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Gromov–Witten theory of Fano orbifold curves, Gamma integral structures and ADE-Toda hierarchies

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We construct an integrable hierarchy in the form of Hirota quadratic equations (HQEs) that governs the Gromov–Witten invariants of the Fano orbifold projective curve $\mathbb{P}^1_{a_1,a_2,a_3}$. The vertex operators in our construction are given in terms of the K–theory of $\mathbb{P}^1_{a_1,a_2,a_3}$ via Iritani's Γ–class modification of the Chern character map. We also identify our HQEs with an appropriate Kac–Wakimoto hierarchy of ADE type. In particular, we obtain a generalization of the famous Toda conjecture about the GW invariants of $\mathbb{P}^1$ to all Fano orbifold curves.

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1 Introduction

Witten's conjecture [51], proved by Kontsevich [38], states that certain intersection numbers on the Deligne–Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$ of Riemann surfaces are governed by the KdV hierarchy. By definition, the intersection numbers on $\overline{\mathcal{M}}_{g,n}$ are the Gromov–Witten (GW) invariants of $X = \mathrm{pt}$. It is natural to ask for a generalization of Witten's conjecture by allowing a more general target manifold X. On the other hand, the examples of integrable hierarchies known at the time of Witten's conjecture were quite isolated, with each example being a separate study, while the possible targets are quite diverse in nature. Nevertheless, Dubrovin and Zhang [16] managed to develop a general theory of integrable systems, based on the notion of a Frobenius manifold and bi-Hamiltonian geometry. Their theory, modulo a certain technical issue which was overcome by Buryak, Posthuma and Shadrin [6; 7], proves the existence of an integrable hierarchy that governs the GW invariants of any manifold with a semisimple quantum cohomology.

There is another approach to integrable systems discovered by M Sato [45] and developed further by Date, Jimbo, Kashiwara and Miwa [14] and by Kac and Wakimoto [36]. The remarkable feature of this approach is that it gives an elegant and very explicit system of differential equations that depends on the root system of an appropriate

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simple Lie algebra. Since the equations are quadratic in the partial derivatives, they are usually called Hirota bilinear equations (HBEs) and sometimes also Hirota quadratic equations (HQEs).

The big motivation for our project is to find out whether the integrable hierarchies in GW theory admit a description in terms of HQEs. There is a natural candidate for root systems, namely the set of exceptional objects in the derived category of X or, assuming Kontsevich's homological mirror symmetry, the set of vanishing cycles of an appropriate Landau–Ginzburg model of X. Constructing HQEs in terms of exceptional objects or vanishing cycles is a very difficult problem, because our knowledge about them is very limited. If $\dim_{\mathbb{C}}(X) > 1$, then only for $X = \mathbb{P}^2$ it is known that the exceptional objects are classified by the solutions of the Markov equations (see Rudakov [42]) and even in this case the structure underlying the exceptional objects seems to be quite sophisticated and mysterious.

Although the vanishing cycles for 1–dimensional orbifolds are well understood, the problem of finding HQEs is still open. The goal of this paper is the case of the 1–dimensional Fano orbifolds. We prove that the corresponding GW theory is governed by certain Kac–Wakimoto integrable hierarchies, which we call *ADE-Toda hierarchies*. Our result yields the first examples of Kac–Wakimoto hierarchies with applications to GW theory that are neither homogeneous nor principal, as well as the first cases where the constructed HQEs govern the GW theory of a *nontoric* target. While the set of vanishing cycles in the Fano case is an affine root system of type ADE, in the non-Fano cases (of 1–dimensional orbifolds with semisimple quantum cohomology) the set of vanishing cycles corresponds to the real roots of a nonaffine Kac–Moody Lie algebra. The generalization of Kac–Wakimoto hierarchies for nonaffine Kac–Moody Lie algebras is a very challenging problem.

1.0.1 Fano orbifold curves By definition a Fano orbifold is a compact complex orbifold with a positive anticanonical bundle. In complex dimension 1, all Fano orbifolds are classified by triplets of positive integers $\boldsymbol{a} = \{a_1, a_2, a_3\}$ where $a_1 \le a_2 \le a_3$ and

$$\chi := \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1 > 0.$$

For each such $\boldsymbol{a}$ the corresponding Fano orbifold $\mathbb{P}^1_{\boldsymbol{a}}$ is topologically $\mathbb{P}^1$ and it has three orbifold points p_k $(k = 1, 2, 3)$ with local isotropy groups $\mathbb{Z}_{a_k}$.[1] The case $a_1 = a_2 = a_3 = 1$ is the smooth curve $\mathbb{P}^1$. It is easy to see that χ is the orbifold Euler characteristic of $\mathbb{P}^1_{\boldsymbol{a}}$.

[1] Using the root construction (see Abramovich, Graber and Vistoli [1] and Cadman [8]) we can construct $\mathbb{P}^1_{\boldsymbol{a}}$ from $\mathbb{P}^1$ by adding $\mathbb{Z}_{a_1}$–, $\mathbb{Z}_{a_2}$– and $\mathbb{Z}_{a_3}$–orbifold points

To each Fano orbifold curve, we can uniquely associate a Dynkin diagram with a Weyl group element σ_b. The triplets $\boldsymbol{a} = \{a_1, a_2, a_3\}$ with $\chi > 0$ are classified by the Dynkin diagrams of types ADE together with a choice of a *branching node*. In the D and E cases there is a unique choice of a branching node, while in the A case any node can be chosen. By removing the branching node we obtain three diagrams of type A_{a_k-1}, $k = 1, 2, 3$. If $a_k = 1$ then the corresponding diagram is empty.

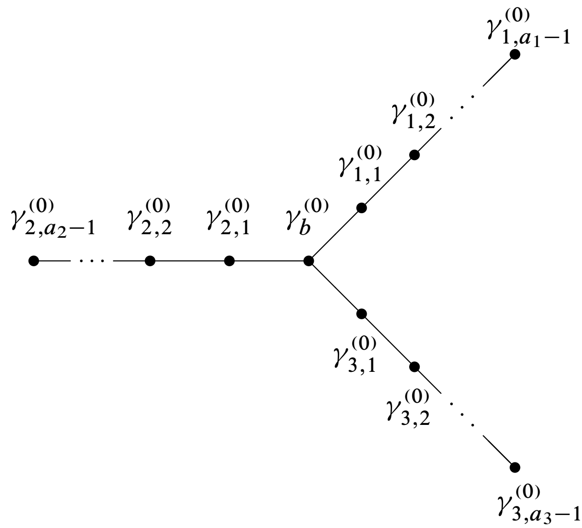

Figure 1: The branching node

We label the p^{th} simple root on the k^{th} branch of the Dynkin diagram by $\gamma^{(0)}_{k,p}$. The unique element σ_b in the Weyl group that is assigned to the triplet $\boldsymbol{a}$ is defined by

$$\sigma_b = \prod_{k=1}^{3} (s^{(0)}_{k,a_k-1} \cdots s^{(0)}_{k,2} s^{(0)}_{k,1}). \tag{1}$$

Here $\mathfrak{h}^{(0)}$ is the Cartan subalgebra of the corresponding simple Lie algebra $\mathfrak{g}^{(0)}$ and $s^{(0)}_{k,p}\colon \mathfrak{h}^{(0)} \to \mathfrak{h}^{(0)}$ is the reflection through the hyperplanes orthogonal to $\gamma^{(0)}_{k,p}$. The automorphism σ_b can be extended to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$. We denote by κ the order of σ_b as an automorphism of $\mathfrak{g}^{(0)}$.

1.0.2 Gromov–Witten theory The main objects in the orbifold GW theory of $\mathbb{P}^1_{\boldsymbol{a}}$ are the moduli spaces $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\boldsymbol{a}}, d)$ of orbifold stable maps f from a domain orbifold genus-g curve Σ with n marked points, to the target orbifold $\mathbb{P}^1_{\boldsymbol{a}}$, such that the homology class of the image of f is d times the fundamental class of the underlying curve of $\mathbb{P}^1_{\boldsymbol{a}}$. The descendant GW invariants (see (8)) are intersection numbers on the moduli space of stable maps, denoted by

$$\langle \phi_1 \psi_1^{k_1}, \dots, \phi_n \psi_n^{k_n} \rangle_{g,n,d},$$

where ψ_j is the j^{th} ψ–class on the moduli space of stable maps and $\phi_j \in H := H_{\mathrm{CR}}(\mathbb{P}^1_{\boldsymbol{a}}, \mathbb{C})$, the Chen–Ruan orbifold cohomology ring of $\mathbb{P}^1_{\boldsymbol{a}}$ with a unit $\mathbf{1} \in H$. As a vector space, H is just the cohomology of the *inertia orbifold* $I\mathbb{P}^1_{\boldsymbol{a}}$ of $\mathbb{P}^1_{\boldsymbol{a}}$. Our main interest is a potential $\mathcal{D}_{\boldsymbol{a}}(\hbar; \boldsymbol{q})$ in a certain *Fock space* (see (121)), defined by the following generating series of GW invariants:

$$\mathcal{D}_{\boldsymbol{a}}(\hbar; \boldsymbol{q}) = \exp\Bigg(\sum_{g,n,d} \hbar^{g-1} \frac{Q^d}{n!} \langle \boldsymbol{q}(\psi_1), \dots, \boldsymbol{q}(\psi_n) \rangle_{g,n,d} \Bigg), \tag{2}$$

where $Q \in \mathbb{C}^*$ is the Novikov variable, $\hbar$ is a formal variable,

$$\boldsymbol{q}(z) := q_0 + q_1 z + q_2 z^2 + \cdots \in H[\![z]\!],$$

and $\mathcal{D}_{\boldsymbol{a}}(\hbar; \boldsymbol{q})$ is obtained from the *total descendant potential* $\mathcal{D}_{\boldsymbol{a}}(\hbar; \boldsymbol{t})$ by a *dilaton shift*

$$q_m = t_m - \delta_{m,1} \mathbf{1}.$$

1.0.3 Mirror symmetry and Γ–conjecture for the Milnor lattice The construction of HQEs for the total descendant potential $\mathcal{D}_{\boldsymbol{a}}(\hbar; \boldsymbol{t})$ is performed by applying the methods developed by Givental [23; 24] to the Landau–Ginzburg (LG) mirror model (see Ishibashi, Shiraishi and Takahashi [32] and Rossi [41]) of $\mathbb{P}^1_{\boldsymbol{a}}$. In order to identify the resulting hierarchy with a Kac–Wakimoto hierarchy we follow the same strategy as in Frenkel, Givental and Milanov [17], ie we verify that the vertex operators in our construction provide a realization of the basic representation of an appropriate affine Kac–Moody Lie algebra. For more details we refer to Section 3.

While in [17; 23; 24] the vertex operators are constructed in terms of period integrals, in this paper we make use of Iritani's integral structure [30] (see also Katzarkov, Kontsevich and Pantev [37]), which allows us to express the vertex operators in our construction in terms of K–theory. This observation seems to be quite general, so we formulate a conjecture for the general case (see Conjecture 8), which we refer to as the Γ–*conjecture for the Milnor lattice*. One of the key results in this paper is that the Γ–conjecture for the Milnor lattice holds for the Fano orbifolds $\mathbb{P}^1_{\boldsymbol{a}}$ (see Section 2.4). This allows us to obtain an identification of the Milnor lattice and the K–ring of $\mathbb{P}^1_{\boldsymbol{a}}$, which leads to very elegant explicit formulas for both the set of vanishing cycles and the corresponding vertex operators.

1.0.4 The σ_b–twisted Kac–Wakimoto hierarchy The HQE of the σ_b–twisted Kac–Wakimoto hierarchy are given by the following bilinear equation for $\tau = (\tau_n(y))_{n\in\mathbb{Z}}$:

$$\operatorname{Res}_{\zeta=0}\frac{d\zeta}{\zeta}\Bigg(\sum_{\alpha\in\Delta^{(0)}} a_\alpha(\zeta)E_\alpha(\zeta)\otimes E_{-\alpha}(\zeta)\Bigg)\tau\otimes\tau$$
$$=\Bigg(\frac{1}{12}\sum_{k=1}^{3}\frac{a_k^2-1}{a_k}+\frac{\chi}{2}\,(\partial_\omega\otimes 1-1\otimes\partial_\omega)^2$$
$$+\sum_{(i,l)\in I_+}\Big(\frac{m_i}{\kappa}+\ell\Big)(y_{i,\ell}\otimes 1-1\otimes y_{i,\ell})(\partial_{y_{i,\ell}}\otimes 1-1\otimes\partial_{y_{i,\ell}})\Bigg)\tau\otimes\tau. \tag{3}$$

Here $\tau_n(y)$ belongs to a certain Fock space $\mathbb{C}[y]$, and:

- $\mathbb{C}[y]$ is the algebra of polynomials on $y=(y_{i,\ell})$, $(i,l)\in I_+ := \mathfrak{I}\backslash\{(00)\}\times\mathbb{Z}_{\geq 0}$, where

$$\mathfrak{I}=\{(k,p)\in\mathbb{Z}^2 \mid 1\leq k\leq 3, 1\leq p\leq a_k-1\}\cup\{(k,p)=(00),(01)\}.$$

- The vector space $\mathbb{C}[y]^{\mathbb{Z}}$ is a direct product of copies of $\mathbb{C}[y]$ indexed by $n\in\mathbb{Z}$. It is equipped with the structure of a module over the algebra of differential operators in e^ω by setting

$$(e^\omega\cdot\tau)_n=\tau_{n-1},\quad (\partial_\omega\cdot\tau)_n=n\tau_n,\quad \tau=(\tau_n)_{n\in\mathbb{Z}}\in\mathbb{C}[y]^{\mathbb{Z}}.$$

- $\Delta^{(0)}$ is the root lattice of $\mathfrak{g}^{(0)}$. For every root $\alpha\in\Delta^{(0)}$, the operator $E_\alpha(\zeta)=E_\alpha^{(0)}(\zeta)E_\alpha^*(\zeta)$ is defined by *vertex operators* $E_\alpha^{(0)}(\zeta)$ and $E_\alpha^*(\zeta)$.
- Let $a_0=1$ and $m_i := (1-p/a_k)\kappa$ for $i=(k,p)\in\mathfrak{I}$. Let $(k,p)^*=(k,a_k-p)$. We fix a basis $\{H_i\}_{i\in\mathfrak{I}}$ of $\mathfrak{h}^{(0)}$ (see Section 3.2) that $H_{00}=H_{01}$ and $\sigma_b(H_i)=e^{2\pi\sqrt{-1}m_i}H_i$, and define

$$E_\alpha^*(\zeta):=\exp\Bigg(\sum_{(i,\ell)\in I_+}(\alpha|H_i)y_{i,\ell}\zeta^{m_i+\ell\kappa}\Bigg)\exp\Bigg(\sum_{(i,\ell)\in I_+}(\alpha|H_{i^*})\frac{\partial}{\partial y_{i,\ell}}\frac{\zeta^{-m_i-\ell\kappa}}{-m_i-\ell\kappa}\Bigg).$$

- ω_b and $\omega_{k,p}$ are the fundamental weights corresponding to $\gamma_b^{(0)}$ and $\gamma_{k,p}^{(0)}$, and

$$E_\alpha^0(\zeta)=\exp((\omega_b|\alpha)\omega)\exp\Bigg(\Bigg((\omega_b|\alpha)\,\chi\,\log\zeta^\kappa\;-\sum_{k=1}^{3}\sum_{p=1}^{a_k-1}\frac{2\pi\sqrt{-1}}{a_k}(\omega_{k,p}|\alpha)\Bigg)\partial_\omega\Bigg).$$

- $a_\alpha(\zeta)$ is a certain coefficient which will be defined in Section 3.3 by (77).

We call the hierarchy in (3) the *ADE-Toda hierarchy corresponding to the triplet* $\boldsymbol{a}$.

1.0.5 The main theorem We fix a basis $\{\phi_i\}_{i\in\mathfrak{I}}$ of H as in Section 2 and let $q_\ell = \sum_i q_\ell^i \phi_i$. Notice that $\phi_{00} = \mathbf{1}$ is the unit. Then we define a sequence of formal power series

$$\tau_n(\hbar; \boldsymbol{q}) = (\kappa^\chi Q)^{\frac{1}{2}n^2} \mathcal{D}_{\boldsymbol{a}}(\hbar; \boldsymbol{q} + n\sqrt{\hbar}\mathbf{1}), \quad n \in \mathbb{Z}. \tag{4}$$

Using the change of variables

$$y_{i,\ell} = \frac{1}{\sqrt{\hbar}} \frac{\kappa^{d_i}}{\sqrt{\kappa C_i}} \frac{q_\ell^i}{m_i(m_i+\kappa)\cdots(m_i+\ell\kappa)}, \quad C_i = \begin{cases} \chi, & i = (01), \\ a_k, & i = (k,p) \neq (01), \end{cases}$$

we identify the sequence of descendant variables q_ℓ^i with the dynamical variables $y_{i,\ell}$ of the ADE-Toda hierarchy. Our main result can be stated as follows.

Theorem 1 *For any Fano orbifold curve $\mathbb{P}^1_{\boldsymbol{a}}$, the sequence $(\tau_n(\hbar; \boldsymbol{q}))_{n\in\mathbb{Z}}$ in (4) is a solution to the corresponding ADE-Toda hierarchy, ie the σ_b–twisted Kac–Wakimoto HQE (3).*

The proof of Theorem 1 follows the idea of the argument of Givental and Milanov [24]. However, one of the greatest achievements of this paper is that we managed to improve the argument in such a way that it will also apply in general for all other orbifolds. Namely, first we used K–theory to obtain explicit formulas for the leading terms of the period mapping. In particular, this simplifies the analysis of the monodromy representation. Second, a certain analyticity property (see Section 4.4) of the so-called phase factors, which was previously proved via the theory of finite reflection groups and their relation to Artin groups, is now proved by arguments applicable in much more general settings, as they rely only on the fact that the Gauss–Manin connection has regular singularities and that the vertex operators are local to each other (in the sense of the theory of vertex operator algebras).

1.0.6 Further questions The variables $q_1^{00}, q_2^{00}, \ldots$ appear as parameters in the differential equations (3) for τ. It is natural to expect that the σ_b–twisted Kac–Wakimoto HQE can be extended to include differential equations in $q_1^{00}, q_2^{00}, \ldots$ as well. For example, for Dynkin diagrams of type A, our hierarchy should agree with a certain reduction of the 2D Toda hierarchy and the required extension was constructed by G Carlet [9] based on the ideas of Carlet, Dubrovin and Zhang [10]. For the type D and E cases, the extension can be constructed using the same idea as in Milanov [39] with a slight necessary modification.

We will see in Section 5 that in the case $\boldsymbol{a} = \{2,2,2\}$ the genus-0 potential is uniquely determined by the integrable hierarchy and the string equation. It is tempting to conjecture that the sequence (4) is the unique solution to the ADE-Toda hierarchy satisfying the string equation. Moreover, we expect that (4) can be identified with an appropriate matrix integral.

Finally, it is very interesting to investigate the relation between the integrable hierarchies obtained by applying Dubrovin and Zhang's construction [16] to the quantum cohomology of $\mathbb{P}^1_{\boldsymbol{a}}$ and the integrable hierarchies in Theorem 1. It is natural to expect that the two approaches yield the same integrable hierarchy. We hope to return to this problem in the near future.

Organization of the paper The rest of this paper is organized as follows.

In Section 2, we recall the orbifold GW theory for Fano projective curves $\mathbb{P}^1_{\boldsymbol{a}}$ and the corresponding LG mirror model. Then we prove that Iritani's integral structure [30] for $\mathbb{P}^1_{\boldsymbol{a}}$ corresponds to the Milnor lattice under mirror symmetry. We also use the period mapping to identify the root system arising from the set of vanishing cycles with an affine root system in the quantum cohomology of $\mathbb{P}^1_{\boldsymbol{a}}$.

In Section 3, we give a Fock-space realization of the basic representations of the affine Lie algebras of ADE type. Then we recall the Kac–Wakimoto hierarchies, construct integrable hierarchies for affine cusp polynomials and show that these hierarchies are related by a Laplace transform (Theorem 28).

In Section 4 we construct another hierarchy (99) and describe its relation with the hierarchies from previous sections; see Proposition 31. Then we show that the ancestor potential of $\mathbb{P}^1_{\boldsymbol{a}}$ satisfies the integrable hierarchy (99). This proves Theorem 1. In Section 5 we consider the example $\boldsymbol{a} = \{2, 2, 2\}$.

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2 Orbifold GW theory of Fano orbifold curves $\mathbb{P}^1_{\boldsymbol{a}}$ and its Γ–integral lattice

The goal in this section is to introduce some of the background on orbifold GW theory, recall the appropriate LG mirror model, and finally prove the Γ–conjecture for the Milnor lattice.

2.1 Orbifold GW theory of $\mathbb{P}^1_{\boldsymbol{a}}$ and its mirror symmetry

Recall the index set

$$\begin{aligned}\mathfrak{I} &:= \mathfrak{I}_{\mathrm{tw}} \cup \{(00),(01)\} \\ &:= \big\{(k,p) \mid 1 \le k \le 3, 1 \le p \le a_k - 1\big\} \cup \{(k,p) = (00),(01)\}.\end{aligned} \tag{5}$$

We fix a basis of the Chen–Ruan orbifold cohomology $H := H_{\mathrm{CR}}(\mathbb{P}^1_{\boldsymbol{a}};\mathbb{C})$ as follows:

$$\phi_{00} = \mathbf{1}, \quad \phi_{01} = P$$

are the unit and the hyperplane class of the underlying $\mathbb{P}^1$, respectively, and

$$\phi_i = \phi_{k,p}, \quad i := (k,p) \in \mathfrak{I}_{\mathrm{tw}}$$

are the units of the corresponding twisted sectors of $\mathbb{P}^1_{\boldsymbol{a}}$. The cohomology degrees of the classes are

$$\deg_{\mathrm{CR}} \phi_i = \frac{p}{a_k}, \quad i = (k,p) \in \mathfrak{I}, \qquad a_0 := 1,$$

where slightly violating the standard conventions we work with complex degree, ie half of the usual real degrees. There is a natural involution $*$ on $\mathfrak{I}$ induced by orbifold Poincaré duality

$$(k,p)^* = (k, a_k - p). \tag{6}$$

The orbifold Poincaré pairing $(-,-)$ on H is nonzero only for the following cases:

$$(\phi_i, \phi_j) = \frac{1}{a_i}\delta_{i,j^*}, \quad \text{where } a_i := a_k \text{ for all } i = (k,p) \in \mathfrak{I}.$$

GW theory studies integrals over moduli spaces of stable maps. In this paper, we will use both the descendant invariants and the ancestor invariants. Let us introduce their definitions for Fano orbifold curves $\mathbb{P}^1_{\boldsymbol{a}}$. For more details on orbifold GW theory we refer to [11] for the analytic approach and to [1] for the algebraic geometry approach. Let $d \in \mathrm{Eff}(\mathbb{P}^1_{\boldsymbol{a}}) \subset H_2(\mathbb{P}^1_{\boldsymbol{a}};\mathbb{Z}) \cong \mathbb{Z}$ be an effective curve class. By choosing the homology class $[\mathbb{P}^1_{\boldsymbol{a}}]$ as a $\mathbb{Z}$–basis of $H_2(\mathbb{P}^1_{\boldsymbol{a}};\mathbb{Z})$ we may identify d with a nonnegative integer.

Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\boldsymbol{a}}, d)$ be the moduli space of stable orbifold maps f from a genus-g nodal orbifold Riemann surface Σ to $\mathbb{P}^1_{\boldsymbol{a}}$, such that $f_*[\Sigma] = d$. In addition, Σ is equipped with n marked points $z_1, \dots, z_n$ that are pairwise distinct and not nodal and the orbifold structure of Σ is nontrivial only at the marked points and the nodes. The moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\boldsymbol{a}}, d)$ has a virtual fundamental cycle $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\boldsymbol{a}}, d)]^{\mathrm{virt}}$. Its homology degree is

$$2\big((3 - \dim \mathbb{P}^1_{\boldsymbol{a}})(g-1) + \chi \cdot d + n\big). \tag{7}$$

The moduli space is naturally equipped with line bundles $\mathcal{L}_j$ formed by the cotangent lines[2] $T^*_{\bar{z}_j}\bar{\Sigma}/\operatorname{Aut}(\bar{\Sigma}, \bar{z}_1, \dots, \bar{z}_n; f)$ and with evaluation map

$$\mathrm{ev}\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\boldsymbol{a}}, d) \to \underbrace{I\mathbb{P}^1_{\boldsymbol{a}} \times \cdots \times I\mathbb{P}^1_{\boldsymbol{a}}}_{n},$$

obtained by evaluating f at the (orbifold) marked points $z_1, \dots, z_n$ and landing at the connected component of the inertia orbifold $I\mathbb{P}^1_{\boldsymbol{a}}$ corresponding to the generator of the automorphism group of the orbifold point z_j (see [11]).

The *descendant orbifold GW invariants* of $\mathbb{P}^1_{\boldsymbol{a}}$ are intersection numbers

$$\langle \phi_1 \psi_1^{k_1}, \dots, \phi_n \psi_n^{k_n} \rangle_{g,n,d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\boldsymbol{a}}, d)]^{\mathrm{virt}}} \mathrm{ev}^*(\phi_1 \otimes \cdots \otimes \phi_n)\, \psi_1^{k_1} \cdots \psi_n^{k_n}, \tag{8}$$

where $\phi_j \in H := H_{\mathrm{CR}}(\mathbb{P}^1_{\boldsymbol{a}}; \mathbb{C})$, $\psi_j = c_1(\mathcal{L}_j)$. The *total descendant potential* is

$$\mathcal{D}_{\boldsymbol{a}}(\hbar; \boldsymbol{t}) = \exp\bigg(\sum_{g,n,d} \hbar^{g-1} \frac{Q^d}{n!} \langle \boldsymbol{t}(\psi_1), \dots, \boldsymbol{t}(\psi_n) \rangle_{g,n,d} \bigg),$$

where $Q \in \mathbb{C}^*$ is called the *Novikov variable*, $\hbar$, $t_0, t_1, \ldots \in H$ are formal variables and $\boldsymbol{t}(z) := t_0 + t_1 z + t_2 z^2 + \cdots$.

Let $\pi\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\boldsymbol{a}}, d) \to \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism and

$$\Lambda_{g,n,d}(\phi_1, \dots, \phi_n) := \pi_*\big([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_{\boldsymbol{a}}, d)]^{\mathrm{virt}} \cap \mathrm{ev}^*(\phi_1 \otimes \cdots \otimes \phi_n)\big).$$

The *ancestor orbifold GW invariants* of $\mathbb{P}^1_{\boldsymbol{a}}$ are intersections numbers over the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ $(2g - 2 + n > 0)$:

$$\langle \phi_1 \bar{\psi}_1^{k_1}, \dots, \phi_n \bar{\psi}_n^{k_n} \rangle_{g,n,d} := \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n,d}(\phi_1, \dots, \phi_n)\, \bar{\psi}_1^{k_1} \cdots \bar{\psi}_n^{k_n}, \tag{9}$$

[2] Here $\bar{\Sigma}$ is the nodal Riemann surface underlying Σ and $\bar{z}_j \in \bar{\Sigma}$ is the i^{th} marked point on $\bar{\Sigma}$.

where $\bar{\psi}_j$ is the j^{th} ψ–class over $\overline{\mathcal{M}}_{g,n}$. We define the total ancestor potential of $\mathbb{P}^1_{\boldsymbol{a}}$ as follows:

$$\mathcal{A}_{\boldsymbol{a}}(\hbar;\boldsymbol{t}) := \exp\left(\sum_{g,n,d} \hbar^{g-1}\frac{Q^d}{n!}\langle \boldsymbol{t}(\bar{\psi}_1),\ldots,\boldsymbol{t}(\bar{\psi}_n)\rangle_{g,n,d}\right). \tag{10}$$

For each element $t \in H$, it is useful to introduce the double bracket notation:

$$\langle\langle \phi_1\,\bar{\psi}_1^{k_1},\ldots,\phi_n\bar{\psi}_n^{k_n}\rangle\rangle_{g,n}(t) := \sum_{k,d}\frac{Q^d}{k!}\langle \phi_1\,\bar{\psi}_1^{k_1},\ldots,\phi_n\bar{\psi}_n^{k_n},t,\ldots,t\rangle_{g,n+k,d} \tag{11}$$

We define a total ancestor potential that depends on the choice of t:

$$\mathcal{A}_t(\hbar;\boldsymbol{t}) = \exp\left(\sum_{g,n}\hbar^{g-1}\frac{1}{n!}\langle\langle \boldsymbol{t}(\bar{\psi}_1),\ldots,\boldsymbol{t}(\bar{\psi}_n)\rangle\rangle_{g,n}(t)\right). \tag{12}$$

According to [21] the total ancestor potential $\mathcal{A}_t(\hbar;\boldsymbol{t})$ and the total descendant potential $\mathcal{D}_{\boldsymbol{a}}(\hbar;\boldsymbol{t})$ are related by the quantized action of a certain symplectic transformation $S_t(z)$ (see Section 2.2.2). We will explain the details of the quantization in the Appendix.

The *quantum cup product* is a family of associative commutative multiplications $\bullet_t$ (or just $\bullet$ if the reference point t is mentioned) in H defined for each $t \in H$ via the correlators

$$(\phi_i \bullet_t \phi_j, \phi_k) = \langle\langle \phi_i,\phi_j,\phi_k\rangle\rangle_{0,3}(t).$$

The degree-0 part of $\bullet_t$ at $t=0$ is called the *Chen–Ruan cup product*. We denote it by

$$\cup_{\mathrm{CR}} = \bullet_{t=0}|_{Q=0}$$

Let t_i, $i \in \mathfrak{I}$ be the corresponding coordinates of ϕ_i. The quantum cup product induces on H a Frobenius structure of conformal dimension 1 with respect to the *Euler vector field*

$$E = \sum_{i\in\mathfrak{I}} d_i t_i \frac{\partial}{\partial t_i} + \chi\frac{\partial}{\partial t_{01}},$$

where

$$d_i = 1 - \deg_{\mathrm{CR}}(\phi_i) := 1 - \frac{p}{a_k}, \quad i = (k,p) \in \mathfrak{I}.$$

2.1.1 Mirror symmetry The Frobenius structure on H arising from quantum cohomology can be identified with the Frobenius structure on a certain deformation space of the *affine cusp polynomial*

$$f_{\boldsymbol{a}}(x) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - \frac{1}{Q}x_1x_2x_3, \quad x = (x_1,x_2,x_3). \tag{13}$$

where $Q \in \mathbb{C}^*$ is the Novikov variable. The isomorphism in the case $a_1 = 1$ was established in [40] and the general case can be found in [41]. According to Ishibashi, Shiraishi and Takahashi (see [32]), the Frobenius structure can be described also in the general framework of K Saito's theory of primitive forms. This is precisely the point of view suitable for our purposes.

Denote the Milnor number of $f_{\boldsymbol{a}}$ (ie the number of critical points of a Morsification of $f_{\boldsymbol{a}}$) by

$$N+1 = a_1 + a_2 + a_3 - 1.$$

Denote the space of a miniversal deformation of the polynomial $f_{\boldsymbol{a}}$ by

$$M = \mathbb{C}^{N+1}.$$

Note that the cardinality of the set $\mathfrak{I}$ is $N+1$, so we can enumerate the coordinates on M via $s = (s_i)_{i \in \mathfrak{I}}$. Recall $\mathfrak{I}_{\mathrm{tw}} = \mathfrak{I} \setminus \{(00), (01)\}$. Given $s \in M$, we put

$$F(x,s) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - \frac{1}{Qe^{s_{01}}} x_1 x_2 x_3 + s_{00} + \sum_{i=(k,p)\in \mathfrak{I}_{\mathrm{tw}}} s_i\, x_k^p .$$

Let $C \subset M \times \mathbb{C}^3$ be the analytic subvariety with structure sheaf

$$\mathcal{O}_C = \mathcal{O}_{M\times\mathbb{C}^3}/(\partial_{x_1}F, \partial_{x_2}F, \partial_{x_3}F);$$

then the *Kodaira–Spencer map*

$$\mathcal{T}_M \to p_* \mathcal{O}_C, \quad \frac{\partial}{\partial s_i} \mapsto \frac{\partial F}{\partial s_i} \ \operatorname{mod} (\partial_{x_1}F, \partial_{x_2}F, \partial_{x_3}F), \tag{14}$$

where $p\colon M\times\mathbb{C}^3 \to M$ is the projection onto the first factor, is an isomorphism which allows us to define an associative, commutative multiplication $\bullet$ on $\mathcal{T}_M$. The main result in [32] is that

$$\omega = \frac{\sqrt{-1}}{Qe^{s_{01}}} dx_1 \wedge dx_2 \wedge dx_3$$

is a *primitive form* in the sense of K Saito (see [43]), which allows us to construct a Frobenius structure on M (see [44]). More precisely, the form ω gives rise to a residue pairing on $\mathcal{O}_C$

$$(\phi_1, \phi_2) = -\frac{1}{Q^2 e^{2 s_{01}}} \operatorname{Res}_{M\times\mathbb{C}^3/M} \frac{\phi_1\phi_2\, dx_1 \wedge dx_2 \wedge dx_3}{\partial_{x_1}F\, \partial_{x_2}F\, \partial_{x_3}F},$$

which via the Kodaira–Spencer isomorphism (14) induces a nondegenerate bilinear form on $\mathcal{T}_M$. Let us form the following family of connections on $\mathcal{T}_M$

$$\nabla = \nabla^{\mathrm{LC}} - \frac{1}{z}\sum_{i\in\mathfrak{I}}(\partial_{s_i}\bullet)\, ds_i,$$

where ∇^{LC} is the Levi-Cevita connection associated with the residue pairing and $\partial_{s_i}\bullet$ is the operator of multiplication by the vector field $\partial/\partial s_i$. Let us also introduce the *oscillatory integrals*

$$J_{\mathcal{A}}(s,z) = (-2\pi z)^{-3/2}\, z d_M \int_{\mathcal{A}_{s,z}} e^{F(x,s)/z}\,\omega \in T_s^* M,$$

where d_M is the de Rham differential on M, and $\mathcal{A}$ is a flat section of the bundle on $M\times\mathbb{C}^*$, whose fiber over a point (s,z) is given by the space of semi-infinite homology cycles

$$H_3\bigl(\mathbb{C}^3, \{x \mid \mathrm{Re}(F(x,s)/z) \ll 0\}; \mathbb{C}\bigr) \cong \mathbb{C}^{N+1}.$$

The fact that ω is primitive means that the connection ∇ is flat for all $z\neq 0$ and that after identifying $\mathcal{T}_M \cong \mathcal{T}_M^*$ via the residue pairing, the oscillatory integrals $J_{\mathcal{A}}$ give rise to flat sections of ∇. Moreover, since the oscillatory integrals are weighted-homogeneous functions if one assigns weights d_i $(i\in\mathfrak{I})$, $1/a_k$ $(1\le k\le 3)$, and χ to s_i, x_k and Q respectively, they satisfy an additional differential equation with respect to z. Let $E\in\mathcal{T}_M$ be the *Euler vector field*

$$E = \sum_{i\in\mathfrak{I}} d_i s_i \frac{\partial}{\partial s_i} + \chi\frac{\partial}{\partial s_{01}}.$$

Note that under the Kodaira–Spencer isomorphism E corresponds to the equivalence class of F in $p_*\mathcal{O}_C$. The oscillatory integrals satisfy the following differential equation:

$$(z\partial_z + E) J_{\mathcal{A}}(t,z) = \theta J_{\mathcal{A}}(t,z), \tag{15}$$

where $\theta\colon \mathcal{T}_M\to\mathcal{T}_M$ is the *Hodge grading operator* defined via

$$\theta(X) = \nabla^{\mathrm{LC}}_X(E) - \tfrac{1}{2}X \tag{16}$$

where the constant $\frac{1}{2}$ is chosen in such a way that θ is *antisymmetric* with respect to the residue pairing: $(\theta(X),Y) = -(X,\theta(Y))$.

The quantum cohomology computed at $t=0$ is isomorphic as a Frobenius algebra with $T_0 M$ (see [32; 41]). The identification has the form

$$\phi_i = x_k^p + \cdots, \quad \phi_{00} = 1, \quad \phi_{01} = \frac{1}{Q}x_1x_2x_3 + \cdots.$$

where $i = (k, p)$ is the index of a twisted class and the dots stand for some polynomials that involve higher-order powers of Q. More precisely, using the Kodaira–Spencer isomorphism we have

$$\phi_i = \partial_{s_i} + \cdots, \quad \phi_{00} = \partial_{s_{00}}, \quad \phi_{01} = \partial_{s_{01}} + \cdots,$$

where the dots stand for some vector fields depending holomorphically on Q near $Q = 0$ and vanishing at $Q = 0$. These additional terms are uniquely fixed by the requirement that the vector fields ϕ_i $(i \in \mathfrak{I})$ are flat, ie the residue pairing is constant independent of Q. On the other hand the flatness of ∇ implies that the residue pairing is flat, therefore we can extend uniquely the isomorphism $H \cong T_0 M$ to an isomorphism

$$TH \cong TM$$

such that the residue pairing coincides with the Poincaré pairing. In other words, the linear coordinates t_i, $i \in \mathfrak{I}$ on H are functions on M such that $t_i(0) = 0$, the vector field $\partial/\partial t_i$ is flat with respect to the Levi-Civita connection, and at $s = 0$ it coincides with ϕ_i. The mirror symmetry for quantum cohomology can be stated as follows.

Theorem 2 [32, Theorem 4.1] *The isomorphism $M \cong H$, $s \mapsto t(s)$ is an isomorphism of Frobenius manifolds, ie $T_s M \cong T_{t(s)} H$ as Frobenius algebras.*

Remark 3 Theorem 2 can be proved also by using the extended J–function of $\mathbb{P}^1_{\boldsymbol{a}}$ (see Section 2.4.3). Namely, it is not hard to derive an identification between the quantum cohomology D–module of $\mathbb{P}^1_{\boldsymbol{a}}$ and the D–module defined by $f_{\boldsymbol{a}}(x)$.

From now on we will make use of the residue pairing to identify $T^*M \cong TM$. Also the flat Levi-Civita connection ∇^{LC} allows us to construct a trivialization $TM \cong M \times T_0 M$, and finally, the Kodaira–Spencer map (14) together with the mirror symmetry isomorphism gives $T_0 M \cong H$. In other words, we have natural trivializations

$$T^*M \cong TM \cong M \times H. \tag{17}$$

2.2 The period integrals, the calibration operator, and higher genus

Givental noticed that certain period integrals (see formula (18) below) in singularity theory play a crucial role in the theory of integrable systems. In this section, we recall Givental's construction as well as some of its basic properties. See [23] for more details.

Put $X = M \times \mathbb{C}^3$ and let

$$\varphi\colon X \to M \times \mathbb{C}, \quad (s, x) \mapsto (s, F(x, s)).$$

The set of all $(s,\lambda)\in M\times\mathbb{C}$ such that the fibers of φ,

$$X_{s,\lambda} := \varphi^{-1}(s,\lambda)$$

is singular is an analytic hypersurface, called *discriminant*. Its complement in $M\times\mathbb{C}$ will be denoted by $(M\times\mathbb{C})'$. The homology and cohomology groups $H_2(X_{s,\lambda};\mathbb{C})$ and $H^2(X_{s,\lambda};\mathbb{C})$, $(s,\lambda)\in(M\times\mathbb{C})'$ form vector bundles over the base $(M\times\mathbb{C})'$. Moreover, the integral structure in the fibers allows us to define a flat connection known as the *Gauss–Manin* connection.

Let us fix the point $(0,1)\in(M\times\mathbb{C})'$ (for $Q\ll 1$) to be our reference point. The vector space

$$\mathfrak{h} = H_2(X_{0,1};\mathbb{C})$$

has a very rich structure, which we would like to recall. Let

$$\Delta\subset\mathfrak{h}$$

be the set of *vanishing cycles*, and $(\cdot\,|\,\cdot)$ be the *negative* of the intersection pairing. The negative sign is chosen so that $(\alpha|\alpha)=2$ for all $\alpha\in\Delta$. The parallel transport with respect to the Gauss–Manin connection induces a monodromy representation

$$\pi_1((M\times\mathbb{C})')\to \mathrm{GL}(\mathfrak{h}).$$

The image

$$W\subset \mathrm{GL}(\mathfrak{h})$$

of the fundamental group under this representation is a subgroup of the group of linear transformations of $\mathfrak{h}$ that preserve the intersection form. The Picard–Lefschetz theory can be applied in our setting as well and W is in fact a reflection group generated by the reflections

$$s_\alpha(x) = x-(\alpha|x)\alpha,\quad \alpha\in\Delta.$$

The reflection s_α is the monodromy transformation along a simple loop that goes around a generic point on the discriminant over which the cycle α vanishes. Finally, recall that the *classical* monodromy $\sigma\in W$ is the monodromy transformation along a big loop around the discriminant. For more details on vanishing homology and cohomology and the Picard–Lefschetz theory we refer to the book [4]. We will see in Proposition 13 below that Δ is an affine root system.

The main objects in our construction are the multivalued analytic functions

$$I_\alpha^{(n)}(t,\lambda) = -\frac{1}{2\pi}\,\partial_\lambda^{n+1}\, d_M \int_{\alpha_{t,\lambda}} d^{-1}\omega, \tag{18}$$

where the value of the RHS depends on the choice of a path avoiding the discriminant, connecting the reference point with (t,λ). The cycle $\alpha_{t,\lambda}$ is obtained from $\alpha \in \mathfrak{h}$ via a parallel transport (along the chosen path), $d^{-1}\omega$ is any holomorphic 2–form η on $\mathbb{C}^3$ such that $\omega = d\eta$, and d_M is the de Rham differential on M. The RHS in (18) defines naturally a cotangent vector in T_t^*M, which via the trivialization (17) is identified with a vector in H.

The period vectors (18) are uniquely defined for all $n \geq -1$. For $n \leq -2$ there is an ambiguity in choosing integration constants, which can be removed using the following differential equations:

$$\partial_{t_i} I_\alpha^{(n)}(t,\lambda) = -\phi_i \bullet I_\alpha^{(n+1)}(t,\lambda), \quad i \in \mathfrak{I}, \tag{19}$$

$$\partial_\lambda I_\alpha^{(n)}(t,\lambda) = I_\alpha^{(n+1)}(t,\lambda), \tag{20}$$

$$(\lambda - E\bullet)\partial_\lambda I_\alpha^{(n)}(t,\lambda) = \left(\theta - n - \tfrac{1}{2}\right) I_\alpha^{(n)}(t,\lambda). \tag{21}$$

Finally, note that the unit vector $\mathbf{1} \in H \cong M$ has coordinates $t_{00} = 1$, $t_i = 0$ for $i \neq (00)$ and that the period vectors have the following translation symmetry:

$$I_\alpha^{(n)}(t,\lambda) = I_\alpha^{(n)}(t - \lambda\mathbf{1}, 0) \quad \text{for all } n \in \mathbb{Z},\ \alpha \in \mathfrak{h}.$$

The oscillatory integrals are related to the period integrals via a Laplace transform along an appropriately chosen path:

$$J_{\mathcal{A}}(t,z) = (-2\pi z)^{-1/2} \int_{u_j}^{\infty} e^{\lambda/z} I_\alpha^{(0)}(t,\lambda)\, d\lambda, \tag{22}$$

where $u_j(t)$ is such that $(t, u_j(t))$ is a point on the discriminant over which the cycle α vanishes. The differential equations (19) are the Laplace transform of $\nabla J_{\mathcal{A}} = 0$, while the equation (21) is the Laplace transform of the differential equation (15). Using equations (20)–(21) we can express $I^{(n)}$ in terms of $I^{(n+1)}$ as long as the operator $\theta - n - \frac{1}{2}$ is invertible. This is the case for $n \leq -2$, which allows us to extend the definition of $I^{(n)}$ to all $n \in \mathbb{Z}$.

2.2.1 Stationary phase asymptotic Let $u_j(t)$, $1 \leq j \leq N+1$ be the critical values of $F(x,t)$. The set

$$M_{\mathrm{ss}} \subset M$$

of all points $t \in M$ such that the critical values $u_j(t)$ form locally near t a coordinate system is open and dense. Let us fix some $t_0 \in M_{\mathrm{ss}}$; then in a neighborhood of t_0 the critical values give rise to a coordinate system in which the pairing and the product $\bullet$ are diagonal, ie

$$\partial/\partial u_j \bullet \partial/\partial u_{j'} = \delta_{j,j'} \partial/\partial u_j, \quad (\partial/\partial u_j, \partial/\partial u_{j'}) = \delta_{j,j'}/\Delta_j,$$

where Δ_j are some multivalued analytic functions on M_{ss}. Following Dubrovin's terminology (see [15]), we refer to u_j as *canonical coordinates*.

