^{\mathrm{op}}$ is a cocartesian fibration.*

**Proof** Since $\overline{\mathfrak{ALG}}_1(\mathcal{C}) \to \mathbf{\Delta}^{\mathrm{op}}$ is a cocartesian fibration, it suffices to show that if $X$ is an object of $\mathfrak{ALG}_1(\mathcal{C})$ over $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, and $X \to \overline{X}$ is a cocartesian morphism in $\overline{\mathfrak{ALG}}_1(\mathcal{C})$ over $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$, then $\overline{X}$ is also in $\mathfrak{ALG}_1(\mathcal{C})$. In other words, we must show that if $X$ is a composite $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$–module, then $(\phi_*)^*X$ is a composite $\mathbf{\Delta}^{\mathrm{op}}_{/[m]}$–module for any map $\phi\colon [m] \to [n]$, ie the counit map $\tau_{m,!}\tau_m^*(\phi_*)^*X \to (\phi_*)^*X$ is an equivalence, where $\tau_m$ is the inclusion $\mathbf{\Lambda}^{\mathrm{op}}_{/[m]} \to \mathbf{\Delta}^{\mathrm{op}}_{/[m]}$. Using the definition of $\tau_{m,!}$ as an operadic left Kan extension and the criterion of Lemma A.53, it suffices to show that for each $\gamma \in \mathbf{\Lambda}^{\mathrm{op}}_{/[m]}$, the natural map

$$\operatorname*{colim}_{\eta\colon \gamma \to \gamma' \in ((\mathbf{\Lambda}_{/[m]})^{\mathrm{act}}_{\gamma/})^{\mathrm{op}}} \Xi(\phi\gamma') \to \Xi(\phi\gamma)$$

is an equivalence, where $\Xi\colon \mathbf{\Delta}^{\mathrm{op,act}}_{/[n]} \to \mathcal{C} \simeq \mathcal{C}^{\otimes}_{[1]}$ denotes the cocartesian pushforward along the unique active maps to $[1]$ of the restriction of $X$ to $\mathbf{\Delta}^{\mathrm{op,act}}_{/[n]}$.

It suffices to consider separately the cases where $\phi$ is either surjective or injective. If $\phi$ is surjective, then the map $\phi_*\colon (\mathbf{\Lambda}_{/[m]})^{\mathrm{act}}_{\gamma/} \to (\mathbf{\Lambda}_{/[n]})^{\mathrm{act}}_{\phi\gamma/}$ gives a factorization of this map as

$$\operatorname*{colim}_{\eta\colon \gamma \to \gamma' \in ((\mathbf{\Lambda}_{/[m]})^{\mathrm{act}}_{\gamma/})^{\mathrm{op}}} \Xi(\phi\gamma') \to \operatorname*{colim}_{\eta\colon \phi\gamma \to \gamma'' \in ((\mathbf{\Lambda}_{/[n]})^{\mathrm{act}}_{\phi\gamma/})^{\mathrm{op}}} \Xi(\gamma'') \to \Xi(\phi\gamma).$$

Here the first map is an equivalence by Proposition 4.37(ii) and the second map is an equivalence since $X$ is composite.

Now suppose $\phi$ is injective. Then the functor $(\mathbf{\Lambda}_{/[m]})^{\mathrm{act}}_{\gamma/} \to (\mathbf{\Lambda}_{/[n]}[\phi])^{\mathrm{act}}_{\phi\gamma/}$ gives a factorization of the map above as

$$\operatorname*{colim}_{\eta\colon \gamma \to \gamma' \in ((\mathbf{\Lambda}_{/[m]})^{\mathrm{act}}_{\gamma/})^{\mathrm{op}}} X(\phi\gamma') \to \operatorname*{colim}_{\eta\colon \phi\gamma \to \gamma'' \in ((\mathbf{\Lambda}_{/[n]}[\phi])^{\mathrm{act}}_{\phi\gamma/})^{\mathrm{op}}} X(\gamma'') \to X(\phi\gamma).$$

Here the first map is an equivalence by Proposition 4.37(i). Moreover, since $X$ is a composite $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$–algebra and the inclusions

$$((\mathbf{\Lambda}_{/[n]})^{\mathrm{act}}_{\phi\gamma/})^{\mathrm{op}} \to ((\mathbf{\Lambda}_{/[n]}[\phi])^{\mathrm{act}}_{\phi\gamma/})^{\mathrm{op}} \to ((\mathbf{\Delta}_{/[n]})^{\mathrm{act}}_{\phi\gamma/})^{\mathrm{op}}$$

are fully faithful, the map

$$\operatorname*{colim}_{\eta\colon \phi\gamma \to \gamma'' \in ((\mathbf{\Lambda}_{/[n]}[\phi])^{\mathrm{act}}_{\phi\gamma/})^{\mathrm{op}}} \Xi(\gamma'') \to \Xi(\phi\gamma)$$

is also an equivalence, since $\Xi$ is a left Kan extension of its restriction to $(\mathbf{\Lambda}_{/[n]})^{\mathrm{act,op}}$ by Lemma A.53. □

Combining Corollary 4.38 with Corollary 4.24, we have proved:

**Theorem 4.39** *Suppose $\mathcal{C}$ is a monoidal $\infty$–category with good relative tensor products. Then the projection $\mathfrak{ALG}_1(\mathcal{C}) \to \mathbf{\Delta}^{\mathrm{op}}$ is a double $\infty$–category.* □

**Definition 4.40** Let $\mathcal{C}$ be a monoidal $\infty$–category with good relative tensor products. Then we define $\mathfrak{Alg}_1(\mathcal{C})$ to be the $(\infty,2)$–category underlying the double $\infty$–category $\mathfrak{ALG}_1(\mathcal{C})$, ie the completion $L_2 U_{\mathrm{Seg}} i\mathfrak{ALG}_1(\mathcal{C})$ of the underlying 2–fold Segal space of $\mathfrak{ALG}_1(\mathcal{C})$ (see Remark 3.15).

## 4.5 The bimodule fibration

Let $\mathcal{C}$ be a monoidal $\infty$–category. We will write $\mathrm{Bimod}(\mathcal{C})$ for the $\infty$–category $\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}_{/[1]}}(\mathcal{C})$ and $\mathrm{Ass}(\mathcal{C})$ for the $\infty$–category $\mathrm{Alg}^1_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{C})$. There is a projection

$$\pi\colon \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Ass}(\mathcal{C})^{\times 2}$$

that sends an $A$–$B$–bimodule $M$ in $\mathcal{C}$ to the pair $(A,B)$. Our goal in this subsection is to analyze this functor, as well as the projection

$$U\colon \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Ass}(\mathcal{C}) \times \mathcal{C} \times \mathrm{Ass}(\mathcal{C})$$

that sends an $A$–$B$–bimodule $M$ in $\mathcal{C}$ to $(A,X,B)$, where $X$ is the object of $\mathcal{C}$ underlying $M$; we will make use of this work below in Section 5.5. Our first task is to prove that the map $U$ is given by restriction along an extendable map of generalized nonsymmetric $\infty$–operads, from which it will follow that $U$ has a left adjoint.

**Definition 4.41** We can identify objects of $\mathbf{\Delta}_{/[1]}$ with lists $(i_0,\dots,i_n)$ where $0 \le i_j \le i_{j+1} \le 1$, and for every $\phi\colon [m]\to[n]$ in $\mathbf{\Delta}$ there is a unique morphism

$$(i_{\phi(0)},\dots,i_{\phi(m)}) \to (i_0,\dots,i_n)$$

over $\phi$ in $\mathbf{\Delta}^{\mathrm{op}}$. Let $\mathcal{U}$ denote the subcategory of $\mathbf{\Delta}_{/[1]}$ containing all the objects and the morphisms $(i_{\phi(0)},\dots,i_{\phi(m)}) \to (i_0,\dots,i_n)$ as before, where, if $t$ is the largest index $j$ such that $i_j = 0$ and $s$ is the largest index $j$ such that $i_{\phi(j)} = 0$, then either $t=-1$, $t=m$ or the image of $\phi$ contains both $s$ and $s+1$.

**Remark 4.42** It is easy to see that the projection $\mathcal{U}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ is a generalized nonsymmetric $\infty$–operad. A $\mathcal{U}$–algebra $A$ in $\mathcal{C}$ contains the information of two associative algebras in $\mathcal{C}$, since we have retained the full subcategories of $\mathbf{\Delta}^{\mathrm{op}}_{/[1]}$ on the objects of the form $(0,\dots,0)$ and $(1,\dots,1)$. The algebra $A$ also determines an object $A(0,1)\in\mathcal{C}$, but we have omitted the maps in $\mathbf{\Delta}^{\mathrm{op}}_{/[1]}$ that describe the action of the two algebras on this object. Indeed, as we will see in Proposition 4.47 below, a $\mathcal{U}$–algebra consists precisely of this information — two associative algebras, and an additional object.

**Lemma 4.43** *Let $i$ denote the inclusion $\mathcal{U} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}_{/[1]}$.*

(i) *$i$ is an extendable morphism of generalized nonsymmetric $\infty$–operads.*

(ii) *For every monoidal $\infty$–category $\mathcal{C}$, the functor $i^*\colon \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Alg}^1_{\mathcal{U}}(\mathcal{C})$ has a left adjoint $i_!$.*

**Proof** The $\infty$–category $\mathcal{U}^{\mathrm{op,act}}_{/X}$ has a final object for every $X \in \mathbf{\Delta}^{\mathrm{op}}_{/[1]}$ (eg $(0,0,1,1)$ is final in $\mathcal{U}^{\mathrm{op,act}}_{/(0,1)}$). This implies that $i$ is extendable and, by Proposition A.50, that operadic left Kan extensions along $i$ always exist. The left adjoint $i_!$ therefore always exists by Corollary A.60. □

Next, we want to prove that the adjunction $i_! \dashv i^*$ is monadic. For this, we need a criterion for the existence of colimits for sifted diagrams of algebras:

**Proposition 4.44** *Suppose $\mathcal{M}$ is a generalized nonsymmetric $\infty$–operad and $\mathcal{C}$ is a monoidal $\infty$–category, and let $K$ be a sifted simplicial set. Then a diagram $p\colon K \to \mathrm{Alg}^1_{\mathcal{M}}(\mathcal{C})$ has a colimit if for every $x \in \mathcal{M}_{[1]}$ the diagram $\mathrm{ev}_x \circ p\colon K \to \mathcal{C}$ has a monoidal colimit in $\mathcal{C}$ (ie it has a colimit that is preserved by tensoring with objects of $\mathcal{C}$). Moreover, if this holds then this colimit is preserved by the forgetful functors $\mathrm{ev}_x$.*

**Proof** This follows from the same argument as in the proof of [16, Theorem A.5.3]. □

**Corollary 4.45** *Let $\mathcal{C}$ be a monoidal $\infty$–category and suppose $K$ is a sifted simplicial set. A diagram $p\colon K \to \mathrm{Bimod}(\mathcal{C})$ has a colimit if the functors $\mathrm{ev}_{(i,j)} \circ p\colon K \to \mathcal{C}$ for $(i,j) = (0,0), (0,1), (1,1)$ all have monoidal colimits in $\mathcal{C}$. Moreover, such colimits are preserved by $i^*\colon \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Alg}^1_{\mathcal{U}}(\mathcal{C})$.*

**Proof** This follows by applying Proposition 4.44 to $\mathrm{Bimod}(\mathcal{C})$ and $\mathrm{Alg}^1_{\mathcal{U}}(\mathcal{C})$. □

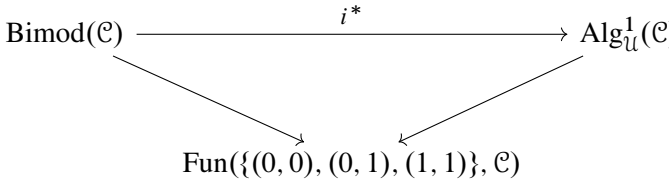
**Corollary 4.46** *For any monoidal $\infty$–category $\mathcal{C}$, the adjunction*

$$i_!\colon \mathrm{Alg}^1_{\mathcal{U}}(\mathcal{C}) \rightleftarrows \mathrm{Bimod}(\mathcal{C}) :i^*$$

*is monadic.*

**Proof** Suppose given a diagram $F\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Bimod}(\mathcal{C})$ that is $i^*$–split in the sense of [31, Definition 4.3.7.2], ie the diagram $i^*F$ extends to a diagram $F'\colon \mathbf{\Delta}^{\mathrm{op}}_{-\infty} \to \mathrm{Alg}^1_{\mathcal{U}}(\mathcal{C})$. A split simplicial object is always a colimit diagram by [28, Lemma 6.1.3.16], so $i^*F$ has a colimit in $\mathrm{Alg}^1_{\mathcal{U}}(\mathcal{C})$. Moreover, for the same reason the underlying diagrams in $\mathcal{C}$ are monoidal colimit diagrams, since tensoring with a fixed object of $\mathcal{C}$ again gives a split simplicial diagram. It then follows from Corollary 4.45 that $F$ has a colimit in $\mathrm{Bimod}(\mathcal{C})$ and this colimit is preserved by $i^*$. The forgetful functors from $\mathrm{Bimod}(\mathcal{C})$

and $\mathrm{Alg}^1_{\mathcal{U}}(\mathcal{C})$ to $\mathrm{Fun}(\{(0,0),(0,1),(1,1)\},\mathcal{C})$ are conservative by [16, Lemma A.5.5]; since the diagram




commutes, it follows that $i^*$ also conservative. The Barr–Beck theorem for $\infty$–categories, ie [31, Theorem 4.7.4.5], now implies that the adjunction $i_! \dashv i^*$ is monadic. □

We now wish to identify the functor $i^*\colon \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Alg}^1_{\mathcal{U}}(\mathcal{C})$ with the projection $U$. To do this, we define $\mathcal{X}$ to be the full subcategory of $\mathbf{\Delta}_{/[1]}$ spanned by the objects $(0)$, $(1)$ and $(0,1)$. The projection $\mathcal{X}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ is a generalized nonsymmetric $\infty$–operad, and the functor $\mathrm{Alg}^1_{\mathcal{X}^{\mathrm{op}}}(\mathcal{C}) \to \mathcal{C}$ given by evaluation at $(0,1)$ is an equivalence for any monoidal $\infty$–category $\mathcal{C}$. We can thus identify the projection $U$ with the map induced by composition with the inclusion $\mathbf{\Delta}^{\mathrm{op}} \amalg_{\{(0)\}} \mathcal{X}^{\mathrm{op}} \amalg_{\{(1)\}} \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}_{/[1]}$.

**Proposition 4.47** *The inclusion $\mathbf{\Delta}^{\mathrm{op}} \amalg_{\{(0)\}} \mathcal{X} \amalg_{\{(1)\}} \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{U}$ is a trivial cofibration in $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{gen}}_{\mathrm{ns}}}$.*

**Proof** We apply Corollary 4.32 — it is clear from the definition of $\mathcal{U}$ as a subset of $\mathbf{\Delta}^{\mathrm{op}}_{/[1]}$ that the required hypotheses hold. □

**Corollary 4.48** *Let $\mathcal{C}$ be a monoidal $\infty$–category. The projection $U\colon \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Ass}(\mathcal{C}) \times \mathcal{C} \times \mathrm{Ass}(\mathcal{C})$ has a left adjoint $F$ such that $UF(A,M,B) \simeq (A, A\otimes M\otimes B, B)$. Moreover, the adjunction $F \dashv U$ is monadic.* □

**Corollary 4.49** *For any $A, B \in \mathrm{Ass}(\mathcal{C})$, let $\mathrm{Bimod}_{A,B}(\mathcal{C})$ denote the fibre of*

$$\pi\colon \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Ass}(\mathcal{C})^{\times 2}$$

*at $(A,B)$. Then:*

(i) *The pullback $U_{A,B}\colon \mathrm{Bimod}_{A,B}(\mathcal{C}) \to \mathcal{C}$ of $U$ has a left adjoint $F_{A,B}$ such that the unit map $M \to U_{A,B}F_{A,B}(M)$ is the map $M \to A\otimes M\otimes B$ given by tensoring with the unit maps of $A$ and $B$.*

(ii) *If $K$ is a sifted simplicial set, then a diagram $p\colon K \to \mathrm{Bimod}_{A,B}(\mathcal{C})$ has a colimit if the underlying diagram $U_{A,B}\circ p\colon K \to \mathcal{C}$ has a monoidal colimit. Moreover, the forgetful functor $U_{A,B}$ detects such colimits.*

(iii) *The adjunction $F_{A,B} \dashv U_{A,B}$ is monadic.*

**Proof** The existence of the adjunction $F_{A,B} \dashv U_{A,B}$ follows from Corollary 4.48 and [31, Proposition 7.3.2.5].

Suppose $p\colon K \to \mathcal{C}$ is as in (ii). Since $K$ is weakly contractible, a constant diagram in $\mathcal{C}$ indexed by $K^{\triangleright}$ is a colimit diagram, and for the same reason it is also a monoidal colimit diagram. The composite diagram $p\colon K \to \mathrm{Bimod}(\mathcal{C})$ therefore has a colimit $K^{\triangleright} \to \mathrm{Bimod}(\mathcal{C})$ by Corollary 4.45, and this factors through $\mathrm{Bimod}_{A,B}(\mathcal{C})$. Since $\mathrm{Bimod}_{A,B}(\mathcal{C})$ is a pullback, and the projections of the diagram to $\mathcal{C}$ and $\mathrm{Ass}(\mathcal{C}) \times \mathcal{C} \times \mathrm{Ass}(\mathcal{C})$ are colimits, it follows that this diagram is also a colimit diagram in $A$–$B$–bimodules. This proves (ii).

Since a $U_{A,B}$–split diagram in $\mathrm{Bimod}_{A,B}(\mathcal{C})$ gives a $U$–split diagram in $\mathcal{C}$, it now follows from Corollary 4.48 that $\mathrm{Bimod}_{A,B}(\mathcal{C})$ has colimits of $U_{A,B}$–split simplicial diagrams and these are preserved by $U_{A,B}$. Since the inclusions $\{A\} \times \mathcal{C} \times \{B\} \hookrightarrow \mathrm{Ass}(\mathcal{C}) \times \mathcal{C} \times \mathrm{Ass}(\mathcal{C})$ and $\mathrm{Bimod}_{A,B}(\mathcal{C}) \hookrightarrow \mathrm{Bimod}(\mathcal{C})$ also detect equivalences, it follows that the adjunction $F_{A,B} \dashv U_{A,B}$ is monadic by [31, Theorem 4.7.4.5]. □

**Corollary 4.50** *Let $\mathcal{C}$ be a monoidal $\infty$–category, and let $I$ be the unit of $\mathcal{C}$ regarded as an associative algebra. Then the projection $U_{I,I}\colon \mathrm{Bimod}_{I,I}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*

**Proof** By Corollary 4.49 the functor $U_{I,I}$ has a left adjoint $F_{I,I}$ and the adjunction $F_{I,I} \dashv U_{I,I}$ is monadic. Moreover, the unit map $M \to U_{I,I}F_{I,I}M$ is the canonical equivalence $M \xrightarrow{\sim} I \otimes M \otimes I$. It follows from [31, Corollary 4.7.4.16] applied to the diagram

$$\begin{array}{ccc} \mathrm{Bimod}_{I,I}(\mathcal{C}) & \xrightarrow{U_{I,I}} & \mathcal{C} \\ & {\scriptstyle U_{I,I}}\searrow \quad \swarrow{\scriptstyle \mathrm{id}} & \\ & \mathcal{C} & \end{array}$$

that $U_{I,I}$ is an equivalence of $\infty$–categories. □

Our next goal is to show that the projection $\pi\colon \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Ass}(\mathcal{C})^{\times 2}$ is a cocartesian fibration if $\mathcal{C}$ has good relative tensor products. This requires some technical preliminary observations:

**Proposition 4.51** *Suppose $p\colon \mathcal{E} \to \mathcal{C}$ is an inner fibration, and that $p$ has a left adjoint $F\colon \mathcal{C} \to \mathcal{E}$. Then a morphism $\phi\colon e \to e'$ in $\mathcal{E}$ is $p$–cocartesian if and only if the commutative square*

$$\begin{array}{ccc} Fp(e) & \xrightarrow{Fp(\phi)} & Fp(e') \\ {\scriptstyle c_e}\downarrow & & \downarrow{\scriptstyle c_{e'}} \\ e & \xrightarrow[\phi]{} & e' \end{array}$$

*is a pushout square, where $c$ denotes the counit of the adjunction.*

**Proof** For any $x \in \mathcal{E}$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(e', x) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(e, x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{E}}(Fpe', x) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(Fpe, x) \\ \downarrow\sim & & \downarrow\sim \\ \mathrm{Map}_{\mathcal{C}}(pe', px) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(pe, px) \end{array}$$

where the vertical composites are equivalent to the maps coming from the functor $p$ by the adjunction identities. The map $\phi$ is thus $p$–cocartesian if and only if the composite square is cartesian for all $x$, and the commutative square

$$\begin{array}{ccc} Fp(e) & \xrightarrow{Fp(\phi)} & Fp(e') \\ {\scriptstyle c_e}\downarrow & & \downarrow{\scriptstyle c_{e'}} \\ e & \xrightarrow[\phi]{} & e' \end{array}$$

is a pushout if and only if the top square is cartesian for all $x$. But since the lower vertical maps are equivalences the bottom square is always cartesian, hence the top square is cartesian if and only if the composite square is. □

**Corollary 4.52** *Suppose $p\colon \mathcal{E} \to \mathcal{C}$ is a categorical fibration between $\infty$–categories, and that $p$ has a left adjoint $F\colon \mathcal{C} \to \mathcal{E}$. Then the following are equivalent:*

(1) *$p$ is a cocartesian fibration.*

(2) *For every $e \in \mathcal{E}$ and every morphism $\phi\colon p(e) \to x$ in $\mathcal{C}$, there is a pushout square*

$$\begin{array}{ccc} Fp(e) & \xrightarrow{F(\phi)} & F(x) \\ {\scriptstyle c_e}\downarrow & & \downarrow{\scriptstyle v} \\ e & \xrightarrow[\bar{\phi}]{} & \overline{x} \end{array}$$

*in $\mathcal{E}$, where $c$ is the counit for the adjunction, such that the composite*

$$x \xrightarrow{u_x} pF(x) \xrightarrow{p(v)} p(\overline{x})$$

*is an equivalence, where $u$ is the unit of the adjunction.*

**Proof** Suppose (2) holds. Given $e \in \mathcal{E}$ and $\phi\colon p(e) \to x$, we must show that there exists a $p$–cocartesian morphism $e \to \phi_! e$ over $\phi$. By assumption, there exists a pushout square

$$\begin{array}{ccc} Fp(e) & \xrightarrow{F(\phi)} & F(x) \\ {\scriptstyle c_e}\downarrow & & \downarrow{\scriptstyle v} \\ e & \xrightarrow[\bar{\phi}]{} & \overline{x} \end{array}$$

in $\mathcal{E}$ such that the composite

$$x \xrightarrow{u_x} pF(x) \xrightarrow{p(v)} p(\overline{x})$$

is an equivalence. The adjunction identities imply that the map $v$ factors as

$$Fx \xrightarrow{F(u_x \circ p(v))} Fp\overline{x} \xrightarrow{c_{\overline{x}}} \overline{x},$$

where the first map is an equivalence, and that the composite

$$Fp(e) \xrightarrow{F\phi} F(x) \xrightarrow{F(u_x \circ p(v))} Fp(\overline{x})$$

is $Fp(\bar{\phi})$. Thus we have a pushout square

$$\begin{array}{ccc} Fp(e) & \xrightarrow{Fp(\bar{\phi})} & Fp(\overline{x}) \\ {\scriptstyle c_e}\downarrow & & \downarrow{\scriptstyle c_{\overline{x}}} \\ e & \xrightarrow[\bar{\phi}]{} & \overline{x} \end{array}$$

which implies that $\bar{\phi}$ is $p$–cocartesian by Proposition 4.51. Since $p$ is a categorical

fibration, by [28, Corollary 2.4.6.5] there exists an equivalence $\overline{x} \to \overline{x}'$ lying over the equivalence $(u_x \circ p(v))^{-1}$ in $\mathcal{C}$, and the composite $e \to \overline{x}'$ is a $p$–cocartesian morphism over $\phi$, which proves (1).

Conversely, if (1) holds, then for any $e \in \mathcal{E}$ and $\phi\colon p(e) \to x$ in $\mathcal{C}$ there exists a $p$–cocartesian morphism $\overline{\phi}\colon e \to \overline{x}$ in $\mathcal{E}$ over $\phi$. By Proposition 4.51 this means we have a pushout square:

$$\begin{array}{ccc} Fp(e) & \xrightarrow{Fp(\overline{\phi})} & Fp(\overline{x}) \\ {\scriptstyle c_e}\downarrow & & \downarrow{\scriptstyle c_{\overline{x}}} \\ e & \xrightarrow[\overline{\phi}]{} & \overline{x} \end{array}$$

But as $\overline{\phi}$ lies over $\phi$, this gives (2). □

**Proposition 4.53** *Suppose $\mathcal{C}$ is a monoidal $\infty$–category with good relative tensor products. Then the restriction $\pi\colon \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Ass}(\mathcal{C})^{\times 2}$ is a cocartesian fibration. Moreover, if $M$ is an $A$–$B$–bimodule and $f\colon A \to A'$ and $g\colon B \to B'$ are morphisms of algebras in $\mathcal{C}$, then the cocartesian pushforward $(f,g)_! M$ is the tensor product $A' \otimes_A M \otimes_B B'$.*

**Proof** Let us first assume that $\mathcal{C}$ has an initial object $\varnothing$ and the monoidal structure is compatible with this (ie $c \otimes \varnothing \simeq \varnothing \otimes c \simeq \varnothing$ for all $c \in \mathcal{C}$). Then the projection $\mathrm{Ass}(\mathcal{C}) \times \mathcal{C} \times \mathrm{Ass}(\mathcal{C}) \to \mathrm{Ass}(\mathcal{C})^{\times 2}$ has a left adjoint, which sends $(A,B)$ to $(A,\varnothing,B)$. By Corollary 4.48 it follows that $\pi$ has a left adjoint $F'$, which sends $(A,B)$ to $F(A,\varnothing,B)$.

