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Characterizing the unit ball by its projective automorphism group

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In this paper we study the projective automorphism group of domains in real, complex, and quaternionic projective space and present two new characterizations of the unit ball in terms of the size of the automorphism group and the regularity of the boundary.

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1 Introduction

Suppose $\mathbb{K}$ is either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$. View $\mathbb{K}^{d+1}$ as a right $\mathbb{K}$–module, and consider the action of $\mathrm{GL}_{d+1}(\mathbb{K})$ on the left. Let $\mathbb{P}(\mathbb{K}^{d+1})$ be the space of $\mathbb{K}$–lines in $\mathbb{K}^{d+1}$ (parametrized on the right). Then $\mathrm{PGL}_{d+1}(\mathbb{K})$ acts on $\mathbb{P}(\mathbb{K}^{d+1})$ by diffeomorphisms.

Given an open set $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$, the *projective automorphism group* is defined to be

$$\mathrm{Aut}(\Omega) = \{\varphi \in \mathrm{PGL}_{d+1}(\mathbb{K}) : \varphi\Omega = \Omega\}.$$

For instance, consider the set

$$\mathcal{B} = \left\{ [1 : z_1 : \cdots : z_d] \in \mathbb{P}(\mathbb{K}^{d+1}) : \sum_{i=1}^{d} |z_i|^2 < 1 \right\} \subset \mathbb{P}(\mathbb{K}^{d+1}).$$

Then $\mathrm{Aut}(\mathcal{B})$ coincides with the image of $U_{\mathbb{K}}(1,d)$ in $\mathrm{PGL}_{d+1}(\mathbb{K})$, and $\mathcal{B}$ is a bounded symmetric domain in the following sense: $\mathcal{B}$ is bounded in an affine chart of $\mathbb{P}(\mathbb{K}^{d+1})$, and $\mathrm{Aut}(\mathcal{B})$ is a simple Lie group which acts transitively on $\mathcal{B}$. Moreover, there is a natural $\mathrm{Aut}(\mathcal{B})$–invariant Riemannian metric g which makes $(\mathcal{B}, g)$ isometric to $\mathbb{K}$–hyperbolic d–space; see, for instance, Mostow [21, Chapter 19].

The main goal of this paper is to provide new characterizations of this symmetric domain. These characterizations will be in terms of the regularity of the boundary and the size of the automorphism group. Notice that $\partial\mathcal{B}$ is real analytic, and $\mathrm{Aut}(\mathcal{B})$ acts transitively on $\mathcal{B}$.

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We will measure the size of $\mathrm{Aut}(\Omega)$ using the *limit set* $\mathcal{L}(\Omega) \subset \partial\Omega$, which is the set of points $x \in \partial\Omega$ such that there exists some $p \in \Omega$ and a sequence $\varphi_n \in \mathrm{Aut}(\Omega)$ with $\varphi_n p \to x$. Since $\mathrm{Aut}(\mathcal{B})$ acts transitively on $\mathcal{B}$, clearly $\mathcal{L}(\mathcal{B}) = \partial\mathcal{B}$.

We will also restrict our attention to a particular class of domains:

Definition 1.1 We call an open set $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ a *proper domain* if Ω is connected and bounded in some affine chart.

We will show that $\mathcal{B}$ is the only proper domain in complex or quaternionic projective space whose boundary is C^1 and whose limit set contains a spanning set.

Theorem 1.2 (see Section 11) *Suppose $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$, and $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary. If there exist $x_1, \dots, x_{d+1} \in \mathcal{L}(\Omega)$ such that*

$$x_1 + \cdots + x_{d+1} = \mathbb{K}^{d+1}$$

(as $\mathbb{K}$–lines), then Ω is projectively isomorphic to $\mathcal{B}$.

Remark 1.3 Theorem 1.2 fails completely in real projective space. In particular, there are many examples of proper domains $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ which have C^1 boundary, $\mathcal{L}(\Omega) = \partial\Omega$, and $\mathrm{Aut}(\Omega)$ is a discrete group which acts properly and cocompactly on Ω. In some of these examples, $\mathrm{Aut}(\Omega)$ is isomorphic to a lattice in $\mathrm{Isom}(\mathbb{H}^d_{\mathbb{R}})$ (see Benoist [2, Section 1.3] for $d > 2$ and Goldman [11] for $d = 2$), while in other examples, $\mathrm{Aut}(\Omega)$ is not quasi-isometric to any symmetric space (see Kapovich [16]). More background on these examples of "divisible sets" in real projective space can be found in the survey papers by Benoist [4], Goldman [12], Marquis [20], and Quint [22].

A subgroup $H \leq \mathrm{Aut}(\Omega)$ acts *cocompactly* on Ω if there exists a compact set $K \subset \Omega$ such that $H \cdot K = \Omega$. When $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω it is straightforward to show that $\mathcal{L}(\Omega) = \partial\Omega$; see Corollary 4.7 below. So Theorem 1.2 implies the following:

Corollary 1.4 *Suppose $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$, and $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary. If $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω, then Ω is projectively isomorphic to $\mathcal{B}$.*

We will show the action of $\mathrm{Aut}(\Omega)$ is proper whenever Ω is a proper domain; see Proposition 4.4 below. In particular, if $\mathrm{Aut}(\Omega)$ is noncompact, then $\mathcal{L}(\Omega) \neq \varnothing$. So Theorem 1.2 also implies:

Corollary 1.5 *Suppose $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$, and $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary. If $\mathrm{Aut}(\Omega)$ is noncompact and the group*

$$G = \{g \in \mathrm{GL}_{d+1}(\mathbb{K}) : [g] \in \mathrm{Aut}(\Omega)\}$$

acts irreducibly on $\mathbb{K}^{d+1}$, then Ω is projectively isomorphic to $\mathcal{B}$.

If $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain, $\partial\Omega$ is a C^1 hypersurface, and for $x \in \partial\Omega$ we define $T_x^{\mathbb{K}}\partial\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ to be the $\mathbb{K}$–hyperplane tangent to $\partial\Omega$ at x. It is reasonable to refer to the set $T_x^{\mathbb{K}}\partial\Omega \cap \partial\Omega$ as the *closed $\mathbb{K}$–face of x* in $\partial\Omega$. Our next result shows that $\mathcal{B}$ is the only set in projective space with C^2 boundary and whose limit set intersects two different closed $\mathbb{K}$–faces.

Theorem 1.6 (see Section 9) *Suppose $\mathbb{K}$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^2 boundary. If there exist $x, y \in \mathcal{L}(\Omega)$ with $T_x^{\mathbb{K}}\partial\Omega \neq T_y^{\mathbb{K}}\partial\Omega$, then Ω is projectively isomorphic to $\mathcal{B}$.*

Remark 1.7 When $d \geq 2$, Theorem 1.6 fails for domains with $C^{1,1}$ boundary (see Section 12) and in the holomorphic setting (see Example 2.10).

Using Proposition 6.1 below, Theorem 1.6 implies:

Corollary 1.8 *Suppose $\mathbb{K}$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^2 boundary. If there exists an element $\varphi \in \mathrm{GL}_{d+1}(\mathbb{K})$ which has eigenvalues of different absolute value, and $[\varphi] \in \mathrm{Aut}(\Omega)$, then Ω is projectively isomorphic to $\mathcal{B}$.*

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2 Some prior results

There is a long history of rigidity results involving the structure of the boundary and the size of $\mathrm{Aut}(\Omega)$. Many previous results make at least one of the following assumptions:

(1) $\mathrm{Aut}(\Omega)$ or a discrete subgroup acts cocompactly on Ω,

(2) $\partial\Omega$ is C^2 and satisfies some curvature condition (for instance, strong convexity or strong pseudoconvexity), or

(3) Ω is convex.

In this brief section, we will survey some of these results in the real projective, the complex projective, and the holomorphic settings.

The real projective setting

As mentioned in Remark 1.3, there are many proper domains $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ with C^1 boundary which admit a cocompact action by $\mathrm{Aut}(\Omega)$. However, rigidity appears if one assumes higher regularity. For instance, Benoist proved the following characterization of the unit ball in real projective space:

Theorem 2.1 [3, Theorem 1.3] *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a proper convex domain, and that there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ which acts cocompactly on Ω. If $\partial\Omega$ is $C^{1,\alpha}$ for all $\alpha \in [0,1)$, then Ω is projectively isomorphic to $\mathcal{B}$.*

Recall that an open bounded set $\Omega \subset \mathbb{R}^d$ is called *strongly convex* if $\Omega = \{x \in \mathbb{R}^d : r(x) < 0\}$ for some C^2 function $r\colon \mathbb{R}^d \to \mathbb{R}$ with $\nabla r \neq 0$ near $\partial\Omega$, and if

$$\mathrm{Hess}_x(r)(v,v) > 0$$

for all $x \in \partial\Omega$ and $v \in T_x\partial\Omega$. A proper domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is called *strongly convex* if it is a strongly convex set in some (hence any) affine chart which contains it as a bounded set. With this terminology, Socié-Méthou proved the following rigidity result:

Theorem 2.2 [24] *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a strongly convex open set. If* $\mathrm{Aut}(\Omega)$ *is noncompact, then Ω is projectively isomorphic to $\mathcal{B}$.*

Remark 2.3 Colbois and Verovic [7] gave an alternative proof with the additional assumption that $\partial\Omega$ is C^3. Later, Jo [15] and Yi [27] proved that it is enough to assume that $\mathcal{L}(\Omega)$ contains a point x where $\partial\Omega$ is strongly convex in a neighborhood of x.

The complex projective setting

The complex projective setting is more rigid than the real projective setting, especially when one assumes that there is a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ which acts cocompactly on Ω.

In $\mathbb{P}(\mathbb{C}^2)$, there do exist nonsymmetric proper domains which admit a cocompact action by a discrete group in $\mathrm{Aut}(\Omega)$. However, if $\partial\Omega$ has very weak regularity, then a result of Bowen implies that Ω must be a symmetric domain:

Theorem 2.4 [5] *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^2)$ is a proper domain and $\partial\Omega$ is a Jordan curve with Hausdorff dimension one. If there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ which acts cocompactly on Ω, then Ω is projectively isomorphic to $\mathcal{B}$.*

In $\mathbb{P}(\mathbb{C}^3)$, the cocompact case is even more rigid, and recent work of Cano and Seade implies the following:

Theorem 2.5 [6] *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^3)$ is a proper domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group which acts cocompactly on Ω. Then Ω is projectively isomorphic to $\mathcal{B}$.*

It is worth noting that Cano and Seade's proof relies on Kobayashi and Ochiai's [18] classification of compact complex surfaces with a projective structure.

In higher dimensions, we proved the following weaker version of Corollary 1.4:

Theorem 2.6 [28] *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a proper $\mathbb{C}$–convex domain and there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ which acts cocompactly on Ω. If $\partial\Omega$ is C^1, then Ω is projectively isomorphic to $\mathcal{B}$.*

Remark 2.7 An open set $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is called $\mathbb{C}$–convex if its intersection with any complex projective line is simply connected. Surprisingly, this weak form of convexity has strong analytic implications. See [1; 13] for more details.

The holomorphic setting

There is also a long history of rigidity results involving bounded domains $\Omega \subset \mathbb{C}^d$ and their biholomorphic automorphism group $\mathrm{Aut}_{\mathrm{hol}}(\Omega)$. We will only mention a few results and refer the reader to the survey articles [14] and [19] for more details.

The most classical is the well known characterization of the unit ball due to Rosay [23] and Wong [26]. Recall that a bounded domain $\Omega \subset \mathbb{C}^d$ is called *strongly pseudoconvex* if Ω has C^2 boundary and the Levi-form at each point in the boundary is positive definite.

Theorem 2.8 (Wong–Rosay ball theorem) *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain. If $\mathrm{Aut}_{\mathrm{hol}}(\Omega)$ is noncompact, then Ω is biholomorphic to the unit ball.*

In fact, it is enough to assume that the limit set contains a point x where $\partial\Omega$ is strongly pseudoconvex in a neighborhood of x; see [23]. Thus one obtains the following characterization of the unit ball:

Corollary 2.9 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain with C^2 boundary. If $\mathrm{Aut}_{\mathrm{hol}}(\Omega)$ acts cocompactly on Ω, then Ω is biholomorphic to the unit ball.*

We should also observe the direct analogue of Theorem 1.6 fails in the holomorphic setting in particular:

Example 2.10 Let $\Omega_0 = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_1) > |z_2|^4\}$. Then for $t \in \mathbb{R}$, $\mathrm{Aut}(\Omega_0)$ contains the biholomorphic map

$$a_t \cdot (z_1, z_2) \to (e^{4t} z_1, e^t z_2).$$

Moreover, Ω_0 is biholomorphic to $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^4 < 1\}$ via the map

$$F\colon \Omega_0 \to \Omega, \quad F(z_1, z_2) = \left(\frac{z_1 - i}{z_1 + i}, \frac{z_2}{2(z_1 + i)^{1/2}} \right).$$

Then $b_t = F \circ a_t \circ F^{-1} \in \mathrm{Aut}(\Omega)$, and so $(1,0), (-1,0) \in \mathcal{L}(\Omega)$. Finally, Ω is not biholomorphic to the unit ball [25].