Remark 4 It is easy to see that the critical variety C of the function F is nonsingular, ie it is a manifold. It can be proved that the projection map $p\colon C \subset M \times \mathbb{C}^3 \to M$ is a finite branched covering of degree $N+1$. The branching points are precisely $M \setminus M_{\mathrm{ss}}$.

Using the canonical coordinates we can construct a trivialization of the tangent bundle

$$\Psi\colon M_0 \times \mathbb{C}^{N+1} \cong TM_0, \quad (t, e_j) \mapsto \Big(t, \sqrt{\Delta_j}\,\frac{\partial}{\partial u_j}\Big).$$

Here $M_0 \subset M_{\mathrm{ss}}$ is an open contractible neighborhood of t_0 and $\{e_j\}$ is the standard basis of $\mathbb{C}^{N+1}$, where the j^{th} component of e_j is 1, while the remaining ones are 0. According to Givental (see [22]), there exists a unique formal asymptotic series $\Psi_t R_t(z) e^{U_t/z}$ that satisfies the same differential equations as the oscillatory integrals $J_{\mathcal{A}}$, where

$$R_t(z) = 1 + \sum_{\ell=1}^{\infty} R_\ell(t) z^\ell, \quad R_\ell(t) \in \mathrm{End}(\mathbb{C}^{N+1}). \tag{23}$$

We will make use of the formal series

$$\boldsymbol{f}_\alpha(t, \lambda; z) = \sum_{n \in \mathbb{Z}} I_\alpha^{(n)}(t, \lambda)\,(-z)^n, \quad \alpha \in \mathfrak{h}. \tag{24}$$

Example 5 Note that for the A_1–singularity $F(t,x) = x^2/2 + t$ we have $u := u_1(t) = t$. Up to a sign there is a unique vanishing cycle. The series (24) will be denoted simply by $\boldsymbol{f}_{A_1}(t, \lambda; z)$. The corresponding period vectors can be computed explicitly:

$$I_{A_1}^{(n)}(u, \lambda) = (-1)^n \frac{(2n-1)!!}{2^{n-1/2}} (\lambda - u)^{-n-1/2}, \quad n \ge 0$$

$$I_{A_1}^{(-n-1)}(u, \lambda) = 2\frac{2^{n+1/2}}{(2n+1)!!} (\lambda - u)^{n+1/2}, \quad n \ge 0.$$

The key lemma (see [23]) is the following.

Lemma 6 *Let $t \in M_{\mathrm{ss}}$ and β be a vanishing cycle vanishing over the point $(t, u_j(t))$. Then for all λ near $u_j := u_j(t)$, we have*

$$\boldsymbol{f}_\beta(t, \lambda; z) = \Psi_t R_t(z)\, e_j\, \boldsymbol{f}_{A_1}(u_j, \lambda; z).$$

An important corollary of Lemma 6 is this remarkable formula due to K Saito [43]:

$$(\alpha|\beta) = (I^{(0)}_\alpha(t,\lambda), (\lambda - E\bullet) I^{(0)}_\beta(t,\lambda)). \tag{25}$$

To prove this formula, first note that the differential equations (19)–(21) imply that the RHS is independent of t and λ. In order to compute the RHS, let us fix $t \in M_{\mathrm{ss}}$ and let λ approach one of the critical values $u_j(t)$ in such a way that the cycle β vanishes over $(t, u_j(t))$. According to Lemma 6 we have

$$I^{(0)}_\beta(t,\lambda) = 2(2(\lambda - u_j))^{-1/2} e_j + O((\lambda - u_j)^{1/2}).$$

Similarly, decomposing $\alpha = \alpha' + (\alpha|\beta)\beta/2$, where α' is invariant with respect to the local monodromy, we get

$$I^{(0)}_\alpha(t,\lambda) = (\alpha|\beta)\,(2(\lambda - u_j))^{-1/2} e_j + O((\lambda - u_j)^{1/2}).$$

It is well known (see [15]) that in canonical coordinates the Euler vector field has the form $E = \sum u_j \partial_{u_j}$. Now it is easy to see that the RHS of (25), up to higher-order terms in $(\lambda - u_j)$, is $(\alpha|\beta)$, and since the latter must be independent of λ the higher-order terms must vanish.

2.2.2 The calibration operator The *calibration* of the Frobenius structure on H is by definition a gauge transformation S of the form

$$S_t(z) = 1 + \sum_{\ell=1}^{\infty} S_\ell(t) z^{-\ell}, \quad S_\ell(t) \in \mathrm{End}(H), \tag{26}$$

such that $\nabla = S d S^{-1}$. In GW theory there is a canonical choice of calibration given by genus-0 descendant invariants as follows (see [21]):

$$(S_t(z)\phi_i, \phi_j) = (\phi_i, \phi_j) + \sum_{\ell=0}^{\infty} \langle\langle \phi_i \psi^\ell, \phi_j \rangle\rangle_{0,2}(t) z^{-\ell-1}. \tag{27}$$

Here

$$\langle\langle \phi_i \psi^\ell, \phi_j \rangle\rangle_{0,2}(t) = \sum_{m\ge 0}\sum_{d\ge 0} \frac{Q^d}{m!} \langle \phi_i \psi^\ell, \phi_j, t, \dots, t\rangle_{0,2+m,d}$$

is defined in (11). It is a general fact in GW theory (see [21]) that

$$S_t(z)^{-1}(\partial_z - z^{-1}\theta + z^{-2} E\bullet) S_t(z) = \partial_z - z^{-1}\theta + z^{-2}\rho, \tag{28}$$

where $\rho = \chi\, P \cup_{\mathrm{CR}}$. By definition the operator ρ acts on H as follows:

$$\rho(\phi_{00}) = \chi \phi_{01}, \qquad \rho(\phi_i) = 0 \quad \text{for } i \in \mathfrak{I} \setminus \{(00)\}. \tag{29}$$

We define a new series

$$\widetilde{f}_\alpha(\lambda; z) := S_t(z)^{-1}\, f_\alpha(t, \lambda; z). \tag{30}$$

Note that the RHS is independent of t. Put

$$\widetilde{f}_\alpha(\lambda; z) = \sum_{n\in\mathbb{Z}} \widetilde{I}^{(n)}_\alpha(\lambda)\,(-z)^n. \tag{31}$$

We will refer to $\widetilde{I}^{(n)}_\alpha(\lambda)$ as the *calibrated* limit of the period vector $I^{(n)}_\alpha(t,\lambda)$.

In our general set up the Novikov variable Q is a fixed nonzero constant. However, it will be useful also to allow Q to vary in a small contractible neighborhood and to study the dependence of the periods and their calibrated limits on Q. By definition $I^{(n)}_\alpha(t,\lambda)$ depend on $Qe^{t_{01}}$, so we simply have

$$Q\partial_Q\, I^{(n)}_\alpha(t,\lambda) = \partial_{t_{01}}\, I^{(n)}_\alpha(t,\lambda).$$

Using the *divisor equation* in GW theory, it is easy to prove (see [21]) that the gauge transformation $S_t(z)$ satisfies the following differential equation:

$$zQ\partial_Q\, S_t(z) = z\partial_{t_{01}} S_t(z) - S_t(z)\,(P\cup_{\mathrm{CR}}).$$

Finally, the gauge identity $\nabla = S\,dS^{-1}$ and the differential equations (19)–(21) imply that the calibrated limit of the period vectors satisfy the following system of differential equations:

$$Q\partial_Q \widetilde{I}^{(n)}_\alpha(\lambda) = -P\cup_{\mathrm{CR}}\, \widetilde{I}^{(n+1)}_\alpha(\lambda) \tag{32}$$

$$\partial_\lambda\, \widetilde{I}^{(n)}_\alpha(\lambda) = \widetilde{I}^{(n+1)}_\alpha(\lambda), \tag{33}$$

$$(\lambda-\rho)\partial_\lambda \widetilde{I}^{(n)}_\alpha(\lambda) = \big(\theta - n - \tfrac{1}{2}\big)\widetilde{I}^{(n)}_\alpha(\lambda). \tag{34}$$

Lemma 7 (a) *Let $\{B_i\}_{i\in\mathfrak{I}}$ be a basis of $\mathfrak{h}^\vee := H^2(X_{0,1};\mathbb{C})$. Then the following formula holds:*

$$\widetilde{I}^{(-1)}_\alpha(\lambda) = \langle B_{00},\alpha\rangle\big(\lambda\mathbf{1} + (\chi\log\lambda - \log Q)\,P\big) + \langle B_{01},\alpha\rangle\,P + \sum_{i\in\mathfrak{I}_{\mathrm{tw}}} \langle B_i,\alpha\rangle\lambda^{d_i}\,\phi_i.$$

(b) *The analytic continuation of $\widetilde{I}^{(n)}_\alpha(\lambda)$ along a closed loop around 0 is $\widetilde{I}^{(n)}_{\sigma(\alpha)}(\lambda)$, where σ is the classical monodromy.*

Proof (a) Recall ρ acts on H by (29), while the operator θ defined in (16) has the form (via (17))

$$\theta(\phi_i) = \big(d_i - \tfrac{1}{2}\big)\,\phi_i, \quad i\in\mathfrak{I}.$$

Note that the H–valued functions that follow the pairings $\langle B_i, \alpha\rangle$ are solutions to the system (32)–(34) with $n=-1$. These solutions are linearly independent, therefore they must give a basis in the space of all solutions.

(b) Now the statement follows, because it is true for $I^{(n)}_\alpha(t,\lambda)$, for $|\lambda| \gg 1$, where

$$I^{(n)}_\alpha(t,\lambda) = \widetilde{I}^{(n)}_\alpha(\lambda) + \sum_{\ell=1}^{\infty} (-1)^\ell S_\ell(t) \widetilde{I}^{(n+\ell)}_\alpha(\lambda). \qquad \square$$

2.2.3 Mirror symmetry in higher genus A Frobenius manifold is called *semisimple* if the multiplication has a semisimple basis. The Frobenius manifold

$$(H, (\cdot,\cdot), \bullet_t, \phi_{00}, E)$$

is isomorphic to the Frobenius manifold constructed from the mirror model of $\mathbb{P}^1_{\boldsymbol{a}}$ [40; 41; 32]; see Theorem 2. Using the mirror model, it is easy to see that $\bullet_t$ is semisimple for generic t.

For any semisimple Frobenius manifold, Givental introduced a higher genus reconstruction formula [22] using the symplectic loop space formalism [21]. Furthermore, he conjectured that the higher genus GW ancestor invariants are uniquely determined from its semisimple quantum cohomology. Teleman [49] has proved this conjecture. In the case of the orbifold $\mathbb{P}^1_{\boldsymbol{a}}$, the Frobenius manifold is semisimple at a generic point $t \in H$. Teleman's higher genus reconstruction theorem [49] implies that the total ancestor potential defined in (12) can be identified with Givental's higher genus reconstruction formula [21] (using the quantization operator $\widehat{\cdot}$)

$$\mathcal{A}_t(\hbar; \boldsymbol{q}(z)) = \widehat{\Psi_t} \widehat{R_t} \widehat{\Psi_t^{-1}} \prod_{j=1}^{N+1} \mathcal{D}_{\mathrm{pt}}(\hbar\Delta_j; {}^j\boldsymbol{q}(z)\sqrt{\Delta_j}) \tag{35}$$
$$\in \mathbb{C}_{\hbar,Q}[\![q_0, q_1 + \mathbf{1}, q_2, \dots]\!]$$

and the total descendant potentials defined in (2) can be identified with

$$\mathcal{D}_{\boldsymbol{a}}(\hbar; \boldsymbol{q}(z)) = e^{F^{(1)}(t)} \widehat{S}_t^{-1} \mathcal{A}_t(\hbar; \boldsymbol{q}(z)), \tag{36}$$

where ${}^j\boldsymbol{q}(z) := \sum_{\ell=0}^{\infty} {}^jq_\ell z^\ell$ and the coefficients ${}^jq_\ell$ are defined by

$$\sum_{j=1}^{N+1} {}^jq_\ell \Psi(e_j) = \sum_{i \in \mathfrak{I}} q^i_\ell \phi_i.$$

Recall that $\mathcal{D}_{\mathrm{pt}}$ is the total descendant potential of a point and the genus-1 primary potential

$$F^{(1)}(t) = \sum_{d,n=0}^{\infty} \frac{Q^d}{n!} \langle t, \ldots, t \rangle_{1,n,d}.$$

For the reader's convenience, we explain the quantization formulas (35) and (36) in the Appendix.

2.3 Iritani's integral structure and mirror symmetry

If X is a compact complex orbifold, then using the K–ring $K(X)$ of orbifold vector bundles on X and a certain Γ–modification of the Chern character map, Iritani has introduced an integral lattice in the Chen–Ruan cohomology group $H_{\mathrm{CR}}(X;\mathbb{C})$ (see [30] and also [37]). If X has semisimple quantum cohomology, then it is expected that X has a LG mirror model and it is natural to conjecture that Iritani's embedding of the K–theoretic lattice coincide with the image of the Milnor lattice via an appropriate period map. In our case, when $X = \mathbb{P}^1_{\boldsymbol{a}}$, we prove the above conjecture by using the same argument as in [30], where the toric case was proved. Moreover, we obtain an explicit identification of the set of vanishing cycles with a certain K–theoretic affine root system.

Let us recall Iritani's construction in the most general case when X is a compact complex orbifold. Let IX be the *inertia orbifold* of X, ie as a groupoid the points of IX are

$$(IX)_0 = \{(x, g) \mid x \in X_0,\ g \in \mathrm{Aut}(x)\}$$

while the arrows from (x', g') to (x'', g'') consists of all arrows $g \in X_1$ from x' to x'' such that $g'' \circ g = g \circ g'$. It is known that IX is an orbifold consisting of several connected components X_v, $v \in T := \pi_0(|IX|)$. Following Iritani, we define a linear map

$$\Psi\colon K(X) \to H^*(IX;\mathbb{C}) = \bigoplus_{v \in T} H^*(X_v;\mathbb{C})$$

via

$$\Psi(V) = (2\pi)^{-\dim_{\mathbb{C}} X/2}\, \widehat{\Gamma}(X) \cup (2\pi\sqrt{-1})^{\deg}\, \mathrm{inv}^*\, \widetilde{\mathrm{ch}}(V). \tag{37}$$

Here $\cup$ is the usual cup product in $H^*(IX;\mathbb{C})$. Let us recall the notation. The linear operator

$$\deg\colon H^*(IX;\mathbb{C}) \to H^*(IX;\mathbb{C})$$

is defined by $\deg(\phi)=r\phi$ if $\phi\in H^{2r}(IX;\mathbb{C})$. The involution inv: $IX\to IX$ inverts all arrows while on the points it acts as $(x,g)\mapsto(x,g^{-1})$. If V is an orbifold vector bundle, then we have an eigenbasis decomposition

$$\mathrm{pr}^*(V)=\bigoplus_{v\in T}V_v=\bigoplus_{v\in T}\bigoplus_{0\le f<1}V_{v,f},$$

where pr: $IX\to X$ is the forgetful map $(x,g)\mapsto x$ and $V_{v,f}$ is the subbundle of $V_v:=\mathrm{pr}^*(V)|_{X_v}$ whose fiber over a point $(x,g)\in(IX)_0$ is the eigenspace of g corresponding to the eigenvalue $e^{2\pi\sqrt{-1}f}$. For $j=1,\dots,l_{v,f}:=\mathrm{rk}(V_{v,f})$, we denote by $\delta_{v,f,j}$ the Chern roots of $V_{v,f}$. Then the Chern character and the Γ–class of V are defined by

$$\widetilde{\mathrm{ch}}(V)=\sum_{v\in T}\sum_{0\le f<1}e^{2\pi\sqrt{-1}f}\,\mathrm{ch}(V_{v,f}),\quad \widehat{\Gamma}(V)=\sum_{v\in T}\prod_{0\le f<1}\prod_{j=1}^{l_{v,f}}\Gamma(1-f+\delta_{v,f,j}),$$

where the value of the Γ–function $\Gamma(1-f+y)$ at $y=\delta_{v,f,j}$ is obtained by first expanding in Taylor's series at $y=0$ and then formally substituting $y=\delta_{v,f,j}$. By definition $\widehat{\Gamma}(X):=\widehat{\Gamma}(TX)$.

2.3.1 The Γ–conjecture for the Milnor lattice We denote by $H_{\mathrm{CR}}(X;\mathbb{C})$ the vector space $H^*(IX;\mathbb{C})$ equipped with the Chen–Ruan cup product $\cup_{\mathrm{CR}}$. We define a shift function $\iota\colon T\to\mathbb{Q}$ by

$$\iota(v)=\sum_{0\le f<1}f\dim_{\mathbb{C}}(TX)_{v,f}.$$

The Chen–Ruan product is graded homogeneous with respect to the grading

$$\deg_{\mathrm{CR}}(\phi)=(r+\iota(v))\phi,\quad \phi\in H^{2r}(X_v;\mathbb{C}).$$

The vector space $H^*(IX;\mathbb{C})$ is equipped with a Poincaré pairing, ie

$$(\phi_1,\phi_2)=\int_{IX}\phi_1\cup\mathrm{inv}^*(\phi_2).$$

This pairing turns both algebras $H^*(IX;\mathbb{C})$ and $H_{\mathrm{CR}}(X;\mathbb{C})$ into Frobenius algebras. Let us point out also that by using the Kawasaki Riemann–Roch formula we can also prove that the map Ψ is compatible (up to a sign) with the natural pairing on $K(X)$ and the Poincaré pairing

$$\chi(V_1\otimes V_2^\vee)=\big(e^{\pi\sqrt{-1}\theta_X}e^{\pi\sqrt{-1}\rho_X}\Psi(V_1),\Psi(V_2)\big),\tag{38}$$

where $\rho_X=c_1(TX)\cup_{\mathrm{CR}}$ and θ_X is the *Hodge grading operator* of X,

$$\theta_X=\tfrac12\dim_{\mathbb{C}}X-\deg_{\mathrm{CR}}.$$

On the other hand, if X has a LG-mirror model, then we can define the calibrated periods $\widetilde{I}^{(-\ell)}_\alpha(\lambda)$ in the same way as in formulas (18) and (30). The main motivation for the above construction is the following conjecture, which is motivated by Iritani's mirror symmetry theorem in [30]. To simplify the formulation we set all Novikov variables to be 1. Using the divisor equation one can recover easily the Novikov variables.

Conjecture 8 (Γ–conjecture for the Milnor lattice) *Given an integral cycle α, there exists a class $V_\alpha \in K(X)$ in the K–theory of vector bundles such that, for all $\ell \gg 0$,*

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\lambda s}\, \widetilde{I}^{(-\ell)}_\alpha(\lambda)\, d\lambda = s^{-\theta_X-\ell-1/2} s^{-\rho_X}\, \Psi(V_\alpha).$$

The conjecture can be refined even further, by saying that if α is a vanishing cycle then V_α can be represented by an exceptional object in the derived category $\mathcal{D}^b(X)$ and that the monodromy transformations of α correspond to certain mutation operations in $\mathcal{D}^b(X)$. See [19] for more discussions.

Next we describe Conjecture 8 in the case of $\mathbb{P}^1_{\boldsymbol{a}}$.

2.3.2 The K–ring of $\mathbb{P}^1_{\boldsymbol{a}}$ Let $\boldsymbol{a} = (a_1, a_2, a_3)$ be a triple of nonnegative integers and put $X = \mathbb{P}^1_{\boldsymbol{a}}$. The orbifold $\mathbb{P}^1_{\boldsymbol{a}}$ can be constructed as follows. Put

$$G = \{t = (t_1, t_2, t_3) \in (\mathbb{C}^*)^3 \mid t_1^{a_1} = t_2^{a_2} = t_3^{a_3}\}.$$

We have

$$\mathbb{P}^1_{\boldsymbol{a}} = [Y_{\boldsymbol{a}}/G], \quad Y_{\boldsymbol{a}} = \big\{y = (y_1, y_2, y_3) \in \mathbb{C}^3 \setminus \{0\} \mid y_1^{a_1} + y_2^{a_2} + y_3^{a_3} = 0\big\},$$

where the quotient is taken in the category of orbifolds, ie it should be viewed as an orbifold groupoid. The K–ring of orbifold vector bundles on $\mathbb{P}^1_{\boldsymbol{a}}$ can be presented as a quotient of the polynomial ring $\mathbb{C}[L_1, L_2, L_3]$ by the relations

$$L = L_1^{a_1} = L_2^{a_2} = L_3^{a_3}, \qquad (1-L_k)(1-L_{k'}) = 0 \quad \text{for } 1 \le k < k' \le 3.$$

Here L is the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ under the natural map $\mathbb{P}^1_{\boldsymbol{a}} \to \mathbb{P}^1$, and the product is given by tensor product of vector bundles. The orbifold vector bundle L_k is the trivial line bundle $Y_{\boldsymbol{a}} \times \mathbb{C}$ equipped with the following G–action

$$G \times L_k \to L_k, \quad (t, y, v) \mapsto (ty, t_k v).$$

It is easy to see that the K–ring is generated by L_1, L_2, L_3, L. The first set of relations follows from the definition of G. To see the remaining ones, note that the coordinate function y_k on $Y_{\boldsymbol{a}}$ gives rise to a section of L_k. The Koszul complex associated with the sections $(y_k, y_{k'})$ is G–equivariant and it gives rise to the exact sequence

$$0 \to L_k^\vee \otimes L_{k'}^\vee \to L_k^\vee \oplus L_{k'}^\vee \to \mathcal{O}_{\mathbb{P}^1_{\boldsymbol{a}}} \to 0.$$

This proves that $(1-L_k)(1-L_{k'}) = 0$.