Moreover, for any $M \in \mathrm{Bimod}(\mathcal{C})$ and any morphism $(f,g)\colon (A,B) \simeq \pi(M) \to (A',B')$, the pushout

$$\begin{array}{ccc} (UF)^n(A,\varnothing,B) & \longrightarrow & (UF)^n(A',\varnothing,B') \\ \downarrow & & \downarrow \\ (UF)^n U(M) & \longrightarrow & X_n \end{array}$$

exists in $\mathrm{Ass}(\mathcal{C}) \times \mathcal{C} \times \mathrm{Ass}(\mathcal{C})$: since $\mathcal{C}$ is compatible with initial objects, the top horizontal morphism can be identified with $(A,\varnothing,B) \to (A',\varnothing,B')$ and the left vertical morphism with $(A,\varnothing,B) \to (A, A^{\otimes n} \otimes M \otimes B^{\otimes n}, B)$, so that $X_n$ is simply $(A', A^{\otimes n} \otimes M \otimes B^{\otimes n}, B')$. We then get a simplicial object $F(X_\bullet)$ in $\mathrm{Bimod}(\mathcal{C})$. Evaluated at $(0,0)$ and $(1,1)$ this is constant at $A'$ and $B'$, respectively, and at $(0,1)$

we get $A' \otimes A^{\otimes \bullet} \otimes M \otimes B^{\otimes \bullet} \otimes B'$. Since $\mathcal{C}$ has good relative tensor products, the colimit of this simplicial diagram exists, is monoidal, and can be identified with the relative tensor product $A' \otimes_A M \otimes_B B'$. It follows from Corollary 4.45 that the diagram $F(X_\bullet)$ has a colimit in $\mathrm{Bimod}(\mathcal{C})$. Moreover, since $F$ is a left adjoint and colimits commute we can identify this colimit as

$$\begin{aligned}|F(X_\bullet)| &\simeq |F(UF)^\bullet U(M) \amalg_{F(UF)^\bullet(A,\varnothing,B)} F(UF)^\bullet(A', \varnothing, B')| \\ &\simeq |F(UF)^\bullet U(M)| \amalg_{|F(UF)^\bullet(A,\varnothing,B)|} |F(UF)^\bullet(A', \varnothing, B')| \\ &\simeq M \amalg_{F(A,\varnothing,B)} F(A', \varnothing, B').\end{aligned}$$

Thus the pushout $M \amalg_{F'(A,B)} F'(A', B')$ exists in $\mathrm{Bimod}(\mathcal{C})$. It then follows from Corollary 4.52 that $\pi$ is a cocartesian fibration, and that the object of $\mathcal{C}$ underlying $(f,g)_! M$ is $A' \otimes_A M \otimes_B B'$.

Now consider a general monoidal $\infty$–category $\mathcal{C}$. By [31, Proposition 4.8.1.10] (or by a direct construction) the $\infty$–category $\mathcal{C}^\triangleleft$ has a monoidal structure that is compatible with the initial object $-\infty$ and such that the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^\triangleleft$ is monoidal. Moreover, this inclusion preserves geometric realizations (and in general colimits other than the initial object), and thus $\mathcal{C}^\triangleleft$ also has good relative tensor products. By our previous argument we then have a cocartesian fibration $\mathrm{Bimod}(\mathcal{C}^\triangleleft) \to \mathrm{Ass}(\mathcal{C}^\triangleleft)^{\times 2}$. The initial object in $\mathcal{C}^\triangleleft$ does not admit an associative algebra structure (since it has no map from the unit), so the inclusion $\mathrm{Ass}(\mathcal{C}) \to \mathrm{Ass}(\mathcal{C}^\triangleleft)$ is an equivalence. We thus wish to show that the restriction of the projection $\mathrm{Bimod}(\mathcal{C}^\triangleleft) \to \mathrm{Ass}(\mathcal{C})^{\times 2}$ to $\mathrm{Bimod}(\mathcal{C})$ is still a cocartesian fibration. For this it suffices to show that if $M$ is an $A$–$B$–bimodule in $\mathcal{C}$ and $f\colon A \to A'$ and $g\colon B \to B'$ are maps of associative algebras, then $(f,g)_! M$ (computed in $\mathrm{Bimod}(\mathcal{C}^\triangleleft)$) is also in $\mathrm{Bimod}(\mathcal{C})$. But this is true since the underlying object of $(f,g)_! M$ is given by a relative tensor product that cannot be the initial object $-\infty$. □

## 5 $\mathbb{E}_n$–algebras and iterated bimodules

In this section we extend the results of Section 4 to the case $n > 1$: if $\mathcal{C}$ is a nice $\mathbb{E}_n$–monoidal $\infty$–category we will construct an $(n+1)$–fold $\infty$–category $\mathfrak{ALG}_n(\mathcal{C})$ of $\mathbb{E}_n$–algebras; we can then define the $(\infty, n+1)$–category $\mathfrak{Alg}_n(\mathcal{C})$ of $\mathbb{E}_n$–algebras in $\mathcal{C}$ as the completion of the underlying $(n+1)$–fold Segal space of $\mathfrak{ALG}_n(\mathcal{C})$.

In order to iterate our results in the case $n = 1$ it is convenient to work with a theory of *$\infty$–operads over $\mathbf{\Delta}^{n,\mathrm{op}}$* (or *$\mathbf{\Delta}^n$–$\infty$–operads*); we will introduce these objects in Section 5.1 (with the more technical results we need delegated to the appendix). Then in Section 5.2 we observe that the definitions of Section 4.2 can be iterated and use this

to define the $\infty$–categories $\mathfrak{ALG}_n(\mathcal{C})_I$ for $I \in \mathbf{\Delta}^{n,\mathrm{op}}$, and in Section 5.3 we prove that these $\infty$–categories satisfy the Segal condition and give a functor $\mathbf{\Delta}^{n,\mathrm{op}} \to \mathrm{Cat}_\infty$. In Section 5.4 we then show that $\mathfrak{ALG}_n(\mathcal{C})$ is a lax monoidal functor in $\mathcal{C}$ and conclude from this that if $\mathcal{C}$ is an $\mathbb{E}_{n+m}$–monoidal $\infty$–category then $\mathfrak{ALG}_n(\mathcal{C})$ inherits an $\mathbb{E}_m$–monoidal structure. Finally, in Section 5.5 we identify the $(\infty,n)$–category of maps from $A$ to $B$ in $\mathfrak{Alg}_n(\mathcal{C})$ with $\mathfrak{Alg}_{n-1}(\mathrm{Bimod}_{A,B}(\mathcal{C}))$.

## 5.1 $\infty$–operads over $\mathbf{\Delta}^{n,\mathrm{op}}$

In this subsection we will introduce the notion of *$\infty$–operads over $\mathbf{\Delta}^{n,\mathrm{op}}$* or *$\mathbf{\Delta}^n$–$\infty$–operads*, which is the setting in which we will iterate the constructions of Section 4.

In Section 3.1 we introduced *$\mathbf{\Delta}^n$–monoids* in an $\infty$–category $\mathcal{C}$ with finite products, by iterating the definition of an associative monoid. Applying this to the $\infty$–category $\mathrm{Cat}_\infty$ of $\infty$–categories, we get a notion of *$\mathbf{\Delta}^n$–monoidal $\infty$–category*. Using the straightening equivalence, we can reinterpret these as certain cocartesian fibrations over $\mathbf{\Delta}^{n,\mathrm{op}}$:

**Definition 5.1** A *$\mathbf{\Delta}^n$–monoidal $\infty$–category* is a cocartesian fibration $\mathcal{C}^\otimes \to \mathbf{\Delta}^{n,\mathrm{op}}$ such that, for any object $I \in \mathbf{\Delta}^{n,\mathrm{op}}$, the functor

$$\mathcal{C}^\otimes_I \to (\mathcal{C}^\otimes_{C_n})^{\times |I|},$$

induced by the cocartesian morphisms over the maps in $|I|$, is an equivalence.

**Remark 5.2** The $\mathbf{\Delta}^n$–monoidal $\infty$–categories can be interpreted as $\infty$–categories equipped with $n$ compatible associative monoid structures, ie as *$n$–tuply monoidal $\infty$–categories*. We will see below in Corollary A.31 that they are also equivalent to *$\mathbb{E}_n$–monoidal $\infty$–categories* as defined in [31], ie to algebras for the $\mathbb{E}_n$–$\infty$–operad in $\mathrm{Cat}_\infty$.

Lurie [31] defines symmetric $\infty$–operads by weakening the definition of a symmetric monoidal $\infty$–category as a cocartesian fibration over $\mathbf{\Gamma}^{\mathrm{op}}$, and above in Definition 4.2 we defined nonsymmetric $\infty$–operads by analogously weakening the definition of a monoidal $\infty$–category as a cocartesian fibration over $\mathbf{\Delta}^{\mathrm{op}}$. Applying the same idea to $\mathbf{\Delta}^n$–monoidal $\infty$–categories gives a definition of *$\mathbf{\Delta}^n$–$\infty$–operads*:

**Definition 5.3** A *$\mathbf{\Delta}^n$–$\infty$–operad* is a functor of $\infty$–categories $\pi\colon \mathcal{O} \to \mathbf{\Delta}^{n,\mathrm{op}}$ such that:

(i) For each inert map $\phi\colon I \to J$ in $\mathbf{\Delta}^{n,\mathrm{op}}$ and every $X \in \mathcal{O}$ such that $\pi(X) = I$, there exists a $\pi$–cocartesian morphism $X \to \phi_! X$ over $\phi$.

(ii) For every $I$ in $\mathbf{\Delta}^{n,\mathrm{op}}$, the functor

$$\mathcal{O}_I \to \mathcal{O}_{C_n}^{\times|I|}$$

induced by the cocartesian morphisms over the inert maps $C_n \to I$ in $\mathbf{\Delta}^{n,\mathrm{op}}$ is an equivalence.

(iii) For every morphism $\phi\colon I \to J$ in $\mathbf{\Delta}^{n,\mathrm{op}}$, $X \in \mathcal{O}_I$ and $Y \in \mathcal{O}_J$, composition with the cocartesian morphisms $Y \to Y_i$ over the inert morphisms $i\colon I \to C_n$ gives an equivalence

$$\mathrm{Map}_{\mathcal{O}}^{\phi}(X,Y) \xrightarrow{\sim} \prod_i \mathrm{Map}_{\mathcal{O}}^{i\circ\phi}(X,Y_i),$$

where $\mathrm{Map}_{\mathcal{O}}^{\phi}(X,Y)$ denotes the subspace of $\mathrm{Map}_{\mathcal{O}}(X,Y)$ of morphisms that map to $\phi$ in $\mathbf{\Delta}^{n,\mathrm{op}}$. (Equivalently, $Y$ is a $\pi$–limit of the $Y_i$.)

**Remark 5.4** We will see in Section A.2 that there is an adjunction between $\mathbf{\Delta}^n$–$\infty$–operads and symmetric $\infty$–operads over $\mathbb{E}_n$. In the case $n = 1$ this adjunction is an equivalence by [31, Proposition 4.7.1.1]. We expect that this is true also for $n > 1$. Thus, $\mathbf{\Delta}^n$–$\infty$–operads should be thought of as a more combinatorial or explicit model for symmetric $\infty$–operads over $\mathbb{E}_n$, where we do not need to deal with configuration spaces of points in $\mathbb{R}^n$.

**Remark 5.5** $\mathbf{\Delta}^n$–$\infty$–operads are a special case of Barwick's notion of *$\infty$–operads over an operator category* as defined in [10]. Specifically, they are $\infty$–operads over the cartesian product $\mathbb{O}^{\times n}$, where $\mathbb{O}$ is the operator category of finite ordered sets.

**Remark 5.6** A $\mathbf{\Delta}^n$–monoidal $\infty$–category as we defined it above is the same thing as a $\mathbf{\Delta}^n$–$\infty$–operad that is also a cocartesian fibration.

To extend the definitions of iterated bimodules from Section 3 to the noncartesian setting, we will need to consider a more general notion than that of $\mathbf{\Delta}^n$–$\infty$–operads. To introduce this, recall that by iterating the definition of category object in $\mathrm{Cat}_\infty$ we can define *$\mathbf{\Delta}^n$–uple $\infty$–categories* (which model $(n+1)$–uple $\infty$–categories) as certain functors from $\mathbf{\Delta}^{n,\mathrm{op}}$ to $\mathrm{Cat}_\infty$. Rephrasing this in terms of cocartesian fibrations, we get the following definition:

**Definition 5.7** A *$\mathbf{\Delta}^n$–uple $\infty$–category* is a cocartesian fibration $\mathcal{M} \to \mathbf{\Delta}^{n,\mathrm{op}}$ such that, for any $I \in \mathbf{\Delta}^{n,\mathrm{op}}$, the functor

$$\mathcal{M}_I \to \lim_{C \to I \in \mathrm{Cell}^{n,\mathrm{op}}_{/I}} \mathcal{M}_C,$$

induced by the cocartesian morphisms over the inert morphisms $C \to I$ in $\mathbf{\Delta}^n$, is an equivalence.

We can now weaken this definition in the same way as that which gave us the definition of $\mathbf{\Delta}^n$–$\infty$–operads from that of $\mathbf{\Delta}^n$–monoidal $\infty$–categories:

**Definition 5.8** A *generalized* $\mathbf{\Delta}^n$–$\infty$–*operad* is a functor of $\infty$–categories $\pi\colon \mathcal{M} \to \mathbf{\Delta}^{n,\mathrm{op}}$ such that:

(i) For every inert morphism $\phi\colon I \to J$ in $\mathbf{\Delta}^{n,\mathrm{op}}$ and every $X \in \mathcal{O}_I$, there exists a $\pi$–cocartesian edge $X \to \phi_! X$ over $\phi$.

(ii) For every $I$ in $\mathbf{\Delta}^{n,\mathrm{op}}$, the functor

$$\mathcal{M}_I \to \lim_{C \to I} \mathcal{M}_C,$$

induced by the cocartesian arrows over the inert maps $C \to I$ in $\mathrm{Cell}^{n,\mathrm{op}}_{/I}$, is an equivalence.

(iii) Given $Y$ in $\mathcal{O}_J$, choose a cocartesian lift $\overline{\eta}\colon (\mathrm{Cell}^{n,\mathrm{op}}_{/I})^{\triangleleft} \to \mathcal{O}$ of the diagram of inert morphisms $J \to C$ with $\overline{\eta}(-\infty) \simeq Y$. Then for any map $\phi\colon I \to J$ in $\mathbf{\Delta}^{n,\mathrm{op}}$ and $X \in \mathcal{O}_I$, the diagram $\overline{\eta}$ induces an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{O}}(X, Y) \simeq \lim_{i\colon C \to I \in \mathrm{Cell}^{n,\mathrm{op}}_{/I}} \mathrm{Map}^{i \circ \phi}_{\mathcal{O}}(X, \overline{\eta}(i)).$$

(Equivalently, any cocartesian lift of the diagram $(\mathrm{Cell}^{n,\mathrm{op}}_{/I})^{\triangleleft} \to \mathbf{\Delta}^{n,\mathrm{op}}$ is a $\pi$–limit diagram in $\mathcal{O}$.)

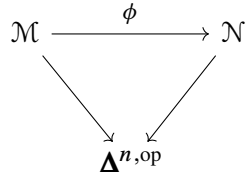
**Remark 5.9** A $\mathbf{\Delta}^n$*–uple* $\infty$*–category* as we defined it above is the same thing as a generalized $\mathbf{\Delta}^n$–$\infty$–operad that is also a cocartesian fibration.

**Definition 5.10** Let $\pi\colon \mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$ be a (generalized) $\mathbf{\Delta}^n$–$\infty$–operad. We say that a morphism $f$ in $\mathcal{M}$ is *inert* if it is cocartesian and $\pi(f)$ is an inert morphism in $\mathbf{\Delta}^{\mathrm{op}}$. We say that $f$ is *active* if $\pi(f)$ is an active morphism in $\mathbf{\Delta}^{\mathrm{op}}$.

**Lemma 5.11** *The active and inert morphisms form a factorization system on any generalized* $\mathbf{\Delta}^n$–$\infty$–*operad.*

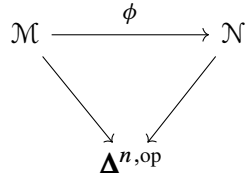
**Proof** This is a special case of [31, Proposition 2.1.2.5]. □

**Definition 5.12** A morphism of (generalized) $\boldsymbol{\Delta}^n$–$\infty$–operads is a commutative diagram




where $\mathcal{M}$ and $\mathcal{N}$ are (generalized) $\boldsymbol{\Delta}^n$–$\infty$–operads, such that $\phi$ carries inert morphisms in $\mathcal{M}$ to inert morphisms in $\mathcal{N}$. We will also refer to a morphism of (generalized) $\boldsymbol{\Delta}^n$–$\infty$–operads $\mathcal{M} \to \mathcal{N}$ as an $\mathcal{M}$–*algebra* in $\mathcal{N}$; we write $\mathrm{Alg}^{\boldsymbol{n}}_{\mathcal{M}}(\mathcal{N})$ for the $\infty$–category of $\mathcal{M}$–algebras in $\mathcal{N}$, defined as a full subcategory of the $\infty$–category of functors $\mathcal{M} \to \mathcal{N}$ over $\boldsymbol{\Delta}^{n,\mathrm{op}}$.

**Definition 5.13** If $\mathcal{M}$ and $\mathcal{N}$ are $\boldsymbol{\Delta}^n$–uple $\infty$–categories, a $\boldsymbol{\Delta}^n$–*uple functor* from $\mathcal{M}$ to $\mathcal{N}$ is a commutative diagram




where $\phi$ preserves all cocartesian morphisms; if $\mathcal{M}$ and $\mathcal{N}$ are in fact $\boldsymbol{\Delta}^n$–monoidal $\infty$–categories we will also refer to $\boldsymbol{\Delta}^n$–uple functors as $\boldsymbol{\Delta}^n$–*monoidal functors*. We write $\mathrm{Fun}^{\otimes,\boldsymbol{n}}(\mathcal{M},\mathcal{N})$ for the $\infty$–category of $\boldsymbol{\Delta}^n$–uple functors, defined as a full subcategory of the $\infty$–category of functors $\mathcal{M} \to \mathcal{N}$ over $\boldsymbol{\Delta}^{n,\mathrm{op}}$.

## 5.2 Iterated bimodules for $\mathbb{E}_n$–algebras and their tensor products

In Section 3 we considered iterated bimodules for $\mathbb{E}_n$–algebras as monoids for the overcategories $\boldsymbol{\Delta}^{n,\mathrm{op}}_{/I}$. Using generalized $\boldsymbol{\Delta}^n$–$\infty$–operads we now have a natural way to extend this definition to the noncartesian setting, because of the following observation:

**Lemma 5.14** *Let $I$ be any object of $\boldsymbol{\Delta}^n$. Then the forgetful functor $\boldsymbol{\Delta}^{n,\mathrm{op}}_{/I} \to \boldsymbol{\Delta}^{n,\mathrm{op}}$ is a $\boldsymbol{\Delta}^n$–uple $\infty$–category.*

**Proof** The forgetful functor $\boldsymbol{\Delta}^{n,\mathrm{op}}_{/I} \to \boldsymbol{\Delta}^{n,\mathrm{op}}$ is the cocartesian fibration associated to the functor

$$\mathrm{Hom}_{\boldsymbol{\Delta}^n}(-, I)\colon \boldsymbol{\Delta}^{n,\mathrm{op}} \to \mathrm{Set}.$$

This fibration is a $\mathbf{\Delta}^n$–uple $\infty$–category if and only if the associated functor satisfies the Segal condition, which it does (for instance since, if $I = ([i_1], \dots, [i_n])$, it is the product of the functors $\mathrm{Hom}_{\mathbf{\Delta}}(-, [i_k])$ which satisfy the Segal condition for $\mathbf{\Delta}^{\mathrm{op}}$). □

By Corollary A.26 $\mathbf{\Delta}^{n,\mathrm{op}}$–algebras in a $\mathbf{\Delta}^n$–monoidal $\infty$–category $\mathcal{C}$ are equivalent to $\mathbb{E}_n$–algebras. To define the $n$–fold category object $\mathfrak{AL}\mathfrak{G}_n(\mathcal{C})$ in $\mathrm{Cat}_\infty$ of $\mathbb{E}_n$–algebras, a natural choice for the $\infty$–category of objects is thus $\mathrm{Alg}^{n}_{\mathbf{\Delta}^{n,\mathrm{op}}}(\mathcal{C})$. Similarly, the $n$ different $\infty$–categories of 1–morphisms are given by

$$\begin{aligned}
\mathfrak{AL}\mathfrak{G}_n(\mathcal{C})_{(1,0,\dots,0)} &:= \mathrm{Alg}^{n}_{\mathbf{\Delta}^{\mathrm{op}}_{/[1]}\times\mathbf{\Delta}^{(n-1),\mathrm{op}}}(\mathcal{C}),\\
\mathfrak{AL}\mathfrak{G}_n(\mathcal{C})_{(0,1,0,\dots,0)} &:= \mathrm{Alg}^{n}_{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}_{/[1]}\times\mathbf{\Delta}^{(n-2),\mathrm{op}}}(\mathcal{C}),\\
&\vdots\\
\mathfrak{AL}\mathfrak{G}_n(\mathcal{C})_{(0,\dots,0,1)} &:= \mathrm{Alg}^{n}_{\mathbf{\Delta}^{(n-1),\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}_{/[1]}}(\mathcal{C}),
\end{aligned}$$

and more generally the $\infty$–categories of commutative $k$–cubes are given by

$$\mathfrak{AL}\mathfrak{G}_n(\mathcal{C})_I := \mathrm{Alg}^{n}_{\mathbf{\Delta}^{n,\mathrm{op}}_{/I}}(\mathcal{C}),$$

where $I = ([i_1], \dots, [i_n])$ with each $i_j$ either 0 are 1 and exactly $k$ 1's. To define the remaining $\infty$–categories $\mathfrak{AL}\mathfrak{G}_n(\mathcal{C})_I$ we must define an appropriate notion of *composite* $\mathbf{\Delta}^{n,\mathrm{op}}_{/I}$–algebras; luckily, there is a natural generalization of our definition in the case $n = 1$:

**Definition 5.15** We say a morphism $(\phi_1, \dots, \phi_n)$ in $\mathbf{\Delta}^n$ is *cellular* if $\phi_i$ is cellular for all $i$. For $I \in \mathbf{\Delta}^n$, we write $\mathbf{\Lambda}^n_{/I}$ for the full subcategory of $\mathbf{\Delta}^n_{/I}$ spanned by the cellular maps.

**Lemma 5.16** *The projection $\mathbf{\Lambda}^{n,\mathrm{op}}_{/I} \to \mathbf{\Delta}^{n,\mathrm{op}}$ is a generalized $\mathbf{\Delta}^n$–$\infty$–operad, and the inclusion $\tau_I\colon \mathbf{\Lambda}^{n,\mathrm{op}}_{/I} \hookrightarrow \mathbf{\Delta}^{n,\mathrm{op}}_{/I}$ is a morphism of generalized $\mathbf{\Delta}^n$–$\infty$–operads.*

**Proof** This is as Lemma 4.14, using the $\mathbf{\Delta}^n$–analogue of Lemma 4.15. □

**Proposition 5.17** *For every $I \in \mathbf{\Delta}^n$, the inclusion $\tau_I\colon \mathbf{\Lambda}^{n,\mathrm{op}}_{/I} \to \mathbf{\Delta}^{n,\mathrm{op}}_{/I}$ is extendable.*

**Proof** We must show that for any $I \in \mathbf{\Delta}^n$ and any map $\xi\colon J \to I$ in $\mathbf{\Delta}^n$, the map

$$(\mathbf{\Lambda}^{n,\mathrm{op}}_{/I})^{\mathrm{act}}_{/\xi} \to \prod_{\phi\colon C_n \to J} (\mathbf{\Lambda}^{n,\mathrm{op}}_{/I})^{\mathrm{act}}_{/\xi\phi}$$

is cofinal, or equivalently that the map

$$(\mathbf{\Lambda}^{n}_{/I})^{\mathrm{act}}_{\xi/} \to \prod_{\phi\colon C_n \to J} (\mathbf{\Lambda}^{n}_{/I})^{\mathrm{act}}_{\xi\phi/}$$

is coinitial. This map decomposes as a product, hence, since a product of coinitial maps is coinitial, this follows from the proof of Proposition 4.16. □

**Definition 5.18** We say a $\mathbf{\Delta}^n$–monoidal $\infty$–category *has good relative tensor products* if it is $\tau_I$–compatible for all $I \in \mathbf{\Delta}^{n,\mathrm{op}}$. Similarly, we say a $\mathbf{\Delta}^n$–monoidal functor is *compatible with relative tensor products* if it is $\tau_I$–compatible for all $I$.