However, we recently proved this variant of Theorem 1.6 in the complex setting:

Theorem 2.11 [29] *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex open set with C^∞ boundary. If there exist $x, y \in \mathcal{L}(\Omega)$ with $T_x^{\mathbb{C}} \partial\Omega \neq T_y^{\mathbb{C}} \partial\Omega$, then Ω is biholomorphic to a domain of the form*

$$\{(z_1, \dots, z_d) \in \mathbb{C}^d : |z_1|^2 + p(z_2, \dots, z_d) < 1\},$$

where p is a polynomial.

Remark 2.12 In [29], we show that p is a "weighted homogeneous polynomial."

Finally we should mention a remarkable theorem due to Frankel:

Theorem 2.13 [10] *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex open set and there exists a discrete group $\Gamma \leq \mathrm{Aut}_{\mathrm{hol}}(\Omega)$ which acts properly discontinuously, freely, and cocompactly on Ω. Then Ω is a bounded symmetric domain.*

3 Preliminaries

Notation

Given some object o we will let $[o]$ be the projective equivalence class of o: for instance, if $v \in \mathbb{K}^{d+1} \setminus \{0\}$, let $[v]$ denote the image of v in $\mathbb{P}(\mathbb{K}^{d+1})$, and if $\phi \in \mathrm{GL}_{d+1}(\mathbb{K})$, let $[\phi]$ denote the image of ϕ in $\mathrm{PGL}_{d+1}(\mathbb{K})$.

For $v, w \in \mathbb{K}^{d+1}$, we define the standard inner product

$$\langle v, w \rangle = {}^t\overline{v} w,$$

where ${}^t\overline{v}$ is the conjugate transpose of v. We let $\|v\| = \sqrt{\langle v, v \rangle}$ be the norm induced by this inner product, and for $T \in \mathrm{End}(\mathbb{K}^{d+1})$ let $\|T\|$ be the associated operator norm. If $\mathbb{K}^{(d+1)*}$ is the $\mathbb{K}$–module of $\mathbb{K}$–linear functions $f\colon \mathbb{K}^{d+1} \to \mathbb{K}$, then define

$$\|f\| = \sup\{|f(z)| : z \in \mathbb{K}^{d+1},\ \|z\| = 1\}.$$

If X is a manifold, we say an open subset $\Omega \subset X$ has C^k boundary if $\partial\Omega$ is a C^k embedded codimension-one submanifold in X.

Quaternions

In this paper, we identify $\mathbb{H}^d$ with the space of $d \times 1$ matrices with entries in $\mathbb{H}$, and we let $\mathbb{H}$ act on $\mathbb{H}^d$ as follows:

$$\begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} \cdot \alpha = \begin{pmatrix} z_1\alpha \\ \vdots \\ z_d\alpha \end{pmatrix}.$$

We then define $\mathrm{GL}_d(\mathbb{H})$ to be the invertible $\mathbb{R}$–linear transformations of $\mathbb{H}^d$ which commute with the above action of $\mathbb{H}$. If $M_d(\mathbb{H})$ is the space of $d \times d$ matrices with entries in $\mathbb{H}$, then we can identify

$$\mathrm{GL}_d(\mathbb{H}) = \mathrm{GL}_{2d}(\mathbb{C}) \cap M_d(\mathbb{H}).$$

Since the quaternions are noncommutative, this identification requires that the scalar multiplication acts on the right while $M_d(\mathbb{H})$ acts on the left.

Now we can define the quaternionic projective space $\mathbb{P}(\mathbb{H}^{d+1})$ to be the quotient

$$\mathbb{P}(\mathbb{H}^{d+1}) = \big\{z \in \mathbb{H}^{d+1} \setminus \{0\}\big\}/\sim,$$

where $z_1 \sim z_2$ if and only if $z_1 = z_2\alpha$ for some nonzero $\alpha \in \mathbb{H}$. Then $\mathrm{GL}_{d+1}(\mathbb{H})$ acts on $\mathbb{P}(\mathbb{H}^{d+1})$, and an element $\varphi \in \mathrm{GL}_{d+1}(\mathbb{H})$ acts trivially if and only if

$$\varphi = \lambda\, \mathrm{Id}$$

for some $\lambda \in \mathbb{R}^*$. So the group

$$\mathrm{PGL}_{d+1}(\mathbb{H}^{d+1}) := \mathrm{GL}_{d+1}(\mathbb{H})/\mathbb{R}^*\, \mathrm{Id}$$

acts faithfully on $\mathbb{P}(\mathbb{H}^{d+1})$.

4 An intrinsic metric and applications

Let $\mathbb{K}^{(d+1)*}$ denote the $\mathbb{K}$–module of $\mathbb{K}$–linear functions $f\colon \mathbb{K}^{d+1} \to \mathbb{K}$; that is, $f(vz) = f(v)z$ for all $v \in \mathbb{K}^{d+1}$ and $z \in \mathbb{K}$. Then let $\mathbb{P}(\mathbb{K}^{(d+1)*})$ be the projective space of lines in $\mathbb{K}^{(d+1)*}$ (parametrized on the right). The *dual set* of $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is the set

$$\Omega^* = \{f \in \mathbb{P}(\mathbb{K}^{(d+1)*}) : \ker f \cap \Omega = \varnothing\}.$$

Given $\varphi \in \mathrm{PGL}_{d+1}(\mathbb{K})$, let ${}^*\varphi \in \mathrm{PGL}(\mathbb{K}^{(d+1)*})$ be the transformation ${}^*\varphi(f) = f \circ \varphi$. We begin by making some observations:

Observation 4.1 (1) If Ω is open, then Ω^* is compact.

(2) If Ω is bounded in an affine chart, then Ω^* has nonempty interior.

(3) If $\varphi \in \mathrm{Aut}(\Omega)$, then ${}^*\varphi \in \mathrm{Aut}(\Omega^*)$.

Now, using the dual, we can define a metric which generalizes the classical Hilbert metric in real projective geometry. For an open set $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$, define the function $C_\Omega\colon \Omega \times \Omega \to \mathbb{R}$ by

$$C_\Omega(p,q) = \sup_{f,g \in \Omega^*} \tfrac{1}{2} \log \left| \frac{f(p)g(q)}{f(q)g(p)} \right|.$$

Since $(\varphi\Omega)^* = {}^*\varphi\Omega^*$, we see that

$$C_\Omega(p,q) = C_{\varphi\Omega}(\varphi p, \varphi q)$$

for all $\varphi \in \mathrm{PGL}_{d+1}(\mathbb{K})$ and $p, q \in \Omega$. Thus C_Ω will be $\mathrm{Aut}(\Omega)$–invariant.

When $\mathbb{K} = \mathbb{R}$ and $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a convex subset, this function C_Ω coincides with the classical Hilbert metric; see, for instance, [17]. In the setting where $\mathbb{K} = \mathbb{C}$ and $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a linearly convex set, this function was introduced by Dubois [9]. For such domains, Dubois proved that C_Ω is a complete metric. Additional properties of the metric C_Ω for linearly convex sets were established in [28]. Finally, we recently constructed an analogue of the metric C_Ω for certain domains in real flag manifolds [30].

Next we will show that C_Ω is a metric generating the standard topology whenever the domain is proper. However, without convexity assumptions, C_Ω may not be a complete metric.

Proposition 4.2 *Suppose $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain. Then C_Ω is an $\mathrm{Aut}(\Omega)$–invariant metric on Ω which generates the standard topology.*

It will be helpful to observe that C_Ω on the unit ball is actually the symmetric metric:

Lemma 4.3 *If*

$$\mathcal{B} = \left\{ [1 : z_1 : \cdots : z_d] \in \mathbb{P}(\mathbb{K}^{d+1}) : \sum_{i=1}^{d} |z_i|^2 < 1 \right\},$$

then $(\mathcal{B}, C_{\mathcal{B}})$ coincides with the model of $\mathbb{K}$–hyperbolic d–space described in [21, Chapter 19]*. In particular, $C_{\mathcal{B}}$ is a complete metric on $\mathcal{B}$ which generates the standard topology.*

Proof Let $d_{\mathbb{K}}$ be the distance on $\mathcal{B}$ described in [21, Chapter 19] and for $t \in (-1, 1)$ let

$$x_t = [1 : t : 0 : \cdots : 0].$$

Now for any $p, q \in \Omega$, there exists $\varphi \in \mathrm{SU}_{\mathbb{K}}(1, d)$ so that $\varphi p = x_0$ and $\varphi_q = x_t$ for some $t \in [0, 1)$. Then since $C_{\mathcal{B}}$ and $d_{\mathbb{K}}$ are $\mathrm{SU}_{\mathbb{K}}(1, d)$–invariant, it is enough to show

$$C_{\mathcal{B}}(x_0, x_t) = d_{\mathbb{K}}(x_0, x_t)$$

when $t \in [0, 1)$. Moreover, when $t \in [0, 1)$, we have

$$d_{\mathbb{K}}(x_0, x_t) = \cosh^{-1} \frac{1}{\sqrt{1 - t^2}} = \tfrac{1}{2} \log \frac{1 + t}{1 - t}$$

by [21, Equation 19.4]. Using the standard inner product, we can identify $\mathbb{K}^{(d+1)*}$ with $\mathbb{K}^{d+1}$ and then view $\mathcal{B}^*$ as a subset of $\mathbb{P}(\mathbb{K}^{d+1})$. Thus

$$\mathcal{B}^* = \left\{ [1 : f_1 : \cdots : f_d] : \sum_{i=1}^{d} |f_i|^2 \leq 1 \right\}.$$

Then for $t \in [0, 1)$, we have

$$C_{\mathcal{B}}(x_0, x_t) = \sup_{f, g \in \mathcal{B}^*} \tfrac{1}{2} \log \frac{1 - t f_1}{1 - t g_1}.$$

This is clearly maximized when $f = [1 : -1 : 0 : \cdots : 0]$ and $g = [1 : 1 : 0 : \cdots : 0]$. So

$$C_{\mathcal{B}}(x_0, x_t) = \tfrac{1}{2} \log \frac{1 + t}{1 - t}$$

when $t \in [0, 1)$, and thus $C_{\mathcal{B}} = d_{\mathbb{K}}$. □

Proof of Proposition 4.2 Suppose that $p, q, r \in \Omega$. Since Ω^* is compact, there exist $f, g \in \Omega^*$ such that

$$C_{\Omega}(p, q) = \tfrac{1}{2} \log \left| \frac{f(p) g(q)}{f(q) g(p)} \right|.$$

Then for $r \in \Omega$,

$$C_\Omega(p,q) = \tfrac{1}{2}\log\left|\frac{f(p)g(q)}{f(q)g(p)}\right| = \tfrac{1}{2}\log\left|\frac{f(p)g(r)}{f(r)g(p)}\right| + \tfrac{1}{2}\log\left|\frac{f(r)g(q)}{f(q)g(r)}\right|$$
$$\le C_\Omega(p,r) + C_\Omega(r,q).$$

So C_Ω satisfies the triangle inequality.

Now fix an affine chart $\mathbb{K}^d$ which contains Ω as a bounded set. Then after rescaling Ω, we may assume that

$$\Omega \subset \mathcal{B} := \{z \in \mathbb{K}^d : \|z\| < 1\}.$$

By the above lemma, $C_{\mathcal{B}}$ is a complete metric which generates the standard topology on $\mathcal{B}$. Moreover, $\mathcal{B}^* \subset \Omega^*$, and so $C_{\mathcal{B}} \le C_\Omega$ on Ω. Then for $p,q \in \Omega$ distinct, we have

$$0 < C_{\mathcal{B}}(p,q) \le C_\Omega(p,q).$$

Thus C_Ω is a metric.