2.3.3 The image of $K(\mathbb{P}^1_{\boldsymbol{a}})$ The connected components of $I\mathbb{P}^1_{\boldsymbol{a}}$ are indexed by $\{(0,0)\} \cup \mathfrak{I}_{\mathrm{tw}}$. Let us denote by $P = c_1(L)$. Then $c_1(L_k) = P/a_k$. By the adjunction formula $TX = L_1L_2L_3L^{-1}$, we get

$$c_1(TX) = \chi P, \quad \chi = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1.$$

Furthermore, note that

$$(L_k)_{k',p,f} = \begin{cases} 0 & \text{if } k \neq k' \text{ and } f \neq 0, \\ 0 & \text{if } k = k' \text{ and } f \neq p/a_{k'}, \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

From here we get that the eigenspace decomposition of TX is

$$(TX)_{k,p,f} = \begin{cases} TX & \text{if } k = 0 \text{ and } f = 0, \\ \mathbb{C} & \text{if } k \neq 0 \text{ and } f = p/a_k, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that for $i = (k,p) \in \mathfrak{I}_{\mathrm{tw}}$, $d_i = d_{k,p} = 1 - p/a_k$, we get the formulas

$$\begin{aligned} \widehat{\Gamma}(X) &= \Gamma(1+\chi P) + \sum_{i \in \mathfrak{I}_{\mathrm{tw}}} \Gamma(d_i)\phi_i, \\ \widetilde{\mathrm{ch}}(L_k^m) &= \mathbf{1} + \frac{m}{a_k} P + \sum_{(j,p) \in \mathfrak{I}_{\mathrm{tw}}} \zeta_j^{mp\delta_{k,j}} \phi_{j,p}, \quad \zeta_j := e^{2\pi\sqrt{-1}/a_j}. \end{aligned}$$

Let us point out that in the above formulas $\mathbf{1}, P \in H^*(X_{0,0})$, while $\phi_{k,p} \in H^0(X_{k,p})$ is the standard generator for the twisted sector. Note that the unit of the algebra $(H^*(IX;\mathbb{C}), \cup)$ is

$$\widetilde{\mathrm{ch}}(\mathcal{O}) = \mathbf{1} + \sum_{i \in \mathfrak{I}_{\mathrm{tw}}} \phi_i.$$

Finally, since

$$(2\pi\sqrt{-1})^{\deg} \mathrm{inv}^* \widetilde{\mathrm{ch}}(L_k^m) = \mathbf{1} + \frac{2\pi\sqrt{-1}m}{a_k} P + \sum_{(j,p) \in \mathfrak{I}_{\mathrm{tw}}} \zeta_j^{-mp\delta_{k,j}} \phi_{j,p},$$

we get the formula

$$(39)\qquad (2\pi)^{1/2}\Psi(L_k^m) = \mathbf{1} + \left(-\gamma\chi + \frac{2\pi\sqrt{-1}m}{a_k}\right)P + \sum_{(j,p)\in\mathfrak{I}_{\mathrm{tw}}} \frac{\Gamma(d_{j,p})}{\zeta_j^{mp\delta_{k,j}}}\phi_{j,p}.$$

2.4 Γ–conjecture for Fano orbifold curves

Now we give a proof[3] of the Γ–conjecture for the Milnor lattice for $\mathbb{P}^1_{\boldsymbol{a}}$. The proof is obtained by applying Iritani's argument from the proof of [31, Theorem 4.11] and [31, Theorem 5.7] and relies on the Γ–conjecture for the Milnor lattice for the Fano toric orbifold (proven in [30])

$$Y := \mathbb{P}^2_{\boldsymbol{a}} = [(\mathbb{C}^3 \setminus \{0\})/G]$$

and the explicit formulas for the J–functions of $X := \mathbb{P}^1_{\boldsymbol{a}}$ and Y. Note that X is a suborbifold of Y.

Remark 9 There is a natural map $p\colon \mathbb{P}^2_{\boldsymbol{a}} \to \mathbb{P}^2$. The above description of $X = \mathbb{P}^1_{\boldsymbol{a}}$ realizes X as the zero locus of a section of the line bundle $p^*\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2_{\boldsymbol{a}}$. Applying the recipe of constructing mirrors of complete intersections in [20], we obtain $f_{\boldsymbol{a}}$ as the mirror of X.

Notice that the line bundles L_k are restrictions of line bundles on Y and the K–ring of Y is the quotient of the polynomial ring $\mathbb{C}[L_1, L_2, L_3]$ by the relations

$$L = L_1^{a_1} = L_2^{a_2} = L_3^{a_3}, \quad (1-L_1)(1-L_2)(1-L_3) = 0.$$

Put $L = p^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $P = c_1(L)$. We have isomorphisms

$$\mathbb{Q} \cong H_2(X;\mathbb{Q}) \cong H_2(Y;\mathbb{Q}), \quad d \mapsto d[\mathbb{P}^1_a]$$

and

$$H^2(Y;\mathbb{Q}) \cong H^2(X;\mathbb{Q}) \cong \mathbb{Q}, \quad \alpha \mapsto \langle \alpha, [\mathbb{P}^1_a]\rangle.$$

The J–function of an orbifold X used by Iritani is

$$J_X(\tau, z) = L(\tau, z)^{-1}\mathbf{1},$$

where $\tau \in H_{\mathrm{CR}}(X)$,

$$L(\tau, z) := S_\tau(-z)e^{-P\log Q/z}$$

and S is the calibration operator (27). Note that this definition differs from Givental's one by a sign and by the exponential factor.

[3] Note that $\mathbb{P}^1_{\boldsymbol{a}}$ is not covered by results in [30; 31].

2.4.1 Combinatorics of the inertia orbifolds The orbifold Y is toric. We describe its stacky fan as follows. Put $b_1 = (a_1, 0)$, $b_2 = (0, a_2)$, $b_3 = (-a_3, -a_3) \in \mathbb{Z}^2$. The fan of Y is

$$\Sigma \cong \{\varnothing, \{k\}, \{k, k'\} \mid 1 \le k, k' \le 3\},$$

where each set I on the RHS determines a cone in $\mathbb{R}^2$ spanned by b_k, $k \in I$. Note that Σ is the fan for $\mathbb{P}^2$. The fan map for Y sends the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{Z}^3$ to $\mathbb{Z}^2$ by

$$\mathbb{Z}^3 \to \mathbb{Z}^2, \quad e_k \mapsto b_k.$$

The connected components of IY are parametrized by

$$\Box(\Sigma) = \Big\{(c_1, c_2, c_3) \mid 0 \le c_k < 1, \ \sum\nolimits_k c_k b_k \in \mathbb{Z}^2 \cap \sigma \text{ for some cones } \sigma \in \Sigma\Big\},$$

where $c \in \Box(\Sigma)$ determines the twisted sector

$$Y_c = \big[\{y \in \mathbb{C}^3 \mid y_k = 0 \text{ if } c_k \neq 0\}/G\big],$$

which has a generic stabilizer given by the cyclic subgroup of G generated by

$$(e^{2\pi\sqrt{-1}c_1}, e^{2\pi\sqrt{-1}c_2}, e^{2\pi\sqrt{-1}c_3}) \in G.$$

The inertia orbifold IX is a suborbifold of IY and the twisted sectors of IX are parametrized by those $c \in \Box(\Sigma)$ for which $\dim(Y_c) > 0$, ie at most one component of c is nonzero.

2.4.2 The J–function of Y Let $\mathbf{1}_c \in H^0(Y_c)$ be the dual of the fundamental class for $c \in \Box(\Sigma)$ and

$$\tau = \tau_1 \mathbf{1}_{(1/a_1,0,0)} + \tau_2 \mathbf{1}_{(0,1/a_2,0)} + \tau_3 \mathbf{1}_{(0,0,1/a_3)}.$$

According to the mirror theorem of [13], the J–function $J_Y(\tau, z)$ depending on τ is equal to the S–extended I–function [13, Definition 28] with

$$S = \{(1,0), (0,1), (-1,-1)\}.$$

This gives

$$J_Y(\tau, z) = e^{P \log Q/z}\Bigg(\sum_{d=0}^{\infty} \sum_{n_1,n_2,n_3=0}^{\infty} \frac{Q^d}{z^{\deg_Y(Q^d)}} \frac{t^n}{n! z^{\deg_Y(t^n)}} J^Y_{d,n}(\tau, z)\Bigg),$$

where we introduced homogeneous parameters $t = (t_1, t_2, t_3)$, whose dependence on τ and Q can be determined from the expansion $J_Y = 1 + \tau/z + \cdots$, the degrees of Q

and t are

$$\deg_Y(Q^d) := \int_d c_1(TY) = d\Big(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\Big), \quad \deg_Y(t_k) := \deg_Y(\tau_k) = 1-1/a_k.$$

Finally, we denoted $n=(n_1,n_2,n_3)$ and we used the standard multi-index notations

$$t^n = t_1^{n_1}t_2^{n_2}t_3^{n_3}, \quad n! = n_1!n_2!n_3!.$$

In order to define the component $J^Y_{d,n}$ let us define $m_k \in \mathbb{Z}$ and $c_k \in \mathbb{Q}$ by

$$\frac{n_k-d}{a_k} = -m_k + c_k, \quad 0 \le c_k < 1.$$

Then we have

$$J^Y_{d,n}(\tau,z) = \frac{\mathbf{1}_c}{z^{\deg_Y(\mathbf{1}_c)}} \prod_{k=1}^{3} \frac{\Gamma(1-c_k+(P/a_k)z^{-1})}{\Gamma(1-c_k+m_k+(P/a_k)z^{-1})},$$

where if $c \notin \Box(\Sigma)$ then we set $\mathbf{1}_c = 0$. In other words we sum over all (d,n) such that at least one of the numbers c_k is 0.

2.4.3 The J–function of X Since $p^*\mathcal{O}_{\mathbb{P}^2}(1)$ is a convex line bundle in the sense of [12, Example B], the J–function of $\mathbb{P}^1_{\boldsymbol{a}}$ can be computed from that of $\mathbb{P}^2_{\boldsymbol{a}}$ using the quantum Lefschetz theorem of [50] and [12].

Using the embedding $j\colon IX \to IY$ we restrict τ and $\mathbf{1}_c$ to $H^*(IX)$. Slightly abusing the notation, we use the same notation for the restrictions. Note that now $\mathbf{1}_c = 0$ if c has more than one nonzero component. The formula for J_X has the same form

$$J_X(\tau,z) = e^{P\log Q/z}\Big(\sum_{d=0}^{\infty}\sum_{n_1,n_2,n_3=0}^{\infty} \frac{Q^d}{z^{\deg_X(Q^d)}}\frac{t^n}{n!\,z^{\deg_X(t^n)}} J^X_{d,n}(\tau,z)\Big),$$

where

$$J^X_{d,n}(\tau,z) = \frac{\mathbf{1}_c}{z^{\deg_X(\mathbf{1}_c)}} \frac{\Gamma(1+d+Pz^{-1})}{\Gamma(1+Pz^{-1})} \prod_{k=1}^{3} \frac{\Gamma(1-c_k+(P/a_k)z^{-1})}{\Gamma(1-c_k+m_k+(P/a_k)z^{-1})}$$

Note that the grading takes the form

$$\deg_X(Q^d) := \int_d c_1(TX) = d\Big(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}-1\Big)$$

while the degrees of t and $\mathbf{1}_c$ do not change, because the restriction map preserves the grading.

2.4.4 The Galois action The Picard group $\mathrm{Pic}(X)$ of isomorphism classes of (topological) orbifold line bundles on X can be presented as a quotient

$$\mathbb{Z}^3 \to \mathrm{Pic}(X), \quad (r_1, r_2, r_3) \mapsto L_1^{r_1} L_2^{r_2} L_3^{r_3}$$

with kernel given by the relations

$$a_1 e_1 = a_2 e_2 = a_3 e_3,$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{Z}^3$. The group $\mathrm{Pic}(X)$ acts naturally on the Milnor fibration via

$$\nu \cdot (x, t) = (\nu \cdot x, \nu \cdot t), \quad \nu = (r_1, r_2, r_3) \in \mathrm{Pic}(X),$$

where

$$(\nu \cdot x)_k = e^{2\pi\sqrt{-1} r_k / a_k} x_k,$$

and the action on the remaining components is defined in such a way that

$$F(\nu \cdot x, \nu \cdot t) = F(x, t),$$

that is,

$$\begin{aligned}
(\nu \cdot t)_{k,p} &= e^{-2\pi\sqrt{-1} r_k p / a_k} t_{k,p}, \quad 1 \le k \le 3,\ 1 \le p \le a_k - 1, \\
(\nu \cdot t)_{01} &= t_{01} + 2\pi\sqrt{-1} \sum_{k=1}^{3} \frac{r_k}{a_k}, \\
(\nu \cdot t)_{00} &= t_{00}.
\end{aligned}$$

Let us fix some $(t, \lambda) \in M \times \mathbb{C}$ with λ sufficiently large. Then for every $\nu = (r_1, r_2, r_3)$ we can construct a path from (t, λ) to $(\nu \cdot t, \lambda)$ as follows. Using the above formulas we let $c \in \mathbb{R}^3$ act on M. As c varies along the straight segment from 0 to $(r_1, r_2, r_3) \in \mathbb{Z}^3 \subset \mathbb{R}^3$ we get a path in M connecting t and $\nu \cdot t$. The parallel transport along this path with respect to the Gauss–Manin connection gives an identification $H_2(X_{\nu\cdot t,\lambda}; \mathbb{Z}) \cong H_2(X_{t,\lambda}; \mathbb{Z})$. Combined with the $\mathrm{Pic}(X)$–action on $\mathbb{C}^3$ we get an action

$$\mathrm{Pic}(X) \times H_2(X_{t,\lambda}; \mathbb{Z}) \to H_2(X_{t,\lambda}; \mathbb{Z}), \quad (\nu, \alpha) \mapsto \nu(\alpha).$$

Following Iritani, we refer to this as the Galois action of $\mathrm{Pic}(X)$ on the Milnor lattice.

Lemma 10 *If the Γ–conjecture for the Milnor lattice is true for some cycle α and $V_\alpha \in K(X)$ is the corresponding K–theoretic vector bundle, then the conjecture is true for all $\nu(\alpha)$, $\nu = (r_1, r_2, r_3) \in \mathrm{Pic}(X)$. Moreover, $V_{\nu(\alpha)} = V_\alpha \otimes L_\nu$, $L_\nu = L_1^{r_1} L_2^{r_2} L_3^{r_3}$.*

Proof Using the vector space decomposition

$$H_{\mathrm{CR}}(X) = H^*(X) \oplus \Big(\bigoplus_{(k,p)\in\mathfrak{I}_{\mathrm{tw}}} H_{\mathrm{CR}}^{p/a_k}(X) \Big),$$

we define a linear operator

$$\theta_\nu \colon H_{\mathrm{CR}}(X) \to H_{\mathrm{CR}}(X), \quad \theta_\nu = \sum_{k=1}^{3} \sum_{p=1}^{a_k-1} \frac{r_k p}{a_k} \mathrm{pr}_{k,p},$$

where $\mathrm{pr}_{k,p}$ is the projection onto the subspace $H_{\mathrm{CR}}^{p/a_k}(X)$. By changing the variables $y = \nu \cdot x$ in the period integrals we get

$$I^{(\ell)}_{\nu(\alpha)}(t,\lambda) = e^{-2\pi\sqrt{-1}\theta_\nu} I^{(\ell)}_\alpha(\nu^{-1}\cdot t,\lambda) \quad \text{for all } \ell\in\mathbb{Z}.$$

On the other hand the calibration operator satisfies

$$S_{\nu^{-1}(t)}(z) = e^{2\pi\sqrt{-1}\theta_\nu} S_t(z) e^{-2\pi\sqrt{-1}\theta_\nu} e^{-2\pi\sqrt{-1}c_1(L_\nu)/z},$$

which can be seen easily by using that if the correlator $\langle \alpha_1\psi_1^{k_1},\dots,\alpha_n\psi_n^{k_n}\rangle_{0,n,d}$ is nonzero then, since we have at least one stable map $f\colon C\to X$, we have

$$\chi(f^*L_\nu) = \int_d c_1(L_\nu) - \sum_{j=1}^{n} \theta_\nu(\alpha_j) \in \mathbb{Z}.$$

Since by definition

$$I^{(\ell)}_\alpha(t,\lambda) = S_t(-\partial_\lambda^{-1})\widetilde{I}^{\ell}_\alpha(\lambda),$$

the above formulas imply that

$$\widetilde{I}^{(\ell)}_{\nu(\alpha)}(\lambda) = e^{-2\pi\sqrt{-1}\theta_\nu} e^{2\pi\sqrt{-1}c_1(L_\nu)\partial_\lambda} \widetilde{I}^{\ell}_\alpha(\lambda).$$

In particular, after taking a Laplace transform, we get

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\lambda s}\widetilde{I}^{(-\ell)}_{\nu(\alpha)}(\lambda) = e^{-2\pi\sqrt{-1}\theta_\nu} e^{2\pi\sqrt{-1}c_1(L_\nu)s} \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\lambda s}\widetilde{I}^{(-\ell)}_\alpha(\lambda).$$

On the other hand, using the definition of Ψ we get

$$\Psi(V\otimes L_\nu) = e^{-2\pi\sqrt{-1}\theta_\nu} e^{2\pi\sqrt{-1}c_1(L_\nu)}\Psi(V).$$

It remains only to notice that $s^{-\theta}P = (Ps)s^{-\theta}$ and that θ_ν commutes with both θ and the Chen–Ruan product multiplication operators. $\square$

The Milnor lattice is known to be unimodular with respect to the K–theoretic bilinear form

$$(\cdot,\cdot)\colon K(X)\otimes_{\mathbb{Z}} K(X)\to\mathbb{Z},\quad (L_1,L_2)=\chi(L_1\otimes L_2^{\vee})$$

(see [31, Section 2]). The above lemma implies that it is enough to prove that the Γ–conjecture holds for the structure sheaf. Indeed, if this is true, then since $K(X)$ is generated by $\mathrm{Pic}(X)$, the Γ conjecture correspondence will embed $K(X)$ into a sublattice of the Milnor lattice. Since both lattices are unimodular, they must coincide.

2.4.5 The central charge Iritani's Γ–conjecture for the Milnor lattice looks different since he works with Lefschetz thimbles. Nevertheless, our formulation is completely equivalent. Here is the reason. Take a Lefschetz thimble $\mathcal{A}$ corresponding to a vanishing cycle α, ie for fixed $(t,z)\in M\times\mathbb{C}^*$ we fix a path C in $\mathbb{C}$ from u_j to ∞ such that $\mathrm{Re}(\lambda/z)>0$ for all $\lambda\in C$ and the cycle $\alpha_{t,\lambda}$ vanishes when λ approaches u_j. In this way we can identify the Milnor lattice with a lattice of Lefschetz thimbles.

We claim that

$$L(t,z)^{-1}\int_{u_j}^{\infty} e^{-\lambda/z} I_\alpha^{(-\ell)}(t,\lambda)\,d\lambda = e^{z^{-1}P\log Q}\int_0^{\infty} e^{-\lambda/z}\widetilde{I}_\alpha^{(-\ell)}(\lambda)\,d\lambda,$$

where $L(t,z)=S_t(-z)e^{-z^{-1}P\log Q}$. Indeed, one can check easily using the quantum differential equations that the LHS is independent of t and Q. On the other hand we have

$$L(t,z)=1-z^{-1}P\log Q+\cdots,\quad u_j=0+\cdots\quad\text{and}\quad I_\alpha^{(-\ell)}(t,\lambda)=\widetilde{I}_\alpha^{(-\ell)}(\lambda)+\cdots,$$

where the dots stand for terms that vanish at $t=Q=0$. So modulo terms that vanish at $t=Q=0$ the LHS coincides with the RHS. Our claim follows.

We define the *central charge* of $V_\alpha\in K(X)$ by

$$Z_X^{(0)}(V_\alpha)(t,z):=\big(L(t,z)z^{\theta}z^{\rho}\Psi(V_\alpha),\mathbf{1}\big).$$

Since we will use the result of Iritani, let us clarify the relation between our notations. In Iritani's notation, the central charge is defined to be

$$Z_X^{(n)}(V)(t,z):=(2\pi z)^{n/2}(2\pi\sqrt{-1})^{-n}\big(L(t,z)z^{\theta}z^{\rho}\Psi(V_\alpha),\mathbf{1}\big),\quad n=\dim_{\mathbb{C}}(X).$$

For the LG models studied in [30] the Γ–conjecture for the central charge is stated as

$$(2\pi\sqrt{-1})^{-n}\int_{\mathcal{A}} e^{-F(x,t)/z}\omega = Z_X^{(n)}(V_\alpha)(t,z).$$

As we see from the LG model that we use, in general one should choose n to be the number of variables in the LG potentials. For the LG models in [30] the number of variables coincides with the dimension of the orbifold, so this difference does not matter.

The identity in the Γ–conjecture for the Milnor lattice is equivalent to

$$\frac{1}{\sqrt{2\pi}}\int_{u_j}^{\infty} e^{-\lambda/z} I_{\alpha}^{(-\ell)}(t,\lambda)\,d\lambda = L(t,z) z^{\theta+\ell+1/2} z^{\rho}\Psi(V_\alpha). \tag{40}$$

The number ℓ must be chosen sufficiently large. We will see that in our case $\ell=1$ works. In general ℓ can be chosen such that the number of variables in the LG potential is $2\ell+1$. Recalling the definition of the period integrals, we transform the LHS into

$$(-z d_M)(2\pi)^{-3/2}\int_{\mathcal{A}} e^{-F(x,t)/z}\omega,$$

where d_M is the de Rham differential on M. In particular, since the Poincaré pairing of the RHS with $\mathbf{1}$ corresponds to contracting the LHS with ∂_{00} we get

$$(2\pi z)^{-3/2}\int_{\mathcal{A}} e^{-F(x,t)/z}\omega = Z_X^{(0)}(V_\alpha)(t,z). \tag{41}$$

In order to prove the Γ–conjecture for the Milnor lattice it is enough to prove that if $V=\mathcal{O}_X$, then we can find an integral cycle $\mathcal{A}$ such that the identity (41) holds for all parameters t of the form

$$t = t_{1,1}\mathbf{1}_{1,1} + t_{2,1}\mathbf{1}_{2,1} + t_{3,1}\mathbf{1}_{3,1}.$$

One can check that the partial derivatives of the LHS and the RHS of (41) with respect to any other parameter $t_{k,p}$ can be expressed in terms the same differential operator involving only $t_{k,1}$, $1\le k\le 3$. Therefore if (41) holds for all t of the above form, then (40) holds also for all such t. As explained above, if the identity (40) holds for a single point $t=t_0$ then it holds for all t and it is equivalent to the identity in our Γ–conjecture, ie the Γ–conjecture holds for the structure sheaf. Recalling Lemma 10 we get that the Γ–conjecture holds for the entire Milnor lattice.