Applying Corollary A.60, we get:

**Proposition 5.19** *Suppose $\mathcal{C}$ is a $\mathbf{\Delta}^n$–monoidal $\infty$–category with good relative tensor products. Then the restriction $\tau_I^*\colon \mathrm{Alg}^n_{\mathbf{\Delta}^{n,\mathrm{op}}_{/I}}(\mathcal{C}) \to \mathrm{Alg}^n_{\mathbf{\Lambda}^{n,\mathrm{op}}_{/I}}(\mathcal{C})$ has a fully faithful left adjoint $\tau_{I,!}$.* □

Next, we observe that the notion of having good relative tensor products has a simple equivalent reformulation:

**Lemma 5.20** *Let $\mathcal{C}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category. The following are equivalent:*

(1) *$\mathcal{C}$ has good relative tensor products.*

(2) *Any one of the underlying monoidal $\infty$–categories of $\mathcal{C}$ (obtained by pulling back along the inclusions $\{[1]\}\times\cdots\times\mathbf{\Delta}^{\mathrm{op}}\times\cdots\times\{[1]\}\hookrightarrow\mathbf{\Delta}^{n,\mathrm{op}}$) has good relative tensor products in the sense of Definition 4.18.*

(3) *Any one of the underlying monoidal $\infty$–categories of $\mathcal{C}$ satisfies the criterion of Lemma 4.19.*

*Moreover, a $\mathbf{\Delta}^n$–monoidal functor is compatible with relative tensor products if and only if any one of its underlying monoidal functors is compatible with relative tensor products.*

**Proof** By definition, we must show that, for any $\mathbf{\Lambda}^{n,\mathrm{op}}_{/I}$–algebra $A$ in $\mathcal{C}$ and any $X \in \mathbf{\Delta}^{n,\mathrm{op}}_{/I}$, the colimit of the induced diagram $(\mathbf{\Lambda}^{n,\mathrm{op}}_{/I})^{\mathrm{act}}_{/X} \to \mathcal{C}$ exists and is preserved tensoring with any object of $\mathcal{C}$ using each of the $n$ tensor products. But since the $\mathbf{\Delta}^n$–monoidal $\infty$–category $\mathcal{C}$ arises from an $\mathbb{E}_n$–monoidal $\infty$–category by Corollary A.31, these $n$ tensor product functors are all equivalent. It therefore suffices to show that if one of the underlying monoidal $\infty$–categories of $\mathcal{C}$ has good relative tensor products, then the colimits above exist in $\mathcal{C}$ and are preserved by tensoring (on either side) with any object of $\mathcal{C}$.

But the category $(\mathbf{\Lambda}^{n,\mathrm{op}}_{/I})^{\mathrm{act}}_{/X}$ decomposes as a product

$$\prod_{k=1}^{n} (\mathbf{\Lambda}^{\mathrm{op}}_{/[i_k]})^{\mathrm{act}}_{/X_k}$$

(where $I = ([i_1], \dots, [i_n])$ and $X = (X_1, \dots, X_n)$), and for each $Y_k \in (\mathbf{\Lambda}^{\mathrm{op}}_{/[i_k]})^{\mathrm{act}}_{/X_k}$ with $k \neq j$, the restriction of the diagram to

$$\{Y_1\} \times \cdots \times (\mathbf{\Lambda}^{\mathrm{op}}_{/[i_j]})^{\mathrm{act}}_{/X_j} \times \cdots \times \{Y_n\} \to \mathcal{C}$$

is obtained by tensoring a number of diagrams associated to $\mathbf{\Lambda}^{\mathrm{op}}_{/[i_j]}$–algebras in $\mathcal{C}$ with some fixed objects. By siftedness, the colimits of these diagrams therefore exist in $\mathcal{C}$, and our desired colimit can be obtained by an iterated colimit of such diagrams. It follows that the colimit over $(\mathbf{\Lambda}^{n,\mathrm{op}}_{/I})^{\mathrm{act}}_{/X}$ does indeed exist, and is preserved under tensoring, as required. Similarly, a $\mathbf{\Delta}^n$–monoidal functor is compatible with relative tensor products if and only if one of its underlying monoidal functors is. □

We can now define the $\infty$–categories $\mathfrak{ALG}_n(\mathcal{C})_I$ for all $I$:

**Definition 5.21** Let $\mathcal{C}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category with good relative tensor products. We say that a $\mathbf{\Delta}^{n,\mathrm{op}}_{/I}$–algebra $M$ in $\mathcal{C}$ is *composite* if the counit map $\tau_{I,!}\tau_I^* M \to M$ is an equivalence, or equivalently if $M$ is in the essential image of the functor $\tau_{I,!}$. We write $\mathfrak{ALG}_n(\mathcal{C})_I$ for the full subcategory of $\mathrm{Alg}^n_{\mathbf{\Delta}^{n,\mathrm{op}}_{/I}}(\mathcal{C})$ spanned by the composite $\mathbf{\Delta}^{n,\mathrm{op}}_{/I}$–algebras.

## 5.3 The $(n+1)$–fold $\infty$–category of $\mathbb{E}_n$–algebras

Our goal in this subsection is to extend the results of Sections 4.3 and 4.4 to the case of $\mathbb{E}_n$–algebras, ie to prove that the $\infty$–categories $\mathfrak{ALG}_n(\mathcal{C})_I$ satisfy the Segal condition and are functorial in $I$. Luckily, it turns out that these results both follow from those in the case $n = 1$ by simple inductions.

We first prove that $\mathfrak{ALG}_n(\mathcal{C})_I$ satisfies the Segal condition. Let $(\mathbf{\Delta}^n_{/I})^{\amalg,\mathrm{op}}$ denote the ordinary colimit

$$\operatorname*{colim}_{I \to C \in (\mathrm{Cell}^n_{I/})^{\mathrm{op}}} (\mathbf{\Delta}^n_{/C})^{\mathrm{op}}$$

in (marked) simplicial sets (over $\mathbf{\Delta}^{n,\mathrm{op}}$). From the structure of $\mathrm{Cell}^n$ it is easy to see that this colimit can be written as an iterated pushout along injective maps of simplicial sets, so this is a homotopy colimit in the generalized $\mathbf{\Delta}^n$–$\infty$–operad model structure of Section A.1. We wish to prove that the inclusion $(\mathbf{\Delta}^n_{/I})^{\amalg,\mathrm{op}} \hookrightarrow \mathbf{\Lambda}^{n,\mathrm{op}}_{/I}$ is a trivial cofibration in this model structure:

**Lemma 5.22** *Suppose $I = ([i_1], \dots, [i_n])$ is an object of $\mathbf{\Delta}^n$. Then the natural map*

$$(\mathbf{\Delta}^n_{/I})^{\amalg,\mathrm{op}} \to \prod_{p=1}^{n} (\mathbf{\Delta}_{/[i_p]})^{\amalg,\mathrm{op}}$$

*is an isomorphism.*

**Proof** The category $(\mathrm{Cell}^n_{I/})^{\mathrm{op}}$ is isomorphic to the product category $\prod_k (\mathrm{Cell}^1_{[i_k]/})^{\mathrm{op}}$, and the functor $(I \to C) \mapsto (\mathbf{\Delta}^n_{/C})^{\mathrm{op}}$ is isomorphic to the product of the functors $([i_k] \to [j]) \mapsto \mathbf{\Delta}^{\mathrm{op}}_{/[j]}$ (where $j = 0$ or $1$). Since the cartesian product of (marked) simplicial sets preserves colimits in each variable, the result follows. □

**Proposition 5.23** *Let $I$ be an object of $\mathbf{\Delta}^n$. The inclusion $(\mathbf{\Delta}^n_{/I})^{\amalg,\mathrm{op}} \to \mathbf{\Lambda}^{n,\mathrm{op}}_{/I}$ is a trivial cofibration in the model category $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_n}$.*

**Proof** Suppose $I = ([i_1], \dots, [i_n])$. By Lemma 5.22 we may identify the inclusion $(\mathbf{\Delta}^n_{/I})^{\amalg,\mathrm{op}} \to \mathbf{\Lambda}^{n,\mathrm{op}}_{/I}$ with the product over $p = 1, \dots, n$ of the inclusions

$$(\mathbf{\Delta}_{/[i_p]})^{\amalg,\mathrm{op}} \hookrightarrow \mathbf{\Lambda}^{\mathrm{op}}_{/[i_p]}.$$

By Proposition A.11 and Corollary A.15, the cartesian product is a left Quillen bifunctor $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_1} \times (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_{n-1}} \to (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_n}$, so by induction it suffices to prove the result in the case $n = 1$, which is Proposition 4.22. □

**Corollary 5.24** *Let $\mathcal{M}$ be a generalized $\mathbf{\Delta}^n$–$\infty$–operad. The restriction map*

$$\mathrm{Alg}^n_{\mathbf{\Lambda}^{n,\mathrm{op}}_{/I}}(\mathcal{M}) \to \lim_{I \to C \in \mathrm{Cell}^{n,\mathrm{op}}_{/I}} \mathrm{Alg}^n_{(\mathbf{\Delta}^n_{/C})^{\mathrm{op}}}(\mathcal{M})$$

*is an equivalence of $\infty$–categories.*

**Proof** Since the model category $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{gen}}_n}$ is enriched in marked simplicial sets and $(\mathbf{\Delta}^n_{/I})^{\amalg,\mathrm{op}} \hookrightarrow \mathbf{\Lambda}^{n,\mathrm{op}}_{/I}$ is a trivial cofibration by Proposition 5.23, for any generalized $\mathbf{\Delta}^n$–$\infty$–operad $\mathcal{M}$ the restriction map

$$\mathrm{Alg}^n_{\mathbf{\Lambda}^{n,\mathrm{op}}_{/I}}(\mathcal{M}) \to \mathrm{Alg}^n_{(\mathbf{\Delta}^n_{/I})^{\amalg,\mathrm{op}}}(\mathcal{M})$$

is a trivial Kan fibration. Moreover, we have an equivalence of $\infty$–categories

$$\mathrm{Alg}^n_{(\mathbf{\Delta}^n_{/I})^{\amalg,\mathrm{op}}}(\mathcal{M}) \simeq \lim_{I \to C \in \mathrm{Cell}^{n,\mathrm{op}}_{/I}} \mathrm{Alg}^n_{(\mathbf{\Delta}^n_{/C})^{\mathrm{op}}}(\mathcal{M})$$

since the colimit $(\mathbf{\Delta}^n_{/I})^{\amalg,\mathrm{op}}$ is a homotopy colimit. □

**Corollary 5.25** *Let $\mathcal{C}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category with good relative tensor products. Then the natural restriction map*

$$\mathfrak{ALG}_n(\mathcal{C})_I \to \lim_{I \to C \in \mathrm{Cell}^{n,\mathrm{op}}_{/I}} \mathfrak{ALG}_n(\mathcal{C})_C$$

*is an equivalence.*

**Proof** This map factors as a composite of the maps

$$\mathfrak{ALG}_n(\mathcal{C})_I \to \mathrm{Alg}^n_{\mathbf{\Lambda}^{n,\mathrm{op}}_{/I}}(\mathcal{C}) \to \lim_{I \to C \in \mathrm{Cell}^{n,\mathrm{op}}_{/I}} \mathfrak{ALG}_n(\mathcal{C})_C,$$

where the first is an equivalence by definition and the second by Corollary 5.24. □

Next we prove that the $\infty$–categories $\mathfrak{ALG}_n(\mathcal{C})_I$ for $I \in \mathbf{\Delta}^{n,\mathrm{op}}$ give a multisimplicial object.

**Definition 5.26** Suppose $\mathcal{C}$ is a $\mathbf{\Delta}^n$–monoidal $\infty$–category. Let $\overline{\mathfrak{ALG}}_n(\mathcal{C}) \to \mathbf{\Delta}^{n,\mathrm{op}}$ denote a cocartesian fibration associated to the functor $\mathbf{\Delta}^{n,\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends $I$ to $\mathrm{Alg}^n_{\mathbf{\Delta}^{n,\mathrm{op}}_{/I}}(\mathcal{C})$. We write $\mathfrak{ALG}_n(\mathcal{C})$ for the full subcategory of $\overline{\mathfrak{ALG}}_n(\mathcal{C})$ spanned by the objects of $\mathfrak{ALG}_n(\mathcal{C})_I$ for all $I$, ie by the composite $\mathbf{\Delta}^{n,\mathrm{op}}_{/I}$–algebras for all $I \in \mathbf{\Delta}^{n,\mathrm{op}}$.

We wish to show that the projection $\mathfrak{ALG}_n(\mathcal{C}) \to \mathbf{\Delta}^{n,\mathrm{op}}$ is a cocartesian fibration. To prove this, we extend the definitions of Section 4.4 in the obvious way:

**Definition 5.27** Suppose $\Phi = (\phi_1, \dots, \phi_n)\colon I \to J$ is an injective morphism in $\mathbf{\Delta}^n$. We say that a morphism $(\alpha_1, \dots, \alpha_n)\colon K \to J$ in $\mathbf{\Delta}^n$ is $\Phi$–*cellular* if $\alpha_i$ is $\phi_i$–cellular for all $i = 1, \dots, n$.

**Definition 5.28** For $I \in \mathbf{\Delta}^n$ and $\Phi\colon J \to I$ an injective morphism in $\mathbf{\Delta}^n$, we write $\mathbf{\Lambda}^n_{/I}[\Phi]$ for the full subcategory of $\mathbf{\Delta}^n_{/I}$ spanned by the $\Phi$–cellular maps to $I$.

**Proposition 5.29** (1) *If $\Phi\colon J \to I$ is an injective morphism in $\mathbf{\Delta}^n$, then for any $\Gamma\colon K \to J$ the map*

$$\Phi_*\colon (\mathbf{\Lambda}^n_{/J})^{\mathrm{act}}_{\Gamma/} \to (\mathbf{\Lambda}^n_{/I}[\Phi])^{\mathrm{act}}_{\Phi\Gamma/}$$

*given by composition with $\Phi$ is coinitial.*

(2) *If $\Phi\colon J \to I$ is a surjective morphism in $\mathbf{\Delta}^n$, then for any $\Gamma\colon K \to J$ the map*

$$(\mathbf{\Lambda}^n_{/J})^{\mathrm{act}}_{\Gamma/} \to (\mathbf{\Lambda}^n_{/I})^{\mathrm{act}}_{\Phi\Gamma}$$

*given by composition with $\Phi$ is coinitial.*

**Proof** Since products of coinitial functors are coinitial, this is immediate from Proposition 4.37. □

**Corollary 5.30** *Suppose $\mathcal{C}$ is a $\mathbf{\Delta}^n$–monoidal $\infty$–category compatible with small colimits. Then the projection $\mathfrak{ALG}_n(\mathcal{C}) \to \mathbf{\Delta}^{n,\mathrm{op}}$ is a cocartesian fibration.*

**Proof** Since $\overline{\mathfrak{ALG}}_n(\mathcal{C}) \to \mathbf{\Delta}^{n,\mathrm{op}}$ is a cocartesian fibration, it suffices to show that if $X$ is an object of $\mathfrak{ALG}_n(\mathcal{C})$ over $I \in \mathbf{\Delta}^{n,\mathrm{op}}$, and $X \to \overline{X}$ is a cocartesian morphism in $\overline{\mathfrak{ALG}}_n(\mathcal{C})$ over $\Phi\colon J \to I$ in $\mathbf{\Delta}^n$, then $\overline{X}$ is also in $\mathfrak{ALG}_n(\mathcal{C})$. This follows by the same argument as in the proof of Corollary 4.38, using Proposition 5.29. □

Combining Corollary 5.30 with Corollary 5.25, we have proved:

**Theorem 5.31** *Let $\mathcal{C}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category with good relative tensor products. Then the projection $\mathfrak{ALG}_n(\mathcal{C}) \to \mathbf{\Delta}^{n,\mathrm{op}}$ is a $\mathbf{\Delta}^n$–uple $\infty$–category.* □

**Remark 5.32** Suppose $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are $\mathbf{\Delta}^n$–monoidal $\infty$–categories with good relative tensor products, and $f^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a $\mathbf{\Delta}^n$–monoidal functor compatible with relative tensor products. Composition with $f^{\otimes}$ induces a functor $f_*\colon \overline{\mathfrak{ALG}}_n(\mathcal{C}) \to \overline{\mathfrak{ALG}}_n(\mathcal{D})$. It follows from Lemma A.63 that this functor takes the full subcategory $\mathfrak{ALG}_n(\mathcal{C})$ into $\mathfrak{ALG}_n(\mathcal{D})$, and so induces a map $f_*\colon \mathfrak{ALG}_n(\mathcal{C}) \to \mathfrak{ALG}_n(\mathcal{D})$ of $(n+1)$–fold $\infty$–categories.

**Definition 5.33** Let $\mathcal{C}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category with good relative tensor products. We write $\mathfrak{Alg}_n(\mathcal{C})$ for the completion $L_n U_{\mathrm{Seg}} i \mathfrak{ALG}_n(\mathcal{C})$ of the underlying $(n+1)$–fold Segal space $U_{\mathrm{Seg}} i \mathfrak{ALG}_n(\mathcal{C})$ of the image of $\mathfrak{ALG}_n(\mathcal{C})$ under the forgetful functor $i\colon \mathrm{Upl}_\infty^{\mathbf{\Delta}^n} \simeq \mathrm{Cat}^n(\mathrm{Cat}_\infty) \to \mathrm{Cat}^{n+1}(\mathcal{S})$. Thus $\mathfrak{Alg}_n(\mathcal{C})$ is a complete $(n+1)$–fold Segal space, ie an $(\infty, n+1)$–category.

## 5.4 Functoriality and monoidal structures

Our goal in this subsection is to show that the $(n+1)$–fold $\infty$–categories $\mathfrak{ALG}_n(\mathcal{C})$ we constructed above are functorial in $\mathcal{C}$, and moreover that this functor is lax monoidal. From this it will follow immediately that if $\mathcal{C}$ is an $\mathbb{E}_{n+m}$–monoidal $\infty$–category with good relative tensor products, then the $(\infty, n+1)$–category $\mathfrak{Alg}_n(\mathcal{C})$ inherits a canonical $\mathbb{E}_m$–monoidal structure. We begin by introducing some notation for the source of our functor:

**Definition 5.34** Let $\widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n}$ denote the $\infty$–category of $\mathbf{\Delta}^n$–monoidal $\infty$–categories and $\mathbf{\Delta}^n$–monoidal functors. We write $\widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}}$ for the subcategory of $\widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n}$ determined by the $\mathbf{\Delta}^n$–monoidal $\infty$–categories with good relative tensor products and the $\mathbf{\Delta}^n$–monoidal functors compatible with these. If $n = 1$ we also denote this by $\widehat{\mathrm{Mon}}_\infty^{\mathrm{GRTP}}$.

**Definition 5.35** Let $\mathrm{Alg}^n \to (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}} \times \widehat{\mathrm{Opd}}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}}$ be defined in the same way as the cocartesian fibration in Section A.7, but allowing the target generalized $\mathbf{\Delta}^n$–$\infty$–operads to be large. Then we define $\overline{\mathfrak{ALG}}_n$ by the pullback square

$$\begin{array}{ccc} \overline{\mathfrak{ALG}}_n & \longrightarrow & \mathrm{Alg}^n \\ \downarrow & & \downarrow \\ \mathbf{\Delta}^{n,\mathrm{op}} \times \widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}} & \longrightarrow & (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}} \times \widehat{\mathrm{Opd}}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}} \end{array}$$

where the bottom horizontal map is the product of $\mathbf{\Delta}^{n,\mathrm{op}}_{/(-)}\colon \mathbf{\Delta}^{n,\mathrm{op}} \to (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}}$ and the forgetful functor from large $\mathbf{\Delta}^n$–monoidal $\infty$–categories with good relative tensor products to large generalized $\mathbf{\Delta}^n$–$\infty$–operads. Write $\mathfrak{ALG}_n$ for the full subcategory of $\overline{\mathfrak{ALG}}_n$ spanned by the objects in $\mathfrak{ALG}_n(\mathcal{C})$ for all $\mathbf{\Delta}^n$–monoidal $\infty$–categories $\mathcal{C}$ in $\widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}}$.

**Proposition 5.36** *The restricted projection $\mathfrak{ALG}_n \to \mathbf{\Delta}^{n,\mathrm{op}} \times \widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}}$ is a cocartesian fibration.*

**Proof** Suppose $X$ is an object of $\mathfrak{ALG}_n$ over $(I,\mathcal{C})$ and $(\Phi,F)\colon (I,\mathcal{C}) \to (J,\mathcal{D})$ is a morphism in $\mathbf{\Delta}^{n,\mathrm{op}} \times \widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}}$. Then it suffices to prove that if $X \to (\Phi,F)_! X$ is a cocartesian morphism in $\overline{\mathfrak{ALG}}_n$, then $(\Phi,F)_! X$ lies in $\mathfrak{ALG}_n$.

It is enough to consider the morphisms $(\Phi,\mathrm{id}_{\mathcal{C}})$ and $(\mathrm{id}_I,F)$ separately. We know that $(\Phi,\mathrm{id}_{\mathcal{C}})_! X$ is in $\mathfrak{ALG}_n$ by Corollary 5.30, and the object $(\mathrm{id},F)_! X$ lies in $\mathfrak{ALG}_n$ by Remark 5.32. □

**Corollary 5.37** *There is a functor $\mathfrak{ALG}_n(-)\colon \widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}} \to \mathrm{Cat}^n(\mathrm{Cat}_\infty)$ that sends $\mathcal{C}$ to $\mathfrak{ALG}_n(\mathcal{C})$.*

**Proof** By Proposition 5.36 there is a functor $\widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}} \times \mathbf{\Delta}^{n,\mathrm{op}} \to \mathrm{Cat}_\infty$, or equivalently

$$\widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}} \to \mathrm{Fun}(\mathbf{\Delta}^{n,\mathrm{op}},\mathrm{Cat}_\infty),$$

associated to the cocartesian fibration $\mathfrak{ALG}_n \to \mathbf{\Delta}^{n,\mathrm{op}} \times \widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}}$. By Corollary 5.25 this functor lands in the full subcategory $\mathrm{Cat}^n(\mathrm{Cat}_\infty)$ of $n$–uple category objects. □

**Lemma 5.38** (i) *The $\infty$–category $\widehat{\mathrm{Mon}}_\infty^{\mathrm{GRTP}}$ has products, and the forgetful functor $\widehat{\mathrm{Mon}}_\infty^{\mathrm{GRTP}} \to \widehat{\mathrm{Mon}}_\infty$ preserves these.*

(ii) *The $\infty$–category $\widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}}$ is equivalent to $\mathrm{Alg}^{n-1}_{\mathbf{\Delta}^{n-1},\mathrm{op}}(\widehat{\mathrm{Mon}}_\infty^{\mathrm{GRTP}})$.*

(iii) *The $\infty$–category $\widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}}$ has products for all $n$, and the forgetful functor $\widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n,\mathrm{GRTP}} \to \widehat{\mathrm{Mon}}_\infty^{\mathbf{\Delta}^n}$ preserves these.*

**Proof** Suppose $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are monoidal $\infty$–categories with good relative tensor products. We will show that the product $(\mathcal{C}\times\mathcal{D})^{\otimes} := \mathcal{C}^{\otimes}\times_{\mathbf{\Delta}^{\mathrm{op}}}\mathcal{D}^{\otimes}$ in $\widehat{\mathrm{Mon}}_{\infty}$ is also a product in the subcategory $\widehat{\mathrm{Mon}}_{\infty}^{\mathrm{GRTP}}$. Thus, we need to prove:

(1) The product $(\mathcal{C}\times\mathcal{D})^{\otimes}$ has good relative tensor products.

(2) If $\mathcal{E}\in\widehat{\mathrm{Mon}}_{\infty}^{\mathrm{GRTP}}$, then a monoidal functor $F\colon\mathcal{E}\to\mathcal{C}\times\mathcal{D}$ is compatible with relative tensor products if and only if the monoidal functors $F_1\colon\mathcal{E}\to\mathcal{C}$ and $F_2\colon\mathcal{E}\to\mathcal{D}$ obtained by composing with the projections are both compatible with relative tensor products.

To prove (1), we use Lemma 4.19. A $\mathbf{\Lambda}^{\mathrm{op}}_{/[2]}$–algebra in $\mathcal{C}\times\mathcal{D}$ is given by a $\mathbf{\Lambda}^{\mathrm{op}}_{/[2]}$–algebra $A$ in $\mathcal{C}$ and an algebra $B$ in $\mathcal{D}$. Moreover, the induced diagram $(\mathbf{\Lambda}^{\mathrm{op}}_{/[2]})^{\mathrm{act}}_{/(0,2)}\to\mathcal{C}\times\mathcal{D}$ is the composite

$$(\mathbf{\Lambda}^{\mathrm{op}}_{/[2]})^{\mathrm{act}}_{/(0,2)}\to(\mathbf{\Lambda}^{\mathrm{op}}_{/[2]})^{\mathrm{act}}_{/(0,2)}\times(\mathbf{\Lambda}^{\mathrm{op}}_{/[2]})^{\mathrm{act}}_{/(0,2)}\to\mathcal{C}\times\mathcal{D}.$$

Since $(\mathbf{\Lambda}^{\mathrm{op}}_{/[2]})^{\mathrm{act}}_{/(0,2)}$ is sifted by Lemma 4.17, this colimit is therefore given by the pair of colimits in $\mathcal{C}$ and $\mathcal{D}$. It follows that $\mathcal{C}\times\mathcal{D}$ has good relative tensor products. Then (2) follows from a similar argument, again using siftedness. This proves (i).

To prove (ii), observe that $\widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP}}$ and $\mathrm{Alg}^{n-1}_{\mathbf{\Delta}^{n-1},\mathrm{op}}(\widehat{\mathrm{Mon}}_{\infty}^{\mathrm{GRTP}})$ can both be identified with subcategories of $\widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n}$, and it follows from Lemma 5.20 that they are the same subcategory. Now (iii) follows by the same argument as for (i), or using the description of limits in $\infty$–categories of algebras from [31, Corollary 3.2.2.4]. □

**Definition 5.39** Let $\mathrm{Alg}^{n,\otimes}\to(\mathrm{Opd}_{\infty}^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}}\times(\widehat{\mathrm{Opd}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{gen}})^{\times}$ be the obvious variant of the cocartesian fibration of generalized symmetric $\infty$–operads defined in Section A.7. Then we define $\overline{\mathfrak{ALG}}{}^{\otimes}_n$ by the pullback square

$$\begin{array}{ccc}\overline{\mathfrak{ALG}}{}^{\otimes}_n & \longrightarrow & \mathrm{Alg}^{n,\otimes}\\ \downarrow & & \downarrow\\ \mathbf{\Delta}^{n,\mathrm{op}}\times\widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP},\times} & \longrightarrow & (\mathrm{Opd}_{\infty}^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}}\times(\widehat{\mathrm{Opd}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{gen}})^{\times}\end{array}$$

where the bottom horizontal map is the product of $\mathbf{\Delta}^{n,\mathrm{op}}_{/(-)}$ and the symmetric monoidal structure on the forgetful functor that arises since this preserves products. Then $\overline{\mathfrak{ALG}}{}^{\otimes}_n\to\mathbf{\Delta}^{n,\mathrm{op}}\times\widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP},\times}$ is a cocartesian fibration of generalized symmetric $\infty$–operads. Write $\mathfrak{ALG}^{\otimes}_n$ for the full subcategory of $\overline{\mathfrak{ALG}}{}^{\otimes}_n$ spanned by the objects corresponding to lists of objects of $\mathfrak{ALG}_n$.