Since Ω^* is compact, the function $C_\Omega \colon \Omega \times \Omega \to \mathbb{R}_{\ge 0}$ is continuous. Thus to show that C_Ω generates the standard topology, it is enough to show: for any $p \in \Omega$ and $U \subset \Omega$ an open neighborhood of p, there exists $\epsilon > 0$ such that

$$\{q \in \Omega : C_\Omega(p,q) < \epsilon\} \subset U.$$

But since $C_{\mathcal{B}}$ generates the standard topology on $\mathcal{B}$, there exists $\epsilon > 0$ such that

$$\{q \in \mathcal{B} : C_{\mathcal{B}}(p,q) < \epsilon\} \subset U.$$

But then

$$\{q \in \Omega : C_\Omega(p,q) < \epsilon\} \subset \{q \in B_R : C_{\mathcal{B}}(p,q) < \epsilon\} \subset U$$

since $C_{\mathcal{B}} \le C_\Omega$ on Ω. So C_Ω generates the standard topology. □

The automorphism group

Proposition 4.4 *Suppose $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain. Then* $\mathrm{Aut}(\Omega) \le \mathrm{PGL}_{d+1}(\mathbb{K})$ *is a closed subgroup and acts properly on Ω.*

Proof We first show that $\mathrm{Aut}(\Omega)$ is a closed subgroup of $\mathrm{PGL}_{d+1}(\mathbb{K})$. Suppose that $\varphi_n \in \mathrm{Aut}(\Omega)$ and $\varphi_n \to \varphi$ in $\mathrm{PGL}_{d+1}(\mathbb{K})$. Let $d_{\mathbb{P}}$ be a distance on $\mathbb{P}(\mathbb{K}^{d+1})$ induced by a Riemannian metric. Then since $\varphi_n \to \varphi$, there exists some $M \ge 1$ so that

$$\frac{1}{M} d_{\mathbb{P}}(p,q) \le d_{\mathbb{P}}(\varphi_n p, \varphi_n q) \le M d_{\mathbb{P}}(p,q)$$

for all $p, q \in \mathbb{P}(\mathbb{K}^{d+1})$ and $n \in \mathbb{N}$. Next define $\delta_\Omega \colon \Omega \to \mathbb{R}_{>0}$ by

$$\delta_\Omega(p) = \inf\{d_{\mathbb{P}}(p, x) : x \in \mathbb{P}(\mathbb{K}^{d+1}) \setminus \Omega\}.$$

Then

$$\frac{1}{M}\delta_\Omega(p) \le \delta_\Omega(\varphi p) \le M\delta_\Omega(p)$$

for $p \in \Omega$. So $\varphi(\Omega) \subset \Omega$. Since $\varphi_n^{-1} \to \varphi^{-1}$, the same argument shows that $\varphi^{-1}(\Omega) \subset \Omega$. Thus $\varphi(\Omega) = \Omega$, and so $\varphi \in \mathrm{Aut}(\Omega)$.

We now show that $\mathrm{Aut}(\Omega)$ acts properly. This argument requires some care because C_Ω may not be a complete metric. Fix a compact set $K \subset \Omega$; we claim that

$$\{\varphi \in \mathrm{Aut}(\Omega) : \varphi K \cap K \neq \varnothing\}$$

is compact. So suppose that $\varphi_n k_n \in K$ for some sequence $\varphi_n \in \mathrm{Aut}(\Omega)$ and $k_n \in K$. By passing to a subsequence we can suppose that $k_n \to k \in K$. Now since C_Ω is a locally compact metric (it generates the standard topology) and $K \subset \Omega$ is compact, there exists some $\delta > 0$ such that the set

$$K_1 = \{q \in \Omega : C_\Omega(K, q) \le 2\delta\}$$

is compact. Next let

$$K_2 = \{q \in \Omega : C_\Omega(k, q) \le \delta\}.$$

Then for large n, we have $\varphi_n(K_2) \subset K_1$. Since φ_n preserves the metric C_Ω and $(K_1, C_\Omega|_{K_1})$ is a complete metric space, we can pass to a subsequence and assume that $\varphi_n|_{K_2}$ converges uniformly to a function $f \colon K_2 \to K_1$. Moreover,

$$C_\Omega(f(p_1), f(p_2)) = \lim_{n\to\infty} C_\Omega(\varphi_n p_1, \varphi_n p_2) = C_\Omega(p_1, p_2)$$

for all $p_1, p_2 \in \Omega$. Since C_Ω is a metric, f is injective. Next pick representatives $\hat{\varphi}_n \in \mathrm{GL}_{d+1}(\mathbb{K})$ of φ_n so that $\|\hat{\varphi}_n\| = 1$. By passing to a subsequence, we may assume that $\hat{\varphi}_n \to \Phi$ in $\mathrm{End}(\mathbb{K}^{d+1})$. Moreover, if $p \in K_2 \setminus \ker \Phi$, then

$$f(p) = \lim_{n\to\infty} \varphi_n p = \Phi(p).$$

Since K_2 has nonempty interior and f is injective, this implies that Φ induces an injective map $\mathbb{P}(\mathbb{K}^{d+1}) \to \mathbb{P}(\mathbb{K}^{d+1})$. Hence $\Phi \in \mathrm{GL}_{d+1}(\mathbb{K})$. Thus $\varphi_n \to [\Phi]$ in $\mathrm{PGL}_{d+1}(\mathbb{K})$, and since $\mathrm{Aut}(\Omega)$ is closed, we see that $[\Phi] \in \mathrm{Aut}(\Omega)$. □

The asymptotic geometry of the intrinsic metric

Proposition 4.5 *Suppose $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain, $p_n, q_n \subset \Omega$ are sequences such that $p_n \to x \in \partial\Omega$ and $q_n \to y \in \partial\Omega$, and*

$$\lim_{n\to\infty} C_\Omega(p_n, q_n) < \infty.$$

Then

$$y \in \bigcap \{\ker f : f \in \Omega^*, f(x) = 0\}.$$

Proof Suppose $f \in \Omega^*$ and $f(x) = 0$. Since Ω^* has nonempty interior, there exists $g \in \Omega^*$ so that $g(x) \neq 0$ and $g(y) \neq 0$. Then

$$C_\Omega(p_n, q_n) \geq \tfrac{1}{2} \log \left| \frac{f(q_n) g(p_n)}{f(p_n) g(q_n)} \right|.$$

Let $\hat{p}_n, \hat{q}_n, \hat{x}, \hat{y} \in \mathbb{K}^{d+1}$ and $\hat{f}, \hat{g} \in \mathbb{K}^{(d+1)*}$ be representatives of $p_n, q_n, x, y \in \mathbb{P}(\mathbb{K}^{d+1})$ and $f, g \in \mathbb{P}(\mathbb{K}^{(d+1)*})$, respectively, normalized so that

$$\|\hat{f}\| = \|\hat{g}\| = \|\hat{p}_n\| = \|\hat{q}_n\| = \|\hat{x}\| = \|\hat{y}\| = 1.$$

Then

$$C_\Omega(p_n, q_n) \geq \tfrac{1}{2} \log \left| \frac{\hat{f}(\hat{q}_n)}{\hat{f}(\hat{p}_n)} \right| + \tfrac{1}{2} \log \left| \frac{\hat{g}(\hat{p}_n)}{\hat{g}(\hat{q}_n)} \right|.$$

Since $f(x) = 0$, we see that $\hat{f}(\hat{p}_n) \to 0$. Since $g(x) \neq 0$ and $g(y) \neq 0$, we see that

$$\log \left| \frac{\hat{g}(\hat{p}_n)}{\hat{g}(\hat{q}_n)} \right|$$

is bounded from above and below. Thus $\hat{f}(\hat{q}_n) \to 0$, and so $y \in \ker f$. □

Proposition 4.6 *Suppose $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain, and $p_n, q_n \subset \Omega$ are sequences such that $p_n \to x \in \overline{\Omega}$. If*

$$\lim_{n \to \infty} C_\Omega(p_n, q_n) = 0,$$

then $q_n \to x$.

Proof Fix an affine chart $\mathbb{K}^d$ which contains Ω as a bounded set. Then after scaling, we may assume that

$$\overline{\Omega} \subset \mathcal{B} = \{z \in \mathbb{C}^d : \|z\| < 1\}.$$

By Lemma 4.3, $C_{\mathcal{B}}$ is a complete metric which generates the standard topology on $\mathcal{B}$. Moreover, $\mathcal{B}^* \subset \Omega^*$, and so $C_{\mathcal{B}} \leq C_\Omega$ on Ω. Then

$$\lim_{n \to \infty} C_{\mathcal{B}}(p_n, q_n) = 0,$$

and so $q_n \to x$. □

Corollary 4.7 *Suppose $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain, and* $\mathrm{Aut}(\Omega)$ *acts cocompactly on Ω. Then $\mathcal{L}(\Omega) = \partial\Omega$.*

Proof Fix $x \in \partial\Omega$ and a sequence $p_n \in \Omega$ so that $p_n \to x$. Now there exists a compact set $K \subset \Omega$ and $\varphi_n \in \mathrm{Aut}(\Omega)$ so that $\varphi_n p_n \in K$. We can pass to a subsequence so that $\varphi_n p_n \to k \in K$. Then

$$\lim_{n\to\infty} C_\Omega(p_n, \varphi_n^{-1}k) = \lim_{n\to\infty} C_\Omega(\varphi_n p_n, k) = 0,$$

and so $\varphi_n^{-1}k \to x$ by the previous proposition. So $x \in \mathcal{L}(\Omega)$. $\square$

5 Limits of automorphisms

We begin this section by introducing a natural quotient of $\mathrm{End}(\mathbb{K}^{d+1}) \setminus \{0\}$. When $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, define

$$\mathrm{PEnd}(\mathbb{K}^{d+1}) := \mathbb{P}(\mathrm{End}(\mathbb{K}^{d+1})).$$

That is, $\mathrm{PEnd}(\mathbb{K}^{d+1})$ is the standard projective space associated to the $\mathbb{K}$–vector space $\mathrm{End}(\mathbb{K}^{d+1})$. When $\mathbb{K} = \mathbb{H}$, let

$$\mathrm{PEnd}(\mathbb{H}^{d+1}) := \big(\mathrm{End}(\mathbb{H}^{d+1}) \setminus \{0\}\big) / \sim,$$

where $T_1 \sim T_2$ if and only if $T_1 = \lambda T_2$ for some $\lambda \in \mathbb{R}^*$. Then $\mathrm{PGL}_{d+1}(\mathbb{K})$ embeds into $\mathrm{PEnd}(\mathbb{K}^{d+1})$, and $\mathrm{PEnd}(\mathbb{K}^{d+1})$ is compact.

Proposition 5.1 *Suppose $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary, and $\varphi_n \in \mathrm{Aut}(\Omega)$ with*

$$\varphi_n p \to x^+ \quad \text{and} \quad \varphi_n^{-1} p \to x^-,$$

where $p \in \Omega$ and $x^+, x^- \in \partial\Omega$. Then

(1) *$\varphi_n q \to x^+$ and $\varphi_n^{-1} q \to x^-$ for all $q \in \Omega$;*

(2) *there exists $f^\pm \in \Omega^*$ such that $\ker f^\pm = T^{\mathbb{K}}_{x^\pm}\partial\Omega$;*

(3) *if $\Phi \in \mathrm{PEnd}(\mathbb{K}^{d+1})$ is the element with $\mathrm{Im}(\Phi) = x^+$ and $\ker \Phi = T^{\mathbb{K}}_{x^-}\partial\Omega$, then $\varphi_n \to \Phi$ as elements of $\mathrm{PEnd}(\mathbb{K}^{d+1})$;*

(4) *if U is a neighborhood of $\overline{\Omega} \cap T^{\mathbb{K}}_{x^-}\partial\Omega$ and V is a neighborhood of x^+, then there exists $N \geq 0$ such that*

$$\varphi_n(\overline{\Omega} \setminus U) \subset V$$

for all $n \geq N$.

Proof Notice that $\varphi_n \to \infty$ in $\mathrm{PGL}_{d+1}(\mathbb{K})$ since $x^+, x^- \in \partial\Omega$ and $\mathrm{Aut}(\Omega) \leq \mathrm{PGL}_{d+1}(\mathbb{K})$ is closed.

We begin by proving part (3). Since $\mathrm{PEnd}(\mathbb{K}^{d+1})$ is compact, it is enough to show that any convergent subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ converges to Φ. By passing to a subsequence, we can assume that φ_n converges. Then let $\widehat{\varphi}_n \in \mathrm{GL}_{d+1}(\mathbb{K})$ be a representative of φ_n such that $\|\varphi_n\| = 1$ and $\widehat{\varphi}_n \to \Phi_+$ in $\mathrm{End}(\mathbb{K}^{d+1})$. We can write

$$\widehat{\varphi}_n = k_{n,1} \begin{pmatrix} a_{n,1} & & \\ & \ddots & \\ & & a_{n,d+1} \end{pmatrix} k_{n,2}$$

for some $k_{n,1}, k_{n,2} \in U_{\mathbb{K}}(d+1)$ and $1 = a_{n,1} \geq \cdots \geq a_{n,d+1}$. By passing to a subsequence, we can suppose that $k_{n,1} \to k_1$ and $k_{n,2} \to k_2$ in $U_{\mathbb{K}}(d+1)$, and the limits

$$\lambda_i^+ := \lim_{n\to\infty} a_{n,i} \quad \text{and} \quad \lambda_i^- := \lim_{n\to\infty} \frac{a_{n,d+1}}{a_{n,i}}$$

exist for $1 \leq i \leq d+1$. Then

$$\Phi_+ = \lim_{n\to\infty} \widehat{\varphi}_n = k_1 \begin{pmatrix} \lambda_1^+ & & \\ & \ddots & \\ & & \lambda_{d+1}^+ \end{pmatrix} k_2.$$

Now $\widehat{\varphi}_{n,-} := a_{n,d+1}\widehat{\varphi}_n^{-1}$ is a representative of φ_n^{-1} which converges in $\mathrm{End}(\mathbb{K}^{d+1})$ to

$$\Phi_- := k_2^{-1} \begin{pmatrix} \lambda_1^- & & \\ & \ddots & \\ & & \lambda_{d+1}^- \end{pmatrix} k_1^{-1}.$$

Next, identify $\mathbb{K}^{(d+1)*}$ with $\mathbb{K}^{d+1}$ using the standard inner product, and using this identification, view Ω^* as a subset of $\mathbb{P}(\mathbb{K}^{d+1})$. Then with this identification,

$$\{{}^t\overline{\varphi} : \varphi \in \mathrm{Aut}(\Omega)\} \leq \mathrm{Aut}(\Omega^*),$$

where ${}^t\overline{\varphi} \in \mathrm{PGL}_{d+1}(\mathbb{K})$ is the standard conjugate transpose of $\varphi \in \mathrm{PGL}_{d+1}(\mathbb{K})$.