2.4.6 The central charge as an oscillatory integral It remains only to prove (41). Following Iritani, it is convenient to rewrite the RHS of (41) in terms of the so-called H–function

$$H_X^{(0)}(t,z) = \widetilde{\mathrm{ch}}(H_K^{(0)}(t,z)),$$

where the $K(X)$–valued function

$$H_K^{(0)}\colon M\times\mathbb{C}^* \to K(X)$$

is defined by the equation

$$\mathbf{1} = L(t,z)z^{\theta}z^{\rho}\Psi(H_K^{(0)}(t,z)).$$

For the central charge $Z_X^{(0)}(V)$ we have

$$\begin{aligned}\big(L(t,z)z^{\theta}z^{\rho}\Psi(V), L(t,-z)(-z)^{\theta}(-z)^{\rho}\Psi(H_K^{(0)}(t,-z))\big)\\ = \big(\Psi(V), e^{\pi\sqrt{-1}\theta}e^{\pi\sqrt{-1}\rho}\Psi(H_K^{(0)})\big),\end{aligned}$$

where we define $(-1)^R := (e^{\pi\sqrt{-1}R})$ for all linear operators R. Recalling (38) and the Kawasaki Riemann–Roch formula we get

$$Z_X^{(0)}(V) = \chi(H_K^{(0)} \otimes V^{\vee}) = \int_{IX} H_X^{(0)}(t,-z) \cup \widetilde{\mathrm{ch}}(V^{\vee}) \cup \widetilde{\mathrm{Td}}(TX),$$

where in the notation of Section 2.3 the Todd class of an orbifold vector bundle is a multiplicative characteristic class defined by

$$\widetilde{\mathrm{Td}}(V) = \sum_{v\in T}\prod_{j=1}^{l_{v,0}} \frac{\delta_{v,0,j}}{1-e^{-\delta_{v,0,j}}} \prod_{0<f<1}\prod_{j=1}^{l_{v,f}} \frac{1}{1-e^{-2\pi\sqrt{-1}f}e^{-\delta_{v,f,j}}}.$$

The proof of formula (41) requires a simple lemma. The main ingredient is a slight modification of the usual Laplace transform defined as follows. Let $f(t,Q;z)$ be any function. Then we define

$$\mathfrak{L}(f)(t,Q;z) = \int_0^{\infty} e^{-\eta} f(t,-\eta z Q; z)\, d\eta.$$

The integral is convergent if for example f depends polynomially on Q and $\log Q$, which is the case that we have. Note that this Laplace transform does not commute with the involution $z \mapsto -z$.

Lemma 11 *Let $j\colon IX \to IY$ be the natural embedding. Then*

$$j_* H_X^{(0)}(t,Q;-z) = (-z/2\pi)^{1/2}\, \widetilde{e}(L) \cup \mathfrak{L}(H_Y^{(0)})(t,Q;-z),$$

where $\widetilde{e}(L) = \sum_{v\in T} e(L_v)$ is the orbifold Euler class of L.

Proof Since $j_*(j^*\alpha) = \widetilde{e}(L) \cup \alpha$ for every $\alpha \in H^*(IY)$, it is enough to prove that

$$\mathfrak{L}(j^* H_Y^{(0)})(t,Q;-z) = (-z/2\pi)^{-1/2}\, H_X^{(0)}(t,Q;-z). \tag{42}$$

We have

$$(2\pi)^{-1/2}\widehat{\Gamma}(X)\cup(2\pi\sqrt{-1})^{\deg}\operatorname{inv}^* H_X^{(0)}(t,Q,z)=z^{-\rho_X}z^{-\theta_X}J_X(t,Q;z),$$
$$(2\pi)^{-1}\widehat{\Gamma}(Y)\cup(2\pi\sqrt{-1})^{\deg}\operatorname{inv}^* H_Y^{(0)}(t,Q,z)=z^{-\rho_Y}z^{-\theta_Y}J_Y(t,Q;z).$$

On the other hand, using the explicit formulas for the J–functions it is easy to check that

$$\mathfrak{L}(j^*J_Y)(t,Q;-z)=(-z)^{-P/z}\Gamma(1-P/z)\cup J_X(t,Q;-z).$$

In order to prove formula (42), it is enough only to recall the identities

$$j^*(-z)^{-\rho_Y}(-z)^{-\theta_Y}=(-z)^{-P-1/2}(-z)^{-\rho_X}(-z)^{-\theta_X}j^*,$$
$$(-z)^{-\rho_X}(-z)^{-\theta_X}(-z)^{-P/z}\Gamma(1-P/z)=(-z)^{P}\Gamma(1+P)(-z)^{-\rho_X}(-z)^{-\theta_X}$$

and $j^*\widehat{\Gamma}(Y)=\widehat{\Gamma}(L)\widehat{\Gamma}(X)$. □

Lemma 11 yields the following relation between the central charges of sheaves on X and Y. Let $V\in K(Y)$. Then

$$Z_X^{(0)}(j^*V)=(-z/2\pi)^{1/2}\mathfrak{L}\big(Z_Y^{(0)}(V-V\otimes L)\big).$$

In particular,

$$Z_X^{(0)}(1)=(-z/2\pi)^{1/2}\mathfrak{L}\big(Z_Y^{(0)}(1-L)\big). \tag{43}$$

Theorem 12 *For a Fano orbifold curve $X=\mathbb{P}^1_{\boldsymbol{a}}$, given a class $L_k^m\in K(X)$ in the K–theory of vector bundles, there exists an integral cycle $\alpha_{k,m}\in\mathfrak{h}$ such that, for all $\ell\gg 0$,*

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\lambda s}\widetilde{I}^{(-\ell)}_{\alpha_{k,m}}(\lambda)\,d\lambda=s^{-\theta_X-\ell-1/2}s^{-\rho_X}\Psi(L_k^m).$$

Proof It is enough to prove (41). Let us look at the corresponding oscillatory integrals. Recall that the LG model of Y is given by the restriction of

$$F_{\mathbb{P}^2}(x,t)=\sum_{k=1}^{3}(x_k^{a_k}+t_{k,1}x_k)$$

to the complex torus $x_1x_2x_3=Q$, while the corresponding primitive form is

$$\omega_{\mathbb{P}^2}=\frac{dx_1\,dx_2\,dx_3}{d(x_1x_2x_3)}.$$

Let us assume now that z and Q are real numbers such that $z>0$ and $Q<0$. Let $\mathcal{C}\subset\mathbb{C}^3$ be the chain

$$\mathcal{C}=\{x\in\mathbb{R}^3 \mid x_k\geq 0,\ k=1,2,3\}.$$

The oscillatory integral

$$(2\pi z)^{-3/2}\int_{\mathcal{C}} e^{-F(x,t)/z}\omega$$

is given by

$$(2\pi z)^{-3/2}(-1)^{1/2}\int_0^\infty e^{-\eta}\int_{\Gamma_{-zQ\eta}} e^{-F_{\mathbb{P}^2}(x,t)/z}\frac{dx_1\,dx_2\,dx_3}{d(x_1x_2x_3)}(-z)\,d\eta,$$

where we presented the chain $\mathcal{C}$ as a family of cycles

$$\Gamma_{-z\eta Q}=\{x\in\mathcal{C}\mid x_1x_2x_3=-z\eta Q\}.$$

and used the Fubini theorem. The Γ–conjecture for Y was proved by Iritani [30]. Moreover, the real cycle $\Gamma_{-z\eta Q}$ corresponds to the structure sheaf $\mathcal{O}_Y$, so the above integral coincides with

$$(-1)^{3/2}z(2\pi z)^{-3/2}\int_0^\infty e^{-\eta}(2\pi z)Z_Y^{(0)}(1)(t,-z\eta Q;z)\,d\eta \\ =(-1)^{3/2}(z/2\pi)^{1/2}\mathfrak{L}(Z_Y^{(0)}(1)).$$

Recalling the argument in Lemma 10 it is easy to see that the analytic continuation around $Q=0$ in the clockwise direction of $\mathfrak{L}(Z_Y^{(0)}(1))$ is $\mathfrak{L}(Z_Y^{(0)}(L))$, therefore the cycle that we are looking for is $\widetilde{\mathcal{C}}-\mathcal{C}$, where $\widetilde{\mathcal{C}}$ is the chain obtained from $\mathcal{C}$ via the monodromy transformation around $Q=0$ in the clockwise direction. More precisely, $\widetilde{\mathcal{C}}$ is the family of cycles $\widetilde{\Gamma}_{-z\eta Q}$ obtained from $\Gamma_{-z\eta Q}$ by the monodromy transformation around $Q=0$. It remains only to notice that the boundaries of $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ are the same. Together with (43), this proves (41). □

2.5 Affine root systems and vanishing cycles

According to Theorem 12 (recall that we have to put $Q=1$) and formula (39), we have

$$\int_0^\infty e^{-\lambda s}\widetilde{I}^{(-\ell)}_{\alpha_{k,m}}(\lambda)\,d\lambda \\ =\frac{\mathbf{1}}{s^{\ell+1}}+\left(\frac{2\pi\sqrt{-1}m}{a_k}-\gamma\chi-\chi\log s\right)\frac{P}{s^\ell}+\sum_{(j,p)\in\mathfrak{I}_{\mathrm{tw}}}\frac{\Gamma(d_j)}{e^{2\pi\sqrt{-1}mp\delta_{k,j}/a_j}}\frac{\phi_{j,p}}{s^{\ell+d_{j,p}}},$$

where $d_{j,p} = 1 - p/a_j$ and γ is Euler's gamma constant, defined by

$$\gamma = \lim_{m\to\infty} H_m - \ln m, \qquad H_m := 1 + \frac{1}{2} + \cdots + \frac{1}{m}.$$

If $\ell \geq 1$, then we can recall the inverse Laplace transform and also the divisor equation (32) to get

$$\widetilde{I}^{(-\ell)}_{\alpha_{k,m}}(\lambda) = \frac{\lambda^\ell}{\ell!}\mathbf{1} + \frac{\lambda^{\ell-1}}{(\ell-1)!}\left(\frac{2\pi\sqrt{-1}m}{a_k} + \chi(\log\lambda - C_{\ell-1})\right)P + \sum_{(j,p)\in\mathfrak{I}_{\mathrm{tw}}} \frac{\lambda^{d_{j,p}+\ell-1}e^{2\pi\sqrt{-1}m\delta_{k,j}d_{j,p}}}{(d_{j,p}+\ell-1)\cdots(d_{j,p})}\phi_{j,p}, \tag{44}$$

where if $\ell = 1$ we set $C_0 := (1/\chi)\log Q$ and if $\ell > 1$ then $C_\ell = C_{\ell-1} + 1/\ell$.

Proposition 13 (1) *The set of vanishing cycles $\Delta \subset \mathfrak{h} = H_2(X_{0,1};\mathbb{C})$ is an affine root system of type $\mathrm{X}_N^{(1)}$, where $N = a_1 + a_2 + a_3 - 2$ and*

$$\mathrm{X} = \begin{cases} \mathrm{A} & \textit{if } a_1 = 1, \\ \mathrm{D} & \textit{if } a_1 = a_2 = 2, \\ \mathrm{E} & \textit{otherwise.} \end{cases}$$

(2) *There exists a basis of simple roots such that the classical monodromy σ is an affine Coxeter transformation.*

Part (1) of Proposition 13 is due to A Takahashi (see [48]), whose proof is based on a standard method developed by Guseĭn-Zade [27; 28] and A'Campo [2; 3]. We give a proof of Proposition 13 based on Iritani's integral structure.

We will be interested in the two maps from the sequence

$$\widetilde{I}^{(n)}(1)\colon \mathfrak{h} \to H, \quad \alpha \mapsto \widetilde{I}^{(n)}_\alpha(1) \tag{45}$$

corresponding to $n = -1$ and $n = 0$. According to Lemma 7 we have

$$\widetilde{I}^{(-1)}(1) = B_{00}\,(\mathbf{1} - (\log Q)P) + B_{01}\,P + \sum_{i\in\mathfrak{I}_{\mathrm{tw}}} B_i\,\phi_i,$$

which proves that the map for $n = -1$ is an isomorphism. Using $\widetilde{I}^{(-1)}(1)$ we equip H with an intersection pairing $(\cdot|\cdot)$, ie

$$(\phi'|\phi'') := (\alpha'|\alpha'') \quad \text{for } \phi' = \widetilde{I}^{(-1)}_{\alpha'}(1),\ \phi'' = \widetilde{I}^{(-1)}_{\alpha''}(1).$$

The period map (45) with $n = 0$ has a 1–dimensional kernel because (using (34))

$$\widetilde{I}^{(0)}(1) = (1-\rho)^{-1}\big(\theta + \tfrac{1}{2}\big)\widetilde{I}^{(-1)}(1) = (1+\rho)(1 - \deg_{\mathrm{CR}})\widetilde{I}^{(-1)}(1),$$

so the kernel is $\mathbb{C}P$. We denote the image of $\widetilde{I}^{(0)}(1)$ by $H^{(0)}$. Let us denote by $r\colon H \to H^{(0)}$ the map defined by $\widetilde{I}^{(0)}(1) = r \circ \widetilde{I}^{(-1)}(1)$, ie

$$r(b) = (1+\rho)(1-\deg_{\mathrm{CR}})(b).$$

According to Saito's formula (25) the intersection pairing on H takes the form

$$(\phi'|\phi'') = (r(\phi'), (1-\rho)r(\phi'')), \quad \phi', \phi'' \in H. \tag{46}$$

It follows that we can push forward the intersection form to a nondegenerate bilinear pairing on $H^{(0)}$, which we denote again by $(\cdot|\cdot)$. More precisely, we define

$$(\phi'|\phi'') = (\phi', (1-\rho)\phi''), \quad \phi', \phi'' \in H^{(0)}.$$

We denote by $\Delta^{(-1)} \subset H$ and $\Delta^{(0)} \subset H^{(0)}$ the images of the set of vanishing cycles, ie

$$\Delta^{(-1)} = \{\widetilde{I}_\alpha^{(-1)}(1) \mid \alpha \in \Delta\}, \quad \Delta^{(0)} = \{\widetilde{I}_\alpha^{(0)}(1) \mid \alpha \in \Delta\}.$$

A straightforward computation with formula (46) implies:

Lemma 14 *Consider $\alpha_{k,m}$ as in (44). Then the cycles $\alpha_{k,m}$ $(1 \le k \le 3,\ m \in \mathbb{Z})$ satisfy*

$$(\alpha_{k,m}|\alpha_{k,n}) = \begin{cases} 2 & \text{if } m = n \pmod{a_k}, \\ 1 & \text{if } m \neq n \pmod{a_k}, \end{cases}$$

and, for $k \neq k'$,

$$(\alpha_{k,m}\alpha_{k',n}) = \begin{cases} 2 & \text{if } m = 0 \pmod{a_k} \text{ and } n = 0 \pmod{a_{k'}}, \\ 0 & \text{if } m \neq 0 \pmod{a_k} \text{ and } n \neq 0 \pmod{a_{k'}}, \\ 1 & \text{otherwise.} \end{cases}$$

2.5.1 The toroidal cycle Let $\Gamma_\varepsilon \subset \mathbb{C}^3$ be the torus

$$\Gamma_\epsilon := \{|x_1| = |x_2| = 1, |x_3| = \varepsilon\}.$$

For sufficiently large ε, Γ_ε does not intersect the Milnor fiber $X_{0,1}$. Hence we have a well-defined cycle

$$[\Gamma_\varepsilon] \in H_3(\mathbb{C}^3 \setminus X_{0,1}; \mathbb{Z}) \cong H_2(X_{0,1}; \mathbb{Z}),$$

where the isomorphism is given by the so called *tube mapping* (see [25; 26] for more details). Let us denote by φ the image of $[\Gamma_\varepsilon]$ under the above isomorphism.

Proposition 15 *We have $I_\varphi^{(-1)}(t,\lambda) = 2\pi\sqrt{-1}\,P$.*

Proof Increasing ε does not change the homology class $[\Gamma_\varepsilon]$, therefore by choosing $\varepsilon \gg 0$ we may arrange that Γ_ε does not intersect the Milnor fiber $X_{t,\lambda}$ for all (t,λ) sufficiently close to $(0,1)$. In particular, the cycle $\varphi_{t,\lambda}$ obtained from φ via a parallel transport with respect to the Gauss–Manin connection coincides with the image of $[\Gamma_\varepsilon]$ via the tube mapping. We have (see [25; 26])

$$I(t,\lambda,Q) := \int_{[\Gamma_\varepsilon]} \frac{\omega}{F(t,x)-\lambda} = 2\pi\sqrt{-1}\int_{\varphi_{t,\lambda}} \frac{\omega}{dF} = 2\pi\sqrt{-1}\,\partial_\lambda \int d^{-1}\omega. \tag{47}$$

Comparing with the definition (18) we get

$$I(t,\lambda,Q) = -(2\pi)^2\sqrt{-1}\,(I^{(-1)}_\varphi(t,\lambda),\mathbf{1}).$$

Using the differential equation (21), we get

$$(\lambda\partial_\lambda + E)I(t,\lambda,Q) = 0. \tag{48}$$

The integral $I(t,\lambda,Q)$ is analytic at $(t,\lambda,Q)=(0,0,0)$ because it has the form

$$\sqrt{-1}\int_{[\Gamma_\varepsilon]} \frac{dx_1\,dx_2\,dx_3}{Qe^{t_{01}}(G(t,x)-\lambda)-x_1x_2x_3},$$

where $G(t,x)$ is a holomorphic function in t and x. However, Equation (48) means that $I(t,\lambda,Q)$ is homogeneous of degree 0 and since the weights of all variables are positive, the integral must be a constant. In particular, we may set $t=Q=\lambda=0$, which gives

$$I(t,\lambda,Q) = -\sqrt{-1}\int_{[\Gamma_\varepsilon]} \frac{dx_1\,dx_2\,dx_3}{x_1x_2x_3} = (2\pi)^3.$$

Note that (47) implies that $I^{(0)}_\varphi(t,\lambda)=0$. Recalling again the differential equation (21), we get

$$I^{(-1)}_\varphi(t,\lambda) = (I^{(-1)}_\varphi(t,\lambda),\mathbf{1})\,P = (2\pi)^{-2}\sqrt{-1}\,I(t,\lambda,Q)\,P = 2\pi\sqrt{-1}\,P. \qquad \square$$

Corollary 16 *The cycle φ corresponds to the skyscraper sheaf $\mathcal{O}_{\mathrm{pt}} := L-\mathcal{O}$, ie*

$$\delta := \widetilde{I}^{(-1)}_{\alpha_{k,a_k}}(1) - \widetilde{I}^{(-1)}_{\alpha_{k,0}}(1) = 2\pi\sqrt{-1}\,P.$$

Proof of Proposition 13(1) The image of the Milnor lattice in H has a $\mathbb{Z}$–basis given by

$$\delta,\quad \gamma_b^{(-1)},\quad \gamma_i^{(-1)}\ \ (i \in \mathfrak{I}_{\mathrm{tw}}),$$

where for $n = 0$ or -1, and $i = (k, p)$, we get

$$\gamma_b^{(n)} := \widetilde{I}^{(n)}_{\alpha_{k,0}}(1) \quad \text{(corresponds to } \mathcal{O}\text{)},$$
$$\gamma_{k,p}^{(n)} := \widetilde{I}^{(n)}_{\alpha_{k,-p}}(1) - \widetilde{I}^{(n)}_{\alpha_{k,-p+1}}(1).$$

It is easy to check that the intersection diagram of the set of cycles $\gamma_b^{(-1)}$, $\gamma_i^{(-1)}$, $i \in \mathfrak{I}_{\mathrm{tw}}$ is given by the Dynkin diagram on Figure 1. As usual, each node has self-intersection 2, each edge means that the intersection of the cycles corresponding to the nodes of the edge is -1, and no edge means that the intersection is 0. It follows that the intersection form of the Milnor lattice is semipositive definite with 1–dimensional kernel. This is possible only if $\Delta^{(-1)}$ is an affine root system. □

In particular we get also that δ is a $\mathbb{Z}$–basis for the imaginary roots and that $\Delta^{(0)} = r(\Delta^{(-1)})$ is a finite root system.

2.5.2 Splitting of the affine root system It is convenient to enumerate the roots $\gamma_b^{(n)}$, $\gamma_i^{(n)}$, $i \in \mathfrak{I}_{\mathrm{tw}}$ also by $\gamma_j^{(n)}$ $(1 \le j \le N)$. The Dynkin diagram on Figure 1 is of type X_N, where $\mathrm{X} = \mathrm{A}$, D or E. Let us denote by $\gamma_0^{(-1)}$ the *affine vertex*, ie the extra node that we have to attach to X_N in order to obtain the corresponding affine Dynkin diagram $\mathrm{X}_N^{(1)}$.

Vectors $\gamma_j^{(0)}$, $1 \le j \le N$, form a basis of simple roots of $\Delta^{(0)}$. Let $W^{(0)}$ be the reflection group generated by $\gamma_j^{(0)}$. It is well known that there exists a group embedding $W^{(0)} \to W$ which is induced by the map

$$s_j^{(0)} := s_{\gamma_j^{(0)}} \mapsto s_j^{(-1)} := s_{\gamma_j^{(-1)}}, \quad 1 \le j \le N.$$

Given $\alpha \in \Delta^{(0)}$, let us define a *lift* $\widetilde{\alpha} \in \Delta^{(-1)}$ as follows:

$$\alpha = \sum_{j=1}^{N} b_j \gamma_j^{(0)} \mapsto \widetilde{\alpha} := \sum_{j=1}^{N} b_j \gamma_j^{(-1)}.$$

Then the root system $\Delta^{(-1)}$ coincides with the set

$$\{\widetilde{\alpha} + n\delta \mid \alpha \in \Delta^{(0)}, n \in \mathbb{Z}\},$$

where $\delta = \gamma_0^{(-1)} + \theta^{(-1)}$ and $\theta \in \Delta^{(0)}$ is the highest root with respect to the basis $\{\gamma_j^{(0)}\}_{j=1}^N$ (see [33]). Following Kac, we will refer to $n\delta$ $(n \in \mathbb{Z})$ as *imaginary roots*. Finally, let us denote by

$$\Lambda^{(-1)} := H_2(X_{0,1}; \mathbb{Z})$$

the root lattice of $\Delta^{(-1)}$. Given $\alpha \in \Lambda^{(-1)}$ such that $|\alpha|^2 := (\alpha|\alpha) \neq 0$, recall that the reflection with respect to α is defined by

$$s_\alpha(x) = x - 2\frac{(\alpha|x)}{(\alpha|\alpha)}\,\alpha.$$

We also define the following translation:

$$T_\alpha(x) := s_{\alpha+\delta}s_\alpha(x) = x + 2\frac{(\alpha|x)}{(\alpha|\alpha)}\,\delta.$$

This definition induces a group embedding $T\colon \Lambda^{(0)} \to W$. Recall that $w\, s_\alpha\, w^{-1} = s_{w(\alpha)}$ for all $w \in W$ and $\alpha \in \Lambda^{(-1)}$ such that $|\alpha|^2 \neq 0$. Therefore, $\Lambda^{(0)}$ is a normal subgroup of W and we have an isomorphism

$$W \cong \Lambda^{(0)} \rtimes W^{(0)}.$$

Let us emphasize that this isomorphism is not canonical — it depends on the choice of a basis of simple roots of $\Delta^{(-1)}$.