**Proposition 5.40** *The restricted projection $\mathfrak{ALG}^{\otimes}_n\to\mathbf{\Delta}^{n,\mathrm{op}}\times\widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP},\times}$ is a cocartesian fibration of generalized symmetric $\infty$–operads.*

**Proof** Since $\mathfrak{ALG}_n^{\otimes}$ is the full subcategory of $\overline{\mathfrak{ALG}}_n^{\otimes}$ determined by the full subcategory $\mathfrak{ALG}_n$ of $\overline{\mathfrak{ALG}}_n \simeq (\overline{\mathfrak{ALG}}_n^{\otimes})_{\langle 1\rangle}$, it is a generalized symmetric $\infty$–operad. Moreover, it is easy to see from Proposition 5.36 and Remark A.68 that $\mathfrak{ALG}_n^{\otimes}$ inherits cocartesian morphisms from $\overline{\mathfrak{ALG}}_n^{\otimes}$. □

**Corollary 5.41** *$\mathfrak{ALG}_n$ defines a lax symmetric monoidal functor $\widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP}} \to \mathrm{Cat}^n(\mathrm{Cat}_\infty)$. In particular, if $\mathcal{C}$ is an $\mathbb{E}_{n+m}$–monoidal $\infty$–category then $\mathfrak{ALG}_n(\mathcal{C})$ inherits a canonical $\mathbb{E}_m$–monoidal structure.*

**Proof** Since $\mathfrak{ALG}_n^{\otimes} \to \mathbf{\Delta}^{n,\mathrm{op}} \times \widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP},\times}$ is a cocartesian fibration of generalized symmetric $\infty$–operads, the associated functor $\mathbf{\Delta}^{n,\mathrm{op}} \times \widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP},\times} \to \mathrm{Cat}_\infty$ is a monoid object. The corresponding functor $\widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP},\times} \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ is then also a monoid object, and lands in the full subcategory $\mathrm{Cat}^n(\mathrm{Cat}_\infty)$ of $n$–fold category objects. This therefore corresponds to a lax monoidal functor $\widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP}} \to \mathrm{Cat}^n(\mathrm{Cat}_\infty)$ by [31, Proposition 2.4.2.5]. □

**Corollary 5.42** *$\mathfrak{Alg}_n$ defines a lax symmetric monoidal functor $\widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP}} \to \mathrm{Cat}_{(\infty,n)}$. In particular, if $\mathcal{C}$ is an $\mathbb{E}_{n+m}$–monoidal $\infty$–category, then $\mathfrak{Alg}_n(\mathcal{C})$ inherits a canonical $\mathbb{E}_m$–monoidal structure.*

**Proof** By definition, $\mathfrak{Alg}_n$ is the composite of the lax monoidal functor

$$\mathfrak{ALG}_n\colon \widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP}} \to \mathrm{Cat}^n(\mathrm{Cat}_\infty)$$

with the inclusion $i\colon \mathrm{Cat}^n(\mathrm{Cat}_\infty) \to \mathrm{Cat}^{n+1}(\mathcal{S})$, the functor $U_{\mathrm{Seg}}\colon \mathrm{Cat}^{n+1}(\mathcal{S}) \to \mathrm{Seg}_n(\mathcal{S})$ that takes an $n$–uple Segal space to its underlying $n$–fold Segal space, and $L_n\colon \mathrm{Seg}_n(\mathcal{S}) \to \mathrm{CSS}_n(\mathcal{S}) \simeq \mathrm{Cat}_{(\infty,n)}$, the completion functor. The functor $L_n U_{\mathrm{Seg}} i$ is symmetric monoidal by Remark 3.15, and so the composite $\widehat{\mathrm{Mon}}_{\infty}^{\mathbf{\Delta}^n,\mathrm{GRTP}} \to \mathrm{Cat}_{(\infty,n)}$ is also lax symmetric monoidal. □

## 5.5 The mapping $(\infty,n)$–categories of $\mathfrak{Alg}_n(\mathcal{C})$

Our goal in this subsection is to prove that if $A$ and $B$ are $\mathbb{E}_n$–algebras in an $\mathbb{E}_n$–monoidal $\infty$–category $\mathcal{C}$, then the $(\infty,n)$–category $\mathfrak{Alg}_n(\mathcal{C})(A,B)$ of maps from $A$ to $B$ in $\mathfrak{Alg}_n(\mathcal{C})$ can be identified with the $(\infty,n)$–category $\mathfrak{Alg}_{n-1}(\mathrm{Bimod}_{A,B}(\mathcal{C}))$ of $\mathbb{E}_{n-1}$–algebras in the $\infty$–category $\mathrm{Bimod}_{A,B}(\mathcal{C})$ of $A$–$B$–bimodules, equipped with a natural $\mathbb{E}_{n-1}$–monoidal structure.

First we will show that in this situation $\mathrm{Bimod}_{A,B}(\mathcal{C})$ does in fact inherit an $\mathbb{E}_{n-1}$–monoidal structure:

**Definition 5.43** Let $\mathcal{C}$ be a $\mathbf{\Delta}^{n+1}$–monoidal $\infty$–category. We write $\mathrm{Bimod}^{\otimes}(\mathcal{C})$ for the internal hom $\mathrm{ALG}^{1,n+1}_{\mathbf{\Delta}^{\mathrm{op}}_{/[1]}}(\mathcal{C})$ and $\mathrm{Ass}^{\otimes}(\mathcal{C})$ for $\mathrm{ALG}^{1,n+1}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{C})$. By Lemma A.75 these are both $\mathbf{\Delta}^n$–monoidal $\infty$–categories, and the natural map $\mathrm{Bimod}^{\otimes}(\mathcal{C}) \to \mathrm{Ass}^{\otimes}(\mathcal{C}) \times_{\mathbf{\Delta}^{n,\mathrm{op}}} \mathrm{Ass}^{\otimes}(\mathcal{C})$ induced by the map of generalized nonsymmetric $\infty$–operads $\mathbf{\Delta}^{\mathrm{op}} \amalg \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}_{/[1]}$ is a $\mathbf{\Delta}^n$–monoidal functor.

**Proposition 5.44** *Let $\mathcal{C}^{\otimes}$ be a $\mathbf{\Delta}^{n+1}$–monoidal $\infty$–category with good relative tensor products. Then the projection $\Pi\colon \mathrm{Bimod}^{\otimes}(\mathcal{C}) \to \mathrm{Ass}^{\otimes}(\mathcal{C}) \times_{\mathbf{\Delta}^{n,\mathrm{op}}} \mathrm{Ass}^{\otimes}(\mathcal{C})$ is a cocartesian fibration of $\mathbf{\Delta}^n$–monoidal $\infty$–categories.*

For the proof we use the following criterion:

**Proposition 5.45** *Suppose given a commutative triangle*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{D} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{C} & \end{array}$$

*of functors between $\infty$–categories such that:*

(1) *$p$ and $q$ are cartesian fibrations.*

(2) *$f$ takes $p$–cartesian edges to $q$–cartesian edges.*

(3) *For each object $c \in \mathcal{C}$ the induced map on fibres $f_c\colon \mathcal{E}_c \to \mathcal{D}_c$ is a cartesian fibration.*

(4) *Suppose given a commutative square*

$$\begin{array}{ccc} \phi^* e' & \xrightarrow{\alpha} & e' \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \phi^* e & \xrightarrow[\delta]{} & e \end{array}$$

*in $\mathcal{E}$ lying over the degenerate square*

$$\begin{array}{ccc} c' & \xrightarrow{\phi} & c \\ {\scriptstyle \mathrm{id}_{c'}}\downarrow & & \downarrow{\scriptstyle\phi} \\ c' & \xrightarrow[\mathrm{id}_c]{} & c \end{array}$$

*in $\mathcal{C}$, where $\alpha$ and $\delta$ are $p$–cartesian edges and $\gamma$ is $f_c$–cartesian. Then $\beta$ is $f_{c'}$–cartesian. (In other words, the induced functor $\phi^*\colon \mathcal{E}_c \to \mathcal{E}_{c'}$ takes $f_c$–cartesian edges to $f_{c'}$–cartesian edges.)*

*Then $f$ is also a cartesian fibration.*

**Proof** By [28, Proposition 2.4.4.3] we must show that $f$–cartesian morphisms exist in $\mathcal{E}$. More precisely, suppose given $e \in \mathcal{E}$ lying over $d \in \mathcal{D}$ and $c \in \mathcal{C}$ (ie $d \simeq f(e)$ and $c \simeq p(e) \simeq q(d)$) and a morphism $\delta\colon d' \to d$ in $\mathcal{D}$ lying over $\gamma\colon c' \to c$ in $\mathcal{C}$. Then we must show that there exists an $f$–cartesian morphism $e' \to e$ over $\delta$.

Since $p$ is a cartesian fibration, there exists a $p$–cartesian morphism $\beta\colon \gamma^* e \to e$ over $\gamma$, and as $f$ takes $p$–cartesian edges to $q$–cartesian edges, its image in $\mathcal{D}$ is a $q$–cartesian edge $f(\beta)\colon \gamma^* d \to d$. There is then an essentially unique factorization of $\delta$ through $f(\beta)$, as

$$d' \xrightarrow{\alpha} \gamma^* d \xrightarrow{f(\beta)} d.$$

Now $\alpha$ is a morphism in $\mathcal{D}_{c'}$, so since $f_{c'}$ is a cartesian fibration there exists an $f_{c'}$–cartesian edge $\epsilon\colon \alpha^*\gamma^* e \to \gamma^* e$. We will show that the composite $\beta \circ \epsilon\colon \alpha^*\gamma^* e \to \gamma^* e \to e$ is an $f$–cartesian morphism over $\delta$.

To see this, we consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Map}_{\mathcal{E}}(x, \alpha^*\gamma^* e) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(x, \gamma^* e) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(x, e) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{D}}(f(x), d') & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(f(x), \gamma^* d) & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(f(x), d) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{C}}(p(x), c') & \xrightarrow[\mathrm{id}]{} & \mathrm{Map}_{\mathcal{C}}(p(x), c') & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(p(x), c)
\end{array}$$

where $x$ is an arbitary object of $\mathcal{E}$. By [28, Proposition 2.4.4.3], to see that $\beta \circ \epsilon$ is $f$–cartesian we must show that the composite of the two upper squares is cartesian. We will prove this by showing that both of the upper squares are cartesian. By construction, $\beta$ is $p$–cartesian and $f(\beta)$ is $q$–cartesian, so the composite of the two right squares and the bottom right square are both cartesian, hence so is the upper right square.

Since a commutative square of spaces is cartesian if and only if the induced maps on all fibres are equivalences, to see that the upper left square is cartesian it suffices to

show that the square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(x, \alpha^*\gamma^* e)_\mu & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(x, \gamma^* e)_\mu \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{D}}(f(x), d')_\mu & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(f(x), \gamma^* d)_\mu \end{array}$$

obtained by taking the fibre at $\mu\colon p(x) \to c'$ is cartesian for every map $\mu$. Now taking $p$– and $q$–cartesian pullbacks along $\mu$ we can (since $f$ takes $p$–cartesian morphisms to $q$–cartesian morphisms) identify this with the square:

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}_{p(x)}}(x, \mu^*\alpha^*\gamma^* e) & \longrightarrow & \mathrm{Map}_{\mathcal{E}_{p(x)}}(x, \mu^*\gamma^* e) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{D}_{p(x)}}(f(x), \mu^* d') & \longrightarrow & \mathrm{Map}_{\mathcal{D}_{p(x)}}(f(x), \mu * \gamma^* d) \end{array}$$

But this is cartesian since by assumption the map $\mu^*\alpha^*\gamma^* e \to \mu^*\gamma^* e$ is $f_{p(x)}$–cartesian (because $\epsilon$ is $f_{c'}$–cartesian). □

**Proof of Proposition 5.44** We know that the projections $\mathrm{Bimod}^{\otimes}(\mathcal{C}) \to \mathbf{\Delta}^{n,\mathrm{op}}$ and $\mathrm{Ass}^{\otimes}(\mathcal{C}) \to \mathbf{\Delta}^{n,\mathrm{op}}$ are cocartesian fibrations, and that the map $\Pi$ preserves cocartesian morphisms. By Proposition 5.45 it thus suffices to check that:

(a) The map on fibres

$$\mathrm{Bimod}^{\otimes}(\mathcal{C})_I \to \mathrm{Ass}^{\otimes}(\mathcal{C})_I^{\times 2}$$

is a cocartesian fibration for all $I \in \mathbf{\Delta}^{n,\mathrm{op}}$.

(b) For every map $\phi\colon I \to J$ in $\mathbf{\Delta}^{n,\mathrm{op}}$ the induced functor

$$\mathrm{Bimod}^{\otimes}(\mathcal{C})_I \to \mathrm{Bimod}^{\otimes}(\mathcal{C})_J$$

takes $\Pi_I$–cocartesian morphisms to $\Pi_J$–cocartesian morphisms.

But by Corollary A.77 we may identify the map $\Pi_I$ with the map $\mathrm{Bimod}(\mathcal{C}^{\otimes}_{(I,\bullet)}) \to \mathrm{Ass}(\mathcal{C}^{\otimes}_{(I,\bullet)})^{\times 2}$, which is a cocartesian fibration by Proposition 4.53; this proves (a). Moreover, the map

$$\mathrm{Bimod}^{\otimes}(\mathcal{C})_I \to \mathrm{Bimod}^{\otimes}(\mathcal{C})_J$$

induced by $\phi$ can be identified with the map $\mathrm{Bimod}(\mathcal{C}^{\otimes}_{I,\bullet}) \to \mathrm{Bimod}(\mathcal{C}^{\otimes}_{J,\bullet})$ induced by composition with $\phi_!\colon \mathcal{C}^{\otimes}_{I,\bullet} \to \mathcal{C}^{\otimes}_{J,\bullet}$. This is a $\mathbf{\Delta}^n$–monoidal functor, and it is compatible

with relative tensor products since $\mathcal{C}^{\otimes}$ has good relative tensor products. The description of the $\Pi_I$–cocartesian morphisms in Proposition 4.53 therefore implies (b). □

**Corollary 5.46** *Let $\mathcal{C}$ be a $\mathbf{\Delta}^{n+1}$–monoidal $\infty$–category with good relative tensor products, and suppose $A$ and $B$ are $\mathbf{\Delta}^{n+1,\mathrm{op}}$–algebras in $\mathcal{C}$. Then we can regard $A$ and $B$ as $\mathbf{\Delta}^{n,\mathrm{op}}$–algebras in $\mathrm{Ass}^{\otimes}(\mathcal{C})$. Define an $\infty$–category $\mathrm{Bimod}^{\otimes}_{A,B}(\mathcal{C})$ by the pullback square:*

$$\begin{array}{ccc} \mathrm{Bimod}^{\otimes}_{A,B}(\mathcal{C}) & \longrightarrow & \mathrm{Bimod}^{\otimes}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathbf{\Delta}^{n,\mathrm{op}} & \xrightarrow[(A,B)]{} & \mathrm{Ass}^{\otimes}(\mathcal{C}) \times_{\mathbf{\Delta}^{n,\mathrm{op}}} \mathrm{Ass}^{\otimes}(\mathcal{C}) \end{array}$$

*Then the projection $\mathrm{Bimod}^{\otimes}_{A,B}(\mathcal{C}) \to \mathbf{\Delta}^{n,\mathrm{op}}$ is a $\mathbf{\Delta}^{n}$–monoidal $\infty$–category with underlying $\infty$–category $\mathrm{Bimod}_{A,B}(\mathcal{C})$.* □

**Lemma 5.47** *Let $\mathcal{C}$ be a $\mathbf{\Delta}^{n+1}$–monoidal $\infty$–category with good relative tensor products, and suppose $A$ and $B$ are $\mathbf{\Delta}^{n+1,\mathrm{op}}$–algebras in $\mathcal{C}$. Then the $\mathbf{\Delta}^{n}$–monoidal $\infty$–category $\mathrm{Bimod}_{A,B}(\mathcal{C})$ has good relative tensor products.*

**Proof** By Lemma 5.20 it suffices to consider the case $n = 1$, in which case we use the criterion of Lemma 4.19. Suppose given an algebra $U\colon \mathbf{\Lambda}^{\mathrm{op}}_{/[2]} \to \mathrm{Bimod}^{\otimes}_{A,B}(\mathcal{C})$. The induced diagram $F\colon (\mathbf{\Lambda}^{\mathrm{op}}_{/[2]})^{\mathrm{act}}_{(0,2)} \to \mathrm{Bimod}_{A,B}(\mathcal{C})$ can be identified with the cocartesian pushforward to the fibre over $(A, B)$ of the corresponding diagram $F'\colon (\mathbf{\Lambda}^{\mathrm{op}}_{/[2]})^{\mathrm{act}}_{(0,2)} \to \mathrm{Bimod}(\mathcal{C})$ for the composite algebra $U'\colon \mathbf{\Lambda}^{\mathrm{op}}_{/[2]} \to \mathrm{Bimod}^{\otimes}(\mathcal{C})$. Projecting the latter diagram to $\mathrm{Ass}(\mathcal{C}) \times \mathrm{Ass}(\mathcal{C})$ gives the simplicial diagrams $A \otimes A^{\otimes \bullet} \otimes A$ and $B \otimes B^{\otimes \bullet} \otimes B$ with colimits $A \otimes_A A \simeq A$ and $B \otimes_B B \simeq B$. To see that $F$ has a colimit, it then suffices by [28, Propositions 4.3.1.9 and 4.3.1.10] to show that $F'$ has a colimit. For this we use Corollary 4.45, since $(\mathbf{\Lambda}^{\mathrm{op}}_{/[2]})^{\mathrm{act}}_{(0,2)}$ is sifted by Lemma 4.17. The projections to $\mathcal{C}$ of this diagram are all relative tensor product diagrams, and so have monoidal colimits since $\mathcal{C}$ has good relative tensor products, so the colimit of $F'$ does exist in $\mathrm{Bimod}(\mathcal{C})$. The colimit in $\mathrm{Bimod}_{A,B}(\mathcal{C})$ is moreover preserved under tensoring with objects of $\mathrm{Bimod}_{A,B}(\mathcal{C})$ by a similar argument, since the tensor product in $A$–$B$–bimodules projects to a relative tensor product in $\mathcal{C}$. □

**Proposition 5.48** *Let $\mathcal{C}$ be a $\mathbf{\Delta}^{n+1}$–monoidal $\infty$–category with good relative tensor products, and suppose $A$ and $B$ are $\mathbf{\Delta}^{n+1,\mathrm{op}}$–algebras in $\mathcal{C}$ and*

$$U\colon \mathbf{\Delta}^{n,\mathrm{op}}_{/I} \to \mathrm{Bimod}^{\otimes}_{A,B}(\mathcal{C})$$

*is a $\mathbf{\Delta}^{n,\mathrm{op}}_{/I}$–algebra in $\mathrm{Bimod}_{A,B}(\mathcal{C})$. Then $U$ is composite if and only if the algebra $U'$ in $\mathrm{Bimod}(\mathcal{C})$ obtained by composing with the inclusion is a composite $\mathbf{\Delta}^{n,\mathrm{op}}_{/I}$–algebra in $\mathrm{Bimod}^{\otimes}(\mathcal{C})$.*

**Proof** For $X \in \mathbf{\Delta}^{n,\mathrm{op}}_{/I}$ we can conclude, using [28, Propositions 4.3.1.9 and 4.3.1.10] as in the proof of Lemma 5.47, that the diagram $\overline{\xi}'\colon ((\mathbf{\Lambda}^{n,\mathrm{op}}_{/I})^{\mathrm{act}}_{/X})^{\triangleright} \to \mathrm{Bimod}(\mathcal{C})$ induced by $U'$ is a colimit diagram if and only if the corresponding diagram $\overline{\xi}$ for $U$, which is obtained as the cocartesian pushforward of $\overline{\xi}'$ to the fibre over $(A,B)$, is a colimit in $\mathrm{Bimod}_{A,B}(\mathcal{C})$ and this is preserved by the functors $(f,g)_!\colon \mathrm{Bimod}_{A,B}(\mathcal{C}) \to \mathrm{Bimod}_{A',B'}(\mathcal{C})$ induced by the cocartesian morphisms over any maps $f\colon A \to A'$ and $g\colon B \to B'$ of associative algebras. On the other hand, we know that $\xi := \overline{\xi}|_{(\mathbf{\Lambda}^{n,\mathrm{op}}_{/I})^{\mathrm{act}}_{/X}}$ does have a colimit in $\mathrm{Bimod}_{A,B}(\mathcal{C})$ whose underlying diagram in $\mathcal{C}$ is a monoidal colimit diagram. Using Corollary 4.45 this implies that this colimit is necessarily preserved by the functors $(f,g)_!$. The two conditions are therefore equivalent, as required. □

**Corollary 5.49** *Let $\mathcal{C}$ be a $\mathbf{\Delta}^{n+1}$–monoidal $\infty$–category compatible with geometric realizations and initial objects, and suppose $A$ and $B$ are $\mathbf{\Delta}^{n+1,\mathrm{op}}$–algebras in $\mathcal{C}$. Then we have a pullback square*

$$\begin{array}{ccc} \mathfrak{ALG}_n(\mathrm{Bimod}_{A,B}(\mathcal{C})) & \longrightarrow & \mathfrak{ALG}_{n+1}(\mathcal{C})_{[1]} \\ \downarrow & & \downarrow \\ * & \xrightarrow[(A,B)]{} & \mathfrak{ALG}_{n+1}(\mathcal{C})^{\times 2}_{[0]} \end{array}$$

*of $n$–uple category objects in $\mathrm{Cat}_\infty$.*

**Proof** By the universal property of the internal hom $\mathrm{ALG}^{n+1,1}_{(-)}(-)$, we can identify the map

$$\overline{\mathfrak{ALG}}_{n+1}(\mathcal{C})_{([1],I)} \to \overline{\mathfrak{ALG}}_{n+1}(\mathcal{C})^{\times 2}_{([0],I)}$$

with

$$\mathrm{Alg}^{n}_{\mathbf{\Delta}^{n,\mathrm{op}}_{/I}}(\mathrm{Bimod}^{\otimes}(\mathcal{C})) \to \mathrm{Alg}^{n}_{\mathbf{\Delta}^{n,\mathrm{op}}_{/I}}(\mathrm{Ass}^{\otimes}(\mathcal{C}) \times_{\mathbf{\Delta}^{n,\mathrm{op}}} \mathrm{Ass}^{\otimes}(\mathcal{C})).$$

Since $\mathrm{Alg}^{n}_{\mathbf{\Delta}^{n,\mathrm{op}}_{/I}}(-)$ preserves limits, and $\mathbf{\Delta}^{n,\mathrm{op}}$ is the final $\mathbf{\Delta}^{n}$–monoidal $\infty$–category, we obtain a pullback square

$$\begin{array}{ccc} \overline{\mathfrak{AL}\mathfrak{S}}_n(\mathrm{Bimod}_{A,B}(\mathcal{C}))_I & \longrightarrow & \overline{\mathfrak{AL}\mathfrak{S}}_{n+1}(\mathcal{C})_{([1],I)} \\ \downarrow & & \downarrow \\ * & \xrightarrow[(A,B)]{} & \overline{\mathfrak{AL}\mathfrak{S}}_{n+1}(\mathcal{C})^{\times 2}_{([0],I)} \end{array}$$

natural in $I$. Proposition 5.48 implies that this restricts to a pullback square

$$\begin{array}{ccc} \mathfrak{AL}\mathfrak{S}_n(\mathrm{Bimod}_{A,B}(\mathcal{C}))_I & \longrightarrow & \mathfrak{AL}\mathfrak{S}_{n+1}(\mathcal{C})_{([1],I)} \\ \downarrow & & \downarrow \\ * & \xrightarrow[(A,B)]{} & \mathfrak{AL}\mathfrak{S}_{n+1}(\mathcal{C})^{\times 2}_{([0],I)} \end{array}$$

and the naturality in $I$ then gives a pullback square of $n$–fold category objects in $\mathrm{Cat}_\infty$, since the inclusion of these into all functors $\mathbf{\Delta}^{n,\mathrm{op}} \to \mathrm{Cat}_\infty$ is a right adjoint and so preserves limits. □

From this we can now prove the main result of this subsection:

**Theorem 5.50** *Let $\mathcal{C}$ be a $\mathbf{\Delta}^{n+1}$–monoidal $\infty$–category with good relative tensor products, and suppose $A$ and $B$ are $\mathbf{\Delta}^{n+1,\mathrm{op}}$–algebras in $\mathcal{C}$. Then the $(\infty,n)$–category $\mathfrak{Alg}_{n+1}(\mathcal{C})(A,B)$ is naturally equivalent to $\mathfrak{Alg}_n(\mathrm{Bimod}_{A,B}(\mathcal{C}))$.*

This will follow from Corollary 5.49 together with the following observation:

**Lemma 5.51** *Suppose $\mathcal{X}$ is an $(n+1)$–fold Segal space and $x$ and $y$ are two objects of $\mathcal{X}$. Then the $n$–fold Segal space $(L_{n+1}\mathcal{X})(x,y)$ of maps in the completion of $\mathcal{X}$ is the completion $L_n(\mathcal{X}(x,y))$ of the $n$–fold Segal space $\mathcal{X}(x,y)$ of maps from $x$ to $y$ in $\mathcal{X}$.*

**Proof** We can write the localization $L_{n+1}\colon \mathrm{Seg}_{n+1}(\mathcal{S}) \to \mathrm{CSS}_{n+1}(\mathcal{S})$ as a composite

$$\mathrm{Seg}_{n+1}(\mathcal{S}) \xrightarrow{L_{n,*}} \mathrm{Seg}(\mathrm{CSS}_n(\mathcal{S})) \xrightarrow{\Lambda} \mathrm{CSS}_{n+1}(\mathcal{S}).$$