Now $\widehat{\psi}_{n,+} := a_{n,d+1}({}^t\overline{\widehat{\varphi}}_n^{-1})$ is a representative of ${}^t\overline{\varphi}_n^{-1}$ that converges in $\mathrm{End}(\mathbb{K}^{d+1})$ to

$$\Psi_+ := k_1 \begin{pmatrix} \lambda_1^- & & \\ & \ddots & \\ & & \lambda_{d+1}^- \end{pmatrix} k_2,$$

and $\widehat{\psi}_{n,-} := {}^t\overline{\widehat{\varphi}_n}$ is a representative of ${}^t\overline{\varphi}_n$ which converges in $\mathrm{End}(\mathbb{K}^{d+1})$ to

$$\Psi_- := k_2^{-1} \begin{pmatrix} \lambda_1^+ & & \\ & \ddots & \\ & & \lambda_{d+1}^+ \end{pmatrix} k_1^{-1}.$$

Next let $m = \max\{j : \lambda_j^+ \neq 0\}$ and $M = \min\{j : \lambda_j^- \neq 0\}$. Then $m < M$ because $\varphi_n \to \infty$ in $\mathrm{PGL}_{d+1}(\mathbb{K})$.

Next let $e_1, \dots, e_{d+1}$ be the standard basis of $\mathbb{K}^{d+1}$. Then Φ_+ maps any open set of $\mathbb{P}(\mathbb{K}^{d+1}) \setminus \ker \Phi_+$ onto an open set of $k_1 \operatorname{Span}\{e_1, \dots, e_m\}$. Moreover,

$$\Phi_+(z) = \lim_{n\to\infty} \varphi_n(z)$$

for any $z \in \mathbb{P}(\mathbb{K}^{d+1}) \setminus \ker \Phi_+$. Since $\mathrm{Aut}(\Omega)$ acts properly on Ω, if $q \in \Omega$, then any limit point of $\varphi_n q$ is in $\partial\Omega$. Thus, since Ω is open, we see that $\partial\Omega$ contains an open subset of $k_1 \operatorname{Span}\{e_1, \dots, e_m\}$. The same argument applied to Ψ_+ implies that Ω^* contains an open subset of $k \operatorname{Span}\{e_M, \dots, e_{d+1}\}$. Now if

$$z_1 \in \partial\Omega \cap k_1 \operatorname{Span}\{e_1, \dots, e_m\} \quad \text{and} \quad z_2 \in \Omega^* \cap k_1 \operatorname{Span}\{e_M, \dots, e_{d+1}\},$$

then $\langle z_1, z_2 \rangle = 0$. So if $H \subset \mathbb{P}(\mathbb{K}^{d+1})$ is the $\mathbb{K}$–hyperplane defined by $H := \ker\langle \cdot, z_2\rangle$, then $z_1 \in H$, and since $z_2 \in \Omega^*$, we see that $H \cap \Omega = \varnothing$. Thus H is tangent to $\partial\Omega$ at z_1, and so $H = T_{z_1}^{\mathbb{K}} \partial\Omega$. Thus $\dim_{\mathbb{K}} k_1 \operatorname{Span}\{e_M, \dots, e_{d+1}\} = 1$, and so $M = d+1$. Applying this argument to Φ_- and Ψ_-, we see that $m = 1$.

Now, since $m = 1$ and $M = d+1$, we have $\operatorname{Im}\Phi_\pm = y^\pm$, $\operatorname{Im}\Psi_\pm = f^\pm$, and $\langle y^\pm, f^\pm \rangle = 0$ for some $y^\pm, f^\pm \in \mathbb{P}(\mathbb{K}^{d+1})$. By the arguments above, $y^\pm \in \partial\Omega$ and $f^\pm \in \Omega^*$. So $T_{y^\pm}^{\mathbb{K}} \partial\Omega = \ker\langle \cdot, f^\pm \rangle$. On the other hand, by construction, $\ker \Phi_\pm = \ker\langle \cdot, f^\mp \rangle$. So $\ker \Phi_\pm \cap \Omega = \varnothing$, and for all $q \in \Omega$, we have

$$y^\pm = \Phi_\pm(q) = \lim_{n\to\infty} \varphi_n^{\pm 1} q.$$

So $y^\pm = x^\pm$, $T_{x^\pm}^{\mathbb{C}} \partial\Omega = \ker\langle \cdot, f^\pm \rangle$, and $\ker \Phi_+ = T_{x^-}^{\mathbb{C}} \partial\Omega$. This proves part (3) and also parts (1) and (2).

Finally, part (4) follows directly from part (3). □

6 The structure of biproximal automorphisms

Suppose $\varphi \in \mathrm{PGL}_{d+1}(\mathbb{K})$, and $\widehat{\varphi} \in \mathrm{GL}_{d+1}(\mathbb{K})$ is a representative of φ. Let

$$\sigma_1(\widehat{\varphi}) \le \sigma_2(\widehat{\varphi}) \le \cdots \le \sigma_{d+1}(\widehat{\varphi})$$

be the absolute values of the eigenvalues (counted with multiplicity) of $\hat{\varphi}$. Then let

$$\sigma_i(\varphi) := \frac{\sigma_i(\hat{\varphi})}{\left(\prod_{j=1}^{d+1} \sigma_j(\hat{\varphi})\right)^{1/(d+1)}}.$$

Since we are considering absolute values, these numbers only depend on φ.

An element $\varphi \in \mathrm{PGL}_{d+1}(\mathbb{K})$ is called *proximal* if $\sigma_d(\varphi) < \sigma_{d+1}(\varphi)$, and it is called *biproximal* if φ and φ^{-1} are proximal. When φ is biproximal, let x_φ^+ and x_φ^- be the eigenlines in $\mathbb{P}(\mathbb{K}^{d+1})$ corresponding to $\sigma_{d+1}(\varphi)$ and $\sigma_1(\varphi)$.

Proposition 6.1 *Suppose $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary, $\varphi \in \mathrm{Aut}(\Omega)$, and $\sigma_{d+1}(\varphi) > \sigma_1(\varphi)$. Then φ is biproximal. Moreover,*

(1) $x_\varphi^+, x_\varphi^- \in \partial\Omega$,

(2) $T_{x_\varphi^+}^{\mathbb{K}} \partial\Omega \cap \partial\Omega = \{x_\varphi^+\}$,

(3) $T_{x_\varphi^-}^{\mathbb{K}} \partial\Omega \cap \partial\Omega = \{x_\varphi^-\}$, *and*

(4) *if $U^+ \subset \overline{\Omega}$ is a neighborhood of x_φ^+ and $U^- \subset \overline{\Omega}$ is a neighborhood of x_φ^-, then there exists $N > 0$ such that, for all $m > N$, we have*

$$\varphi^m(\partial\Omega \setminus U^-) \subset U^+ \quad \text{and} \quad \varphi^{-m}(\partial\Omega \setminus U^+) \subset U^-.$$

Proof Since $\sigma_{d+1}(\varphi) > \sigma_1(\varphi)$, we have $\varphi^n \to \infty$ in $\mathrm{PGL}_{d+1}(\mathbb{K})$. So by fixing $p \in \Omega$, we can find $n_k \to \infty$ such that

$$\varphi^{n_k} p \to x^+ \quad \text{and} \quad \varphi^{-n_k} p \to x^-$$

for some $x^+, x^- \in \partial\Omega$. By Proposition 5.1, φ^{n_k} converges in $\mathrm{PEnd}(\mathbb{K}^{d+1})$ to an element Φ where $\mathrm{Im}(\Phi) = x^+$ and $\ker \Phi = T_{x^-}^{\mathbb{K}} \partial\Omega$. Moreover, there exists $f^\pm \in \Omega^*$ with $\ker f^\pm = T_{x^\pm}^{\mathbb{K}} \partial\Omega$.

By considering the Jordan block decomposition of φ, we see that x^+ is an eigenline of φ with corresponding eigenvalue having absolute value $\sigma_{d+1}(\varphi)$, and f^- is an eigenline of ${}^*\varphi$ with corresponding eigenvalue having absolute value $\sigma_1(\varphi)$. Applying this argument to φ^{-1} shows that x^- is an eigenline of φ with corresponding eigenvalue having absolute value $\sigma_1(\varphi)$, and f^+ is an eigenline of ${}^*\varphi$ with corresponding eigenvalue having absolute value $\sigma_{d+1}(\varphi)$.

Now since $\sigma_1(\varphi) \neq \sigma_{d+1}(\varphi)$, we see that $f^+ \neq f^-$. Then $f^+(x^-) \neq 0$, for otherwise,

$$\ker f^- = T_{x^+}^{\mathbb{K}} \partial\Omega = \ker f^+,$$

which is impossible. Similarly, $f^-(x^+) \neq 0$.

So φ preserves the subspaces x^+, x^-, and $\ker f^+ \cap \ker f^-$. If $v_1, \dots, v_{d+1}$ is a basis of $\mathbb{K}^{d+1}$ with $\mathbb{K}v_1 = x^+$, $\mathbb{K}v_2 = x^-$, and $\ker f^+ \cap \ker f^- = \mathrm{Span}_{\mathbb{K}}(v_3, \dots, v_{d+1})$, then with respect to this basis, φ is represented by a matrix of the form

$$\begin{pmatrix} \lambda^+ & & \\ & \lambda^- & \\ & & A \end{pmatrix} \in \mathrm{GL}_{d+1}(\mathbb{K}).$$

Since $\mathrm{Im}(\Phi) = x^+$, we see that $\|A\| < |\lambda^+|$, and applying this argument to φ^{-1} shows that $\|A^{-1}\| < |\lambda^-|^{-1}$. Thus φ is biproximal, and $x^\pm = x^\pm_\varphi$.

Next we claim that $\partial\Omega \cap T^{\mathbb{K}}_{x^+}\partial\Omega = \{x^+\}$. Suppose that $z \in \partial\Omega \cap T^{\mathbb{K}}_{x^+}\partial\Omega$; then either $z = x^+$ or $z = [z_1 : 0 : z_2 : \cdots : z_d]$ with $z_j \neq 0$ for some $2 \le j \le d$. In the latter case, there exist $m_i \to \infty$ such that $\varphi^{-m_i} z \to w$ and $w = [0 : 0 : w_2 : \cdots : w_d]$. But then $w \in \partial\Omega \cap T^{\mathbb{K}}_{x^+}\partial\Omega \cap T^{\mathbb{K}}_{x^-}\partial\Omega$, which is impossible since $\partial\Omega$ is C^1. So we have a contradiction, and so $z = x^+$. Applying this argument to φ^{-1} shows that $\partial\Omega \cap T^{\mathbb{K}}_{x^-}\partial\Omega = \{x^-\}$.

Finally, part (4) follows part (4) of Proposition 5.1. □

7 Finding biproximal elements

Theorem 7.1 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary. If there exist $x, y \in \mathcal{L}(\Omega)$ such that $T^{\mathbb{K}}_x\partial\Omega \neq T^{\mathbb{K}}_y\partial\Omega$, then $\mathrm{Aut}(\Omega)$ contains a biproximal element.*

We begin the proof of Theorem 7.1 with two lemmas.

Lemma 7.2 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary, and let $\varphi_n \in \mathrm{Aut}(\Omega)$ with*

$$\varphi_n p \to x^+ \quad \text{and} \quad \varphi_n^{-1} p \to x^-,$$

where $p \in \Omega$ and $x^+, x^- \in \partial\Omega$. If $T^{\mathbb{K}}_{x^+}\partial\Omega \neq T^{\mathbb{K}}_{x^-}\partial\Omega$, then φ_n is biproximal for n large enough. Moreover, $x^+_{\varphi_n} \to x^+$ and $x^-_{\varphi_n} \to x^-$.

Given distinct $x, y \in \mathbb{P}(\mathbb{K}^{d+1})$, let $L(x, y)$ be the projective line containing x and y.

Proof We first claim that, for n large enough, φ_n has fixed points $x_n^+, x_n^- \in \overline{\Omega}$. Fix compact neighborhoods $U^\pm$ of $x^\pm$ with the following properties:

(1) $U^{\pm} \cap T^{\mathbb{K}}_{x^{\mp}} \partial\Omega = \varnothing$,
(2) $U^{\pm} \cap \overline{\Omega}$ is topologically a closed ball,
(3) there exists a compact set $K \subset \Omega$ such that, if $y^+ \in U^+$ and $y^- \in U^-$, then $L(y^+, y^-) \cap K \neq \varnothing$.

Since $x^{\pm} \notin T^{\mathbb{C}}_{x^{\mp}} \partial\Omega$, part (1) holds for small enough neighborhoods. Since $\partial\Omega$ is a C^1 hypersurface, it is always possible to shrink a neighborhood so that part (2) holds. Finally, since

$$T^{\mathbb{K}}_{x^+} \partial\Omega \neq T^{\mathbb{K}}_{x^-} \partial\Omega,$$

the projective line $L(x^+, x^-)$ is transverse to $\partial\Omega$ at x^+ and x^-, so part (3) holds for small enough neighborhoods.