2.5.3 The Coxeter transformation Put $\sigma_b := \sigma_b^{(0)}$, where

$$\sigma_b^{(\ell)} = \prod_{k=1}^{3}(s^{(\ell)}_{k,a_k-1}\cdots s^{(\ell)}_{k,2}s^{(\ell)}_{k,1}) \in \mathrm{Aut}(\Delta^{(\ell)}), \quad \ell = -1, 0. \tag{49}$$

Note that while the order of the reflections that enter each factor of the above product is important, the order in which the three factors are arranged is irrelevant since they pairwise commute.

Proposition 17 *The automorphism of $\Delta^{(0)}$ induced by the action of the classical monodromy σ coincides with σ_b.*

Proof The analytic continuation in λ around $\lambda = 0$ of the period $\widetilde{I}^{(-1)}_{\alpha_{k,m}}(\lambda)$ is equivalent to tensoring the line bundle L_k^m by $TX = L_1L_2L_3L^{-1}$ and then taking the corresponding periods. Using

$$(L_k - 1)L_{k'} = L_k - 1 \quad \text{for} \quad k \neq k'$$

it is easy to check that

$$\begin{aligned}(L_k^{-m} - L_k^{-m+1})TX &= L_k^{-m+1} - L_k^{-m+2} \quad \text{for all } m \in \mathbb{Z},\\ TX^{-1} &= 1 + L_1^{-1} - 1 + L_2^{-1} - 1 + L_3^{-1} - 1 + L - 1,\\ (L_k^{-1} - 1)TX &= 1 - L_k^{-(a_k-1)} + 1 - L.\end{aligned}$$

According to the above remark, the classical monodromy acts as follows:

$$\begin{aligned}\sigma(\gamma_{k,p}) &= \gamma_{k,p-1}, \quad (k,p)\in\mathfrak{I}_{\mathrm{tw}},\\ \sigma^{-1}(\gamma_b) &= \gamma_b+\gamma_{1,1}+\gamma_{2,1}+\gamma_{3,1}+\delta,\\ \sigma(\gamma_{k,1}) &= -\gamma_{1,1}-\cdots-\gamma_{k,a_k-1}-\delta.\end{aligned}$$

It remains only to check that the action of $\sigma_b^{(-1)}$ is given by the same formulas modulo the imaginary root δ. □

It is known that up to a translation the affine Coxeter transformation coincides with σ_b (see [47; 46]), so part (2) of Proposition 13 follows from Proposition 17.

2.5.4 Calibrated periods in terms of the finite root system For $0\le j\le N$, let $\omega_j^{(-1)}\in H^\vee$ be the fundamental weights of $\Delta^{(-1)}$, ie

$$\langle\omega_j^{(-1)},\gamma_m^{(-1)}\rangle=\delta_{j,m}.$$

Using the intersection form we identify $H^{(0)}$ and its dual. For $1\le j\le N$, let $\omega_j^{(0)}\in H^{(0)}$ be the fundamental weights of $\Delta^{(0)}$, ie

$$(\omega_j^{(0)}|\gamma_m^{(0)})=\delta_{j,m},\quad 1\le j,m\le N.$$

We have the relation

$$\langle\omega_j^{(-1)},\widetilde{\alpha}\rangle=(\omega_j^{(0)}|r(\widetilde{\alpha}))-k_j\langle\omega_0^{(-1)},\widetilde{\alpha}\rangle,\quad \widetilde{\alpha}\in\Delta,$$

where k_j $(1\le j\le N)$ are the Kac labels defined by

$$\delta=\gamma_0^{(-1)}+\sum_{j=1}^N k_j\gamma_j^{(-1)}.$$

In terms of the fundamental weights, the splitting of the affine root system from the previous section can be stated also as the following isomorphism:

$$\Delta^{(-1)}\cong\Delta^{(0)}\times\mathbb{Z},\quad \widetilde{\alpha}\mapsto(\alpha,n),\quad \alpha=r(\widetilde{\alpha}),\quad n=\langle\omega_0^{(-1)},\widetilde{\alpha}\rangle.$$

Lemma 18 *The following identity holds:*

$$\omega_b^{(0)}+\sum_{m=1}^{a_k-1}(\zeta_k^{mp}-\zeta_k^{(m-1)p})\omega_{k,m}^{(0)}=a_k\phi_{k,p^*},\quad 1\le k\le 3.$$

Proof We have explicit formulas for the simple roots

$$\gamma_b^{(0)} = \mathbf{1} + \chi P + \sum_{k=1}^{3} \sum_{p=1}^{a_k-1} \phi_{k,p}, \quad \gamma_{k,m}^{(0)} = \sum_{p=1}^{a_k-1} (\zeta_k^{mp} - \zeta_k^{(m-1)p})\phi_{k,p}.$$

It remains only to check that the LHS and the RHS have the same intersection pairing with the above set of simple roots of $\Delta^{(0)}$. □

Let κ be a positive constant whose value will be specified later on. Put

$$H_0 := H_{00} := H_{01} := (\kappa\chi)^{1/2}\omega_b^{(0)}, \quad H_i := (\kappa a_i)^{1/2}\phi_i, \quad i \in \mathfrak{I}_{\mathrm{tw}}. \tag{50}$$

Note that according to Lemma 7 $\{H_i\}_{i\in\mathfrak{I}}$ is a σ_b–eigenbasis of $H^{(0)}$ with $\sigma_b(H_i) = e^{-2\pi\sqrt{-1}d_i} H_i$ in which the intersection form takes the form

$$(H_i|H_j) = \kappa\delta_{i,j^*}, \quad i, j \in \mathfrak{I}, \tag{51}$$

where for $i = j = 00 \in \mathfrak{I}$ we used that $\omega_b^{(0)} = \chi^{-1}\mathbf{1} + P$. Finally, put

$$\rho_b = -\sum_{(k,p)\in\mathfrak{I}_{\mathrm{tw}}} \frac{1}{a_k}\omega_{k,p}^{(0)}.$$

Proposition 19 *Let $\widetilde{\alpha} = (\alpha, n) \in \Delta^{(0)} \times \mathbb{Z} \cong \Delta$ be a vanishing cycle. Then the corresponding calibrated periods are given by the following formulas:*

$$\begin{aligned}
\widetilde{I}^{(\ell)}_{\widetilde{\alpha}}(\lambda) &= (-1)^\ell \ell!(\alpha|\omega_b^{(0)})\chi\lambda^{-\ell-1}P \\
&\qquad + \sum_{i\in\mathfrak{I}_{\mathrm{tw}}} (\alpha|H_{i^*})(d_i-1)\cdots(d_i-\ell)\lambda^{d_i-\ell-1}\sqrt{a_i/\kappa}\,\phi_i, \\
\widetilde{I}^{(0)}_{\widetilde{\alpha}}(\lambda) &= (\alpha|\omega_b^{(0)})\mathbf{1} + (\alpha|\omega_b^{(0)})\chi\lambda^{-1}P + \sum_{i\in\mathfrak{I}_{\mathrm{tw}}} (\alpha|H_{i^*})\lambda^{d_i-1}\sqrt{a_i/\kappa}\,\phi_i, \\
\widetilde{I}^{(-1-\ell)}_{\widetilde{\alpha}}(\lambda) &= (\alpha|\omega_b^{(0)})\frac{\lambda^{\ell+1}}{(\ell+1)!}\mathbf{1} + \frac{\lambda^\ell}{\ell!}\Big((\alpha|\omega_b^{(0)})\chi(\log\lambda - C_\ell) + 2\pi\sqrt{-1}(n+(\rho_b|\alpha))\Big)P \\
&\qquad + \sum_{i\in\mathfrak{I}_{\mathrm{tw}}} (\alpha|H_{i^*})\sqrt{a_i/\kappa}\frac{\lambda^{d_i+\ell}}{d_i(d_i+1)\cdots(d_i+\ell)}\phi_i,
\end{aligned}$$

where $\ell \geq 1$ and C_ℓ $(\ell \geq 1)$ are constants defined recursively by

$$C_\ell = C_{\ell-1} + \frac{1}{\ell}, \quad C_0 = \frac{1}{\chi}\log Q.$$

Proof It is enough to check the statement for the following basis of the Milnor lattice:

$$\gamma_b^{(-1)}, \quad \delta, \quad \gamma_i^{(-1)} \ (i = (k,p) \in \mathfrak{I}_{\mathrm{tw}}).$$

Let us check the last identity for $\widetilde{\alpha} = \gamma_{k,p}^{(-1)}$, ie $n = 0$ and $\alpha = \gamma_{k,p}^{(0)}$. Recalling the explicit formulas for $\widetilde{I}_{\alpha_{k,p}}^{(-1-\ell)}(\lambda)$ and $\gamma_{k,p} = \alpha_{k,-p} - \alpha_{k,-p+1}$, we get (recall that $d_{k,m} = 1 - m/a_k$)

$$\widetilde{I}_{\gamma_{k,p}}^{(-1-\ell)}(\lambda) = -\frac{2\pi\sqrt{-1}}{a_k}\frac{\lambda^\ell}{\ell!}P + \sum_{m=1}^{a_k-1} \frac{\zeta_k^{mp} - \zeta_i^{m(p-1)}}{(\ell + d_{k,m})\cdots(1 + d_{k,m})}\lambda^{\ell + d_{k,m}}\phi_{k,m}.$$

On the other hand, by definition $H_{i*}\sqrt{a_i/\kappa} = a_i\phi_{i*}$, so the identity follows from Lemma 18. The remaining two cases are proved in the same way. □

3 ADE-Toda hierarchies

3.1 Twisted realization of the affine Lie algebra

Let $\mathfrak{g}^{(0)}$ be a simple Lie algebra of type ADE with an invariant bilinear form $(\cdot|\cdot)$, normalized in such a way that all roots have length $\sqrt{2}$. By definition, the affine Kac–Moody algebra corresponding to $\mathfrak{g}$ is the vector space

$$\mathfrak{g} := \mathfrak{g}^{(0)}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

equipped with a Lie bracket defined by the following relations: for $X, Y \in \mathfrak{g}^{(0)}$,

$$[Xt^n, Yt^m] := [X, Y]t^{n+m} + n\delta_{n,-m}(X|Y)K,$$
$$[d, Xt^n] := n(Xt^n), \quad [K, \mathfrak{g}^{(0)}] := 0.$$

We fix a Cartan subalgebra $\mathfrak{h}^{(0)} \subset \mathfrak{g}^{(0)}$ and a basis γ_b, γ_i $(i \in \mathfrak{I})$ of simple roots such that the corresponding Dynkin diagram has the standard shape with γ_b corresponding to the branching node. If the root system is of type A, then we choose any of the nodes to be a branching node and we have (at most) two instead of three branches. Let us define $\sigma_b := \sigma_b^{(0)}$ by formula (49).

Let $\Delta^{(0)} \subset \mathfrak{h}^{(0)}$ be the root system of $\mathfrak{g}^{(0)}$, ie

$$\mathfrak{g}^{(0)} = \bigoplus_{\alpha \in \Delta^{(0)}} \mathfrak{g}_\alpha^{(0)}.$$

The Lie algebra $\mathfrak{g}^{(0)}$ can be constructed in terms of the root system via the so-called Frenkel–Kac construction [18]. Let $\Lambda^{(0)} \subset \mathfrak{h}^{(0)}$ be the root lattice. There exists a bimultiplicative function

$$\epsilon\colon \Lambda^{(0)} \times \Lambda^{(0)} \to \{\pm 1\}$$

satisfying

$$\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)}, \quad \epsilon(\alpha, \alpha) = (-1)^{|\alpha|^2/2},$$

where $|\alpha|^2 := (\alpha|\alpha)$. The map $(\alpha,\beta)\mapsto \varepsilon(\sigma_b(\alpha),\sigma_b(\beta))$ is another bimultiplicative function satisfying the above properties. It is known that all bimultiplicative functions of the above form are equivalent (see [34, Corollary 5.5]). Hence there exists a function $\upsilon\colon \Lambda^{(0)}\to\{\pm 1\}$ such that

$$\upsilon(\alpha)\upsilon(\beta)\varepsilon(\alpha,\beta)=\upsilon(\alpha+\beta)\varepsilon(\sigma_b(\alpha),\sigma_b(\beta)). \tag{52}$$

There exists a set of root vectors

$$A_\alpha\in\mathfrak{g}^{(0)}_\alpha \tag{53}$$

such that

$$\begin{aligned}
[A_\alpha,A_{-\alpha}]&=\epsilon(\alpha,-\alpha)\alpha,\\
[A_\alpha,A_\beta]&=\epsilon(\alpha,\beta)A_{\alpha+\beta} && \text{if } (\alpha|\beta)=-1,\\
[A_\alpha,A_\beta]&=0 && \text{if } (\alpha|\beta)\geq 0.
\end{aligned}$$

We can extend σ_b to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$ via

$$\sigma_b(A_\alpha)=\upsilon(\alpha)^{-1}A_{\sigma_b(\alpha)},\quad \alpha\in\Delta^{(0)}.$$

Let us denote by κ the order of the extended automorphism $\sigma_b\colon\mathfrak{g}^{(0)}\to\mathfrak{g}^{(0)}$. Clearly we have $\kappa=|\sigma_b|$ or $2|\sigma_b|$. Since $(\cdot|\cdot)$ is both $\mathfrak{g}^{(0)}$–invariant (with respect to the adjoint representation) and $W^{(0)}$–invariant, we have

$$(A_\alpha|A_{-\alpha}):=\epsilon(\alpha,-\alpha),\quad (A_\alpha|A_\beta):=(A_\alpha|H)=0\quad\text{for all } \beta\neq-\alpha,\ H\in\mathfrak{h}^{(0)}.$$

Put $\eta=e^{2\pi\sqrt{-1}/\kappa}$. We extend the action of σ_b to the affine Lie algebra $\mathfrak{g}$ by

$$\sigma_b\cdot(X\otimes t^n)=\sigma_b(X)\otimes(\eta^{-1}t)^n,\quad \sigma_b\cdot K=K,\quad \sigma_b\cdot d=d.$$

Let

$$\mathfrak{g}^{\sigma_b}\subset\mathfrak{g}$$

be the Lie subalgebra of σ_b–fixed points. According to Kac (see [33, Theorem 8.6]), $\mathfrak{g}^{\sigma_b}\cong\mathfrak{g}$. Let us recall the isomorphism. The fixed points subspace $(\mathfrak{g}^{(0)})^{\sigma_b}$ contains a Cartan subalgebra $\widetilde{\mathfrak{h}}^{(0)}$. We have a corresponding decomposition into root subspaces

$$\mathfrak{g}^{(0)}=\widetilde{\mathfrak{h}}^{(0)}\oplus\Big(\bigoplus_{\widetilde{\alpha}\in\widetilde{\Delta}^{(0)}}\mathfrak{g}^{(0)}_{\widetilde{\alpha}}\Big),$$

where $\widetilde{\Delta}^{(0)}\subset\widetilde{\mathfrak{h}}^{(0)}$ are the corresponding roots. Note that since the root subspaces are 1–dimensional, they must be eigen-subspaces of σ_b. Therefore, by choosing a set of simple roots $\widetilde{\alpha}_j$, $j=1,2,\dots,N$ in $\widetilde{\Delta}^{(0)}$ we can uniquely define an integral vector

$s = (s_1, \dots, s_N)$, $0 \le s_j < \kappa$ such that the eigenvalue of the eigensubspace $\mathfrak{g}^{(0)}_{\tilde{\alpha}_j}$ is η^{s_j}. Put

$$\rho_s \colon \tilde{\mathfrak{h}}^{(0)} \to \tilde{\mathfrak{h}}^{(0)}, \quad \rho_s = \sum_{j=1}^{N} s_j \tilde{\omega}_j,$$

where $\tilde{\omega}_j \in \tilde{\mathfrak{h}}^{(0)}$ $(1 \le j \le N)$ are the fundamental weights corresponding to the simple roots $\tilde{\alpha}_j$ $(1 \le j \le N)$, ie $(\tilde{\omega}_j | \tilde{\alpha}_{j'}) = \delta_{j,j'}$. The isomorphism

$$\Phi \colon \mathfrak{g} \to \mathfrak{g}^{\sigma_b}$$

is defined as follows:

$$\Phi(Xt^n) = t^{n\kappa + \mathrm{ad}_{\rho_s}} X + \delta_{n,0}, (\rho_s | X) K, \tag{54}$$

$$\Phi(K) = \kappa K,$$

$$\Phi(d) = \kappa^{-1}\big(d - \rho_s - \tfrac{1}{2}(\rho_s|\rho_s)K\big), \tag{55}$$

where

$$t^{\mathrm{ad}_{\rho_s}} X = \exp(\log t \ \mathrm{ad}_{\rho_s}) X.$$

Note that the RHS is single-valued in t and σ_b–invariant in X, because

$$\exp(2\pi\sqrt{-1}\mathrm{ad}_{\rho_s/\kappa}) = \sigma_b.$$

Finally, we make a remark on κ. There is no a canonical way to extend σ_b to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$. Therefore, the value of κ depends on our choice of the cocycle $\epsilon(\alpha, \beta)$ and the corresponding sign-function $\upsilon(\alpha)$. We will see however that replacing κ by $m\kappa$, where m is a positive integer, does not change the HQEs, so we may assume that κ is a sufficiently large integer such that $\sigma_b^\kappa = 1$. For the sake of completeness, let us fix an extension that seems natural for our purposes. Put $\omega_{k,0} = \omega_b$ and $\omega_{k,a_k} = 0$ and define

$$\mathrm{SF}(\alpha, \beta) = \sum_{k=1}^{3} \sum_{p=0}^{a_k - 1} (\omega_{k,p} | \alpha)(\omega_{k,p} - \omega_{k,p+1} | \beta). \tag{56}$$

Since $\mathrm{SF}(\alpha,\beta) + \mathrm{SF}(\beta,\alpha) = (\alpha|\beta)$, the bimultiplicative function $\epsilon(\cdot,\cdot) = (-1)^{\mathrm{SF}(\cdot,\cdot)}$ is an acceptable choice for the Frenkel–Kac construction. Note that

$$\upsilon(\alpha) = (-1)^{\sum_{k=1}^{3} (\omega_b|\alpha)(\omega_{k,1}|\alpha)} \tag{57}$$

satisfies formula (52), so we get an explicit formula for an extension of σ_b to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$. Moreover, since

$$\prod_{m=1}^{|\sigma_b|} \upsilon(\sigma_b^m(\alpha)) = (-1)^{\chi|\sigma_b|},$$

we get that $\kappa = |\sigma_b|$ if $\chi|\sigma_b|$ is even and $\kappa = 2|\sigma_b|$ if $\chi|\sigma_b|$ is odd. Notice that $|\sigma_b| = \mathrm{lcm}(a_1, a_2, a_3)$, the least common multiple of a_1, a_2, a_3.

Remark 20 The notation SF is motivated from the notion of a Seifert form in singularity theory (see [4; 5]). We do not claim that (56) is a Seifert form, although it would be interesting to investigate whether definition (56) can be interpreted as a linking number between α and β.

3.2 The Kac–Peterson construction

Following [35], we would like to recall the realization of the basic level 1 representation of the affine Lie algebra $\mathfrak{g}$ corresponding to the automorphism σ_b. The idea is to construct a representation of the Lie algebra $\mathfrak{g}^{\sigma_b}$ on a Fock space, which induces via the isomorphism Φ the basic level-1 representation.

Fix a σ_b–eigenbasis $\{H_i\}_{i\in\{0\}\sqcup\mathfrak{I}_{\mathrm{tw}}}$ of $\mathfrak{h}^{(0)}$. It is convenient to define $H_{00} := H_{01} := H_0$ and to assume that the basis is normalized so that $(H_i|H_{j^*}) = \kappa\delta_{i,j}$ (compare with (51)). Put

$$m_{00} := 0, \quad m_{01} := \kappa, \quad m_i := d_{i^*}\kappa \ \ (i \in \mathfrak{I}_{\mathrm{tw}}),$$

so that $e^{-2\pi\sqrt{-1}\,d_i} = \eta^{m_i}$ is the eigenvalue corresponding to the eigen vector H_i. The elements

$$H_{i,\ell} := H_i t^{m_i+\ell\kappa} \quad (i \in \mathfrak{I}, \ell \in \mathbb{Z})$$

generate a *Heisenberg* Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}^{\sigma_b}$, ie the following commutation relations hold:

$$[H_{i,\ell}, H_{j,m}] = (m_i + \ell\kappa)\delta_{i,j^*}\delta_{\ell+m,-1}\kappa K.$$

Let us also fix a $\mathbb{C}$–linear basis of $\mathfrak{s}$

$$H_0 := H_{00}, \quad H_{i,\ell}, \quad H_{i^*,-\ell-1}, \quad K \ \ ((i,\ell) \in I_+), \tag{58}$$

where the index set is defined by

$$I_+ = \{(i,\ell) \mid i \in \mathfrak{I} \setminus \{(00)\}, \ell \in \mathbb{Z}_{\geq 0}\}. \tag{59}$$

Let $\mathfrak{S}$ be the subgroup of the affine Kac–Moody Lie group generated by the lifts of the loops

$$h_{\alpha,\beta} = \exp(\alpha \log t^\kappa + 2\pi\sqrt{-1}\,\beta), \tag{60}$$

where $\alpha, \beta \in \mathfrak{h}^{(0)}$ are such that

$$\sigma_b(\alpha) = \alpha, \quad \sigma_b(\beta) - \beta + \alpha \in \Lambda^{(0)}.$$

Let us point out that under the analytical continuation around $t = 0$, the loop $h_{\alpha,\beta}$ gains the factor $e^{2\pi\sqrt{-1}\kappa\alpha}$. The latter must be 1 because

$$\kappa\alpha = (\alpha + \sigma_b(\beta) - \beta) + \sigma_b(\alpha + (\sigma_b(\beta) - \beta)) + \cdots + \sigma_b^{\kappa-1}(\alpha + (\sigma_b(\beta) - \beta)) \in \Lambda^{(0)}.$$

It follows that $h_{\alpha,\beta}$ is single-valued and σ_b–invariant, ie it defines an element of the affine Kac–Moody loop group acting on $\mathfrak{g}^{\sigma_b}$ by conjugation. The main result of Kac and Peterson [35] is the following: *the basic representation of $\mathfrak{g}^{\sigma_b}$ remains irreducible when restricted to the pair* $(\mathfrak{s}, \mathfrak{S})$.