By [29, Theorem 1.2.13], the natural map $\mathcal{Y} \to \Lambda\mathcal{Y}$ is fully faithful and essentially surjective for all $\mathcal{Y} \in \mathrm{Seg}(\mathrm{CSS}_n(\mathcal{S}))$; in particular, we have a pullback square:

$$\begin{array}{ccc} \mathcal{Y}_1 & \longrightarrow & \Lambda\mathcal{Y}_1 \\ \downarrow & & \downarrow \\ \mathcal{Y}_0^{\times 2} & \longrightarrow & \Lambda\mathcal{Y}_0^{\times 2} \end{array}$$

Applying this to $L_{n,*}\mathcal{X}$ we see that we have an equivalence

$$(L_{n,*}\mathcal{X})(x,y) \xrightarrow{\sim} (\Lambda L_{n,*}\mathcal{X})(x,y) \simeq (L_{n+1}\mathcal{X})(x,y).$$

The $n$–fold Segal space $(L_{n,*}\mathcal{X})(x,y)$ is defined by the pullback square:

$$\begin{array}{ccc} (L_{n,*}\mathcal{X})(x,y) & \longrightarrow & L_n\mathcal{X}_1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & L_n\mathcal{X}_0^{\times 2} \end{array}$$

But by [19, Lemma 7.10] the functor $L_n$ preserves pullbacks over constant diagrams, so this fibre is equivalent to $L_n(\mathcal{X}(x,y))$, which completes the proof. □

**Proof of Theorem 5.50** Let $U_{\mathrm{Seg}}^{n+1}\colon \mathrm{Cat}^{n+1}(\mathcal{S}) \to \mathrm{Seg}_{n+1}(\mathcal{S})$ denote the right adjoint to the inclusion, and let $i_n\colon \mathrm{Cat}^n(\mathrm{Cat}_\infty) \to \mathrm{Cat}^{n+1}(\mathcal{S})$ denote the inclusion, which is also a right adjoint. By Corollary 5.49 we then have a pullback square

$$\begin{array}{ccc} U_{\mathrm{Seg}}^{n+1} i_n \mathfrak{ALG}_n(\mathrm{Bimod}_{A,B}(\mathcal{C})) & \longrightarrow & U_{\mathrm{Seg}}^{n+1} i_n (\mathfrak{ALG}_{n+1}(\mathcal{C})_{[1]}) \\ \downarrow & & \downarrow \\ * & \longrightarrow & U_{\mathrm{Seg}}^{n+1} i_n (\mathfrak{ALG}_{n+1}(\mathcal{C})_{[0]})^{\times 2} \end{array}$$

of $(n+1)$–fold Segal spaces. This factors through the pullback square

$$\begin{array}{ccc} (U_{\mathrm{Seg}}^{n+2} i_{n+1} \mathfrak{ALG}_{n+1}(\mathcal{C}))_{[1]} & \longrightarrow & U_{\mathrm{Seg}}^{n+1} i_n (\mathfrak{ALG}_{n+1}(\mathcal{C})_{[1]}) \\ \downarrow & & \downarrow \\ (U_{\mathrm{Seg}}^{n+2} i_{n+1} \mathfrak{ALG}_{n+1}(\mathcal{C}))_{[0]}^{\times 2} & \longrightarrow & U_{\mathrm{Seg}}^{n+1} i_n (\mathfrak{ALG}_{n+1}(\mathcal{C})_{[0]})^{\times 2} \end{array}$$

and so we may identify $U_{\mathrm{Seg}}^{n+1} i_n \mathfrak{ALG}_n(\mathrm{Bimod}_{A,B}(\mathcal{C}))$ with the $(n+1)$–fold Segal space of maps from $A$ to $B$ in the $(n+2)$–fold Segal space $U_{\mathrm{Seg}}^{n+2} i_{n+1} \mathfrak{ALG}_{n+1}(\mathcal{C})$. By Lemma 5.51 it follows that the completion

$$\mathfrak{Alg}_n(\mathrm{Bimod}_{A,B}(\mathcal{C})) \simeq L_{n+1} U_{\mathrm{Seg}}^{n+1} i_n \mathfrak{ALG}_n(\mathrm{Bimod}_{A,B}(\mathcal{C}))$$

is equivalent to the mapping $(\infty, n+1)$–category

$$\mathfrak{Alg}_{n+1}(\mathcal{C})(A,B) \simeq (L_{n+1} U_{\mathrm{Seg}}^{n+2} i_{n+1} \mathfrak{ALG}_{n+1}(\mathcal{C}))(A,B).$$ □

**Corollary 5.52** *Let $\mathcal{C}$ be a $\mathbf{\Delta}^{n+1}$–monoidal $\infty$–category with good relative tensor products, and write $I$ for the unit of the monoidal structure, regarded as a (trivial) $\mathbb{E}_{n+1}$–algebra in $\mathcal{C}$. Then we have an equivalence*

$$\mathfrak{Alg}_{n+1}(\mathcal{C})(I, I) \simeq \mathfrak{Alg}_n(\mathcal{C}).$$

**Proof** By Theorem 5.50 there is an equivalence

$$\mathfrak{Alg}_{n+1}(\mathcal{C})(I, I) \simeq \mathfrak{Alg}_n(\mathrm{Bimod}_{I,I}(\mathcal{C})).$$

But it follows from Corollary 4.50 and the definition of $\mathrm{Bimod}^{\otimes}_{I,I}(\mathcal{C})$ that the natural map $\mathrm{Bimod}^{\otimes}_{I,I}(\mathcal{C}) \to \mathcal{C}^{\otimes}$ is a $\mathbf{\Delta}^{n+1}$–monoidal equivalence. □

**Remark 5.53** Applying Corollary 5.52 inductively, we see that if $\mathcal{C}$ is an $\mathbb{E}_{n+m}$–monoidal $\infty$–category, then $\mathfrak{Alg}_n(\mathcal{C})$ is the endomorphism $(\infty, n+1)$–category of the identity $m$–morphism of the unit $I$ in the $(\infty, n+m+1)$–category $\mathfrak{Alg}_{n+m}(\mathcal{C})$. Thus $\mathfrak{Alg}_n(\mathcal{C})$ inherits an $\mathbb{E}_m$–monoidal structure (see [19, Section 10] for more details). It is intuitively plausible that this is the same as the $\mathbb{E}_m$–monoidal structure we constructed in Section 5.4, but at the moment we are unable to prove this.

# Appendix: Higher algebra over $\mathbf{\Delta}^n$

In this appendix we discuss the more technical results we need about $\mathbf{\Delta}^n$–$\infty$–operads. Many of these are slight variants of results proved for symmetric $\infty$–operads in [31], with essentially the same proofs, and when this is the case we have not included proofs here. Much of the material in this section is also a special case either of results of [10] or of unpublished work of Barwick and Schommer-Pries.

## A.1 The $\infty$–category of $\mathbf{\Delta}^n$–$\infty$–operads

It is clear from the definition of morphisms of (generalized) $\mathbf{\Delta}^n$–$\infty$–operads that the $\infty$–category of these objects should be regarded as a subcategory of the slice $\infty$–category $(\mathrm{Cat}_\infty)_{/\mathbf{\Delta}^{n,\mathrm{op}}}$. In this subsection we will define model categories that describe the $\infty$–categories of $\mathbf{\Delta}^n$–$\infty$–operads and generalized $\mathbf{\Delta}^n$–$\infty$–operads, using Lurie's theory of *categorical patterns*, which is a machine for constructing nice model structures for certain subcategories of such slice $\infty$–categories. We will use these model structures to give an explicit model for a key $\infty$–categorical colimit of generalized $\mathbf{\Delta}^n$–$\infty$–operads in Section 4.3 and Section 5.2. We begin by recalling the definition of a categorical pattern and Lurie's main results concerning them:

**Definition A.1** A categorical pattern $\mathfrak{P} = (\mathcal{C}, S, \{p_\alpha\})$ consists of

- an $\infty$–category $\mathcal{C}$,
- a marking of $\mathcal{C}$, ie a collection $S$ of 1–simplices in $\mathcal{C}$ that includes all the degenerate ones,
- a collection of diagrams of $\infty$–categories $p_\alpha\colon K_\alpha^{\triangleleft} \to \mathcal{C}$ such that $p_\alpha$ takes every edge in $K_\alpha^{\triangleleft}$ to a marked edge of $\mathcal{C}$.

**Remark A.2** Lurie's definition of a categorical pattern in [31, Appendix B] is more general than this: in particular, he includes the data of a *scaling* of the simplicial set $\mathcal{C}$, ie a collection $T$ of 2–simplices in $\mathcal{C}$ that includes all the degenerate ones. In all the examples we consider, however, the scaling consists of *all* 2–simplices of the simplicial set $\mathcal{C}$. We restrict ourselves to this special case as it gives a clearer description of the $\mathfrak{P}$–fibrant objects, and also simplifies the notation.

From a categorical pattern, Lurie constructs a model category that encodes the $\infty$–category of $\mathfrak{P}$–fibrant objects, in the following sense:

**Definition A.3** Suppose $\mathfrak{P} = (\mathcal{C}, S, \{p_\alpha\})$ is a categorical pattern. A map of simplicial sets $X \to \mathcal{C}$ is $\mathfrak{P}$–*fibrant* if the following criteria are satisfied:

(1) The underlying map $\pi\colon Y \to \mathcal{C}$ is an inner fibration. (In particular, $Y$ is an $\infty$–category.)

(2) $Y$ has all $\pi$–cocartesian edges over the morphisms in $S$.

(3) For every $\alpha$, the cocartesian fibration $\pi_\alpha\colon Y \times_{\mathcal{C}} K_\alpha^{\triangleleft} \to K_\alpha^{\triangleleft}$, obtained by pulling back $\pi$ along $p_\alpha$, is classified by a limit diagram $K_\alpha^{\triangleleft} \to \mathrm{Cat}_\infty$.

(4) For every $\alpha$, any cocartesian lift $s\colon K_\alpha^{\triangleleft} \to Y$ of $p_\alpha$ is a $\pi$–limit diagram.

**Theorem A.4** (Lurie, [31, Theorem B.0.20]) *Let $\mathfrak{P} = (\mathcal{C}, S, \{p_\alpha\})$ be a categorical pattern, and let $\overline{\mathcal{C}}$ denote the marked simplicial set $(\mathcal{C}, S)$. There is a unique left proper combinatorial simplicial model structure on the category $(\mathrm{Set}_\Delta^+)_{/\overline{\mathcal{C}}}$ such that:*

(1) *The cofibrations are the morphisms whose underlying maps of simplicial sets are monomorphisms. In particular, all objects are cofibrant.*

(2) *An object $(X, T) \to \overline{\mathcal{C}}$ is fibrant if and only if $X \to \mathcal{C}$ is $\mathfrak{P}$–fibrant and $T$ is precisely the collection of cocartesian morphisms over the morphisms in $S$.*

*We denote the category $(\mathrm{Set}_\Delta^+)_{/\overline{\mathcal{C}}}$ equipped with this model structure by $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}}$.* □

**Definition A.5** We will make use of the following categorical patterns:

(i) Let $\mathfrak{O}_n$ be the categorical pattern

$$(\mathbf{\Delta}^{n,\mathrm{op}}, I_n, \{p_I\colon K_I^{\triangleleft} \to \mathbf{\Delta}^{n,\mathrm{op}}\}),$$

where $I_n$ is the set of inert morphisms in $\mathbf{\Delta}^{n,\mathrm{op}}$ and, for $I \in \mathbf{\Delta}^n$, we write $K_I$ for the set of inert morphisms $I \to C_n$ in $\mathbf{\Delta}^{n,\mathrm{op}}$ and $p_I$ for the functor $K_I^{\triangleleft} \to \mathbf{\Delta}^{n,\mathrm{op}}$ associated to the inclusion $K_I \hookrightarrow (\mathbf{\Delta}^{n,\mathrm{op}})_{I/}$. It is immediate from Definition 5.3 that a map $Y \to \mathbf{\Delta}^{n,\mathrm{op}}$ is $\mathfrak{O}_n$–fibrant precisely if it is a $\mathbf{\Delta}^n$–$\infty$–operad.

(ii) Let $\mathfrak{M}_n$ denote the categorical pattern

$$(\mathbf{\Delta}^{n,\mathrm{op}}, A_n, \{p_I\colon K_I^{\triangleleft} \to \mathbf{\Delta}^{n,\mathrm{op}}\}),$$

where $A_n$ denotes the set of all morphisms in $\mathbf{\Delta}^{n,\mathrm{op}}$. Then a map $Y \to \mathbf{\Delta}^{n,\mathrm{op}}$ is $\mathfrak{M}_n$–fibrant precisely if $Y \to \mathbf{\Delta}^{n,\mathrm{op}}$ is a $\mathbf{\Delta}^n$–monoidal $\infty$–category.

(iii) Let $\mathfrak{O}_n^{\mathrm{gen}}$ be the categorical pattern

$$(\mathbf{\Delta}^{n,\mathrm{op}}, I_n, \{(\mathrm{Cell}^{n,\mathrm{op}}_{/I})^{\triangleleft} \to \mathbf{\Delta}^{n,\mathrm{op}}\}).$$

It is immediate from Definition 5.8 that a map $Y \to \mathbf{\Delta}^{n,\mathrm{op}}$ is $\mathfrak{O}_n^{\mathrm{gen}}$–fibrant if and only if $Y \to \mathbf{\Delta}^{n,\mathrm{op}}$ is a generalized $\mathbf{\Delta}^n$–$\infty$–operad.

(iv) Let $\mathfrak{U}_n$ denote the categorical pattern

$$(\mathbf{\Delta}^{n,\mathrm{op}}, N\mathbf{\Delta}^{n,\mathrm{op}}_1, \{(\mathrm{Cell}^{n,\mathrm{op}}_{/I})^{\triangleleft} \to \mathbf{\Delta}^{n,\mathrm{op}}\}).$$

Then a map $Y \to \mathbf{\Delta}^{n,\mathrm{op}}$ is $\mathfrak{U}_n$–fibrant if and only if $Y \to \mathbf{\Delta}^{n,\mathrm{op}}$ is a $\mathbf{\Delta}^n$–uple $\infty$–category.

**Definition A.6** The $\infty$–category $\mathrm{Opd}_\infty^{\mathbf{\Delta}^n}$ of $\mathbf{\Delta}^n$–$\infty$–operads is the $\infty$–category associated to the simplicial model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_n}$, ie the coherent nerve of its simplicial subcategory of fibrant objects. Thus the objects of $\mathrm{Opd}_\infty^{\mathbf{\Delta}^n}$ can be identified with $\mathbf{\Delta}^n$–$\infty$–operads. Moreover, since the maps between these in $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_n}$ are precisely the maps that preserve inert morphisms, it is also easy to see that the space of maps from $\mathcal{O}$ to $\mathcal{P}$ in $\mathrm{Opd}_\infty^{\mathbf{\Delta}^n}$ is equivalent to the subspace of $\mathrm{Map}_{\mathbf{\Delta}^{n,\mathrm{op}}}(\mathcal{O},\mathcal{P})$ given by the components corresponding to inert-morphism-preserving maps, as expected. This justifies calling $\mathrm{Opd}_\infty^{\mathbf{\Delta}^n}$ the *$\infty$–category of $\mathbf{\Delta}^n$–$\infty$–operads*.

**Remark A.7** This $\infty$–category of $\mathbf{\Delta}^n$–$\infty$–operads is a special case of the $\infty$–categories of $\infty$–operads over an operator category constructed by Barwick [10, Theorem 8.15].

**Definition A.8** Similarly, applying Theorem A.4 to the categorical patterns $\mathfrak{M}_n$, $\mathfrak{O}_n^{\mathrm{gen}}$ and $\mathfrak{U}_n$ gives simplicial model categories $(\mathrm{Set}_\Delta^+)_{\mathfrak{M}_n}$, $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_n^{\mathrm{gen}}}$ and $(\mathrm{Set}_\Delta^+)_{\mathfrak{U}_n}$ whose

fibrant objects are, respectively, $\mathbf{\Delta}^n$–monoidal $\infty$–categories, generalized $\mathbf{\Delta}^n$–$\infty$–operads, and $\mathbf{\Delta}^n$–uple $\infty$–categories. We write $\mathrm{Mon}_\infty^{\mathbf{\Delta}^n}$, $\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}}$ and $\mathrm{Upl}_\infty^{\mathbf{\Delta}^n}$ for the $\infty$–categories associated to these simplicial model categories, and refer to them as the *$\infty$–categories of $\mathbf{\Delta}^n$–monoidal $\infty$–categories*, *generalized $\mathbf{\Delta}^n$–$\infty$–operads* and *$\mathbf{\Delta}^n$–uple $\infty$–categories*.

**Definition A.9** The morphisms in $\mathrm{Mon}_\infty^{\mathbf{\Delta}^n}$ are the (strong) $\mathbf{\Delta}^n$–monoidal functors between $\mathbf{\Delta}^n$–monoidal $\infty$–categories. We write $\mathrm{Mon}_\infty^{\mathbf{\Delta}^n,\mathrm{lax}}$ for the $\infty$–category of $\mathbf{\Delta}^n$–monoidal $\infty$–categories and lax $\mathbf{\Delta}^n$–monoidal functors, ie the full subcategory of $\mathrm{Opd}_\infty^{\mathbf{\Delta}^n}$ spanned by the $\mathbf{\Delta}^n$–monoidal $\infty$–categories.

We now show that taking cartesian products gives left Quillen bifunctors relating $\mathbf{\Delta}^n$–$\infty$–operads for varying $n$. This will allow us to reduce the proofs of the technical results needed in Section 5 to the case where $n = 1$. First we introduce some notation and recall a result of Lurie:

**Definition A.10** Suppose $\mathfrak{P} = (\mathcal{C}, S, \{p_\alpha\colon K_\alpha^\triangleleft \to \mathcal{C}\})$ and $\mathfrak{Q} = (\mathcal{D}, T, \{q_\beta\colon L_\beta^\triangleleft \to \mathcal{D}\})$ are categorical patterns. The *product categorical pattern* $\mathfrak{P} \times \mathfrak{Q}$ is given by

$$\big(\mathcal{C} \times \mathcal{D}, S \times T, \{p_\alpha \times \{d\} : d \in \mathcal{D}\} \cup \{\{c\} \times q_\beta : c \in \mathcal{C}\}\big).$$

**Proposition A.11** (Lurie, [31, Remark B.2.5]) *Suppose $\mathfrak{P}$ and $\mathfrak{Q}$ are categorical patterns. The cartesian product is a left Quillen bifunctor*

$$(\mathrm{Set}_\Delta^+)_{\mathfrak{P}} \times (\mathrm{Set}_\Delta^+)_{\mathfrak{Q}} \to (\mathrm{Set}_\Delta^+)_{\mathfrak{P}\times\mathfrak{Q}}. \qquad \square$$

**Definition A.12** Let us say that a categorical pattern $\mathfrak{P} = (\mathcal{C}, S, D)$ is *objectwise* if the set of diagrams $D$ is of the form $\{p_x\colon K_x^\triangleleft \to \mathcal{C} : x \in \mathcal{C}\}$, where $p_x(-\infty) = x$. We say that $\mathfrak{P}$ is *reduced* if moreover $K_c$ has an initial object for every $c$ in the image of $p_x|_{K_x}$ for any $x$. If $\mathfrak{P} = (\mathcal{C}, S, \{p_x\colon K_x^\triangleleft \to \mathcal{C}\})$ and $\mathfrak{Q} = (\mathcal{D}, T, \{q_y\colon L_y^\triangleleft \to \mathcal{C}\})$ are objectwise categorical patterns, we let $\mathfrak{P} \boxtimes \mathfrak{Q}$ be the objectwise categorical pattern

$$\Big(\mathcal{C} \times \mathcal{D}, S \times T, \big\{(K_x \times L_y)^\triangleleft \to K_x^\triangleleft \times L_y^\triangleleft \xrightarrow{p_x \times q_y} \mathcal{C} \times \mathcal{D} : (x, y) \in \mathcal{C} \times \mathcal{D}\big\}\Big).$$

**Proposition A.13** *Suppose $\mathfrak{P}$ and $\mathfrak{Q}$ are objectwise reduced categorical patterns. Then the model category structures $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}\times\mathfrak{Q}}$ and $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}\boxtimes\mathfrak{Q}}$ on $(\mathrm{Set}_\Delta^+)_{/(\mathcal{C}\times\mathcal{D}, S\times T)}$ are identical.*

For the proof we make use of the following obvious observation:

**Lemma A.14** *Suppose given a commutative square*

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{i_1} & \mathcal{C}_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ \mathcal{C}_2 & \xrightarrow[j_2]{} & \mathcal{C} \end{array}$$

*of $\infty$–categories where all the maps are fully faithful such that every object of $\mathcal{C}$ is contained in the essential image of either $\mathcal{C}_1$ or $\mathcal{C}_2$. If $\pi\colon \mathcal{X} \to \mathcal{Y}$ is an inner fibration of $\infty$–categories and $\phi\colon \mathcal{C} \to \mathcal{X}$ is a functor, then $\phi$ is a $\pi$–right Kan extension of $\phi|_{\mathcal{C}_0}$ along $j_1 i_1 \simeq j_2 i_2$ if and only if $\phi|_{\mathcal{C}_1}$ is a $\pi$–right Kan extension of $\phi|_{\mathcal{C}_0}$ along $j_1$ and $\phi|_{\mathcal{C}_2}$ is a $\pi$–right Kan extension of $\phi|_{\mathcal{C}_0}$ along $j_2$.* □

**Proof of Proposition A.13** By the uniqueness statement in Theorem A.4 it is enough to check that the fibrant objects are the same in the two model structures. Supposing $Y \to \mathcal{C}$ is an inner fibration with all cocartesian morphisms over the morphisms in $S$, we are interested in the following conditions:

(1) For all $(x, y) \in \mathcal{C} \times \mathcal{D}$, the cocartesian fibration $(K_x \times L_y)^{\triangleleft} \times_{\mathcal{C}} Y \to (K_x \times L_y)^{\triangleleft}$ is classified by a limit diagram.

(1′) For all $(x, y) \in \mathcal{C} \times \mathcal{D}$, the cocartesian fibrations $(K_x^{\triangleleft} \times \{y\}) \times_{\mathcal{C}} Y \to K_x^{\triangleleft}$ and $(\{x\} \times L_y^{\triangleleft}) \times_{\mathcal{C}} Y \to L_y^{\triangleleft}$ are classified by limit diagrams.

(2) For all $(x, y) \in \mathcal{C} \times \mathcal{D}$, any cocartesian section $s\colon (K_x \times L_y)^{\triangleleft} \to Y$ is a $\pi$–limit diagram.

(2′) For all $(x, y) \in \mathcal{C} \times \mathcal{D}$, any cocartesian sections $s\colon K_x^{\triangleleft} \times \{y\} \to Y$ and $t\colon \{x\} \times L_y^{\triangleleft} \to Y$ are $\pi$–limit diagrams.

We must show that (1) and (1′) are equivalent, and that (2) and (2′) are equivalent.

To see that (1) implies (1′), let $\phi\colon K_x^{\triangleleft} \times L_y^{\triangleleft} \to \mathrm{Cat}_\infty$ be a diagram classified by the cocartesian fibration $(K_x^{\triangleleft} \times L_y^{\triangleleft}) \times_{\mathcal{C}\times\mathcal{D}} Y \to K_x^{\triangleleft} \times L_y^{\triangleleft}$ for some $(x, y) \in \mathcal{C} \times \mathcal{D}$. We now wish to apply Lemma A.14 to the square:

$$\begin{array}{ccc} K_x \times L_y & \longrightarrow & (K_x \times L_y)^{\triangleleft} \\ \downarrow & & \downarrow \\ (K_x^{\triangleleft} \times L_y) \amalg_{K_x \times L_y} (K_x \times L_y^{\triangleleft}) & \longrightarrow & K_x^{\triangleleft} \times L_y^{\triangleleft} \end{array}$$

By assumption $\phi|_{(K_x \times L_y)^{\triangleleft}}$ is a right Kan extension of $\phi|_{K_x \times L_y}$, so it remains to prove that the restriction of $\phi$ to $(K_x^{\triangleleft} \times L_y) \amalg_{K_x \times L_y} (K_x \times L_y^{\triangleleft})$ is a right Kan extension of

$\phi|_{K_x\times L_y}$. In other words, we must show that for any $z\in L_y$ the object $\phi(-\infty,z)$ is a limit of $\phi|_{(K_x\times L_y)_{(-\infty,z)/}}$, and that for any $w\in K_x$ the object $\phi(w,-\infty)$ is a limit of $\phi|_{(K_x\times L_y)_{(w,-\infty)/}}$. The inclusion $K_x\times\{z\}\to(K_x\times L_y)_{(-\infty,z)/}$ is coinitial, so it suffices to prove that the restriction of $\phi$ to $K_x^{\triangleleft}\times\{z\}$ is a limit diagram. Since the categorical pattern is reduced, by assumption the $\infty$–category $L_z$ has an initial object, and so there is a coinitial map $K_x\times\{z\}\to K_x\times L_z$. Moreover, the restriction of $\phi$ to $K_x^{\triangleleft}\times\{z\}$ is also the restriction of the analogous functor $(K_z\times L_z)^{\triangleleft}\to\mathrm{Cat}_\infty$, which is a limit diagram by assumption. Thus $\phi(-\infty,z)$ is indeed the limit of $\phi|_{K_x\times\{z\}}$, and similarly $\phi(w,-\infty)$ is the limit of $\phi|_{\{w\}\times L_y}$. It then follows from Lemma A.14 that $\phi$ is a right Kan extension of $\phi|_{K_x\times L_y}$.