Now by part (4) of Proposition 5.1, there exists $N \geq 0$ such that

$$\varphi_n(U^{\pm} \cap \overline{\Omega}) \subset U^{\pm} \cap \overline{\Omega}$$

for all $n \geq N$. So by the Brouwer fixed point theorem, for n large enough, φ_n has a fixed point $x_n^{\pm} \in U^{\pm} \cap \overline{\Omega}$.

Now fix points $k_n \in K \cap L(x_n^+, x_n^-)$. Since $K \subset \Omega$ is compact, and $\varphi_n^{\pm 1} q \to x^{\pm}$ for all $q \in \Omega$, we see that

$$\varphi_n k_n \to x^+ \quad \text{and} \quad \varphi_n^{-1} k_n \to x^-.$$

So for large n, the ratios of the absolute values of the eigenvalues of φ_n corresponding to the lines x_n^+ and x_n^- must be different. So for large n, $\sigma_{d+1}(\varphi_n) > \sigma_1(\varphi_n)$. Thus φ_n is biproximal by Proposition 6.1. Then by part (4) of Proposition 6.1, we see that $x_n^{\pm} = x_{\varphi_n}^{\pm}$.

Finally, we can choose U^+ and U^- to be arbitrary small neighborhoods of x^+ and x^-, which implies that $x_{\varphi_n}^+ \to x^+$ and $x_{\varphi_n}^- \to x^-$. □

Lemma 7.3 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary, and $\varphi_n, \phi_m \in \mathrm{Aut}(\Omega)$ with*

$$\varphi_n p \to x^+, \quad \varphi_n^{-1} p \to x^-, \quad \phi_m p \to y^+, \quad \text{and} \quad \phi_m^{-1} p \to y^-,$$

where $p \in \Omega$ and $x^+, x^-, y^+, y^- \in \partial\Omega$. If

$$\{T^{\mathbb{K}}_{x^+} \partial\Omega, T^{\mathbb{K}}_{x^-} \partial\Omega\} \cap \{T^{\mathbb{K}}_{y^+} \partial\Omega, T^{\mathbb{K}}_{y^-} \partial\Omega\} = \varnothing,$$

then $\gamma_k := \varphi_k \phi_k^{-1}$ is biproximal for k large enough. Moreover,

$$\gamma_k p \to x^+ \quad \textit{and} \quad \gamma_k^{-1} p \to y^+.$$

Proof Fix compact neighborhoods $U^{\pm}$ of $x^{\pm}$ and $V^{\pm}$ of $y^{\pm}$ so that

$$(U^{+} \cup U^{-}) \cap (T^{\mathbb{K}}_{y^+}\partial\Omega \cup T^{\mathbb{K}}_{y^-}\partial\Omega) = \varnothing$$

and

$$(V^{+} \cup V^{-}) \cap (T^{\mathbb{K}}_{x^+}\partial\Omega \cup T^{\mathbb{K}}_{x^-}\partial\Omega) = \varnothing.$$

By Proposition 5.1, there exists $N \geq 0$ such that $\varphi_n^{-1} p \in U^{-}$ and $\varphi_n(V^{+} \cup V^{-}) \subset U^{+}$ for all $n \geq N$, and there exists $M \geq 0$ such that $\phi_m^{-1} p \in V^{-}$ and $\phi_m(U^{+} \cup U^{-}) \subset V^{+}$ for all $m \geq M$. Then if $k \geq \max\{M, N\}$ and $\gamma_k := \varphi_k \phi_k^{-1}$, we see that $\gamma_k p \in U^{+}$ and $\gamma_k^{-1} p \in V^{+}$.

Since U^{+} and V^{+} can be chosen to be arbitrary small neighborhoods of x^{+} and y^{+}, respectively, we see that

$$\gamma_k p \to x^{+} \quad \text{and} \quad \gamma_k^{-1} p \to y^{+}.$$

Finally, since

$$T^{\mathbb{K}}_{x^+}\partial\Omega \neq T^{\mathbb{K}}_{y^+}\partial\Omega,$$

Lemma 7.2 implies that γ_k is biproximal for large k. □

Proof of Theorem 7.1 Fix sequences $\varphi_n, \phi_m \in \mathrm{Aut}(\Omega)$ so that

$$\varphi_n p \to x \quad \text{and} \quad \phi_m p \to y$$

for some $p \in \Omega$. By passing to a subsequence, we may suppose that

$$\varphi_n^{-1} p \to x^{-} \quad \text{and} \quad \phi_m^{-1} p \to y^{-}$$

for some $x^{-}, y^{-} \in \partial\Omega$.

Now by Lemma 7.2, if $T^{\mathbb{K}}_{x}\partial\Omega \neq T^{\mathbb{K}}_{x^-}\partial\Omega$, then φ_n is biproximal for large n, and if $T^{\mathbb{K}}_{y}\partial\Omega \neq T^{\mathbb{K}}_{y^-}\partial\Omega$, then ϕ_m is biproximal for large m.

So suppose that $T^{\mathbb{K}}_{x}\partial\Omega = T^{\mathbb{K}}_{x^-}\partial\Omega$ and $T^{\mathbb{K}}_{y}\partial\Omega = T^{\mathbb{K}}_{y^-}\partial\Omega$. Then

$$\{T^{\mathbb{K}}_{x}\partial\Omega, T^{\mathbb{K}}_{x^-}\partial\Omega\} \cap \{T^{\mathbb{K}}_{y}\partial\Omega, T^{\mathbb{K}}_{y^-}\partial\Omega\} = \varnothing,$$

and so $\varphi_k \phi_k^{-1}$ is biproximal for large k by Lemma 7.3. □

8 Rescaling with biproximal elements

Definition 8.1 If $\mathbb{K}$ is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, let $\mathbb{K}_P$ be the *purely imaginary numbers* in $\mathbb{K}$: that is, $\mathbb{R}_P = \{0\}$, $\mathbb{C}_P = i\mathbb{R}$, and $\mathbb{H}_P = i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$.

Suppose that Ω is a proper domain with C^1 boundary. If $\varphi \in \mathrm{Aut}(\Omega)$ is biproximal, then we have the following standard form. First let $H^{\pm}$ be the $\mathbb{K}$–tangent hyperplane at $x_\varphi^{\pm}$. Then make a change of coordinates so that

(1) $x_\varphi^+ = [1:0:\cdots:0]$,

(2) $x_\varphi^- = [0:1:0:\cdots:0]$,

(3) $H^+ \cap H^- = \{[0:0:z_2:\cdots:z_d] : z_2,\dots,z_d \in \mathbb{K}\}$.

With respect to these coordinates, φ is represented by a matrix of the form

$$\begin{pmatrix} \lambda & & \\ & \mu & \\ & & A \end{pmatrix} \in \mathrm{GL}_{d+1}(\mathbb{K}),$$

where A is a $(d-1)\times(d-1)$ matrix. Since

$$H^- = \{[0:z_1:\cdots:z_d] : z_1,\dots,z_d \in \mathbb{K}\},$$

and $\Omega \cap H^- = \varnothing$, we see that Ω is contained in the affine chart

$$\mathbb{K}^d = \{[1:z_1:\cdots:z_d] : z_1,\dots,z_d \in \mathbb{K}\}.$$

In this affine chart, x_φ^+ corresponds to 0, and $T_0^{\mathbb{K}}\partial\Omega = \{0\}\times\mathbb{K}^{d-1}$. Then by a projective transformation, we may assume that

(4) $T_0\partial\Omega = \mathbb{K}_P \times \mathbb{K}^{d-1}$.

Since $\partial\Omega$ is C^1, there exist open neighborhoods $V \subset \mathbb{K}_P$ of 0, $W \subset \mathbb{R}$ of 0, $U \subset \mathbb{K}^{d-1}$ of 0, and a C^1 function $F\colon V \times U \to W$ such that

(5) $\partial\Omega \cap \mathcal{O} = \{(z_1,\dots,z_d) \in \mathcal{O} : \mathrm{Re}(z_1) = F(\mathrm{Im}(z_1), z_2,\dots,z_d)\}$,

where $\mathcal{O} = (V+W)\times U$. By another projective transformation, we can assume

(6) $\Omega \cap \mathcal{O} = \{(z_1,\dots,z_d) \in \mathcal{O} : \mathrm{Re}(z_1) > F(\mathrm{Im}(z_1), z_2,\dots,z_d)\}$.

Next, if $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$, let $G \le \mathrm{GL}_2(\mathbb{K})$ be the closed group generated by

$$\left\{\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} : \mathrm{Re}(w) = 0\right\} \quad \text{and} \quad \left\{\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} : \mathrm{Re}(w) = 0\right\}.$$

Then by Proposition A.2, the image of G in $\mathrm{PGL}_2(\mathbb{K})$ coincides with

$$\mathrm{Aut}_0(\{[1:z] \in \mathbb{P}(\mathbb{K}^2) : \mathrm{Re}(z) > 0\}).$$

Theorem 8.2 *Choosing coordinates as indicated above, the function* F *extends to* $\mathbb{K}_P \times \mathbb{K}^{d-1}$*, and*

$$\Omega = \{(z_1, \dots, z_d) \in \mathbb{K}^d : \operatorname{Re}(z_1) > F(0, z_2, \dots, z_d)\}.$$

Moreover, if $\mathbb{K}$ *is either* $\mathbb{C}$ *or* $\mathbb{H}$*, then for* $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$*, the projective transformation defined by*

$$\psi_h \cdot [z_1, \dots, z_d] = [az_1 + bz_2 : cz_1 + dz_2 : z_3 : \dots : z_{d+1}]$$

is in $\operatorname{Aut}_0(\Omega)$.

Remark 8.3 A special case of the above theorem, namely when $\mathbb{K} = \mathbb{C}$ and Ω is a $\mathbb{C}$–convex set, was established in [28, Theorem 6.1].

Proof We can assume $\mathcal{O}$ is bounded. Then by Proposition 5.1, we can replace φ with a power of φ so that $\varphi(\mathcal{O}) \subset \mathcal{O}$.

We first claim that $F(x, z) = F(0, z)$ for $(x, z) \in V \times U$. Notice that, with our choice of coordinates, φ acts by

$$\varphi \cdot (z_1, z) = (\mu z_1 \lambda^{-1}, Az\lambda^{-1}),$$

where λ, μ, and A are as above. Since φ is biproximal,

$$\mu^n z_1 \lambda^{-n} \to 0 \quad \text{and} \quad A^n z \lambda^{-n} \to 0$$

as $n \to \infty$ for any $z_1 \in \mathbb{K}$ and $z \in \mathbb{K}^{d-1}$.

Since φ preserves $T_0 \partial\Omega = \mathbb{K}_P \times \mathbb{K}^{d-1}$, we see that $\mu \mathbb{K}_P \lambda^{-1} = \mathbb{K}_P$. We claim that $\mu\lambda^{-1} \in \mathbb{R}$. When $\mathbb{K} = \mathbb{R}$, there is nothing to prove. When $\mathbb{K} = \mathbb{C}$, since

$$i\mathbb{R} = \mu(i\mathbb{R})\lambda^{-1} = (\mu\lambda^{-1})i\mathbb{R},$$

this is obvious. So assume $\mathbb{K} = \mathbb{H}$. Then since $\lambda^{-1} = \overline{\lambda}/|\lambda|^2$, we see that $\mu \mathbb{H}_P \overline{\lambda} = \mathbb{H}_P$. So if $z \in \mathbb{H}_P$, then

$$0 = 2\operatorname{Re}(\mu z \overline{\lambda}) = \mu z \overline{\lambda} + \lambda \overline{z}\, \overline{\mu} = \mu z \overline{\lambda} - \lambda z \overline{\mu},$$

which implies that

$$\lambda^{-1} \mu z = z \overline{\lambda^{-1} \mu}.$$

Now suppose that $\lambda^{-1}\mu = a + bi + cj + dk$. If we plug in $z = i$, we see that

$$-b = \operatorname{Re}(\lambda^{-1}\mu i) = \operatorname{Re}(i \overline{\lambda^{-1}\mu}) = b,$$

so $b = 0$. Plugging in $z = j$ and $z = k$ shows that $c = d = 0$. Thus $\lambda^{-1}\mu \in \mathbb{R}$. So $\lambda^{-1} \in \mathbb{R} \cdot \mu^{-1}$, and so $\mu\lambda^{-1} \in \mathbb{R}$.