Let us recall the construction of the representation. Let us denote by

$$\pi_0\colon \mathfrak{h}^{(0)} \to \mathfrak{h}_0^{(0)} \quad \text{and} \quad \pi_*\colon \mathfrak{h}^{(0)} \to (\mathfrak{h}_0^{(0)})^\perp$$

the orthogonal projections of $\mathfrak{h}^{(0)}$ onto $\mathfrak{h}_0^{(0)} := \mathbb{C}\, H_0$ and $(\mathfrak{h}_0^{(0)})^\perp$ respectively. Given $x \in \mathfrak{h}^{(0)}$, let

$$x_0 := \pi_0(x), \quad x_* := \pi_*(x).$$

Let $\mathfrak{s}_- \subset \mathfrak{s}$ be the Lie subalgebra of $\mathfrak{s}$ spanned by the vectors $H_{i^*,-\ell-1}, (i,\ell) \in I_+$. The basic representation can be realized on the following vector space:

$$V_x = S^*(\mathfrak{s}_-) \otimes \mathbb{C}[e^\omega]e^{x\omega}, \tag{61}$$

where x is a complex number and $\omega := \pi_0(\gamma_b)$. The first factor of the tensor product in (61) is the symmetric algebra on $\mathfrak{s}_-$, and the second one is isomorphic to the group algebra of the lattice $\pi_0(\Lambda^{(0)}) = \mathbb{Z}\,\pi_0(\gamma_b)$. We will refer to $|0\rangle := 1 \otimes e^{x\omega}$ as the *vacuum vector*. Slightly abusing the notation, we define the operator

$$\partial_\omega := \frac{\partial}{\partial\omega} - x,$$

acting on V_x, so that $\partial_\omega |0\rangle = 0$.

Put

$$X_\alpha(\zeta) = \sum_{n\in\mathbb{Z}} A_{\alpha,n}\zeta^{-n} = \frac{1}{\kappa}\sum_{m=1}^{\kappa}\sum_{n\in\mathbb{Z}} \eta^{-nm}(\sigma_b^m(A_\alpha)t^n)\zeta^{-n}, \quad \alpha \in \Delta^{(0)},$$

where A_α appears in (53). Let $E^*_\alpha(\zeta)$ be the *vertex operator*

$$E^*_\alpha(\zeta) = \exp\Bigg(\sum_{(i,\ell)\in I_+} (\alpha|H_i)H_{i^*,-\ell-1}\frac{\zeta^{m_i+\ell\kappa}}{m_i+\ell\kappa}\Bigg) \times \exp\Bigg(\sum_{(i,\ell)\in I_+} (\alpha|H_{i^*})H_{i,\ell}\frac{\zeta^{-m_i-\ell\kappa}}{-m_i-\ell\kappa}\Bigg). \tag{62}$$

Lemma 21 *There are operators C_α, $\alpha\in\Delta^{(0)}$, independent of ζ, that commute with all basis vectors (58) of $\mathfrak{s}$ different from H_0, such that*

$$X_\alpha(\zeta) = X^0_\alpha(\zeta)E^*_\alpha(\zeta),$$

where

$$X^0_\alpha(\zeta) = \zeta^{\kappa|\alpha_0|^2/2}C_\alpha\zeta^{\kappa\,\alpha_0}, \quad \alpha_0 := \pi_0(\alpha). \tag{63}$$

Proof After a direct computation we get

$$[H_{i,\ell}, X_\alpha(\zeta)] = (\alpha|H_i)\zeta^{m_i+\ell\kappa}X_\alpha(\zeta).$$

It follows that $X_\alpha(\zeta) = X^0_\alpha(\zeta)E^*_\alpha(\zeta)$, where $X^0_\alpha(\zeta)$ is an operator commuting with all $H_{i,\ell}\neq H_0$.

After a direct computation we get the following commutation relations:

$$\begin{aligned} h_{\alpha,\beta}(-d)h^{-1}_{\alpha,\beta} &= -d+\kappa\alpha+\tfrac{1}{2}|\alpha|^2\kappa^2K,\\ h_{\alpha,\beta}A_{\gamma,n}h^{-1}_{\alpha,\beta} &= e^{2\pi\sqrt{-1}(\beta|\gamma)}A_{\gamma,n+\kappa(\alpha|\gamma)}+\delta_{n,0}(\alpha|A_\gamma)\kappa K, \end{aligned}$$

and the $h_{\alpha,\beta}$ commute with the Heisenberg algebra $\mathfrak{s}$ apart from the relation

$$h_{\alpha,\beta}H_0h^{-1}_{\alpha,\beta} = H_0+(\alpha|H_0)\kappa K.$$

Here $h_{\alpha,\beta}$ are given in (60). In order to determine the dependence on ζ of $X^0_\alpha(\zeta)$ we first have to notice that

$$-d = \tfrac{1}{2}|\rho_s|^2K+\tfrac{1}{2}H_0^2+\sum_{(i,\ell)\in I_+} H_{i^*,-l-1}H_{i,\ell}, \tag{64}$$

where $H_0 = H_{00} = H_{01}$. Indeed, if we decompose the basic representation into a direct sum of weight subspaces of $\mathfrak{s}$, then using the above commutation relations, we get that the LHS of (64) is an operator that preserves these weight subspaces while the difference of the LHS and the RHS commutes with $\mathfrak{s}$ and $\mathfrak{S}$. The formula follows up to the constant term $\frac{1}{2}|\rho_s|^2K$, which is fixed by examining the action of the operator $d\in\mathfrak{g}$ on the vacuum vector. Using formula (54) for $Xt^n=\rho_s$ we get that ρ_s (viewed as an element of $\mathfrak{g}^{\sigma_b}$) acts on the vacuum by the scalar $-|\rho_s|^2/\kappa$; then since the RHS of formula (55) acts by 0 on the vacuum, we get that $d \in \mathfrak{g}^{\sigma_b}$ acts by the scalar

$$-|\rho_s|^2/\kappa + \tfrac{1}{2}|\rho_s|^2/\kappa = -\tfrac{1}{2}|\rho_s|^2/\kappa.$$

Since we have

$$[d, X_\alpha(\zeta)] = -\zeta\partial_\zeta X_\alpha(\zeta), \quad [d, E^*_\alpha(\zeta)] = -\zeta\partial_\zeta E^*_\alpha(z),$$

we easily get $-\zeta\partial_\zeta X^0_\alpha = [d, X^0_\alpha]$. On the other hand, $X^0_\alpha(\zeta)$ commutes with $H_{i,\ell}$ for all i,ℓ, except

(65) $$[H_0, X^0_\alpha(\zeta)] = (\alpha|H_0)X^0_\alpha.$$

It follows that

$$\zeta\partial_\zeta X^0_\alpha = \kappa\big(X^0_\alpha\alpha_0 + \tfrac{1}{2}|\alpha_0|^2 X^0_\alpha\big).$$

Solving the above equation we get formula (63). □

Lemma 22 *The operators C_α in (63) satisfy the commutation relation*

(66) $$C_\alpha C_\beta = \epsilon(\alpha,\beta)B^{-1}_{\alpha,\beta}C_{\alpha+\beta},$$

where

$$B_{\alpha,\beta} = \kappa^{-(\alpha|\beta)}\prod_{m=1}^{\kappa-1}(1-\eta^m)^{(\sigma_b^m(\alpha)|\beta)}.$$

Proof Let us assume first that $\alpha \neq -\beta$ are two roots. After a direct computation we get that the commutator $[X_\alpha(\zeta), X_\beta(w)]$ is given by the following formula:

$$\frac{1}{\kappa}\sum_{m=1}^{\kappa}\Bigg(\prod_{j=1}^{m-1} \upsilon^{-1}(\sigma_b^j(\beta))\Bigg)\epsilon(\alpha,\sigma_b^m(\beta))\,\delta(\eta^{-m}\zeta, w)\, w\, X_{\alpha+\sigma_b^m(\beta)}(\zeta),$$

where $\delta(x,y) := \sum_{n\in\mathbb{Z}} x^n y^{-n-1}$ is the formal delta function. On the other hand,

$$E^*_\alpha(\zeta)E^*_\beta(w) = \prod_{m=1}^{\kappa}\Big(1-\eta^m\frac{w}{\zeta}\Big)^{(\sigma_b^m(\alpha)|\beta)} : E^*_\alpha(\zeta)E^*_\beta(w):,$$

where : : is the standard normal ordering in the Heisenberg group — all annihilation operators $H_{i,\ell}$ must be moved to the right. Substituting in the above commutator

$X_\gamma(\zeta) = X^0_\gamma(\zeta)E^*_\gamma(\zeta)$ we get that the following two expressions are equal:

$$\prod_{m=1}^{\kappa}\Big(1-\eta^m\frac{w}{\zeta}\Big)^{(\sigma_b^m(\alpha)|\beta)} X^0_\alpha(\zeta)X^0_\beta(w) - \prod_{m=1}^{\kappa}\Big(1-\eta^m\frac{\zeta}{w}\Big)^{(\sigma_b^m(\beta)|\alpha)} X^0_\beta(w)X^0_\alpha(\zeta) \tag{67}$$

and

$$\frac{1}{\kappa}\sum_{m=1}^{\kappa}\Big(\prod_{j=1}^{m-1}\upsilon^{-1}(\sigma_b^j(\beta))\Big)\epsilon(\alpha,\sigma_b^m(\beta))\delta(\eta^{-m}\zeta,w)\,w\,X^0_{\alpha+\sigma_b^m(\beta)}(\zeta). \tag{68}$$

Both formulas have the form $i_{\zeta,w}P_1(\zeta,w)-i_{\zeta,w}P_2(\zeta,w)$, where P_1 and P_2 are some rational functions and $i_{\zeta,w}$ (resp. $i_{w,\zeta}$) means the Laurent series expansion in the region $|\zeta|>|w|$ (resp. $|w|<|\zeta|$). Since $P_1=P_2$ for the second expression, the same must be true for the first one, ie

$$\prod_{m=1}^{\kappa}\Big(1-\eta^m\frac{w}{\zeta}\Big)^{(\sigma_b^m(\alpha)|\beta)} X^0_\alpha(\zeta)X^0_\beta(w) = \prod_{m=1}^{\kappa}\Big(1-\eta^m\frac{\zeta}{w}\Big)^{(\sigma_b^m(\beta)|\alpha)} X^0_\beta(w)X^0_\alpha(\zeta).$$

Recalling formula (63) and (65), the above equality implies

$$C_\alpha C_\beta = \prod_{m=1}^{\kappa}(-\eta^m)^{(\alpha|\sigma_b^m(\beta))}C_\beta C_\alpha. \tag{69}$$

Using (69) we can easily write (67) as a sum of formal delta functions. Comparing with (68) we get (66). □

Lemma 23 *Let ω_b, ω_i, $i=(k,p)\in\mathfrak{I}_{\mathrm{tw}}$, be the fundamental weights corresponding to the basis of simple roots γ_b, γ_i, $i\in\mathfrak{I}_{\mathrm{tw}}$. Then*

$$(\omega_i|\chi\omega_b)=d_i,\quad \pi_0(\gamma_b)=\chi\omega_b,\quad \pi_*(\gamma_b)=-\sum_{i\in\mathfrak{I}_{\mathrm{tw}}}d_i\gamma_i.$$

Proof Let $\{\varepsilon_{k,p}\}_{p=1}^{a_k}$ be the standard basis of $\mathbb{C}^{a_k}$ for any fixed $k=1,2,3$. The root system of type A_{a_k-1} is given by $\{\varepsilon_{k,p}-\varepsilon_{k,q}\}$ and the standard choice of simple roots is $\gamma_{k,p}=\varepsilon_{k,p}-\varepsilon_{k,p+1}$, $1\le p\le a_k-1$. Note that the fundamental weights corresponding to the basis of simple roots are

$$\widetilde{\omega}_{k,p}=\Big(1-\frac{p}{a_k}\Big)(\varepsilon_{k,1}+\cdots+\varepsilon_{k,p})-\frac{p}{a_k}(\varepsilon_{k,p+1}+\cdots+\varepsilon_{k,a_k}).$$

It follows that the pairing between the fundamental weights is

$$(\widetilde{\omega}_{k,p}|\widetilde{\omega}_{k,q})=\min(p,q)-pq/a_k.$$

In particular, we have

$$\widetilde{\omega}_{k,p} = \Big(1 - \frac{p}{a_k}\Big)\gamma_1 + (\text{terms involving only } \gamma_2, \dots, \gamma_{a_k-1}). \tag{70}$$

In our settings, the roots $\{\gamma_{k,p}\}_{p=1}^{a_k-1}$ give rise to a subroot system of type A_{a_k-1}. Let us denote by $\widetilde{\omega}_{k,p}$ the corresponding fundamental weights. Note that

$$\omega_{k,p} = \widetilde{\omega}_{k,p} - (\widetilde{\omega}_{k,p}|\gamma_b)\,\omega_b,$$

so the first formula of the lemma follows from (70) and

$$(\gamma_b|\gamma_{k,p}) = -\delta_{p,1}, \quad (\widetilde{\omega}_{k,p}|\omega_b) = 0.$$

The other two identities follow easily from the first one. □

Using formula (66) we define C_α for all α in the root lattice $\Lambda^{(0)}$; then formula (69) still holds. Finally, a similar argument gives us that

$$C_\alpha C_{-\alpha} = \epsilon(\alpha,-\alpha)\, B_{\alpha,-\alpha}^{-1}, \quad \text{ie } C_0 = 1. \tag{71}$$

Lemma 24 *Let c_α ($\alpha \in \Lambda^{(0)}$) be operators defined by*

$$C_\alpha = c_\alpha \exp((\omega_b|\alpha)\omega)\exp(2\pi\sqrt{-1}\,(\rho_b|\alpha)\partial_\omega). \tag{72}$$

Then $[c_\alpha, c_\beta] = 0$.

Proof Note that by definition, the commutator $C_\alpha C_\beta C_\alpha^{-1} C_\beta^{-1}$ is given by the following formula:

$$\prod_{m=1}^{\kappa} (-\eta^m)^{(\alpha|\sigma_b^m(\beta))} = e^{\pi\sqrt{-1}\,(\alpha_0|\beta)} e^{2\pi\sqrt{-1}\,((1-\sigma_b)^{-1}\alpha_*|\beta)}.$$

On the other hand, using (72), the commutator becomes

$$c_\alpha c_\beta c_\alpha^{-1} c_\beta^{-1} \exp 2\pi\sqrt{-1}\,\big((\rho_b|\alpha)(\omega_b|\beta) - (\rho_b|\beta)(\omega_b|\alpha)\big). \tag{73}$$

Recall that σ_b is a composition of 3 matrices $\sigma_k^{(0)}, k = 1,2,3$ whose action on the subspace with basis $\{\gamma_{k,1}, \dots, \gamma_{k,a_k-1}\}$ is represented by the matrix

$$\sigma_k^{(0)} = \begin{bmatrix} -1 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 1 \\ -1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

It is easy to check that the $(p,q)^{\text{th}}$ entry is given by

$$[(1-\sigma_k^{(0)})^{-1}]_{pq} = \frac{p}{a_k} - \varepsilon_{pq}, \quad \varepsilon_{pq} = \begin{cases} 0 & \text{if } p \le q, \\ 1 & \text{if } p > q. \end{cases} \tag{74}$$

A straightforward computation using formula (74) and Lemma 23 yields

$$\begin{aligned} ((1-\sigma_k^{(0)})^{-1}\gamma_{k,p}|\gamma_b) &= -\frac{1}{a_k}, \\ ((1-\sigma_k^{(0)})^{-1}\gamma_{k,p}|\gamma_{k,q}) &= \delta_{p,q} - \delta_{p+1,q}, \\ ((1-\sigma_b)^{-1}(\gamma_b)_*|\gamma_{k,q}) &= \frac{1}{a_k} \quad (\operatorname{mod} \mathbb{Z}), \\ ((1-\sigma_b)^{-1}(\gamma_b)_*|\gamma_b) &= 1 - \tfrac{1}{2}\chi. \end{aligned}$$

Using the above formulas we get

$$((1-\sigma_b)^{-1}\pi_*(\alpha)|\beta) = (\rho_b|\alpha)\,(\omega_b|\beta) - (\rho_b|\beta)\,(\omega_b|\alpha) - \tfrac{1}{2}(\alpha_0|\beta_0) \quad (\operatorname{mod} \mathbb{Z}).$$

For the commutator we get

$$C_\alpha C_\beta C_\alpha^{-1} C_\beta^{-1} = \exp\bigl(2\pi\sqrt{-1}\bigl((\rho_b|\alpha)(\omega_b|\beta) - (\rho_b|\beta)(\omega_b|\alpha)\bigr)\bigr).$$

Comparing with (73) we get $c_\alpha c_\beta c_\alpha^{-1} c_\beta^{-1} = 1$. □

Lemma 24 implies that the operators c_α can be represented by scalars, ie we can find complex numbers c_α, $\alpha \in \Lambda^{(0)}$ such that

$$c_\alpha c_\beta = \epsilon(\alpha,\beta) B_{\alpha,\beta}^{-1} e^{-2\pi\sqrt{-1}(\rho_b|\beta)(\omega_b|\alpha)} c_{\alpha+\beta}. \tag{75}$$

For example, we can choose c_{α_i} arbitrarily for the simple roots α_i and then use formula (75) to define the remaining constants.

The level-1 basic representation can be realized on V_x as follows. Let us represent the Heisenberg algebra $\mathfrak{s}$ on $\mathbb{C}[e^\omega]e^{x\omega}$ by letting all generators act trivially, except for $H_0 \mapsto (H_0|\gamma_b)\,\partial_\omega$. The latter is forced by the commutation relation

$$[H_0, C_\alpha] = (\alpha|H_0)C_\alpha = (\omega_b|\alpha)(H_0|\gamma_b)C_\alpha.$$

In this way V_x naturally becomes an $\mathfrak{s}$–module. Furthermore, put

$$E_\alpha^0(\zeta) = \exp((\omega_b|\alpha)\omega)\exp\bigl(((\omega_b|\alpha)\chi\log\zeta^\kappa + 2\pi\sqrt{-1}(\rho_b|\alpha))\partial_\omega\bigr) \tag{76}$$

and $E_\alpha(\zeta) = E_\alpha^0(\zeta)E_\alpha^*(\zeta)$, where $E_\alpha^*(\zeta)$ is defined by formula (62). Thus the representation of the Heisenberg algebra $\mathfrak{s}$ on V_x can be lifted to a representation of the affine Lie algebra $\mathfrak{g}^{\sigma_b}$ as follows:

$$\begin{aligned} X_\alpha(\zeta) &\mapsto c_\alpha \zeta^{\kappa\,|\alpha_0|^2/2} E_\alpha(\zeta), \quad \alpha \in \Delta^{(0)}, \\ K &\mapsto 1/\kappa, \\ d &\mapsto -\tfrac{1}{2}|\rho_s|^2/\kappa - \tfrac{1}{2}H_0^2 - \sum_{(i,\ell)\in I_+} H_{i^*,-\ell-1} H_{i,\ell}. \end{aligned}$$

3.3 The Kac–Wakimoto hierarchy

Following Kac and Wakimoto (see [36]), we can define an integrable hierarchy in the Hirota form whose solutions are parametrized by the orbit of the vacuum vector $|0\rangle$ of the affine Kac–Moody group. A vector $\tau \in V_x$ belongs to the orbit if and only if $\Omega_x\,(\tau\otimes\tau) = 0$, where Ω_x is the operator representing the bilinear Casimir operator

$$\sum_{\alpha\in\Delta^{(0)}} \sum_n \frac{A_{\alpha,n}\otimes A_{-\alpha,-n}}{(A_\alpha|A_{-\alpha})} + K\otimes d + d\otimes K + \frac{H_0\otimes H_0}{\kappa} + \sum_{(i,\ell)\in I_+} \left(\frac{H_{i,\ell}\otimes H_{i^*,-\ell-1} + H_{i^*,-\ell-1}\otimes H_{i,\ell}}{\kappa} \right).$$

On the other hand, we have

$$\sum_n \frac{A_{\alpha,n}\otimes A_{-\alpha,-n}}{(A_\alpha|A_{-\alpha})} = \mathrm{Res}_{\zeta=0} \frac{d\zeta}{\zeta} a_\alpha(\zeta) E_\alpha(\zeta)\otimes E_{-\alpha}(\zeta),$$

where the coefficients a_α can be computed explicitly thanks to formula (75), ie

(77) $$a_\alpha(\zeta) = B_{\alpha,\alpha}\,\zeta^{\kappa\,|\alpha_0|^2}\, e^{2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}.$$

We identify the symmetric algebra $S^*(\mathfrak{s}_-)$ with the Fock space $\mathbb{C}[y]$, where $y = (y_{i,\ell})$ is a sequence of formal variables indexed by $(i,\ell)\in I_+$ as defined in (59), by identifying $H_{i^*,-\ell-1} = (m_i+\ell\kappa)y_{i,\ell}$. Then (note that $(H_0|\gamma_b) = (\kappa\chi)^{1/2}$) we have

$$H_{i,\ell} = \frac{\partial}{\partial y_{i,\ell}}, \quad H_0 = (\kappa\chi)^{1/2}\,\partial_\omega, \quad K = 1/\kappa,$$

$$d = -\frac{|\rho_s|^2}{2\kappa} - \frac{\kappa\chi}{2}\partial_\omega^2 - \sum_{(i,\ell)\in I_+} (m_i+\ell\kappa) y_{i,\ell}\partial_{y_{i,\ell}}.$$

The elements in V_x can also be thought of as sequences of polynomials as follows:

$$V_x \cong \mathbb{C}[y]^{\mathbb{Z}}, \quad \sum_{n\in\mathbb{Z}} \tau_n(y) e^{(n+x)\omega} \mapsto \tau := (\tau_n(y))_{n\in\mathbb{Z}}.$$

The above isomorphism turns $\mathbb{C}[y]^{\mathbb{Z}}$ into a module over the algebra of differential operators in e^{ω}:

$$(e^{\omega}\cdot\tau)_n=\tau_{n-1},\quad (\partial_{\omega}\cdot\tau)_n=n\tau_n.$$

The HQEs of the σ_b–twisted Kac–Wakimoto hierarchy will assume the form (3) stated in Section 1.0.4 provided we prove the following identity:

Lemma 25 *We have the identities*

$$|\rho_s|^2/\kappa^2=\frac{1}{12}\sum_{k=1}^{3}\frac{a_k^2-1}{a_k}=\tfrac{1}{2}\operatorname{tr}\big(\tfrac{1}{4}+\theta\theta^{\mathrm{T}}\big),$$

where θ is the Hodge grading operator (16).