Now considering the factorization

$$K_x\times L_y\hookrightarrow K_x^{\triangleleft}\times L_y\hookrightarrow(K_x^{\triangleleft}\times L_y)^{\triangleleft}\to K_x^{\triangleleft}\times L_y^{\triangleleft}$$

we see that $\phi|_{(K_x^{\triangleleft}\times L_y)^{\triangleleft}}$ is a limit of $\phi|_{K_x^{\triangleleft}\times L_y}$. Since the inclusion $\{-\infty\}\times L_y\hookrightarrow K_x^{\triangleleft}\times L_y$ is coinitial, it follows that $\phi|_{\{-\infty\}\times L_y^{\triangleleft}}$ is a limit diagram. Similarly, $\phi|_{K_x^{\triangleleft}\times\{-\infty\}}$ is a limit diagram, which proves $(1')$.

Conversely, to see that $(1')$ implies $(1)$ we consider the square:

$$\begin{array}{ccc} K_x\times L_y & \longrightarrow & (K_x^{\triangleleft}\times L_y)\amalg_{K_x\times L_y}(K_x\times L_y^{\triangleleft}) \\ \downarrow & & \downarrow \\ (K_x^{\triangleleft}\times L_y)^{\triangleleft} & \longrightarrow & K_x^{\triangleleft}\times L_y^{\triangleleft} \end{array}$$

Let $\phi$ be as above; then it follows from $(1')$ that $\phi|_{(K_x^{\triangleleft}\times L_y)\amalg_{K_x\times L_y}(K_x\times L_y^{\triangleleft})}$ is a right Kan extension of $\phi|_{K_x\times L_y}$ and $\phi|_{K_x^{\triangleleft}\times L_y}$ is a right Kan extension of $\phi|_{K_x\times L_y}$. Since $\{-\infty\}\times L_y\hookrightarrow K_x^{\triangleleft}\times L_y$ is coinitial, $(1')$ also implies that $\phi|_{(K_x^{\triangleleft}\times L_y)^{\triangleleft}}$ is a right Kan extension of $\phi|_{K_x^{\triangleleft}\times L_y}$, and so by Lemma A.14 it follows that $\phi$ is a right Kan extension of $\phi|_{K_x\times L_y}$. But then $\phi|_{(K_x\times L_y)^{\triangleleft}}$ is also a right Kan extension of $\phi|_{K_x\times L_y}$, which proves $(1)$.

It follows by the same argument, applied to a cocartesian section $\phi\colon K_x^{\triangleleft}\times L_y^{\triangleleft}\to Y$, that $(2)$ is equivalent to $(2')$. □

Applying this to the categorical patterns we're interested in, we get:

**Corollary A.15**

(i) *The model categories* $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_n\times\mathfrak{O}_m}$ *and* $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{n+m}}$ *are identical.*

(ii) *The model categories* $(\mathrm{Set}_\Delta^+)_{\mathfrak{M}_n\times\mathfrak{M}_m}$ *and* $(\mathrm{Set}_\Delta^+)_{\mathfrak{M}_{n+m}}$ *are identical.*

(iii) *The model categories* $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_n^{\mathrm{gen}}\times\mathfrak{O}_m^{\mathrm{gen}}}$ *and* $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_{n+m}^{\mathrm{gen}}}$ *are identical.*

(iv) *The model categories* $(\mathrm{Set}^+_\Delta)_{\mathfrak{U}_n\times\mathfrak{U}_m}$ *and* $(\mathrm{Set}^+_\Delta)_{\mathfrak{U}_{n+m}}$ *are identical.*

**Proof** The categorical patterns $\mathfrak{O}_n$, $\mathfrak{M}_n$, $\mathfrak{O}_n^{\mathrm{gen}}$ and $\mathfrak{U}_n$ are all objectwise reduced, and we have identifications $\mathfrak{O}_{n+m}=\mathfrak{O}_n\boxtimes\mathfrak{O}_m$, $\mathfrak{M}_{n+m}=\mathfrak{M}_n\boxtimes\mathfrak{M}_m$, $\mathfrak{O}_{n+m}^{\mathrm{gen}}=\mathfrak{O}_n^{\mathrm{gen}}\boxtimes\mathfrak{O}_m^{\mathrm{gen}}$ and $\mathfrak{U}_{n+m}=\mathfrak{U}_n\boxtimes\mathfrak{U}_m$. The result is therefore immediate from Proposition A.13 □

**Corollary A.16** *The cartesian product defines left Quillen bifunctors*

$$\begin{aligned}(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_n}\times(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_m}&\to(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_{n+m}},\\ (\mathrm{Set}^+_\Delta)_{\mathfrak{M}_n}\times(\mathrm{Set}^+_\Delta)_{\mathfrak{M}_m}&\to(\mathrm{Set}^+_\Delta)_{\mathfrak{M}_{n+m}},\\ (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_n^{\mathrm{gen}}}\times(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_m^{\mathrm{gen}}}&\to(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_{n+m}^{\mathrm{gen}}},\\ (\mathrm{Set}^+_\Delta)_{\mathfrak{U}_n}\times(\mathrm{Set}^+_\Delta)_{\mathfrak{U}_m}&\to(\mathrm{Set}^+_\Delta)_{\mathfrak{U}_{n+m}}.\end{aligned}$$

**Proof** Combine Corollary A.15 with Proposition A.11. □

Finally, we recall a useful result on functoriality of categorical pattern model structures:

**Definition A.17** Suppose $\mathfrak{P}=(\mathcal{C},S,\{p_\alpha\})$ and $\mathfrak{Q}=(\mathcal{D},T,\{q_\beta\})$ are categorical patterns. A *morphism of categorical patterns* $f\colon\mathfrak{P}\to\mathfrak{Q}$ is a functor $f\colon\mathcal{C}\to\mathcal{D}$ such that $f(S)\subseteq f(T)$ and $f\circ p_\alpha$ lies in $\{q_\beta\}$ for all $\alpha$.

**Proposition A.18** (Lurie, [31, Proposition B.2.9]) *Suppose $f\colon\mathfrak{P}\to\mathfrak{Q}$ is a morphism of categorical patterns. Then composition with $f$ gives a left Quillen functor*

$$f_!\colon(\mathrm{Set}^+_\Delta)_{\mathfrak{P}}\to(\mathrm{Set}^+_\Delta)_{\mathfrak{Q}}.$$

## A.2 $\boldsymbol{\Delta^n}$–$\boldsymbol{\infty}$–operads and symmetric $\boldsymbol{\infty}$–operads

In this subsection we will relate $\boldsymbol{\Delta}^n$–$\infty$–operads to the symmetric $\infty$–operads studied in [31]. We first recall some definitions:

**Definition A.19** For $n$ a nonnegative integer, let $\langle n\rangle$ denote the set $\{0,1,\dots,n\}$, regarded as a pointed set with base point 0. A morphism $f\colon\langle n\rangle\to\langle m\rangle$ of finite pointed sets is *inert* if $f^{-1}(i)$ has a single element for every $i\neq 0$, and *active* if $f^{-1}(0)=\{0\}$. Recall that the inert and active morphisms form a factorization system on $\boldsymbol{\Gamma}^{\mathrm{op}}$.

**Definition A.20** A *symmetric $\infty$–operad* is a functor of $\infty$–categories $\pi\colon \mathcal{O} \to \mathbf{\Gamma}^{\mathrm{op}}$ such that:

(i) For every inert morphism $\phi\colon \langle m\rangle \to \langle n\rangle$ in $\mathbf{\Gamma}^{\mathrm{op}}$ and every $X \in \mathcal{O}_{\langle n\rangle}$ there exists a $\pi$–cocartesian morphism $X \to \phi_! X$ over $\phi$.

(ii) Let $\rho_i\colon \langle n\rangle \to \langle 1\rangle$, $i = 1, \dots, n$, denote the (inert) map that sends $i$ to 1 and every other element of $\langle n\rangle$ to 0. For every $\langle n\rangle \in \mathbf{\Gamma}^{\mathrm{op}}$ the functor

$$\mathcal{O}_{\langle n\rangle} \to (\mathcal{O}_{\langle 1\rangle})^{\times n}$$

induced by the cocartesian arrows over the maps $\rho_i$ is an equivalence of $\infty$–categories.

(iii) For every morphism $\phi\colon \langle n\rangle \to \langle m\rangle$ in $\mathbf{\Gamma}^{\mathrm{op}}$ and $Y \in \mathcal{O}_{\langle m\rangle}$, composition with cocartesian morphisms $Y \to Y_i$ over the inert morphisms $\rho_i$ gives an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{O}}(X, Y) \xrightarrow{\sim} \prod_i \mathrm{Map}^{\rho_i \circ \phi}_{\mathcal{O}}(X, Y_i),$$

where $\mathrm{Map}^{\phi}_{\mathcal{O}}(X, Y)$ denotes the subspace of $\mathrm{Map}_{\mathcal{O}}(X, Y)$ of morphisms that map to $\phi$ in $\mathbf{\Delta}^{\mathrm{op}}$.

**Definition A.21** Let $\mathfrak{O}_\Sigma$ denote the categorical pattern $\big(\mathbf{\Gamma}^{\mathrm{op}}, I_\Sigma, \{p_{\langle n\rangle}\colon P^{\triangleleft}_{\langle n\rangle} \to \mathbf{\Gamma}^{\mathrm{op}}\}\big)$, where $\mathbf{\Gamma}^{\mathrm{op}}$ is the category of finite pointed sets, $I_\Sigma$ denotes the set of inert morphisms in $\mathbf{\Gamma}^{\mathrm{op}}$, and $P_{\langle n\rangle}$ is the set of inert morphisms $\langle n\rangle \to \langle 1\rangle$ in $\mathbf{\Gamma}^{\mathrm{op}}$.

**Definition A.22** The $\mathfrak{O}_\Sigma$–fibrant objects are precisely the symmetric $\infty$–operads, and we write $\mathrm{Opd}^{\Sigma}_{\infty}$ for the $\infty$–category associated to the model category $(\mathrm{Set}^{+}_{\Delta})_{\mathfrak{O}_\Sigma}$.

**Definition A.23** Let $u^1\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Gamma}^{\mathrm{op}}$ be the functor defined as in [31, Construction 4.1.2.5] (this is the same as the functor introduced by Segal [38]). Recall that this sends $[n]$ to $\langle n\rangle$, and a map $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$ to the map $\langle m\rangle \to \langle n\rangle$ given by

$$u^1(\phi)(i) = \begin{cases} j & \text{if } \phi(j-1) < i \le \phi(j), \\ 0 & \text{if no such } j \text{ exists.} \end{cases}$$

This takes inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$ to inert morphisms in $\mathbf{\Gamma}^{\mathrm{op}}$, and moreover induces a morphism of categorical patterns from $\mathfrak{O}_1$ to $\mathfrak{O}_\Sigma$. Let $\mu\colon \mathbf{\Gamma}^{\mathrm{op}} \times \mathbf{\Gamma}^{\mathrm{op}} \to \mathbf{\Gamma}^{\mathrm{op}}$ be the functor defined in [31, Notation 2.2.5.1]; this takes $(\langle m\rangle, \langle n\rangle)$ to $\langle mn\rangle$ and takes a morphism $(f\colon \langle m\rangle \to \langle m'\rangle, g\colon \langle n\rangle \to \langle n'\rangle)$ to the morphism $\mu(f, g)$ given by

$$\mu(f,g)(an + b - n) = \begin{cases} 0 & \text{if } f(a) = 0 \text{ or } g(b) = 0, \\ f(a)n' + g(b) - n' & \text{otherwise.} \end{cases}$$

The functor $\mu$ induces a morphism of categorical patterns $\mathfrak{O}_\Sigma \times \mathfrak{O}_\Sigma \to \mathfrak{O}_\Sigma$. We then inductively define $u^n\colon \mathbf{\Delta}^{n,\mathrm{op}} \to \mathbf{\Gamma}^{\mathrm{op}}$ to be the composite

$$\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{n-1,\mathrm{op}} \xrightarrow{u^1 \times u^{n-1}} \mathbf{\Gamma}^{\mathrm{op}} \times \mathbf{\Gamma}^{\mathrm{op}} \xrightarrow{\mu} \mathbf{\Gamma}^{\mathrm{op}},$$

so that $u^n$ is a morphism of categorical patterns $\mathfrak{O}_n \to \mathfrak{O}_\Sigma$ for all $n$. Thus $u^n$ induces adjoint functors

$$u^n_!\colon \mathrm{Opd}_\infty^{\mathbf{\Delta}^n} \rightleftarrows \mathrm{Opd}_\infty^{\Sigma} :u^{n,*}.$$

Moreover, since the induced Quillen functors are enriched in marked simplicial sets we get equivalences

$$\mathrm{Alg}^n_{\mathcal{O}}(u^{n,*}\mathcal{P}) \simeq \mathrm{Alg}^{\Sigma}_{u^n_!\mathcal{O}}(\mathcal{P}),$$

where $\mathcal{O}$ is a $\mathbf{\Delta}^n$–$\infty$–operad and $\mathcal{P}$ is a symmetric $\infty$–operad.

**Remark A.24** The Quillen adjunction $u^n_! \dashv u^{n,*}$ is a special case of the adjunctions arising from morphisms of operator categories that are discussed in [10, Proposition 8.18].

By Corollary A.16 and Proposition A.18 we then have a commutative diagram of left Quillen functors

$$\begin{array}{ccccc}
(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_1} \times (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_n} & \xrightarrow{\times} & (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_1 \times \mathfrak{O}_n} & \xrightarrow{\cong} & (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_{n+1}} \\
\downarrow{\scriptstyle u^1_! \times u^n_!} & & \downarrow{\scriptstyle (u^1 \times u^n)_!} & & \downarrow{\scriptstyle u^{n+1}_!} \\
(\mathrm{Set}^+_\Delta)_{\mathfrak{O}_\Sigma} \times (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_\Sigma} & \xrightarrow{\times} & (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_\Sigma \times \mathfrak{O}_\Sigma} & \xrightarrow{\mu_!} & (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_\Sigma}
\end{array}$$

where the left horizontal functors are given by the cartesian products. The Boardman–Vogt tensor product of symmetric $\infty$–operads, as defined in [31, Section 2.2.5], is the functor of $\infty$–categories induced by the composite functor along the bottom of this diagram. On the level of $\infty$–categories we have therefore proved the following:

**Proposition A.25** *There is a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Opd}_\infty^{\mathbf{\Delta}} \times \mathrm{Opd}_\infty^{\mathbf{\Delta}^n} & \xrightarrow{\times} & \mathrm{Opd}_\infty^{\mathbf{\Delta}^{n+1}} \\
\downarrow{\scriptstyle u^1_! \times u^n_!} & & \downarrow{\scriptstyle u^{n+1}_!} \\
\mathrm{Opd}_\infty^{\Sigma} \times \mathrm{Opd}_\infty^{\Sigma} & \xrightarrow{\otimes} & \mathrm{Opd}_\infty^{\Sigma}
\end{array}$$

Invoking the Dunn–Lurie additivity theorem, we get:

**Corollary A.26** *The symmetric $\infty$–operad $u^n_!(\mathbf{\Delta}^{n,\mathrm{op}})$ is equivalent to $\mathbb{E}_n$.*

**Proof** Applying Proposition A.25 we have an equivalence

$$u^n_!(\mathbf{\Delta}^{n,\mathrm{op}}) \simeq u^1_!(\mathbf{\Delta}^{\mathrm{op}}) \otimes u^{n-1}_!(\mathbf{\Delta}^{n-1,\mathrm{op}}).$$

By [31, Proposition 4.1.2.10 and Example 5.1.0.7], the symmetric $\infty$–operad $u^1_!(\mathbf{\Delta}^{\mathrm{op}})$ is equivalent to $\mathbb{E}_1$, so by induction we have an equivalence $u^n_!(\mathbf{\Delta}^{n,\mathrm{op}}) \simeq \mathbb{E}_1^{\otimes n}$. Now [31, Theorem 5.1.2.2] says that the symmetric $\infty$–operad $\mathbb{E}_1^{\otimes n}$ is equivalent to $\mathbb{E}_n$, which completes the proof. □

**Corollary A.27** *Let $\mathcal{O}$ be a symmetric $\infty$–operad. Then there is a natural equivalence*

$$\mathrm{Alg}^{\Sigma}_{\mathbb{E}_n}(\mathcal{O}) \simeq \mathrm{Alg}^n_{\mathbf{\Delta}^{n,\mathrm{op}}}(u^{n,*}\mathcal{O}).$$

## A.3 $\mathbf{\Delta}^n$–monoid objects

We will now observe that $\mathbf{\Delta}^n$–algebras in a cartesian monoidal $\infty$–category are equivalent to the $\mathbf{\Delta}^n$–monoids we discussed above in Section 3. More generally, we can define $\mathcal{O}$–monoids for any generalized $\mathbf{\Delta}^n$–$\infty$–operad $\mathcal{O}$ as an equivalent way of describing $\mathcal{O}$–algebras in a cartesian monoidal $\infty$–category:

**Definition A.28** Let $\mathcal{C}$ be an $\infty$–category with finite products and $\mathcal{O}$ a generalized $\mathbf{\Delta}^n$–$\infty$–operad. An *$\mathcal{O}$–monoid* in $\mathcal{C}$ is a functor $F\colon \mathcal{O} \to \mathcal{C}$ such that for every $I \in \mathbf{\Delta}^{n,\mathrm{op}}$ and $X \in \mathcal{O}_I$, the map $F(X) \to \prod_{i \in |I|} F(X_i)$, induced by the cocartesian morphisms $X \to X_i$ over $i$, is an equivalence. We write $\mathrm{Mon}^n_{\mathcal{O}}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$ spanned by the $\mathcal{O}$–monoids.

**Proposition A.29** *Suppose $\mathcal{C}$ is an $\infty$–category with finite products, and let $\mathcal{C}^{\times}$ denote the cartesian symmetric monoidal structure on $\mathcal{C}$ constructed in* [31, Section 2.4.1]. *If $\mathcal{O}$ is a (generalized) $\mathbf{\Delta}^n$–$\infty$–operad, then there is a natural equivalence $\mathrm{Mon}^n_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}^n_{\mathcal{O}}(u^{n,*}\mathcal{C}^{\times})$.*

**Proof** This is the same as the proof of [31, Proposition 2.4.2.5]. □

**Corollary A.30** *Let $\mathcal{C}$ be an $\infty$–category with finite products. Then there is a natural equivalence*

$$\mathrm{Mon}^{\Sigma}_{\mathbb{E}_n}(\mathcal{C}) \simeq \mathrm{Mon}^n_{\mathbf{\Delta}^{n,\mathrm{op}}}(\mathcal{C}).$$

**Proof** Combine Corollary A.27 with Proposition A.29 and [31, Proposition 2.4.2.5] — this gives a natural equivalence

$$\mathrm{Mon}^{\Sigma}_{\mathbb{E}_n}(\mathcal{C}) \simeq \mathrm{Alg}^{\Sigma}_{\mathbb{E}_n}(\mathcal{C}^{\times}) \simeq \mathrm{Alg}^n_{\mathbf{\Delta}^{n,\mathrm{op}}}(u^{n,*}\mathcal{C}^{\times}) \simeq \mathrm{Mon}^n_{\mathbf{\Delta}^{n,\mathrm{op}}}(\mathcal{C}).$$ □

**Corollary A.31** *The $\infty$–category $\mathrm{Mon}^{\mathbf{\Delta}^n}_{\infty}$ of $\mathbf{\Delta}^n$–monoidal $\infty$–categories is equivalent to the $\infty$–category $\mathrm{Mon}^{\Sigma,\mathbb{E}_n}_{\infty}$ of $\mathbb{E}_n$–monoidal $\infty$–categories.*

**Proof** This is just the special case of Corollary A.30 where $\mathcal{C} = \mathrm{Cat}_{\infty}$. □

## A.4 Weak operadic colimits

Suppose $\mathcal{C}$ is a $\mathbf{\Delta}^n$–monoidal $\infty$–category. Then by Corollary A.31 $\mathcal{C}$ is equivalently an $\mathbb{E}_n$–monoidal $\infty$–category. Moreover, if $\mathcal{O}$ is a $\mathbf{\Delta}^n$–$\infty$–operad, then $\mathcal{O}$–algebras in $\mathcal{C}$, regarded as $\mathbf{\Delta}^n$–monoidal, are equivalent to $u^n_!\mathcal{O}$–algebras in $\mathcal{C}$, regarded as $\mathbb{E}_n$–monoidal, by the results of Section A.2. If $f\colon \mathcal{O} \to \mathcal{P}$ is a morphism of $\mathbf{\Delta}^n$–$\infty$–operads, this means that we can apply the results of [31, Section 3.1.3] to $u^n_! f$ to conclude that if $\mathcal{C}$ is well-behaved, then the functor $f^*\colon \mathrm{Alg}^n_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C})$ has a left adjoint $f_!$.

In [31, Section 3.1.3], such left adjoint functors are constructed by forming certain concrete colimit diagrams. However, as we do not have any explicit understanding of the symmetric $\infty$–operad $u^n_!\mathcal{O}$, the results of [31] do not allow us to understand what the functor $f_!$ does for $f$ a morphism of $\mathbf{\Delta}^n$–$\infty$–operads. For the results of Sections 4 and 5 this is insufficient — in fact, we need an explicit description of such a left adjoint for certain maps of *generalized* $\mathbf{\Delta}^n$–$\infty$–operads, which introduces another inexplicit construction, namely the localization functor from generalized $\mathbf{\Delta}^n$–$\infty$–operads to $\mathbf{\Delta}^n$–$\infty$–operads, before we can apply the results from [31]. For this reason, we will in the next couple of subsections discuss analogues of many of the results in [31, Sections 3.1.1–3.1.3] in the setting of generalized $\mathbf{\Delta}^n$–$\infty$–operads. Luckily, these results can generally be obtained by minor variations of the arguments from [31], and when this is the case we have not included complete details.

In this section we consider the analogue, in the setting of $\mathbf{\Delta}^n$–$\infty$–operads, of the *weak operadic colimits* introduced in [31, Section 3.1.1]. However, unlike in [31, Section 3.1.1], we will not consider *relative* weak operadic colimits, as these are not needed in this paper.

**Remark A.32** In [31, Section 3.1.1], weak operadic colimits are considered as a preliminary to a notion of *operadic colimits*. These do not have a straightforward analogue in the $\mathbf{\Delta}^n$–context. Instead, we will introduce a notion we call a *monoidal colimit*, which is an adequate substitute such that the required arguments from [31] still go through.

**Notation A.33** Suppose $\mathcal{O}$ is a $\mathbf{\Delta}^n$–$\infty$–operad; we denote the subcategory of $\mathcal{O}$ containing only the active morphisms by $\mathcal{O}^{\mathrm{act}}$. If $p\colon K \to \mathcal{O}^{\mathrm{act}}$ is a diagram, we write $\mathcal{O}^{\mathrm{act}}_{C_n, p/}$ for the $\infty$–category $\mathcal{O}_{C_n} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{p/}$ — thus an object of $\mathcal{O}^{\mathrm{act}}_{C_n, p/}$ consists of a cone $K^{\triangleright} \to \mathcal{O}^{\mathrm{act}}$ that restricts to $p$ on $K$ and with the image of the cone point in the fibre over $C_n$.

**Definition A.34** Suppose $\mathcal{O}$ is a $\mathbf{\Delta}^n$–$\infty$–operad, let $\overline{p}\colon K^{\triangleright} \to \mathcal{O}^{\mathrm{act}}$ be a diagram, and set $p := \overline{p}|_K$. We say that $\overline{p}$ is a *weak operadic colimit diagram* if the evident forgetful map

$$\mathcal{O}^{\mathrm{act}}_{C_n,\overline{p}/} \to \mathcal{O}^{\mathrm{act}}_{C_n,p/}$$

is an equivalence of $\infty$–categories.

**Remark A.35** If the image of the cone point of $K^{\triangleright}$ under $\overline{p}$ lies over $C_n$, then $\overline{p}$ is itself an object of $\mathcal{O}^{\mathrm{act}}_{C_n,p/}$, so $\overline{p}$ is a weak operadic colimit diagram if and only if it is a final object of $\mathcal{O}^{\mathrm{act}}_{C_n,p/}$.

**Remark A.36** It follows from [28, Proposition 2.1.2.1] that the map $\mathcal{O}^{\mathrm{act}}_{C_n,\overline{p}/} \to \mathcal{O}^{\mathrm{act}}_{C_n,p/}$ is always a left fibration. By [28, Proposition 2.4.4.6] it is therefore an equivalence of $\infty$–categories if and only if it is a trivial Kan fibration.

**Remark A.37** Suppose $K^{\triangleright} \to \mathcal{O}^{\mathrm{act}}$ is a weak operadic colimit diagram and $L \to K$ is a cofinal map. Then the composite $L^{\triangleright} \to K^{\triangleright} \to \mathcal{O}^{\mathrm{act}}$ is also a weak operadic colimit diagram.