Now for $x \in V$ and $z \in U$, we have

$$\varphi \cdot (x + F(x,z), z) = (\mu x \lambda^{-1} + \mu F(x,z)\lambda^{-1}, Az\lambda^{-1}).$$

Since $\varphi(\mathcal{O}) \subset \mathcal{O}$, $\mu x \lambda^{-1} \in \mathbb{K}_P$, and $\mu F(x,z)\lambda^{-1} = \mu\lambda^{-1}F(x,z) \in \mathbb{R}$, we see that

$$F(\mu x \lambda^{-1}, Az\lambda^{-1}) = \mu\lambda^{-1}F(x,z).$$

Differentiating F in the x direction yields

$$(\nabla_x F)(x,z) = (\nabla_x F)(\mu x\lambda^{-1}, Az\lambda^{-1}),$$

and repeated applications of the above formula show

$$(\nabla_x)F(x,z) = (\nabla_x F)(\mu^n x \lambda^{-n}, A^n z \lambda^{-n})$$

for all $n > 0$. Taking the limit as n goes to infinity proves that $(\nabla_x F)(x,z) = (\nabla_x F)(0,0)$. Since $(\nabla_x F)(0,0) = 0$ we then see that $F(x,z) = F(0,z)$ for all $(x,z) \in V \times U$.

Now for $(x,z) \in \mathbb{K}_P \times \mathbb{K}^{d-1}$, there exists $N > 0$ such that $\varphi^N \cdot (x,z) \in V \times U$. Then we define

$$F(x,z) := \mu^N \lambda^{-N} F(\mu^N x \lambda^{-N}, A^N z \lambda^{-N}).$$

Notice that this definition does not depend on the choice of N; that is, if $\varphi^M \cdot (x,z) \in V \times U$ then

$$\mu^M \lambda^{-M} F(\mu^M x \lambda^{-M}, A^M z \lambda^{-M}) = \mu^N \lambda^{-N} F(\mu^N x \lambda^{-N}, A^N z \lambda^{-N}).$$

So we see that F extends to a function defined on $\mathbb{K}_P \times \mathbb{K}^{d-1}$. Moreover, this function is clearly C^1. With this extension,

$$\bigcup_{n \in \mathbb{N}} \varphi^{-n}(\mathcal{O} \cap \Omega) = \{(z_1, \dots, z_d) \in \mathbb{K}^d : \mathrm{Re}(z_1) > F(0, z_2, \dots, z_d)\},$$

and thus $\bigcup_{n \in \mathbb{N}} \varphi^{-n}(\mathcal{O} \cap \Omega) = \Omega$ by Proposition 6.1. This proves the first part of the theorem.

Now assume that $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$. Then for $w \in \mathbb{K}_P$, define the projective map u_w by $u_w \cdot (z_1, \dots, z_d) = (z_1 + w, z_2, \dots, z_d)$. Since $F(x,z) = F(0,z)$, we see that $u_w \in \mathrm{Aut}_0(\Omega)$ for all $w \in \mathbb{K}_P$. Also, u_w corresponds to the matrix

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$$

in the action of $\mathrm{GL}_2(\mathbb{K})$ defined in the statement of the theorem.

The same argument starting with φ^{-1} instead of φ (that is, viewing Ω as a subset of the affine chart $\{[z_1 : 1 : z_2 : \cdots : z_d]\}$) shows that $\mathrm{Aut}_0(\Omega)$ contains the group of automorphisms corresponding to the matrices

$$\left\{ \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} : w \in \mathbb{K}_P \right\}$$

in the action of $\mathrm{GL}_2(\mathbb{K})$ defined in the statement of the theorem. □

We end the section with three corollaries of Theorem 8.2. If $t \in \mathbb{R}$ and we consider the matrix

$$h = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in G$$

in the statement of Theorem 8.2, then we have the following:

Corollary 8.4 *Suppose $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$ and Ω is a proper domain with C^1 boundary. If $\varphi \in \mathrm{Aut}(\Omega)$ is biproximal, then there exists a one-parameter subgroup $\psi_t \in \mathrm{SL}_{d+1}(\mathbb{K})$ of biproximal elements such that $[\psi_t] \in \mathrm{Aut}_0(\Omega)$, and*

(1) $(\psi_t)|_{x_\varphi^+} = e^t \,\mathrm{Id}\,|_{x_\varphi^+}$,

(2) $(\psi_t)|_{x_\varphi^-} = e^{-t} \,\mathrm{Id}\,|_{x_\varphi^-}$,

(3) $(\psi_t)|_{H^+ \cap H^-} = \mathrm{Id}\,|_{H^+ \cap H^-}$, *where* $H^\pm = T^{\mathbb{K}}_{x_\varphi^\pm} \partial\Omega$.

Proposition A.2 and Theorem 8.2 also imply the following:

Corollary 8.5 *Suppose $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$, Ω is a proper domain with C^1 boundary, $\varphi \in \mathrm{Aut}(\Omega)$ is biproximal, and L is the projective line containing x_φ^+ and x_φ^-. Then for all $x, y \in L \cap \partial\Omega$, there exists $\varphi_{xy} \in \mathrm{Aut}_0(\Omega)$ such that $\varphi_{xy}(x) = y$.*

Finally, Theorem 8.2 also implies the following:

Corollary 8.6 *Suppose $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$, Ω is a proper domain with C^1 boundary, and $\mathrm{Aut}(\Omega)$ contains a biproximal element. If $x_1, \dots, x_n \in \partial\Omega$, then there exists a biproximal element $\gamma \in \mathrm{Aut}_0(\Omega)$ so that*

$$\{x_1, \dots, x_n\} \cap \{x_\gamma^+, x_\gamma^-\} = \varnothing.$$

Proof Suppose $\varphi \in \mathrm{Aut}_0(\Omega)$ is biproximal, and L is the projective line containing x_φ^+ and x_φ^-. Fix distinct $x, y \in L \cap \partial\Omega$ so that

$$\{x_1, \dots, x_n\} \cap \{x, y\} = \varnothing.$$

By Theorem 8.2 and Proposition A.2, there exists $g_1 \in \mathrm{Aut}_0(\Omega)$ such that $g_1 x_\varphi^- = x_\varphi^-$ and $g_1 x_\varphi^+ = x$. For the same reason, there exists $g_2 \in \mathrm{Aut}_0(\Omega)$ such that $g_2 x = x$ and $g_2 x_\varphi^- = y$.

Then $\gamma := (g_2 g_1)\varphi(g_2 g_1)^{-1}$ is biproximal, $x_\gamma^+ = x$, and $x_\gamma^- = y$. □

9 Proof of Theorem 1.6

For this section, suppose that $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain, $\partial\Omega$ is C^2, and there exist $x, y \in \mathcal{L}(\Omega)$ such that

$$T_x^{\mathbb{K}} \partial\Omega \neq T_y^{\mathbb{K}} \partial\Omega.$$

First, by Theorem 7.1, we see there exists a biproximal element $\varphi \in \mathrm{Aut}(\Omega)$. Let $H^\pm = T_{x_\varphi^\pm}^{\mathbb{K}} \partial\Omega$. Pick coordinates so that

(1) $x_\varphi^+ = [1:0:\cdots:0]$,

(2) $x_\varphi^- = [0:1:0:\cdots:0]$,

(3) $H^+ \cap H^- = \{[0:0:z_2:\cdots:z_d] : z_2, \dots, z_d \in \mathbb{K}\}$.

For the rest of the proof, identify $\mathbb{K}^d$ with the affine chart

$$\{[1:z_1:z_2:\cdots:z_d] : z_1, \dots, z_d \in \mathbb{K}\}.$$

Then by Theorem 8.2, there exists a C^2 function $F \colon \mathbb{K}^{d-1} \to \Omega$ such that

$$\Omega = \{(z_1, z_2, \dots, z_d) : \mathrm{Re}(z_1) > F(z_2, \dots, z_d)\}.$$

Notice that $F(0) = 0$. Moreover, by Proposition 6.1,

$$T_{x_\varphi^+}^{\mathbb{K}} \partial\Omega \cap \partial\Omega = \{x_\varphi^+\},$$

and so $F(z) > 0$ for all $z \in \mathbb{K}^{d-1} \setminus \{0\}$.

The real case

Suppose that $\mathbb{K} = \mathbb{R}$. With respect to these coordinates, φ is represented by a matrix of the form

$$\begin{pmatrix} \lambda & & \\ & 1/\lambda & \\ & & A \end{pmatrix} \in \mathrm{GL}_{d+1}(\mathbb{R})$$

with $\lambda > 1$. And so

$$\lambda^{2n} F\left(\frac{1}{\lambda^n} A^n x\right) = F(x)$$

for all $x \in \mathbb{R}^{d-1}$ and $n \in \mathbb{N}$.

We first claim that, up to a change of coordinates, $A \in O(d-1)$. Since F is C^2, there exists $C > 0$ such that $F(x) \leq C\|x\|^2$ for all x sufficiently close to 0. Thus for n large enough,

$$F(x) \leq \lambda^{2n} \left\| \frac{A^n}{\lambda^n} x \right\|^2 = \|A^n x\|^2.$$

Since F is positive on $\mathbb{R}^{d-1} \setminus \{0\}$, this implies that

$$\inf_{n \in \mathbb{N}} \inf_{\|x\|=1} \|A^n x\| > 0.$$

Since

$$\|A^{-n}\| = \frac{1}{\inf_{\|x\|=1} \|A^n x\|},$$

we see that $\|A^{-n}\|$ is uniformly bounded. Applying the same argument to φ^{-1} shows that $\|A^n\|$ is uniformly bounded. Thus $\{A^n : n \in \mathbb{Z}\} \leq \mathrm{GL}_{d-1}(\mathbb{R})$ is a bounded group. Hence, up to a change of coordinates, $A \in O(d-1)$.

Now we can fix $n_k \to \infty$ so that $A^{n_k} \to \mathrm{Id}_{d-1}$. Then for $x \in \mathbb{R}^{d-1}$,

$$F(x) = \lim_{k \to \infty} \lambda^{2n_k} F\left(\frac{1}{\lambda^{n_k}} A^{n_k} x\right) = \tfrac{1}{2} \operatorname{Hess}(F)_0(x,x)$$

since F is C^2. Since $F(x) > 0$ for all nonzero x, we then see that $\operatorname{Hess}(F)_0$ is positive definite, and hence up to a change of coordinates, we see that

$$F(x_2, \dots, x_d) = \frac{1}{2} \sum_{i=2}^{d} x_i^2,$$

and so

$$\Omega = \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d : \operatorname{Re}(x_1) > \sum_{i=2}^{d} x_i^2 \right\}.$$

The complex and quaternionic case

Now suppose that $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$. In this case, we can assume that $\mathrm{Aut}_0(\Omega)$ contains the transformation

$$[z_1, \dots, z_d] \to [az_1 + bz_2 : cz_1 + dz_2 : z_3 : \cdots : z_{d+1}]$$

when $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. We claim that

$$F(z) = \frac{1}{|w|^2} F(zw) \tag{1}$$

for $w \in \mathbb{K} \setminus \{0\}$. First, if $t \in \mathbb{R}$, then the transformation

$$a_t \cdot [z_1, \dots, z_d] = [e^{-t} z_1 : e^t z_2 : z_3 : \cdots : z_{d+1}]$$

is in $\mathrm{Aut}(\Omega)$ and acts on the affine chart $\mathbb{K}^d$ by

$$a_t \cdot (z_1, \dots, z_d) = (e^{2t} z_1, e^t z_2, \dots, e^t z_d),$$

so

$$F(z) = \frac{1}{e^{2t}} F(e^t z). \tag{2}$$

Next, if $w \in \mathbb{K}_P$, then the transformation

$$u_w \cdot [z_1, \dots, z_d] = [z_1 + w z_2 : z_2 : z_3 : \cdots : z_{d+1}]$$

is in $\mathrm{Aut}(\Omega)$ and acts on the affine chart $\mathbb{K}^d$ by

$$u_w \cdot (z_1, z_2, \dots, z_d) = (z_1(1 + w z_1)^{-1}, z_2(1 + w z_1)^{-1}, \dots, z_d(1 + w z_1)^{-1}).$$

Notice that

$$\mathrm{Re}(z_1(1 + w z_1)^{-1}) = \frac{1}{|1 + w z_1|^2} \mathrm{Re}(z_1(1 - \overline{z}_1 w)) = \frac{\mathrm{Re}(z_1)}{|1 + w z_1|^2},$$

so if we apply $u_{w/F(z)}$ to the point $(F(z), z) \in \partial\Omega$, we see that

$$F(z) = |1 + w|^2 F(z(1 + w)^{-1}) \tag{3}$$

for all $w \in \mathbb{K}_P$. Combining equations (2) and (3), we see that (1) holds for all $w \in \mathbb{K}$ with $\mathrm{Re}(w) > 0$. On the other hand, any $w \in \mathbb{K} \setminus \{0\}$ can be written as $z = w_1 w_2$ where $\mathrm{Re}(w_1), \mathrm{Re}(w_2) > 0$. So (1) holds for all $w \in \mathbb{K} \setminus \{0\}$.

Now, since F is C^2 and $F(z) = e^{2t} F(e^{-t} z)$, we see that

$$F(z) = \tfrac{1}{2} \mathrm{Hess}(F)_0(z, z)$$

for all $z \in \mathbb{K}^{d-1}$. Since $T_0^{\mathbb{K}} \partial\Omega \cap \partial\Omega = \{0\}$, we have $F(z) > 0$ for all $z \in \mathbb{K}^{d-1}$, and so the Hessian of F is positive definite.

Now let $r = \dim_{\mathbb{R}} \mathbb{K}$, and identify $\mathbb{K}^{d-1}$ with $\mathbb{R}^{r(d-1)}$ in the obvious way. For $w \in \mathbb{K}$, let $M(w) \in \mathrm{GL}_{r(d-1)}(\mathbb{R})$ denote the action by scalar multiplication by w (on the right); that is,

$$M(w) z = z w.$$

Notice that ${}^t M(w) = M(\overline{w})$. Now, under this identification, there exists a symmetric $r(d-1) \times r(d-1)$ real matrix A such that

$$\mathrm{Hess}(F)_0(z_1, z_2) = {}^t z_1 A z_2.$$

Since $F(z) > 0$ when $z \neq 0$, we see that $A \in \mathrm{GL}_{r(d-1)}(\mathbb{R})$. Since $F(zw) = |w|^2 F(z)$ for $w \in \mathbb{K}$, we see that

$$M(\overline{w}) A M(w) = |w|^2 A.$$

So

$$A M(w) = |w|^2 M(\overline{w})^{-1} A.$$

But $|w|^2 M(\overline{w})^{-1} = M(w)$. Thus A is $\mathbb{K}$–linear. Hence A can be viewed as a matrix in $\mathrm{GL}_{d-1}(\mathbb{K})$. Now, since ${}^t A = A$ as a matrix in $\mathrm{GL}_{r(d-1)}(\mathbb{R})$, we see that ${}^t \overline{A} = A$ as a matrix in $\mathrm{GL}_{d-1}(\mathbb{K})$. Moreover, A is positive definite. Thus there exists $g \in \mathrm{GL}_{d-1}(\mathbb{K})$ so that

$${}^t \overline{g} A g = \mathrm{Id}_{d-1}.$$

Thus, up to a change of coordinates,

$$F(z_2, \dots, z_d) = \sum_{i=2}^{d} |z_i|^2 \quad \text{and} \quad \Omega = \left\{ (z_1, \dots, z_d) \in \mathbb{K}^d : \mathrm{Re}(z_1) > \sum_{i=2}^{d} |z_i|^2 \right\}.$$

This completes the proof of Theorem 1.6. □

10 The structure of the limit set

Proposition 10.1 *Suppose $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$, and $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary. If there exist $x, y \in \mathcal{L}(\Omega)$ such that $T_x^{\mathbb{K}} \partial\Omega \neq T_y^{\mathbb{K}} \partial\Omega$, then the limit set $\mathcal{L}(\Omega) \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a closed C^∞ submanifold of $\mathbb{P}(\mathbb{K}^{d+1})$, and $\mathrm{Aut}_0(\Omega)$ acts transitively on $\mathcal{L}(\Omega)$.*

The fact that $\mathcal{L}(\Omega)$ is a C^∞ submanifold of $\mathbb{P}(\mathbb{K}^{d+1})$ will follow from a general fact about the orbits of Lie groups:

Lemma 10.2 *Suppose G is a connected Lie group acting smoothly on a smooth manifold M. Then an orbit $G \cdot m$ is an embedded smooth submanifold of M if and only if $G \cdot m$ is locally closed in M.*

Here, smooth means C^∞, and for a proof, see [8, Theorem 15.3.7].

Lemma 10.3 *Suppose $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$, and $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary. If $x^+, x^- \in \partial\Omega$, $T_{x^+}^{\mathbb{K}} \partial\Omega \neq T_{x^-}^{\mathbb{K}} \partial\Omega$, and there exist biproximal elements $\varphi_n \in \mathrm{Aut}(\Omega)$ such that*

$$x_{\varphi_n}^+ \to x^+ \quad \text{and} \quad x_{\varphi_n}^- \to x^-,$$

then there exists $\varphi \in \mathrm{Aut}(\Omega)$ biproximal such that $x^+ = x_\varphi^+$ and $x^- = x_\varphi^-$.

Proof First let $H_n^{\pm} = T^{\mathbb{K}}_{x^{\pm}_{\varphi_n}} \partial\Omega$ and $H^{\pm} = T^{\mathbb{K}}_{x^{\pm}} \partial\Omega$. Since $H^+ \neq H^-$, we then have $H_n^+ \cap H_n^- \to H^+ \cap H^-$ in the space of $(d-1)$–planes in $\mathbb{K}^{d+1}$. By Corollary 8.4, we can assume that $\varphi_n \in \mathrm{GL}_{d+1}(\mathbb{K})$, and

(1) $(\varphi_n)|_{x^+_{\varphi_n}} = 2\,\mathrm{Id}\,|_{x^+_{\varphi_n}}$,

(2) $(\varphi_n)|_{x^-_{\varphi_n}} = \frac{1}{2}\,\mathrm{Id}\,|_{x^-_{\varphi_n}}$,

(3) $(\varphi_n)|_{H_n^+ \cap H_n^-} = \mathrm{Id}\,|_{H_n^+ \cap H_n^-}$.

Since $H_n^+ \cap H_n^- \to H^+ \cap H^-$, we see that φ_n converges to $\varphi \in \mathrm{GL}_{d+1}(\mathbb{K})$, where

(1) $(\varphi)|_{x^+} = 2\,\mathrm{Id}\,|_{x^+}$,

(2) $(\varphi)|_{x^-} = \frac{1}{2}\,\mathrm{Id}\,|_{x^-}$,

(3) $(\varphi)|_{H^+ \cap H^-} = \mathrm{Id}\,|_{H^+ \cap H^-}$.

Then, since $\mathrm{Aut}(\Omega)$ is closed, $[\varphi] \in \mathrm{Aut}(\Omega)$. □

Lemma 10.4 *Suppose $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$, $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary, and there exist $x, y \in \mathcal{L}(\Omega)$ such that $T^{\mathbb{K}}_x \partial\Omega \neq T^{\mathbb{K}}_y \partial\Omega$. Then for any $z \in \mathcal{L}(\Omega)$,*

$$T^{\mathbb{K}}_z \partial\Omega \cap \partial\Omega = \{z\}.$$

Proof By definition, there exists $\phi_m \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ so that $\phi_m p \to z$. By passing to a subsequence, we can assume that $\phi_m^{-1} p \to z^-$ for some $z^- \in \partial\Omega$. Now by Theorem 7.1 and Corollary 8.6, there exists $\gamma \in \mathrm{Aut}(\Omega)$ biproximal such that $\{z, z^-\} \cap \{x^+_\gamma, x^-_\gamma\} = \varnothing$. Proposition 6.1 implies that

$$T^{\mathbb{K}}_{x^{\pm}_\gamma} \partial\Omega \cap \partial\Omega = \{x^{\pm}_\gamma\},$$

and so

$$\{T^{\mathbb{K}}_z \partial\Omega, T^{\mathbb{K}}_{z^-} \partial\Omega\} \cap \{T^{\mathbb{K}}_{x^+_\gamma} \partial\Omega, T^{\mathbb{K}}_{x^-_\gamma} \partial\Omega\} = \varnothing.$$

Then by Lemma 7.3, there exist biproximal elements $\gamma_k \in \mathrm{Aut}(\Omega)$ such that

$$x^+_{\gamma_k} \to z \quad \text{and} \quad x^-_{\gamma_k} \to x^+_\gamma.$$

So by the previous lemma, there exists a biproximal element $\varphi \in \mathrm{Aut}(\Omega)$ such that $x^+_\varphi = z$. But then, by Proposition 6.1,

$$T^{\mathbb{K}}_z \partial\Omega \cap \partial\Omega = \{z\}.$$ □

Lemma 10.5 *Suppose $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$, $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain with C^1 boundary, and there exist $x, y \in \mathcal{L}(\Omega)$ such that $T_x^{\mathbb{K}} \partial\Omega \neq T_y^{\mathbb{K}} \partial\Omega$. Then for all $x^+, y^+ \in \mathcal{L}(\Omega)$ distinct, there exists a biproximal element $\varphi \in \operatorname{Aut}(\Omega)$ such that $x^+ = x_\varphi^+$ and $y^+ = x_\varphi^-$.*

Proof By definition, there exist $\phi_m, \varphi_n \in \operatorname{Aut}(\Omega)$ and $p, q \in \Omega$ such that $\phi_m p \to x^+$ and $\varphi_n q \to y^+$. Then $\varphi_n p \to y^+$ by Proposition 5.1. By passing to a subsequence, we may suppose that $\phi_m^{-1} p \to x^-$ and $\varphi_n^{-1} p \to y^-$ for some $x^-, y^- \in \partial\Omega$.

If $\{x^+, x^-\} \cap \{y^+, y^-\} = \varnothing$, then Lemma 10.4 implies that

$$\{T_{x^+}^{\mathbb{K}} \partial\Omega, T_{x^-}^{\mathbb{K}} \partial\Omega\} \cap \{T_{y^+}^{\mathbb{K}} \partial\Omega, T_{y^-}^{\mathbb{K}} \partial\Omega\} = \varnothing.$$

Then the lemma follows from Lemma 7.3 and Lemma 10.3.

Next consider the case in which $x^- = y^+$. Since $y^+ \neq x^+$, Lemma 10.4 implies that $T_{x^+}^{\mathbb{K}} \partial\Omega \neq T_{y^+}^{\mathbb{K}} \partial\Omega$. Then Lemma 7.2 implies that ϕ_m is biproximal for large m. The lemma then follows from Lemma 10.3.

When $y^- = x^+$, the same argument can be used to show that φ_n is biproximal for large n. Then the lemma follows from Lemma 10.3.

It remains to consider the case when $x^- = y^-$. Now by Theorem 7.1 and Corollary 8.6, there exists a biproximal element $\gamma \in \operatorname{Aut}(\Omega)$ such that

$$\{x^+, x^-, y^+, y^-\} \cap \{x_\gamma^+, x_\gamma^-\} = \varnothing.$$

Then since $T_{x_\gamma^\pm}^{\mathbb{C}} \partial\Omega \cap \partial\Omega = \{x_\gamma^\pm\}$, this implies that

$$\{T_{x^+}^{\mathbb{K}} \partial\Omega, T_{x^-}^{\mathbb{K}} \partial\Omega\} \cap \{T_{x_\gamma^+}^{\mathbb{K}} \partial\Omega, T_{x_\gamma^-}^{\mathbb{K}} \partial\Omega\} = \varnothing.$$

So by Lemma 7.3 and Lemma 10.3, there exists a biproximal element ϕ such that $x_\phi^+ = x^+$ and $x_\phi^- = x_\gamma^+$. Now,

$$\phi^n p \to x^+, \quad \phi^{-n} p \to x_\gamma^-, \quad \varphi_m p \to y^+, \quad \text{and} \quad \varphi_m^{-1} p \to y^-.$$

Also, $\{x^+, x_\gamma^-\} \cap \{y^+, y^-\} = \varnothing$. So

$$\{T_{x^+}^{\mathbb{K}} \partial\Omega, T_{x_\gamma^-}^{\mathbb{K}} \partial\Omega\} \cap \{T_{y^+}^{\mathbb{K}} \partial\Omega, T_{y^-}^{\mathbb{K}} \partial\Omega\} = \varnothing,$$

and the lemma follows from Lemma 7.3 and Lemma 10.3. □

Proof of Proposition 10.1 We first observe that $\mathcal{L}(\Omega)$ is closed. Suppose $x_n \in \mathcal{L}(\Omega)$ and $x_n \to x$. Then there exist $\varphi_{n,m} \in \operatorname{Aut}(\Omega)$ and $p_n \in \Omega$ such that

$$\lim_{m \to \infty} \varphi_{n,m} p_n = x_n.$$

Now fix $p \in \Omega$; then by Proposition 5.1,

$$\lim_{m\to\infty} \varphi_{n,m} p = x_n.$$

Then there exists $m_n \to \infty$ such that

$$\lim_{n\to\infty} \varphi_{n,m_n} p = x.$$

So $\mathcal{L}(\Omega)$ is closed.

Now if $x, y \in \mathcal{L}(\Omega)$ are distinct, then there exists a biproximal element $\varphi \in \mathrm{Aut}(\Omega)$ such that $x = x_\varphi^+$ and $y = x_\varphi^-$. Then by Corollary 8.5, $y \in \mathrm{Aut}_0(\Omega) \cdot x$. Since $x, y \in \mathcal{L}(\Omega)$ were arbitrary, we see that $\mathrm{Aut}_0(\Omega)$ acts transitively on $\mathcal{L}(\Omega)$.

Now $\mathrm{Aut}_0(\Omega) \leq \mathrm{PGL}_{d+1}(\mathbb{C})$, being a closed subgroup (see Proposition 4.4), is a Lie subgroup, and it acts smoothly on $\mathbb{P}(\mathbb{K}^{d+1})$. Since $\mathcal{L}(\Omega) = \mathrm{Aut}_0(\Omega) \cdot x$ for any $x \in \mathcal{L}(\Omega)$, we see from Lemma 10.2 that $\mathcal{L}(\Omega)$ is a C^∞ submanifold of $\mathbb{P}(\mathbb{K}^{d+1})$. □

11 Proof of Theorem 1.2

For this section, suppose that $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ is a proper domain, $\partial\Omega$ is a C^1 hypersurface, and the limit set spans $\mathbb{K}^{d+1}$. Since the limit set spans, there exist $x, y \in \mathcal{L}(\Omega)$ such that $x \notin T_y^{\mathbb{K}} \partial\Omega$. Then $T_x^{\mathbb{K}} \partial\Omega \neq T_y^{\mathbb{K}} \partial\Omega$, and so $\mathrm{Aut}(\Omega)$ contains a biproximal element by Theorem 7.1.