**Proposition A.38** *Let $\mathcal{O}$ be a $\mathbf{\Delta}^n$–$\infty$–operad. A diagram $\overline{p}\colon K^{\triangleright} \to \mathcal{O}^{\mathrm{act}}$ is a weak operadic colimit if and only if for every $n > 0$ and every diagram*

$$\begin{array}{ccc} K \star \partial\Delta^n & \xrightarrow{\overline{f}_0} & \mathcal{O}^{\mathrm{act}} \\ \downarrow & \overset{\overline{f}}{\nearrow} & \downarrow \\ K \star \Delta^n & \xrightarrow[f]{} & (\mathbf{\Delta}^{n,\mathrm{op}})^{\mathrm{act}} \end{array}$$

*there exists an extension $\overline{f}$ of $\overline{f}_0$.*

**Proof** This is the same as the proof of [31, Proposition 3.1.1.7]. □

**Proposition A.39** *Let $\mathcal{O}$ be a $\mathbf{\Delta}^n$–$\infty$–operad, let $\overline{h}\colon \Delta^1 \times K^{\triangleright} \to \mathcal{O}^{\mathrm{act}}$ be a natural transformation from $\overline{h}_0 := \overline{h}|_{\{0\}\times K^{\triangleright}}$ to $\overline{h}_1 := \overline{h}|_{\{1\}\times K^{\triangleright}}$. Suppose that:*

(1) *For every vertex $x \in K^{\triangleright}$, the restriction $\overline{h}|_{\Delta^1\times\{x\}}$ is a cocartesian edge of $\mathcal{O}$.*

(2) *The composite $\Delta^1 \times \{\infty\} \to \mathcal{O} \to \mathbf{\Delta}^{n,\mathrm{op}}$ is an identity morphism. (Equivalently, the restriction $\overline{h}|_{\Delta^1\times\{\infty\}}$ is an equivalence in $\mathcal{O}$.)*

*Then $\overline{h}_0$ is a weak operadic colimit diagram if and only if $\overline{h}_1$ is a weak operadic colimit diagram.*

**Proof** This is the same as the proof of [31, Proposition 3.1.1.15]. □

Applied to a $\mathbf{\Delta}^n$–monoidal $\infty$–category $\mathcal{C}^{\otimes}$, this lets us reduce the question of whether a diagram in $(\mathcal{C}^{\otimes})^{\mathrm{act}}$ is a weak operadic colimit diagram to whether a diagram in a fibre $\mathcal{C}^{\otimes}_I$ is a weak operadic colimit diagram:

**Corollary A.40** *Let $\mathcal{C}^{\otimes}$ be $\mathbf{\Delta}^n$–monoidal $\infty$–category, and suppose $\overline{p}\colon K^{\triangleright} \to (\mathcal{C}^{\otimes})^{\mathrm{act}}$ is a diagram lying over $\overline{q}\colon K^{\triangleright} \to \mathbf{\Delta}^{n,\mathrm{op}}$. Take $\overline{p}'$ to be the cocartesian pushforward to the fibre over $\overline{q}(\infty)$. Then $\overline{p}$ is a weak operadic colimit diagram if and only if $\overline{p}'$ is a weak operadic colimit diagram.*

**Proposition A.41** *Let $\mathcal{C}^{\otimes}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category, and let $\overline{p}\colon K^{\triangleright} \to \mathcal{C}^{\otimes}_I$ be a diagram in the fibre over some $I \in \mathbf{\Delta}^{n,\mathrm{op}}$. Then $\overline{p}$ is a weak operadic colimit diagram if and only if, for $m\colon I \to C_n$ the unique active map in $\mathbf{\Delta}^{n,\mathrm{op}}$, the composite*

$$K^{\triangleright} \xrightarrow{\overline{p}} \mathcal{C}^{\otimes}_I \xrightarrow{m_!} \mathcal{C}$$

*is a colimit diagram in $\mathcal{C}$.*

**Proof** This is the same as the proof of [31, Proposition 3.1.1.16]. □

**Definition A.42** Let $\mu^j$ denote the map $(\mathrm{id}, \dots, d_1, \dots, \mathrm{id})\colon ([1], \dots, [2], \dots, [1]) \to ([1], \dots, [1])$ in $\mathbf{\Delta}^{n,\mathrm{op}}$ (with $d_1$ in the $j^{\mathrm{th}}$ place), and let $\mathcal{C}^{\otimes}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category. We say a diagram $\overline{p}\colon K^{\triangleright} \to \mathcal{C}$ is a *monoidal colimit diagram* if, for every $x \in \mathcal{C}$ and every $j = 1, \dots, n$, the composite

$$K^{\triangleright} \times \{x\} \to \mathcal{C} \times \mathcal{C} \simeq \mathcal{C}^{\otimes}_{([1],\dots,[2],\dots,[1])} \xrightarrow{\mu^j_!} \mathcal{C}$$

is a colimit diagram. More generally, if $\overline{p}\colon K^{\triangleright} \to (\mathcal{C}^{\otimes})^{\mathrm{act}}$ is a diagram with $\overline{p}(\infty)$ in $\mathcal{C}^{\otimes}_{C_n}$, then we say that $\overline{p}$ is a *monoidal colimit diagram* if the cocartesian pushforward to a diagram $\overline{p}'\colon K^{\triangleright} \to \mathcal{C}^{\otimes}_{C_n}$ is a monoidal colimit diagram in the first sense.

**Proposition A.43** *Let $\mathcal{O}$ be a $\mathbf{\Delta}^n$–$\infty$–operad. Suppose given, for some $I \in \mathbf{\Delta}^{n,\mathrm{op}}$, a finite collection of simplicial sets $K_i$ for $i \in |I|$ and diagrams $\overline{p}_i\colon K_i^{\triangleright} \to \mathcal{O}_{C_n}$. Suppose the product diagram*

$$\prod_{i \in |I|} K_i^{\triangleright} \to \prod_{i \in |I|} \mathcal{O}_{C_n} \simeq \mathcal{O}_I$$

*is such that, for every $i$ and every choice of $k_j \in K_j^{\triangleright}$ for $j \neq i$, the diagram*

$$K_i^{\triangleright} \xrightarrow{\sim} \{k_1\} \times \cdots \times K_i^{\triangleright} \times \cdots \times \{k_n\} \to \mathcal{O}_I \hookrightarrow \mathcal{O}^{\mathrm{act}}$$

*is a weak operadic colimit diagram. Then the composite*

$$\Big(\prod_{i=1}^n K_i\Big)^{\triangleright} \to \prod_{i=1}^n K_i^{\triangleright} \to \mathcal{O}_I \hookrightarrow \mathcal{O}^{\mathrm{act}}$$

*is also a weak operadic colimit diagram.*

**Proof** This is the same as the proof of [31, Proposition 3.1.1.8]. □

**Corollary A.44** *Let $\mathcal{C}^{\otimes}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category. Suppose given, for some $I \in \mathbf{\Delta}^{n,\mathrm{op}}$, a finite collection of simplicial sets $K_i$ for $i \in |I|$ and monoidal colimit diagrams $\bar{p}_i\colon K_i^{\triangleright} \to \mathcal{C}$. Then the composite diagram*

$$\left(\prod_{i=1}^{n} K_i\right)^{\triangleright} \to \prod_{i=1}^{n} K_i^{\triangleright} \to \prod_{i\in|I|} \mathcal{C} \simeq \mathcal{C}_I^{\otimes} \hookrightarrow (\mathcal{C}^{\otimes})^{\mathrm{act}}$$

*is a weak operadic colimit diagram. Moreover, the cocartesian pushforward of this diagram to $\mathcal{C}^{\otimes}_{C_n}$ is a monoidal colimit diagram.* □

## A.5 Operadic left Kan extensions

In this section we introduce the notion of *operadic left Kan extensions* in the $\mathbf{\Delta}^n$–setting. We then use the results of the previous section to give two key results: first, we will see that operadic left Kan extensions have a lifting property that will allow us to conclude, in the next section, that they can be used to construct adjoints, and second we consider an existence result for operadic left Kan extensions.

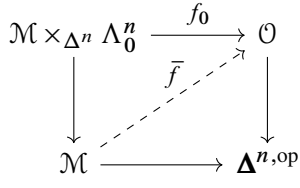
**Definition A.45** If $\mathcal{C}$ is an $\infty$–category, a *$\mathcal{C}$–family* of generalized $\mathbf{\Delta}^n$–$\infty$–operads is a morphism $\mathcal{M} \to \mathbf{\Delta}^{n,\mathrm{op}} \times \mathcal{C}$ of generalized $\mathbf{\Delta}^n$–$\infty$–operads.

**Definition A.46** Suppose $\mathcal{M} \to \mathbf{\Delta}^{n,\mathrm{op}} \times \Delta^1$ is a $\Delta^1$–family of generalized $\mathbf{\Delta}^n$–$\infty$–operads between $\mathcal{A} := \mathcal{M}_0$ and $\mathcal{B} := \mathcal{M}_1$. If $\mathcal{O}$ is a $\mathbf{\Delta}^n$–$\infty$–operad, an algebra $\mathcal{M} \to \mathcal{O}$ is an *operadic left Kan extension* if, for every $X \in \mathcal{B}$, the diagram

$$(\mathcal{A}^{\mathrm{act}}_{/X})^{\triangleright} \to \mathcal{M} \to \mathcal{O}$$

is a weak operadic colimit diagram.

**Proposition A.47** *For $n > 1$, let $\mathcal{M} \to \mathbf{\Delta}^{n,\mathrm{op}} \times \Delta^n$ be a $\Delta^n$–family of generalized $\mathbf{\Delta}^n$–$\infty$–operads and let $\mathcal{O}$ be a $\mathbf{\Delta}^n$–$\infty$–operad. Suppose we are given a commutative diagram of generalized $\mathbf{\Delta}^n$–$\infty$–operads*




*such that the restriction of $\bar{f}_0$ to $\mathcal{M} \times_{\Delta^n} \Delta^{\{0,1\}}$ is an operadic left Kan extension. Then there exists an extension $\bar{f}$ of $\bar{f}_0$.*

**Proof** This is the same as the proof of [31, Theorem 3.1.2.3(B)]. (Note that when [31, Proposition 3.1.1.7] is invoked in step (1) in the proof it is sufficient for the diagram to be a weak operadic colimit.) □

**Corollary A.48** *Suppose $\mathcal{M} \to \mathbf{\Delta}^{n,\mathrm{op}} \times \Delta^1$ is a $\Delta^1$–family of generalized $\mathbf{\Delta}^n$–$\infty$–operads between $\mathcal{A} := \mathcal{M}_0$ and $\mathcal{B} := \mathcal{M}_1$, and let $\mathcal{O}$ be a $\mathbf{\Delta}^n$–$\infty$–operad. Suppose that $n > 0$ and that we are given a diagram*

$$\begin{array}{ccc} (\mathcal{A} \times \Delta^n) \amalg_{(\mathcal{A} \times \partial\Delta^n)} (\mathcal{M} \times \partial\Delta^n) & \xrightarrow{f_0} & \mathcal{O} \\ \downarrow & \overset{f}{\nearrow} & \downarrow \\ \mathcal{M} \times \Delta^n & \longrightarrow & \mathbf{\Delta}^{n,\mathrm{op}} \end{array}$$

*of generalized $\mathbf{\Delta}^n$–$\infty$–operads. If the restriction of $f_0$ to $\mathcal{M} \times \{0\}$ is an operadic left Kan extension, then there exists an extension $f$ of $f_0$ that is a map of generalized $\mathbf{\Delta}^n$–$\infty$–operads.*

**Proof** This is the same as the proof of [31, Lemma 3.1.3.16]. □

**Definition A.49** We say a $\Delta^1$–family of generalized $\mathbf{\Delta}^n$–$\infty$–operads $\mathcal{M} \to \mathbf{\Delta}^{n,\mathrm{op}} \times \Delta^1$ is *extendable* if for every object $B \in \mathcal{M}_1$, lying over $I \in \mathbf{\Delta}^{n,\mathrm{op}}$, with inert projections $B \to B_i$ over $i \in |I|$, the map $\mathcal{M}^{\mathrm{act}}_{0,/B} \to \prod_{i \in |I|} \mathcal{M}^{\mathrm{act}}_{0,/B_i}$ is cofinal.

**Proposition A.50** *Let $\mathcal{M} \to \mathbf{\Delta}^{n,\mathrm{op}} \times \Delta^1$ be an extendable $\Delta^1$–family of generalized $\mathbf{\Delta}^n$–$\infty$–operads and let $\mathcal{C}^{\otimes}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category. Suppose given a diagram*

$$\begin{array}{ccc} \mathcal{M}_0 & \xrightarrow{f_0} & \mathcal{C}^{\otimes} \\ \downarrow & \overset{f}{\nearrow} & \downarrow \\ \mathcal{M} & \longrightarrow & \mathbf{\Delta}^{n,\mathrm{op}} \end{array}$$

*such that, for every $x \in \mathcal{M}_{C_n,1}$, the diagram*

$$\mathcal{M}^{\mathrm{act}}_{0,/x} \to \mathcal{M}_0 \xrightarrow{f_0} \mathcal{C}^{\otimes}$$

*can be extended to a* monoidal *colimit diagram lifting the map $(\mathcal{M}^{\mathrm{act}}_{0,/x})^{\triangleright} \to \mathcal{M} \to \mathbf{\Delta}^{n,\mathrm{op}}$. Then there exists an extension $f\colon \mathcal{M} \to \mathcal{C}^{\otimes}$ of $f_0$ that is an operadic left Kan extension.*

**Proof** This is essentially the same as the proof of [31, Theorem 3.1.2.3(A)], with a slight difference in step (1): To extend the functor to the 0–simplices of $\mathcal{M}_1$ we use the monoidal colimits that exist by assumption. Then for the construction of the

higher-dimensional simplices we need to show that the maps $\delta\colon (\mathcal{M}^{\mathrm{act}}_{0,/B})^{\triangleright} \to \mathcal{C}^{\otimes}$ are weak operadic colimits. If $B$ lies over $I \in \mathbf{\Delta}^{n,\mathrm{op}}$, let

$$\delta'\colon (\mathcal{M}^{\mathrm{act}}_{0,/B})^{\triangleright} \to \mathcal{C}^{\otimes}_{I}$$

denote the cocartesian pushforward along the active maps to I; by Corollary A.40 it suffices to show that $\delta'$ is a weak operadic colimit. Choose cocartesian morphisms $B \to B_i$ over the inert maps $i\colon I \to C_n$. Then $\delta'$ factors as

$$(\mathcal{M}^{\mathrm{act}}_{0,/B})^{\triangleright} \to \left(\prod_i \mathcal{M}^{\mathrm{act}}_{0,/B_i}\right)^{\triangleright} \to \prod_i (\mathcal{M}^{\mathrm{act}}_{0,/B_i})^{\triangleright} \xrightarrow{\prod \delta_i} \prod_i \mathcal{C} \simeq \mathcal{C}^{\otimes}_{I}.$$

The map $\mathcal{M}^{\mathrm{act}}_{0,/B} \to \prod_i \mathcal{M}^{\mathrm{act}}_{0,/B_i}$ is cofinal since $\mathcal{M}$ is extendable, so it suffices to show that the map from $\left(\prod_i \mathcal{M}^{\mathrm{act}}_{0,/B_i}\right)^{\triangleright}$ is a weak operadic colimit diagram. This follows from Corollary A.44, since the maps $\delta_i$ are monoidal colimit diagrams. □

**Definition A.51** Suppose $\mathcal{C}^{\otimes}$ is a $\mathbf{\Delta}^n$–monoidal $\infty$–category. If $\mathcal{K}$ is some class of simplicial sets we say that $\mathcal{C}^{\otimes}$ is *compatible with $\mathcal{K}$–indexed colimits* if

(1) the underlying $\infty$–category $\mathcal{C}$ has $\mathcal{K}$–indexed colimits;

(2) for $j = 1, \dots, n$, the functor $\mu^j_!\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{K}$–indexed colimits in each variable.

**Corollary A.52** *Let $\mathcal{M} \to \mathbf{\Delta}^{n,\mathrm{op}} \times \Delta^1$ be an extendable $\Delta^1$–family of generalized $\mathbf{\Delta}^n$–$\infty$–operads and let $\mathcal{C}^{\otimes}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category that is compatible with $\mathcal{M}^{\mathrm{act}}_{0,/x}$–indexed colimits for all $x \in \mathcal{M}_{C_n,1}$. Suppose given a diagram:*

$$\begin{array}{ccc} \mathcal{M}_0 & \xrightarrow{f_0} & \mathcal{C}^{\otimes} \\ \downarrow & \overset{f}{\nearrow} & \downarrow \\ \mathcal{M} & \longrightarrow & \mathbf{\Delta}^{n,\mathrm{op}} \end{array}$$

*Then there exists an extension $f\colon \mathcal{M} \to \mathcal{C}^{\otimes}$ of $f_0$ that is an operadic left Kan extension.* □

We end this subsection with the following observation, which will be useful for recognizing operadic left Kan extension:

**Lemma A.53** *Let $i\colon \mathcal{A} \to \mathcal{B}$ be a morphism of generalized $\mathbf{\Delta}^n$–$\infty$–operads, let $\mathcal{C}^{\otimes}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category, and suppose given a $\mathcal{B}$–algebra $B$ in $\mathcal{C}$ and a morphism $A \to i^*B$ of $\mathcal{A}$–algebras. Choose a factorization of the induced map $\phi\colon \mathcal{A} \times \Delta^1 \amalg_{\mathcal{A}\times\{1\}} \mathcal{B} \to \mathcal{C}^{\otimes}$ through a $\Delta^1$–family of generalized $\mathbf{\Delta}^n$–$\infty$–operads $\mathcal{M}$. Then the following are equivalent:*

(1) *The map $\mathcal{M} \to \mathcal{C}^{\otimes}$ is an operadic left Kan extension.*

(2) *Choose a cocartesian pushforward*

$$\phi' \colon \mathcal{A}^{\mathrm{act}} \times \Delta^1 \amalg_{\mathcal{A}^{\mathrm{act}} \times \{1\}} \mathcal{B}^{\mathrm{act}} \to \mathcal{C} \simeq \mathcal{C}^{\otimes}_{C_n}$$

*of the restriction of $\phi$ to the subcategories of active maps, along the unique active maps to $C_n$. Then $\phi'$ is a left Kan extension in the sense of* [28, Definition 4.3.3.2].

**Proof** This is immediate from the description of weak operadic colimits in $\mathbf{\Delta}^n$–monoidal $\infty$–categories from Corollary A.40 and Proposition A.41. □

## A.6 Free algebras

**Definition A.54** Suppose $i \colon \mathcal{O} \to \mathcal{P}$ is a morphism of generalized $\mathbf{\Delta}^n$–$\infty$–operads and $\mathcal{C}^{\otimes}$ is a $\mathbf{\Delta}^n$–monoidal $\infty$–category. If $\bar{A}$ is a $\mathcal{P}$–algebra in $\mathcal{C}$ and $\phi \colon A \to i^*\bar{A}$ is a morphism of $\mathcal{O}$–algebras, then we say that $\phi$ exhibits $\bar{A}$ as the *free $\mathcal{P}$–algebra* generated by $A$ along $i$ if for every $\mathcal{P}$–algebra $B$ the composite

$$\mathrm{Map}_{\mathrm{Alg}^n_{\mathcal{P}}(\mathcal{C})}(\bar{A}, B) \to \mathrm{Map}_{\mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C})}(i^*\bar{A}, i^*B) \to \mathrm{Map}_{\mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C})}(A, i^*B)$$

is an equivalence.

**Lemma A.55** *Let $i \colon \mathcal{O} \to \mathcal{P}$ be a morphism of generalized $\mathbf{\Delta}^n$–$\infty$–operads and let $\mathcal{C}^{\otimes}$ be a $\mathbf{\Delta}^n$–monoidal $\infty$–category. If for every $\mathcal{O}$–algebra $A$ in $\mathcal{C}$ there exists a $\mathcal{P}$–algebra $\bar{A}$ and a morphism $A \to i^*\bar{A}$ that exhibits $\bar{A}$ as the free $\mathcal{P}$–algebra generated by $A$ along $i$, then the functor $i^* \colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ induced by composition with $i$ admits a left adjoint $i_!$ such that the unit morphism $A \to i^*i_!A$ exhibits $i_!A$ as the free $\mathcal{P}$–algebra generated by $A$ along $i$ for all $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.*

**Proof** Apply [28, Lemma 5.2.2.10] to the cocartesian fibration associated to $i^*$. □

**Definition A.56** Suppose $i \colon \mathcal{O} \to \mathcal{P}$ is a morphism of generalized $\mathbf{\Delta}^n$–$\infty$–operads and $\mathcal{C}^{\otimes}$ is a $\mathbf{\Delta}^n$–monoidal $\infty$–category. If $\bar{A}$ is a $\mathcal{P}$–algebra in $\mathcal{C}$ and $\phi \colon A \to i^*\bar{A}$ is a morphism of $\mathcal{O}$–algebras, we have an induced diagram

$$(\mathcal{O} \times \Delta^1) \amalg_{\mathcal{O} \times \{1\}} \mathcal{P} \to \mathcal{C}^{\otimes}.$$

Choose a factorization of this as

$$(\mathcal{O} \times \Delta^1) \amalg_{\mathcal{O} \times \{1\}} \mathcal{P} \hookrightarrow \mathcal{M} \to \mathcal{C}^{\otimes}$$

such that the first map is inner anodyne and $\mathcal{M}$ is a $\Delta^1$–family of generalized $\mathbf{\Delta}^n$–$\infty$–operads. We say that $\phi$ exhibits $\bar{A}$ as an *operadic left Kan extension* of $A$ along $i$ if the map $\mathcal{M} \to \mathcal{C}^{\otimes}$ is an operadic left Kan extension.

**Proposition A.57** *Suppose $i\colon \mathcal{O} \to \mathcal{P}$ is a morphism of generalized $\boldsymbol{\Delta}^n$–$\infty$–operads and $\mathcal{C}^{\otimes}$ is a $\boldsymbol{\Delta}^n$–monoidal $\infty$–category. If $\bar{A}$ is a $\mathcal{P}$–algebra in $\mathcal{C}$ and $\phi\colon A \to i^*\bar{A}$ is a morphism of $\mathcal{O}$–algebras that exhibits $\bar{A}$ as an operadic left Kan extension of $A$ along $i$, then $\phi$ exhibits $\bar{A}$ as the free $\mathcal{P}$–algebra generated by $A$ along $i$.*

**Proof** This is the same as the proof of [31, Proposition 3.1.3.2], using Corollary A.48. □

**Corollary A.58** *Let $i\colon \mathcal{O} \to \mathcal{P}$ be a morphism of generalized $\boldsymbol{\Delta}^n$–$\infty$–operads and let $\mathcal{C}^{\otimes}$ be a $\boldsymbol{\Delta}^n$–monoidal $\infty$–category. If for every $\mathcal{O}$–algebra $A$ in $\mathcal{C}$ there exists a $\mathcal{P}$–algebra $\bar{A}$ and a morphism $A \to i^*\bar{A}$ that exhibits $\bar{A}$ as the operadic left Kan extension of $A$ along $i$, then the functor $i^*\colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ induced by composition with $i$ admits a left adjoint $i_!$ such that the unit morphism $A \to i^*i_!A$ exhibits $i_!A$ as the operadic left Kan extension of $A$ along $i$ for all $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. Moreover, if $i$ is fully faithful, then so is $i_!$.*

**Proof** Combine Proposition A.57 with Lemma A.55. The full faithfulness follows from the description of operadic left Kan extensions in terms of colimits: it is immediate from this that if $i$ is fully faithful then the unit morphism $A \to i^*i_!A$ is an equivalence. □

**Definition A.59** Let $i\colon \mathcal{O} \to \mathcal{P}$ be an extendable morphism of generalized $\boldsymbol{\Delta}^n$–$\infty$–operads. We say that a $\boldsymbol{\Delta}^n$–monoidal $\infty$–category $\mathcal{C}^{\otimes}$ is *$i$–compatible* if, for every $\mathcal{O}$–algebra $A$ in $\mathcal{C}$ and every $x \in \mathcal{P}_{C_n}$, the diagram

$$\mathcal{O}^{\mathrm{act}}_{/x} \to \mathcal{O} \xrightarrow{A} \mathcal{C}^{\otimes}$$

can be extended to a monoidal colimit diagram.

**Corollary A.60** *Let $i\colon \mathcal{O} \to \mathcal{P}$ be an extendable morphism of generalized $\boldsymbol{\Delta}^n$–$\infty$–operads. If $\mathcal{C}^{\otimes}$ is a $\boldsymbol{\Delta}^n$–monoidal $\infty$–category that is $i$–compatible, then the functor $i^*\colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ admits a left adjoint $i_!$ such that the unit morphism $A \to i^*i_!A$ exhibits $i_!A$ as the operadic left Kan extension of $A$ along $i$ for all $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.*

**Proof** Combine Corollary A.58 with Proposition A.50. □

**Corollary A.61** *Let $i\colon \mathcal{O} \to \mathcal{P}$ be an extendable morphism of generalized $\boldsymbol{\Delta}^n$–$\infty$–operads. If $\mathcal{C}^{\otimes}$ is a $\boldsymbol{\Delta}^n$–monoidal $\infty$–category that is compatible with $\mathcal{O}^{\mathrm{act}}_{/p}$–indexed colimits for all $p \in \mathcal{P}_{C_n}$, then the functor $i^*\colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ admits a left adjoint $i_!$ such that the unit morphism $A \to i^*i_!A$ exhibits $i_!A$ as the operadic left Kan extension of $A$ along $i$ for all $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.*

**Proof** Combine Corollary A.58 with Corollary A.52. □

We will also need an observation on the functoriality of free algebras, requiring some terminology:

**Definition A.62** Let $i\colon \mathcal{O} \to \mathcal{P}$ be an extendable morphism of generalized $\mathbf{\Delta}^n$–$\infty$–operads. If $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are $i$–compatible $\mathbf{\Delta}^n$–monoidal $\infty$–categories, we say that a $\mathbf{\Delta}^n$–monoidal functor $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is *$i$–compatible* if, for every $\mathcal{O}$–algebra $A$ in $\mathcal{C}$ and every $x \in \mathcal{P}_{C_n}$, the underlying functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves the (monoidal) colimit of the diagram $\mathcal{O}^{\mathrm{act}}_{/x} \to \mathcal{C}$.

**Lemma A.63** *Suppose $i\colon \mathcal{O} \to \mathcal{P}$ is an extendable morphism of generalized $\mathbf{\Delta}^n$–$\infty$–operads, $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are $i$–compatible $\mathbf{\Delta}^n$–monoidal $\infty$–categories and $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is an $i$–compatible $\mathbf{\Delta}^n$–monoidal functor. Then we have a commutative diagram:*

$$\begin{array}{ccc} \mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C}) & \xrightarrow{F_*} & \mathrm{Alg}^n_{\mathcal{O}}(\mathcal{D}) \\ {\scriptstyle i_!}\downarrow & & \downarrow{\scriptstyle i_!} \\ \mathrm{Alg}^n_{\mathcal{P}}(\mathcal{C}) & \xrightarrow[F_*]{} & \mathrm{Alg}^n_{\mathcal{P}}(\mathcal{D}) \end{array}$$

**Proof** We must show that for every $\mathcal{O}$–algebra $A$ in $\mathcal{C}$, the map $F_*A \to F_* i^* i_! A \simeq i^* F_* i_! A$ exhibits $F_* i_! A$ as the free algebra generated by $F_* A$ along $i$. This follows from Proposition A.57 and the assumption that $F$ is $i$–compatible, since this implies that $F_* i_! A$ is a left operadic Kan extension of $F_* A$. □

## A.7 Monoidal properties of the algebra functor

In this subsection we observe that the cartesian product of generalized $\mathbf{\Delta}^n$–$\infty$–operads leads to natural monoidal structures on $\infty$–categories of algebras.

**Definition A.64** For any categorical pattern $\mathfrak{P}$, the model category $(\mathrm{Set}^+_{\Delta})_{\mathfrak{P}}$ is enriched in marked simplicial sets by Proposition A.11. The enriched Yoneda functor therefore gives a right Quillen bifunctor

$$\mathfrak{H}_{\mathfrak{P}}\colon (\mathrm{Set}^+_{\Delta})^{\mathrm{op}}_{\mathfrak{P}} \times (\mathrm{Set}^+_{\Delta})_{\mathfrak{P}} \to \mathrm{Set}^+_{\Delta}.$$

Applied to $\mathfrak{P} = \mathfrak{O}^{\mathrm{gen}}_n$, this induces at the level of $\infty$–categories a functor

$$\mathrm{Alg}^n_{(-)}(-)\colon (\mathrm{Opd}^{\mathbf{\Delta}^n,\mathrm{gen}}_{\infty})^{\mathrm{op}} \times \mathrm{Opd}^{\mathbf{\Delta}^n,\mathrm{gen}}_{\infty} \to \mathrm{Cat}_{\infty}.$$

We write $\mathrm{Alg}^n \to (\mathrm{Opd}^{\mathbf{\Delta}^n,\mathrm{gen}}_{\infty})^{\mathrm{op}} \times \mathrm{Opd}^{\mathbf{\Delta}^n,\mathrm{gen}}_{\infty}$ for an associated cocartesian fibration.

**Definition A.65** As $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}}$ is a (marked simplicially enriched) symmetric monoidal model category with respect to the cartesian product, the functor $\mathfrak{H}_{\mathfrak{P}}$ is lax symmetric monoidal with respect to the cartesian product. Thus, for $\mathfrak{P} = \mathfrak{O}_n^{\mathrm{gen}}$ it induces on the level of $\infty$–categories a lax symmetric monoidal functor

$$((\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}})^{\amalg} \times_{\mathbf{\Gamma}^{\mathrm{op}}} (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\times} \to \mathrm{Cat}_\infty^{\times},$$

where $((\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}})^{\amalg}$ is the symmetric monoidal structure on $(\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}}$ given by the cartesian product in $\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}}$, since this is the cocartesian monoidal structure on the opposite $\infty$–category. Using [31, Proposition 2.4.2.5] this corresponds to a functor

$$\phi\colon ((\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}})^{\amalg} \times_{\mathbf{\Gamma}^{\mathrm{op}}} (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\times} \to \mathrm{Cat}_\infty$$

that is a $((\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}})^{\amalg} \times_{\mathbf{\Gamma}^{\mathrm{op}}} (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\times}$–monoid in $\mathrm{Cat}_\infty$ (ie it satisfies the relevant Segal conditions). Let

$$\mathrm{Alg}^{n,\boxtimes} \to ((\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}})^{\amalg} \times_{\mathbf{\Gamma}^{\mathrm{op}}} (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\times}.$$

be the cocartesian fibration associated to $\phi$; since the functor $\phi$ satisfies the Segal conditions, this is a cocartesian fibration of generalized symmetric $\infty$–operads.

This construction describes the "external product" that combines algebras $A\colon \mathcal{O} \to \mathcal{O}'$ and $B\colon \mathcal{P} \to \mathcal{P}'$ to $A \boxtimes B := A \times_{\mathbf{\Delta}^{n,\mathrm{op}}} B\colon \mathcal{O} \times_{\mathbf{\Delta}^{n,\mathrm{op}}} \mathcal{O}' \to \mathcal{P} \times_{\mathbf{\Delta}^{n,\mathrm{op}}} \mathcal{P}'$. Since we are considering the cocartesian symmetric monoidal structure on $(\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}}$, by [31, Example 2.4.3.5] there is a morphism of generalized symmetric $\infty$–operads $\alpha\colon \mathbf{\Gamma}^{\mathrm{op}} \times (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}} \to ((\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}})^{\amalg}$. (Informally, this takes $(\langle n\rangle, \mathcal{O})$ to the list $(\mathcal{O},\dots,\mathcal{O})$ with $n$ copies of $\mathcal{O}$.) We define $\mathrm{Alg}^{n,\otimes}$ by the pullback square:

$$\begin{array}{ccc} \mathrm{Alg}^{n,\otimes} & \longrightarrow & \mathrm{Alg}^{n,\boxtimes} \\ \downarrow & & \downarrow \\ (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}} \times (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\times} & \xrightarrow[\alpha \times_{\mathbf{\Gamma}^{\mathrm{op}}} \mathrm{id}]{} & ((\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}})^{\amalg} \times_{\mathbf{\Gamma}^{\mathrm{op}}} (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\times} \end{array}$$

Then the projection $\pi\colon \mathrm{Alg}^{n,\otimes} \to (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}} \times (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\times}$ is again a cocartesian fibration of generalized symmetric $\infty$–operads. Over $\mathcal{O} \in (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}}$ this describes the "half-internalized" tensor product of $\mathcal{O}$–algebras given by, for $A\colon \mathcal{O} \to \mathcal{P}$ and $B\colon \mathcal{O} \to \mathcal{Q}$,

$$A \otimes B\colon \mathcal{O} \xrightarrow{\Delta} \mathcal{O} \times_{\mathbf{\Delta}^{n,\mathrm{op}}} \mathcal{O} \xrightarrow{A \boxtimes B} \mathcal{P} \times_{\mathbf{\Delta}^{n,\mathrm{op}}} \mathcal{Q}.$$

The functor associated to the cocartesian fibration $\pi$ is a $(\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}} \times (\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\times}$–monoid in $\mathrm{Cat}_\infty$, or equivalently a lax symmetric monoidal functor

$$\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}} \to \mathrm{Fun}((\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}})^{\mathrm{op}}, \mathrm{Cat}_\infty).$$

Similarly, pulling back $\pi$ along an arbitrary functor in the first variable, we get:

**Proposition A.66** *Let $F\colon \mathcal{C} \to \mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}}$ be any functor of $\infty$–categories. Then the functor*

$$\mathrm{Alg}^n_{F(-)}(-)\colon \mathcal{C}^{\mathrm{op}} \times \mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}} \to \mathrm{Cat}_\infty$$

*induces a lax symmetric monoidal functor $\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)$.* □

**Corollary A.67** *Suppose $\mathcal{O}$ is a generalized $\mathbf{\Delta}^n$–$\infty$–operad and $\mathcal{C}$ is an $\mathbb{E}_{n+m}$–monoidal $\infty$–category. Then $\mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C})$ is an $\mathbb{E}_m$–monoidal $\infty$–category.*

**Proof** By Proposition A.66, applied to the functor $\{\mathcal{O}\} \to \mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}}$, there is a lax symmetric monoidal functor $\mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}} \to \mathrm{Cat}_\infty$, which sends $\mathcal{P}$ to $\mathrm{Alg}^n_{\mathcal{O}}(\mathcal{P})$. The forgetful functor

$$\mathrm{Mon}_\infty^{\mathbf{\Delta}^n} \to \mathrm{Opd}_\infty^{\mathbf{\Delta}^n,\mathrm{gen}}$$

preserves products, so we get a lax symmetric monoidal functor $\mathrm{Mon}_\infty^{\mathbf{\Delta}^n} \to \mathrm{Cat}_\infty$, and hence a functor

$$\mathrm{Mon}_\infty^{\Sigma,\mathbb{E}_{n+m}} \simeq \mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n+m}}(\mathrm{Cat}_\infty) \simeq \mathrm{Alg}^{\Sigma}_{\mathbb{E}_m}(\mathrm{Mon}_\infty^{\mathbf{\Delta}^n}) \to \mathrm{Alg}^{\Sigma}_{\mathbb{E}_m}(\mathrm{Cat}_\infty) \simeq \mathrm{Mon}_\infty^{\Sigma,\mathbb{E}_m},$$

which sends an $\mathbb{E}_{n+m}$–monoidal $\infty$–category $\mathcal{C}$ to a natural $\mathbb{E}_m$–monoidal structure on $\mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C})$. □

**Remark A.68** Let $i\colon \mathcal{O} \to \mathcal{P}$ be an extendable morphism of generalized $\mathbf{\Delta}^n$–$\infty$–operads, and let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be $i$–compatible $\mathbf{\Delta}^n$–monoidal $\infty$–categories. If the $\infty$–categories $\mathcal{O}^{\mathrm{act}}_{/P}$ are all sifted, then the description of free algebras in terms of weak operadic colimits implies that there is a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C}) \times \mathrm{Alg}^n_{\mathcal{O}}(\mathcal{D}) & \longrightarrow & \mathrm{Alg}^n_{\mathcal{O}\times_{\mathbf{\Delta}^n,\mathrm{op}}\mathcal{O}}(\mathcal{C}\times\mathcal{D}) & \longrightarrow & \mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C}\times\mathcal{D}) \\
\downarrow{\scriptstyle i_!\times i_!} & & \downarrow{\scriptstyle (i\times_{\mathbf{\Delta}^n,\mathrm{op}} i)_!} & & \downarrow{\scriptstyle i_!} \\
\mathrm{Alg}^n_{\mathcal{P}}(\mathcal{C}) \times \mathrm{Alg}^n_{\mathcal{P}}(\mathcal{D}) & \longrightarrow & \mathrm{Alg}^n_{\mathcal{P}\times_{\mathbf{\Delta}^n,\mathrm{op}}\mathcal{P}}(\mathcal{C}\times\mathcal{D}) & \longrightarrow & \mathrm{Alg}^n_{\mathcal{P}}(\mathcal{C}\times\mathcal{D})
\end{array}$$

In other words, $i_!(A \otimes B) \simeq i_!A \otimes i_!B$ where $\otimes$ denotes the "half-internalized" tensor product of algebras. If $\mathcal{C}$ is a $\mathbf{\Delta}^{n+1}$–monoidal $\infty$–category such that its tensor product,

regarded as a $\mathbf{\Delta}^n$–monoidal functor $\mathcal{C}^{\otimes} \times_{\mathbf{\Delta}^{n,\mathrm{op}}} \mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}$, is $i$–compatible, then by Lemma A.63 we get a commutative square:

$$\begin{array}{ccc} \mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C}) \times \mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C}) & \xrightarrow{\otimes} & \mathrm{Alg}^n_{\mathcal{O}}(\mathcal{C}) \\ \downarrow{\scriptstyle i_! \times i_!} & & \downarrow{\scriptstyle i_!} \\ \mathrm{Alg}^n_{\mathcal{P}}(\mathcal{C}) \times \mathrm{Alg}^n_{\mathcal{P}}(\mathcal{C}) & \xrightarrow[\otimes]{} & \mathrm{Alg}^n_{\mathcal{P}}(\mathcal{C}) \end{array}$$

## A.8 $\mathbf{\Delta}^n$–uple envelopes

It is immediate from the definition of the model categories $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{gen}}_n}$ and $(\mathrm{Set}^+_\Delta)_{\mathfrak{U}_n}$ that the identity is a left Quillen functor $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{gen}}_n} \to (\mathrm{Set}^+_\Delta)_{\mathfrak{U}_n}$. On the level of $\infty$–categories, this means that the inclusion $\mathrm{Upl}^n_\infty \to \mathrm{Opd}^{\mathbf{\Delta}^n,\mathrm{gen}}_\infty$ has a left adjoint. In this subsection we observe that the arguments of [31, Section 2.2.4] give an explicit description of this left adjoint.

**Definition A.69** Let $\mathrm{Act}(\mathbf{\Delta}^{n,\mathrm{op}})$ be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathbf{\Delta}^{n,\mathrm{op}})$ spanned by the active morphisms. If $\mathcal{M}$ is a generalized $\mathbf{\Delta}^n$–$\infty$–operad, we define $\mathrm{Env}_n(\mathcal{M})$ to be the fibre product

$$\mathcal{M} \times_{\mathrm{Fun}(\{0\}, \mathbf{\Delta}^{n,\mathrm{op}})} \mathrm{Act}(\mathbf{\Delta}^{n,\mathrm{op}}).$$

We will refer to $\mathrm{Env}_n(\mathcal{M})$ as the *$\mathbf{\Delta}^n$–uple envelope* of $\mathcal{M}$ — this terminology is justified by the next results:

**Proposition A.70** *The map $\mathrm{Env}_n(\mathcal{M}) \to \mathbf{\Delta}^{n,\mathrm{op}}$ induced by evaluation at $1$ in $\Delta^1$ is a $\mathbf{\Delta}^n$–uple $\infty$–category.*

**Proof** This is the same as the proof of [31, Proposition 2.2.4.4]. □

**Proposition A.71** *Suppose $\mathcal{N}$ is a $\mathbf{\Delta}^n$–uple $\infty$–category and $\mathcal{M}$ a generalized $\mathbf{\Delta}^n$–$\infty$–operad. The inclusion $\mathcal{M} \to \mathrm{Env}_n(\mathcal{M})$ induces an equivalence*

$$\mathrm{Fun}^{\otimes,n}(\mathrm{Env}_n(\mathcal{M}), \mathcal{N}) \to \mathrm{Alg}^n_{\mathcal{M}}(\mathcal{N}).$$

**Proof** This is the same as the proof of [31, Proposition 2.2.4.9]. □

**Lemma A.72** *Suppose $\mathcal{O}$ is a generalized $\mathbf{\Delta}^n$–$\infty$–operad and $\mathcal{P}$ is a generalized $\mathbf{\Delta}^m$–$\infty$–operad. There is a natural equivalence*

$$\mathrm{Env}_n(\mathcal{O}) \times \mathrm{Env}_m(\mathcal{P}) \simeq \mathrm{Env}_{n+m}(\mathcal{O} \times \mathcal{P}).$$

**Proof** This is immediate from the definition. □

## A.9 The internal Hom

In this subsection we observe, following [10, Section 9], that if $\mathcal{O}$ is a generalized $\mathbf{\Delta}^n$–$\infty$–operad and $\mathcal{P}$ is a generalized $\mathbf{\Delta}^{m+n}$–$\infty$–operad, then the $\infty$–category $\mathrm{Alg}^n_{\mathcal{O}}(\mathcal{P})$ has a natural generalized $\mathbf{\Delta}^m$–$\infty$–operad structure. When $\mathcal{P}$ is a $\mathbf{\Delta}^{m+n}$–monoidal $\infty$–category we will prove that this makes $\mathrm{Alg}^n_{\mathcal{O}}(\mathcal{P})$ a $\mathbf{\Delta}^m$–monoidal $\infty$–category, and that this structure agrees with that we described in Section A.7.

**Definition A.73** By Corollary A.16, the cartesian product gives a left Quillen bifunctor

$$(\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{gen}}_n} \times (\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{gen}}_m} \to (\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{gen}}_{n+m}}.$$

It therefore induces a right Quillen bifunctor

$$\mathrm{ALG}^{n,m}_{(-)}(-)\colon (\mathrm{Set}^+_\Delta)^{\mathrm{op}}_{\mathfrak{O}^{\mathrm{gen}}_n} \times (\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{gen}}_{m+n}} \to (\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{gen}}_m}.$$

Similarly, there is a right Quillen bifunctor

$$\mathrm{FUN}^{\otimes,n,m}_{(-)}(-)\colon (\mathrm{Set}^+_\Delta)^{\mathrm{op}}_{\mathfrak{U}_n} \times (\mathrm{Set}^+_\Delta)_{\mathfrak{U}_{m+n}} \to (\mathrm{Set}^+_\Delta)_{\mathfrak{U}_m},$$

right adjoint to the cartesian product.

On the level of $\infty$–categories, these right Quillen bifunctors induce functors

$$\mathrm{ALG}^{n,m}_{(-)}(-)\colon \mathrm{Opd}^{\mathbf{\Delta}^n,\mathrm{gen}}_\infty \times \mathrm{Opd}^{\mathbf{\Delta}^{n+m},\mathrm{gen}}_\infty \to \mathrm{Opd}^{\mathbf{\Delta}^m,\mathrm{gen}}_\infty,$$
$$\mathrm{FUN}^{\otimes,n,m}_{(-)}(-)\colon \mathrm{Upl}^{\mathbf{\Delta}^n}_\infty \times \mathrm{Upl}^{\mathbf{\Delta}^{n+m}}_\infty \to \mathrm{Upl}^{\mathbf{\Delta}^m}_\infty,$$

with the universal property that there are natural equivalences of $\infty$–categories

$$\mathrm{Alg}^m_{\mathcal{O}}(\mathrm{ALG}^{n,m}_{\mathcal{P}}(\mathcal{Q})) \simeq \mathrm{Alg}^{n+m}_{\mathcal{O}\times\mathcal{P}}(\mathcal{Q}),$$

where $\mathcal{O}$ is a generalized $\mathbf{\Delta}^m$–$\infty$–operad, $\mathcal{P}$ is a generalized $\mathbf{\Delta}^n$–$\infty$–operad, and $\mathcal{Q}$ is a generalized $\mathbf{\Delta}^{m+n}$–$\infty$–operad, and

$$\mathrm{Fun}^{\otimes,m}(\mathcal{L}, \mathrm{FUN}^{\otimes,n,m}(\mathcal{M},\mathcal{N})) \simeq \mathrm{Fun}^{\otimes,m+n}(\mathcal{L}\times\mathcal{M},\mathcal{N}),$$

where $\mathcal{L}$ is a $\mathbf{\Delta}^m$–uple $\infty$–category, $\mathcal{M}$ is a $\mathbf{\Delta}^n$–uple $\infty$–category, and $\mathcal{N}$ is a $\mathbf{\Delta}^{m+n}$–uple $\infty$–category.

**Lemma A.74** (i) *If $\mathcal{O}$ is a $\mathbf{\Delta}^{n+m}$–$\infty$–operad, then $\mathrm{ALG}^{n,m}_{\mathcal{M}}(\mathcal{O})$ is a $\mathbf{\Delta}^n$–$\infty$–operad for any generalized $\mathbf{\Delta}^n$–$\infty$–operad $\mathcal{M}$.*

(ii) *If $\mathcal{C}^\otimes$ is a $\mathbf{\Delta}^{n+m}$–monoidal $\infty$–category, then $\mathrm{FUN}^{\otimes,n,m}(\mathcal{M},\mathcal{C}^\otimes)$ is a $\mathbf{\Delta}^m$–monoidal $\infty$–category for any $\mathbf{\Delta}^n$–uple $\infty$–category $\mathcal{M}$.*

**Proof** We will prove (i); the proof of (ii) is similar. Suppose $C_S \neq C_n$ is a cell of $\mathbf{\Delta}^{m,\mathrm{op}}$. Then we have

$$\mathrm{ALG}^{n,m}_{\mathcal{M}}(\mathcal{O})_{C_S} \simeq \mathrm{Alg}^{m}_{\{C_S\}}(\mathrm{ALG}^{n,m}_{\mathcal{M}}(\mathcal{O})) \simeq \mathrm{Alg}^{n+m}_{\{C_S\}\times\mathcal{M}}(\mathcal{O}),$$

which is contractible if $\mathcal{O}$ is a $\mathbf{\Delta}^{m+n}$–$\infty$–operad. □

**Lemma A.75** *Suppose $\mathcal{M}$ is a $\mathbf{\Delta}^{n+m}$–uple $\infty$–category. Then there is a natural equivalence*

$$\mathrm{ALG}^{n,m}_{\mathcal{O}}(\mathcal{M}) \simeq \mathrm{FUN}^{\otimes,n,m}(\mathrm{Env}_n(\mathcal{O}), \mathcal{M})$$

*for all generalized $\mathbf{\Delta}^n$–$\infty$–operads $\mathcal{O}$. In particular, $\mathrm{ALG}^{n,m}_{\mathcal{O}}(\mathcal{M})$ is a $\mathbf{\Delta}^m$–uple $\infty$–category.*

**Proof** Using Lemma A.72, we have natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Opd}^{\mathbf{\Delta}^m,\mathrm{gen}}_{\infty}}(\mathcal{P}, \mathrm{ALG}^{n,m}_{\mathcal{O}}(\mathcal{M})) &\simeq \mathrm{Map}_{\mathrm{Opd}^{\mathbf{\Delta}^{m+n},\mathrm{gen}}_{\infty}}(\mathcal{P}\times\mathcal{O}, \mathcal{M}) \\
&\simeq \mathrm{Map}_{\mathrm{Upl}^{\mathbf{\Delta}^{n+m}}_{\infty}}(\mathrm{Env}_{n+m}(\mathcal{P}\times\mathcal{O}), \mathcal{M}) \\
&\simeq \mathrm{Map}_{\mathrm{Upl}^{\mathbf{\Delta}^{n+m}}_{\infty}}(\mathrm{Env}_m(\mathcal{P})\times\mathrm{Env}_n(\mathcal{O}), \mathcal{M}) \\
&\simeq \mathrm{Map}_{\mathrm{Upl}^{\mathbf{\Delta}^{m}}_{\infty}}(\mathrm{Env}_m(\mathcal{P}), \mathrm{FUN}^{\otimes,n,m}(\mathrm{Env}_n(\mathcal{O}), \mathcal{M})) \\
&\simeq \mathrm{Map}_{\mathrm{Opd}^{\mathbf{\Delta}^{m},\mathrm{gen}}_{\infty}}(\mathcal{P}, \mathrm{FUN}^{\otimes,n,m}(\mathrm{Env}_n(\mathcal{O}), \mathcal{M})).
\end{aligned}$$ □

If $\mathcal{C}^{\otimes}$ is a $\mathbf{\Delta}^{n+m}$–monoidal $\infty$–category, combining Lemmas A.74 and A.75 we see that $\mathrm{ALG}^{n,m}_{\mathcal{O}}(\mathcal{C})$ is a $\mathbf{\Delta}^m$–monoidal $\infty$–category for any generalized $\mathbf{\Delta}^n$–$\infty$–operad $\mathcal{O}$; the underlying $\infty$–category of this is $\mathrm{Alg}^{n}_{\mathcal{O}}(\mathcal{C})$. On the other hand, we saw in Corollary A.67 that $\mathrm{Alg}^{n}_{\mathcal{O}}(\mathcal{C})$ inherits an $\mathbb{E}_m$–monoidal structure from the lax monoidal functoriality of $\mathrm{Alg}^{n}_{\mathcal{O}}(-)$; let us denote the resulting $\mathbf{\Delta}^m$–monoidal $\infty$–category by $\mathrm{Alg}^{n,\otimes}_{\mathcal{O}}(\mathcal{C})$. We will now show that these two $\mathbb{E}_m$–monoidal structures agree:

**Proposition A.76** *Let $\mathcal{C}^{\otimes}$ be a $\mathbf{\Delta}^{n+m}$–monoidal $\infty$–category, $\mathcal{O}$ a generalized $\mathbf{\Delta}^n$–$\infty$–operad and $\mathcal{M}$ a $\mathbf{\Delta}^m$–uple $\infty$–category. Then we have a natural equivalence*

$$\mathrm{Map}_{\mathrm{Upl}^{\mathbf{\Delta}^m}_{\infty}}(\mathcal{M}, \mathrm{Alg}^{n,\otimes}_{\mathcal{O}}(\mathcal{C})) \simeq \mathrm{Map}_{\mathrm{Upl}^{\mathbf{\Delta}^{m+n}}_{\infty}}(\mathcal{M}\times\mathrm{Env}_n(\mathcal{O}), \mathcal{C}^{\otimes}).$$

**Proof** We may identify $\mathrm{Upl}^{\mathbf{\Delta}^m}_{\infty}$ with a full subcategory of the $\infty$–category of cocartesian fibrations over $\mathbf{\Delta}^{m,\mathrm{op}}$, which is equivalent to $\mathrm{Fun}(\mathbf{\Delta}^{m,\mathrm{op}}, \mathrm{Cat}_{\infty})$; under this

equivalence $\mathcal{M}$ corresponds to a functor $\mu\colon \mathbf{\Delta}^{m,\mathrm{op}} \to \mathrm{Cat}_\infty$. If $\gamma\colon \mathbf{\Delta}^{m,\mathrm{op}} \to \mathrm{Mon}_{\mathbf{\Delta}^n}^\infty$ is the $\mathbf{\Delta}^m$–monoid corresponding to $\mathcal{C}^\otimes$, then we have a natural equivalence

$$\begin{aligned}\mathrm{Map}_{\mathrm{Upl}_\infty^{\mathbf{\Delta}^m}}(\mathcal{M}, \mathrm{Alg}_{\mathcal{O}}^{n,\otimes}(\mathcal{C})) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{m,\mathrm{op}},\mathrm{Cat}_\infty)}(\mu, \mathrm{Alg}_{\mathcal{O}}^{n}(\gamma))\\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{m,\mathrm{op}},\mathrm{Cat}_\infty)}(\mu, \mathrm{Fun}^{\otimes,n}(\mathrm{Env}_n(\mathcal{O}), \gamma))\\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{m,\mathrm{op}},\mathrm{Cat}_{\infty/\mathbf{\Delta}^{n,\mathrm{op}}}^{\mathrm{cocart}})}(\mu \times \mathrm{Env}_n(\mathcal{O}), \gamma)\\ &\simeq \mathrm{Map}_{\mathrm{Upl}_\infty^{\mathbf{\Delta}^{n+m}}}(\mathcal{M} \times \mathrm{Env}_n(\mathcal{O}), \mathcal{C}^\otimes). \qquad \square\end{aligned}$$

Combining this with Lemma A.75, we get:

**Corollary A.77** *Let $\mathcal{C}^\otimes$ be a $\mathbf{\Delta}^{n+m}$–monoidal $\infty$–category and $\mathcal{O}$ a generalized $\mathbf{\Delta}^n$–$\infty$–operad. Then the $\mathbb{E}_m$–monoidal $\infty$–categories $\mathrm{ALG}_{\mathcal{O}}^{n,m}(\mathcal{C})$ and $\mathrm{Alg}_{\mathcal{O}}^{n,\otimes}(\mathcal{C})$ are naturally equivalent.* $\square$

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