Now fix a biproximal element $\varphi \in \mathrm{Aut}_0(\Omega)$, and let $H^\pm = T_{x_\varphi^\pm}^{\mathbb{K}} \partial\Omega$. Pick coordinates so that

(1) $x_\varphi^+ = [1 : 0 : \cdots : 0]$,

(2) $x_\varphi^- = [0 : 1 : 0 : \cdots : 0]$,

(3) $H^+ \cap H^- = \{[0 : 0 : z_2 : \cdots : z_d] : z_2, \dots, z_d \in \mathbb{K}\}$.

For the rest of the proof, identify $\mathbb{K}^d$ with the affine chart

$$\{[1 : z_1 : z_2 : \cdots : z_d] : z_1, \dots, z_d \in \mathbb{K}\}.$$

Then by Theorem 8.2, there exists a C^1 function $F\colon \mathbb{K}^{d-1} \to \Omega$ such that

$$\Omega = \{(z_1, z_2, \dots, z_d) : \mathrm{Re}(z_1) > F(z_2, \dots, z_d)\}.$$

By Corollary 8.4, for $t \in \mathbb{R}$,

$$\psi_t := \begin{pmatrix} e^t & & \\ & e^{-t} & \\ & & \mathrm{Id}_{d-1} \end{pmatrix} \in \mathrm{Aut}_0(\Omega).$$

Now there exist $x_1, \dots, x_{d-1} \in \mathcal{L}(\Omega)$ such that (as $\mathbb{K}$–lines)

$$x_\varphi^+ + x_\varphi^- + x_1 + \cdots + x_{d-1} = \mathbb{K}^{d+1}.$$

By Proposition 6.1,

$$H^- \cap \partial\Omega = T_{x_\varphi^+}^{\mathbb{K}} \partial\Omega \cap \partial\Omega = \{x_\varphi^-\},$$

and so $x_1, \dots, x_{d-1}$ are contained in our fixed affine chart.

Now we claim that

$$T_0\mathcal{L}(\Omega) = \mathbb{K}_P \times \mathbb{K}^{d-1} = T_0\partial\Omega.$$

Notice that the second equality is true by definition.

For $1 \le i \le d-1$, let L_i be the $\mathbb{K}$–line in $\mathbb{K}^d$ which contains 0 and x_i.

Now fix some i. By Lemma 10.5 and Theorem 8.2, $L_i \cap \partial\Omega$ is projectively equivalent to a half space, and thus in the affine chart, $\mathbb{K}^d$ is either a half space or a open ball in L_i; see Observation A.1. Since $F(z_2, \dots, z_d) > 0$ for all nonzero $(z_2, \dots, z_d)$, we see that $L_i \cap \partial\Omega$ must be an open ball in the affine chart. Moreover, by Theorem 8.2, $L_i \cap \partial\Omega \subset \mathcal{L}(\Omega)$. Now since $L_i \cap \partial\Omega$ is a sphere, we can pick $a_1, \dots, a_r \in L_i \cap \partial\Omega$ so that $r = \dim_{\mathbb{R}} \mathbb{K}$ and, as elements of this affine chart,

$$\operatorname{Span}_{\mathbb{R}}\{a_1, \dots, a_r\} = L_i.$$

Let $P\colon \mathbb{K}^d \to \mathbb{K}^d$ be the projection

$$P(z_1, \dots, z_d) = (0, z_2, \dots, z_d).$$

Now if $z \in \mathcal{L}(\Omega)$, then

$$\psi_t(z) = (e^{-2t} z_1, e^{-t} z_2, \dots, e^{-t} z_d) \in \mathcal{L}(\Omega),$$

and

$$\lim_{t \to \infty} \frac{1}{e^{-t}} \psi_t(z) = (0, z_2, \dots, z_d) = P(z).$$

Since $\mathcal{L}(\Omega)$ is a submanifold, this implies that $P(z) \in T_0\mathcal{L}(\Omega)$. Thus we see that $P(a_1), \dots, P(a_r) \subset T_0\mathcal{L}(\Omega)$. Hence $P(L_i) \subset T_0\mathcal{L}(\Omega)$.

Since i was arbitrary, we then see that

$$P(L_1 + \cdots + L_{d-1}) \subset T_0\mathcal{L}(\Omega).$$

But since

$$x_\varphi^+ + x_\varphi^- + x_1 + \cdots + x_{d-1} = \mathbb{K}^{d+1}$$

as $\mathbb{K}$–lines, we have

$$\{0\} \times \mathbb{K}^{d-1} = P(L_1 + \cdots + L_{d-1}) \subset T_0\mathcal{L}(\Omega).$$

Using Theorem 8.2, we see that $\mathbb{K}_P \times \{0\} \subset \mathcal{L}(\Omega)$, and so

$$T_0\mathcal{L}(\Omega) = \mathbb{K}_P \times \mathbb{K}^{d-1} = T_0\partial\Omega.$$

Thus $\mathcal{L}(\Omega) \subset \partial\Omega$ is an open and closed submanifold of $\partial\Omega$. Since $\partial\Omega$ is connected, this implies that $\mathcal{L}(\Omega) = \partial\Omega$.

Then since $\mathcal{L}(\Omega)$ is a C^∞ submanifold of $\mathbb{P}(\mathbb{K}^{d+1})$, we see that $\partial\Omega$ is C^∞, and so the theorem follows from Theorem 1.6. □

12 An example

In this section, for $d \geq 2$, we construct a nonsymmetric proper domain $\Omega \subset \mathbb{P}(\mathbb{K}^{d+1})$ with $C^{1,1}$ boundary so that there exist $x, y \in \mathcal{L}(\Omega)$ with $T_x^{\mathbb{K}}\partial\Omega \neq T_y^{\mathbb{K}}\partial\Omega$.

Let $S = \{z \in \mathbb{K}^{d-1} : \|z\| = 1\}$ be the unit sphere, and let $\hat{F}\colon S \to \mathbb{R}_{>0}$ be a $C^{1,1}$ function (which is not C^2). Then define the function $F\colon \mathbb{K}^{d-1} \to \mathbb{R}_{\geq 0}$ by

$$F(z) = \begin{cases} \hat{F}(z/\|z\|) & \text{if } z \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that F is $C^{1,1}$ on $\mathbb{K}^{d-1} \setminus \{0\}$.

Then consider the domain

$$\Omega = \{[1 : z_1 : \cdots : z_d] \in \mathbb{P}(\mathbb{K}^{d+1}) : \mathrm{Im}(z_1) > (|z_2|^2 + \cdots + |z_d|^2)F(z_2, \ldots, z_d)\}.$$

Clearly $\partial\Omega$ is $C^{1,1}$ away from $[1:0:\cdots:0]$ and $[0:1:0:\cdots:0]$. Since F is bounded, $\partial\Omega$ is $C^{1,1}$ at $[1:0:\cdots:0]$. Moreover, if we consider the projective map

$$T([z_0 : z_1 : z_2 : \cdots : z_d]) = [z_1 : -z_0 : z_2 : \cdots : z_d],$$

then

$$T(\Omega) = \{[1 : w_1 : \cdots : w_d] \in \mathbb{P}(\mathbb{K}^{d+1}) : \\ \mathrm{Im}(w_1) > (|w_2|^2 + \cdots + |w_d|^2)F(-w_2/w_1, \ldots, -w_d/w_1)\}.$$

Thus, since F is bounded, $T(\Omega)$ is $C^{1,1}$ at $[1:0:0:\cdots:0]$. Hence Ω is $C^{1,1}$ at $[0:1:0:\cdots:0]$.

Notice that $\mathrm{Aut}(\Omega)$ contains the transformation

$$[z_0 : z_1 : z_2 : \cdots : z_d] \to [e^t z_0 : e^{-t} z_1 : z_2 : \cdots : z_d]$$

for any $t \in \mathbb{R}$. Thus $[1:0:\cdots:0], [0:1:0:\cdots:0] \in \mathcal{L}(\Omega)$. Finally, Ω is not projectively isomorphic to $\mathcal{B}$ because $\partial\Omega$ is not C^2.

Appendix: Möbius transformations

In this section, we review the basic properties of Möbius transformations when $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{H}$. All these facts are well known when $\mathbb{K}=\mathbb{C}$.

We can identify $\mathbb{P}(\mathbb{K}^2)$ with $\overline{\mathbb{K}}=\mathbb{K}\cup\{\infty\}$ via the map

$$[z_1:z_2]\to\begin{cases} z_1(z_2)^{-1} & \text{if } z_2\neq 0,\\ \infty & \text{otherwise.}\end{cases}$$

With this identification, $\mathrm{PGL}_2(\mathbb{K})$ acts on $\overline{\mathbb{K}}$ by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot z=(az+b)(cz+d)^{-1}.$$

As in the complex case, Möbius transformations map spheres and hyperplanes to spheres and hyperplanes.

Observation A.1 With the above action, $\mathrm{PGL}_2(\mathbb{K})$ maps spheres and hyperplanes to spheres and hyperplanes.

Proof Every sphere and half plane can be described as a set of the form

$$\{z\in\mathbb{K}:|z-a|=R|z-b|\}$$

for some $a,b\in\mathbb{K}$ and $R>0$. Moreover, every set of this form is a sphere or half plane. A calculation shows that Möbius transformations map a set of this form to a set of this form. □

Let

$$\mathcal{H}_+=\{z\in\mathbb{K}:\mathrm{Re}(z)>0\}.$$

Now $\mathcal{H}_+$ is projectively equivalent to the unit ball by the Möbius transformation

$$z\to(z-1)(z+1)^{-1}.$$

In particular, $\mathrm{Aut}(\mathcal{H}_+)$ is isomorphic with

$$\mathrm{Aut}(\{|z|<1\})=\mathrm{PU}_{\mathbb{K}}(1,1)=\{\varphi\in\mathrm{PGL}(\mathbb{K}^2):Q\circ\varphi=Q\},$$

where $Q(z)=|z_1|^2-|z_2|^2$. The next proposition follows from the basic geometry of rank one symmetric spaces of noncompact type, but we provide an elementary proof.

Proposition A.2 (1) *If $x\in\partial\mathcal{H}_+\subset\overline{\mathbb{K}}$, then the group*

$$P_x=\{\varphi\in\mathrm{Aut}_0(\mathcal{H}_+):\varphi x=x\}$$

acts transitively on $\mathcal{H}_+$.

(2) $\mathrm{Aut}_0(\mathcal{H}_+)$ *acts transitively on* $\partial\mathcal{H}_+$.

(3) $\mathrm{Aut}_0(\mathcal{H}_+)$ *is generated by the two subgroups*

$$U = \left\{ \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} : \mathrm{Re}(w) = 0 \right\} \quad \textit{and} \quad V = \left\{ \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} : \mathrm{Re}(w) = 0 \right\}.$$

Proof A direct calculation shows that

$$P_\infty = \left\{ \begin{pmatrix} \lambda & w \\ 0 & \bar{\lambda}^{-1} \end{pmatrix} : \lambda, w \in \mathbb{K},\ \lambda \neq 0,\ \mathrm{Re}(w) = 0 \right\}.$$

Then P_∞ clearly acts on transitively on $\mathcal{H}_+$ and $\partial\mathcal{H}_+ \setminus \{\infty\}$. Since

$$P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} P_\infty \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we see that P_0 acts transitively on $\partial\mathcal{H}_+ \setminus \{0\}$. Since $\mathrm{Aut}_0(\mathcal{H}_+)$ contains P_0 and P_∞, this implies part (2). Then since $\mathrm{Aut}_0(\mathcal{H}_+)$ acts transitively on the boundary, we see that every group P_x is conjugate to P_∞. Then since P_∞ acts transitively on $\mathcal{H}_+$, we have part (1).

It remains to prove part (3). Let G be the closed group generated by U and V. If $\mathrm{Re}(u) = \mathrm{Re}(w) = 0$, then

$$\left[\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \right] = \begin{pmatrix} wu & 0 \\ 0 & -uw \end{pmatrix} = \begin{pmatrix} wu & 0 \\ 0 & \overline{wu} \end{pmatrix}.$$

So the Lie algebra of G contains

$$\left\{ \begin{pmatrix} \lambda & w \\ u & -\bar{\lambda} \end{pmatrix} : \lambda, w, u \in \mathbb{K},\ \mathrm{Re}(w) = \mathrm{Re}(u) = 0 \right\}.$$

In particular, G contains P_∞ and P_0. This implies that G acts transitively on the boundary. Now suppose $\varphi \in \mathrm{Aut}_0(\mathcal{H}_+)$. Since G acts transitively on $\partial\mathcal{H}_+$, there exists $\gamma \in G$ such that $(\gamma\varphi)(0) = 0$. Then $\gamma\varphi \in P_0 \subset G$, which implies that $\varphi \in G$. □

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