Proof Since $\tau=|0\rangle$ is a solution to the hierarchy, we must have

$$|\rho_s|^2/\kappa^2=\sum_{\alpha:(\omega_b|\alpha)=0}a_\alpha(\zeta).$$

Let $\alpha\in\Delta^{(0)}$ be such that $(\omega_b|\alpha)=0$. Then formula (77) reduces simply to

$$a_\alpha(\zeta)=B_{\alpha,\alpha}=\kappa^{-2}\prod_{m=1}^{\kappa-1}(1-\eta^m)^{(\sigma_b^m(\alpha)|\alpha)}.$$

Recall the notation in the proof of Lemma 23. We claim that α must belong to one of the root subsystems $\Delta_k^{(0)}$ of type A_{a_k-1} corresponding to the legs of the Dynkin diagram for some k. Indeed, let us write α as a linear combination $\sum_{k,p}c_{k,p}\gamma_{k,p}$ for some integers $c_{k,p}$. If this linear combination involves a simple root $\gamma_{k,p}$ for some k, then using reflections $s_{k,p}$ with $p>1$ we can transform α to a cycle α' such that the decomposition of α' as a sum of simple roots will involve $\gamma_{k,1}$. Moreover, we still have $(\omega_b|\alpha')=0$. In other words, we may assume that $c_{k,1}\neq 0$ as long as $c_{k,p}\neq 0$ for some p. However, since $(\alpha|\gamma_b)=-\sum_k c_{k,1}$ and the coefficients $c_{k,p}$ have the same sign (depending on whether α is a positive or a negative root) we get that there is precisely one k for which $c_{k,1}\neq 0$.

Assume that $\alpha\in\Delta_k^{(0)}$. Then since σ_b is a product of the Coxeter transformations $\sigma_{k'}=\cdots s_{k',2}s_{k',1}$, in the above formula for a_α only σ_k contributes and since the order of σ_k is a_k, after a short computation we get

$$a_\alpha(\zeta)=a_k^{-2}\prod_{m=1}^{a_k-1}(1-\eta_k^m)^{(\sigma_k^m(\alpha)|\alpha)},\quad \eta_k=e^{2\pi\sqrt{-1}/a_k}.$$

These are precisely the coefficients of the principal Kac–Wakimoto hierarchy of type A_{a_k-1}. Let ρ_k be the sum of the fundamental weights of $\Delta_k^{(0)}$. It is well known that $|\rho_k|^2 = (a_k-1)a_k(a_k+1)/12$. According to [17] we have

$$|\rho_s|^2/\kappa^2 = \sum_{\alpha\in\Delta_k^{(0)}} a_\alpha(\zeta) = |\rho_k|^2/a_k^2 = \frac{1}{12}\sum_{k=1}^{3}\Big(a_k-\frac{1}{a_k}\Big).$$

It remains only to notice (using $\theta^{\mathrm{T}}=-\theta$) that

$$\tfrac12\,\mathrm{tr}\big(\tfrac14+\theta\,\theta^{\mathrm{T}}\big)=\tfrac12\,\mathrm{tr}\big(\tfrac12+\theta\big)\big(\tfrac12-\theta\big)=\frac12\sum_{i\in\mathfrak{I}_{\mathrm{tw}}} d_i(1-d_i)=\frac{1}{12}\sum_{k=1}^{3}\Big(a_k-\frac{1}{a_k}\Big). \quad\square$$

3.4 Formal discrete Laplace transform

Let $\alpha\in\Delta^{(0)}$ and $\widetilde{\alpha}\in\Delta$ be as in Section 2.5.4. We would like to compare the vertex operators $E_\alpha(\zeta)$ and

$$\widetilde{\Gamma}^{\widetilde{\alpha}}(\lambda) := e^{(\widetilde{f}_{\widetilde{\alpha}}(\lambda;z))^\wedge},$$

where $(-)^\wedge$ is the quantization operation defined in Section 5 and

$$\widetilde{f}_{\widetilde{\alpha}}(\lambda;z)=\sum_{n\in\mathbb{Z}}\widetilde{I}^{(n)}_{\widetilde{\alpha}}(\lambda)(-z)^n;$$

see (31). Using the formulas for the calibrated periods from Section 2.5.4 we get

$$\widetilde{\Gamma}^{\widetilde{\alpha}}(\lambda)=U_{\widetilde{\alpha}}(\lambda)\widetilde{\Gamma}^{\widetilde{\alpha}}_0(\lambda)\widetilde{\Gamma}^{\widetilde{\alpha}}_*(\lambda),$$

where (we dropped the superscript and set $\omega_b:=\omega_b^{(0)}$)

$$U_{\widetilde{\alpha}}(\lambda)=\exp\Big(\sum_{\ell=1}^{\infty}\big((\omega_b|\alpha)\chi(\log\lambda-C_\ell)+2\pi\sqrt{-1}(n+(\rho_b|\alpha))\big)\frac{\lambda^\ell}{\ell!}q_\ell^{00}/\sqrt{\hbar}\Big),$$

$$\widetilde{\Gamma}^{\widetilde{\alpha}}_0(\lambda)=\exp\big(\big((\omega_b|\alpha)\chi(\log\lambda-C_0)+2\pi\sqrt{-1}(n+(\rho_b|\alpha))\big)q_0^{00}/\sqrt{\hbar}\big) \times\exp\Big(-(\omega_b|\alpha)\sqrt{\hbar}\frac{\partial}{\partial q_0^{00}}\Big),$$

$$\widetilde{\Gamma}^{\widetilde{\alpha}}_*(\lambda)=\exp\Big(\sum_{(i,\ell)\in I_+}(\alpha|H_i)\zeta^{m_i+\ell\kappa}\,y_{i,\ell}\Big)\exp\Big(\sum_{(i,\ell)\in I_+}(\alpha|H_{i^*})\frac{\zeta^{-m_i-\ell\kappa}}{-m_i-\ell\kappa}\frac{\partial}{\partial y_{i,\ell}}\Big),$$

where $\lambda = \zeta^\kappa/\kappa$, and we use the change of variables

(78) $$y_{01,\ell} = \frac{1}{\sqrt{\hbar}} \frac{\kappa^{d_{01}}}{\sqrt{\kappa\chi}} \frac{q_\ell^{01}}{m_{01}(m_{01}+\kappa)\cdots(m_{01}+\ell\kappa)},$$

(79) $$y_{i,\ell} = \frac{1}{\sqrt{\hbar}} \frac{\kappa^{d_i}}{\sqrt{\kappa a_i}} \frac{q_\ell^{i}}{m_{i}(m_{i}+\kappa)\cdots(m_{i}+\ell\kappa)}, \quad (i,\ell) \in \mathfrak{I}_{\mathrm{tw}} \times \mathbb{Z}_{\geq 0}.$$

Comparing with (62) and (76) we get that $\widetilde{\Gamma}_*^{\widetilde{\alpha}}(\lambda) = E_\alpha^*(\zeta)$ and that $\widetilde{\Gamma}_0^{\widetilde{\alpha}}(\lambda)$ is a Laplace transform of $E_\alpha^0(\zeta)$. We make the last statement precise as follows. Put

$$\widehat{V} := \mathbb{C}_\hbar[\![y, x, q_1^{00}+1, q_2^{00}, \dots]\!]^{\mathbb{Z}}.$$

The space $\widehat{V}$ contains a completion of the basic representation V_x. It has also some additional variables q_ℓ^{00}, $\ell \geq 1$ which will be treated as parameters. Just like before, we identify the elements of $\widehat{V}$ with formal Fourier series

$$f = (f_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} f_n e^{(n+x)\omega}.$$

Given $f(\hbar;\boldsymbol{q}) \in \mathbb{C}_\hbar[\![\boldsymbol{q}]\!]$ satisfying the condition

(80) $$f(\hbar;\boldsymbol{q})|_{q_0^{00}=x\sqrt{\hbar}} \in \mathbb{C}_\hbar[\![\boldsymbol{q}]\!] \quad \text{for all } x \in \mathbb{C},$$

define the formal Laplace transform of f depending on a parameter C $(C \neq 0)$

$$\mathcal{F}_C(f(q_0^{00},\dots)) := \sum_{n\in\mathbb{Z}} f((x+n)\sqrt{\hbar},\dots)e^{(n+x)\omega} C^{\frac{1}{2}n^2} \in \widehat{V},$$

where the dots stand for the remaining $\boldsymbol{q}$–variables on which f depends. It is easy to check that

(81) $$\mathcal{F}_C \circ q_0^{00}/\sqrt{\hbar} = \frac{\partial}{\partial\omega} \circ \mathcal{F}_C$$

and

(82) $$\mathcal{F}_C \circ e^{-m\sqrt{\hbar}\partial/\partial q_0^{00}} = e^{m\omega} C^{\frac{1}{2}m^2 + m\partial_\omega} \circ \mathcal{F}_C,$$

where we recall that $\partial_\omega = \dfrac{\partial}{\partial\omega} - x$.

Lemma 26 *Let $C = \kappa^\chi e^{\chi C_0}$. Then*

$$E_\alpha^0(\zeta)\mathcal{F}_C = \mathcal{F}_C e^{-AB-\frac{1}{2}B^2\log C} e^{Ax}\, \widetilde{\Gamma}_0^{\widetilde{\alpha}},$$

where

$$A = (\omega_b|\alpha)\,\chi\,(\log\lambda - C_0) + 2\pi\sqrt{-1}(n + (\rho_b|\alpha)), \quad B = (\omega_b|\alpha).$$

Proof Using (81) and (82) we get that the vertex operators in q_0^{00} transform as follows:

$$\mathcal{F}_C\, e^{Aq_0^{00}/\sqrt{\hbar}} e^{-B\sqrt{\hbar}\partial/\partial q_0^{00}} = e^{AB+\frac{1}{2}B^2\log C} e^{Ax} e^{B\omega} e^{(A+B\log C)\partial_\omega} \mathcal{F}_C.$$

On the other hand, after a straightforward computation, we get

$$e^{AB+\frac{1}{2}B^2\log C} = \zeta^{\kappa|\alpha_0|^2} e^{-\frac{|\alpha_0|^2}{2\chi}(2\chi(C_0+\log\kappa)-\log C)} e^{2\pi\sqrt{-1}(\omega_b|\alpha)(\rho_b|\alpha)}$$

and

$$A+B\log C = (\omega_b|\alpha)\big(\chi\log\zeta^\kappa+\log C-\chi(C_0+\log\kappa)\big)+2\pi\sqrt{-1}(n+(\rho_b|\alpha)). \tag{83}$$

Furthermore, note that since the operator $e^{2\pi\sqrt{-1}\partial_\omega}$ acts as the identity on $\widehat{V}$, the integer n in (83) may be set to 0. Finally, it remains only to compare with (76) and to recall our assumption

$$\log C = \chi(C_0+\log\kappa). \tag{84}$$

□

3.5 Integrable hierarchies for the affine cusp polynomials

For every root $\alpha\in\Delta^{(0)}\subset H^{(0)}$ we fix an arbitrary lift $\widetilde{\alpha}\in\Delta\subset\mathfrak{h}$ (see Section 2.5.4). The subset of affine roots obtained in this way will be denoted by Δ'. Following the construction of Givental and Milanov in [24] we introduce the Casimir-like operator

$$\widetilde{\Omega}_{\Delta'}(\lambda) = -\tfrac{1}{2}\lambda^2\bigg(\sum_{m=1}^N :\big(\widetilde{\phi}_m(\lambda)\otimes_{\mathfrak{a}}1-1\otimes_{\mathfrak{a}}\widetilde{\phi}_j(\lambda)\big)\big(\widetilde{\phi}^m(\lambda)\otimes_{\mathfrak{a}}1-1\otimes_{\mathfrak{a}}\widetilde{\phi}^m(\lambda)\big):\bigg) \\ +\sum_{\widetilde{\alpha}\in\Delta'}\widetilde{b}_{\widetilde{\alpha}}(\lambda)\widetilde{\Gamma}^{\widetilde{\alpha}}(\lambda)\otimes_{\mathfrak{a}}\widetilde{\Gamma}^{-\widetilde{\alpha}}(\lambda)-\tfrac{1}{2}\operatorname{tr}\big(\tfrac{1}{4}+\theta\theta^{\mathrm{T}}\big),$$

where the notation is as follows. Let $\{\widetilde{\alpha}_m\}_{m=1}^N$ and $\{\widetilde{\alpha}^m\}_{m=1}^N$ be two sets of vectors in $\mathfrak{h}$ such that under the projection $\widetilde{I}^{(0)}(1)\colon\mathfrak{h}\to H^{(0)}$ they project to bases dual with respect to the intersection form $(\cdot|\cdot)$, ie $(\widetilde{\alpha}_j|\widetilde{\alpha}^m)=\delta_{j,m}$. Then

$$\widetilde{\phi}_m(\lambda) = (\partial_\lambda\,\widetilde{f}_{\widetilde{\alpha}_m}(\lambda;z))^\wedge,\quad \widetilde{\phi}^m(\lambda) = (\partial_\lambda\,\widetilde{f}_{\widetilde{\alpha}^m}(\lambda;z))^\wedge,\quad 1\le m\le N.$$

The tensor product is over the polynomial algebra $\mathfrak{a}:=\mathbb{C}_\hbar[q_1^{00},q_2^{00},\dots]$, which in particular means that almost all terms that involve $\log\lambda$ cancel.

The first sum in the definition of $\widetilde{\Omega}_{\Delta'}$ is monodromy invariant around $\lambda=\infty$ and hence it expands in only integral powers of λ. In fact one can check that the corresponding coefficients give rise to a representation of the Virasoro algebra, which can be identified with an instance of the so-called *coset Virasoro construction*.[4] After a straightforward

[4] We are thankful to B Bakalov for this remark.

computation using the formulas for the periods from Section 2.5.4, we get the following formula for the coefficient in front of λ^{-2} (ie the L_0–Virasoro operator):

$$\frac{\chi}{2\hbar}(q_0^{00}\otimes_{\mathfrak{a}}1-1\otimes_{\mathfrak{a}}q_0^{00})^2+\sum_{(i,\ell)\in I_+}\Big(\frac{m_i}{\kappa}+\ell\Big)(q_\ell^i\otimes_{\mathfrak{a}}1-1\otimes_{\mathfrak{a}}q_\ell^i)(\partial_{q_\ell^i}\otimes_{\mathfrak{a}}1-1\otimes_{\mathfrak{a}}\partial_{q_\ell^i}).$$

The coefficients $\widetilde{b}_{\widetilde{\alpha}}$ are defined in terms of the vertex operators $\widetilde{\Gamma}^{\widetilde{\alpha}}(\lambda)$ as follows:

$$\widetilde{b}_{\widetilde{\alpha}}^{-1}(\lambda)=\lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^2\widetilde{B}_{\widetilde{\alpha},-\widetilde{\alpha}}(\lambda,\mu),\quad \widetilde{\alpha},\widetilde{\beta}\in\Delta, \tag{85}$$

where $\widetilde{B}_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,\mu)$ is the phase factor from the composition of the following two vertex operators:

$$\widetilde{\Gamma}^{\widetilde{\alpha}}(\lambda)\widetilde{\Gamma}^{\widetilde{\beta}}(\mu)=\widetilde{B}_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,\mu):\widetilde{\Gamma}^{\widetilde{\alpha}}(\lambda)\widetilde{\Gamma}^{\widetilde{\beta}}(\mu):.$$

After a straightforward computation as in Section 3.2, we get

$$\widetilde{B}_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,\mu)=\mu^{-(\alpha_0|\beta_0)}e^{C_0(\alpha_0|\beta_0)-2\pi\sqrt{-1}(\omega_b|\alpha)\,(\rho_b|\beta)}\times\prod_{m=1}^{\kappa}\big(1-\eta^m(\mu/\lambda)^{1/\kappa}\big)^{(\sigma_b^m(\alpha)|\beta)}. \tag{86}$$

We are interested in the following system of Hirota quadratic equations: for every integer $n\in\mathbb{Z}$,

$$\mathrm{Res}_{\lambda=\infty}\frac{d\lambda}{\lambda}\big(\widetilde{\Omega}_{\Delta'}(\lambda)\,(\tau\otimes_{\mathfrak{a}}\tau)\big)\big|_{q_0^{00}\otimes 1-1\otimes q_0^{00}=n\sqrt{\hbar}}=0 \tag{87}$$

where $\tau\in\mathbb{C}_\hbar[\![q_0,q_1+1,q_2,\dots]\!]$. The operator $\widetilde{\Omega}_{\Delta'}(\lambda)$ is multivalued near $\lambda=\infty$: the analytic continuation around $\lambda=\infty$ corresponds to a monodromy transformation of each cycles $\widetilde{\alpha}\in\Delta'$ of the type $\widetilde{\alpha}\mapsto\sigma_b(\widetilde{\alpha})+n_{\widetilde{\alpha}}\varphi$, where $n_{\widetilde{\alpha}}\in\mathbb{Z}$. Using Proposition 15 we get that the analytic continuation transforms $\widetilde{\Omega}_{\Delta'}(\lambda)$ by permuting the cycles $\widetilde{\alpha}$ and multiplying each vertex operator term by

$$e^{2\pi\sqrt{-1}n_{\widetilde{\alpha}}(q_0^{00}\otimes 1-1\otimes q_0^{00})}.$$

Therefore the 1–form in (87) is invariant with respect to the analytic continuation near $\lambda=\infty$. Moreover, for the same reason the equations (87) are independent of the choice of a lift Δ' of $\Delta^{(0)}$.

Remark 27 The Hirota quadratic equations (87) are a straightforward generalization of the construction of Givental and Milanov [24] (see also [17], where the coefficients $\widetilde{b}_{\widetilde{\alpha}}$ were interpreted in terms of the vertex operators) of integrable hierarchies for simple singularities.

The following is the main result of this section.

Theorem 28 *If τ is a solution to the Hirota quadratic equations (87), then $\mathcal{F}_C(\tau)$ with $C=\kappa^\chi Q$ is a tau-function of the σ_b–twisted Kac–Wakimoto hierarchy.*

Proof We just have to find the Laplace transform of the Hirota quadratic equations (3) of the Kac–Wakimoto hierarchy. Let $\alpha\in\Delta^{(0)}$ and $\widetilde{\alpha}\in\Delta$ be as in Section 2.5.4. Using Lemma 26 we get

$$\bigl(a_\alpha(\zeta)\,E_\alpha(\zeta)\otimes E_{-\alpha}(\zeta)\bigr)(\mathcal{F}_C\otimes\mathcal{F}_C)=(\mathcal{F}_C\otimes\mathcal{F}_C)\bigl(b_{\widetilde{\alpha}}(\lambda)\,\widetilde{\Gamma}^{\widetilde{\alpha}}(\lambda)\otimes_{\mathfrak{a}}\widetilde{\Gamma}^{-\widetilde{\alpha}}(\lambda)\bigr),$$

where the coefficient $b_{\widetilde{\alpha}}$ is given by

$$a_\alpha(\zeta)\,\zeta^{-2\kappa|\alpha_0|^2}e^{\frac{|\alpha_0|^2}{\chi}\log C}e^{-4\pi\sqrt{-1}(\omega_b|\alpha)(\rho_b|\alpha)}.$$

Recalling formula (77) and $\lambda=\zeta^\kappa/\kappa$ we get

$$b_{\widetilde{\alpha}}(\lambda)=B_{\alpha,\alpha}\,\lambda^{-|\alpha_0|^2}\,e^{|\alpha_0|^2C_0}\,e^{-2\pi\sqrt{-1}(\omega_b|\alpha)(\rho_b|\alpha)}. \tag{88}$$

Using (85) and (86), it is not hard to verify that $b_{\widetilde{\alpha}}(\lambda)=\widetilde{b}_{\widetilde{\alpha}}(\lambda)$.

In other words, $\mathcal{F}_C(\tau)$ is a solution to the Kac–Wakimoto hierarchy if τ satisfies the following equations:

$$\operatorname{Res}_{\lambda=\infty}\frac{d\lambda}{\lambda}\bigl((\mathcal{F}_C\otimes\mathcal{F}_C)\widetilde{\Omega}_{\Delta'}(\lambda)(\tau\otimes_{\mathfrak{a}}\tau)\bigr)=0.$$

Comparing the coefficients in front of $e^{(n'+x)\omega}\otimes e^{(n''+x)\omega}$ we get (87) with $n=n'-n''$. $\square$

4 The main theorem

4.1 Vertex operators

The symplectic loop space formalism in GW theory was introduced by Givental [21]. We apply this natural framework to describe and investigate further the Hirota quadratic equations (87). In this section, we again adopt the notation that α,β are in the affine root system Δ.

Recall the series (24). We are interested in the vertex operators

$$\Gamma^\alpha(t,\lambda)=\,:e^{\widehat{f}^\alpha(t,\lambda)}:,\quad \alpha\in\Delta, \tag{89}$$

and their *phase factors* $B_{\alpha,\beta}(t,\lambda,\mu)$ defined by

$$\Gamma^{\alpha}(t,\lambda)\Gamma^{\beta}(t,\mu) = B_{\alpha,\beta}(t,\lambda,\mu) : \Gamma^{\alpha}(t,\lambda)\Gamma^{\beta}(t,\mu) : \quad \alpha,\beta \in \Delta,$$

where $:\cdot:$ is the usual normal ordering — move all differentiation operators to the right of the multiplication operators. Note that

$$B_{\alpha,\beta}(t,\lambda,\mu) := e^{\Omega(\boldsymbol{f}_{\alpha}(t,\lambda;z)_+,\boldsymbol{f}_{\beta}(t,\mu;z)_-)}. \tag{90}$$

The action of the vertex operators on the Fock space is not well defined in general. We would like to recall the conjugation laws from [23] and to make sense of the vertex operator action on the Fock space.

4.1.1 Vertex operators at infinity

Let us fix $t \in M$ and expand the vertex operators $\Gamma^{\alpha}(t,\lambda)$ in a neighborhood of $\lambda = \infty$. By definition (see (30)) we have $f_{\alpha}(t,\lambda;z) = S_t\,\widetilde{\boldsymbol{f}}_{\alpha}(\lambda;z)$. Using formula (124), it is easy to prove that

$$\widetilde{\Gamma}^{\alpha}(\lambda)\,\widehat{S}_t^{-1} = e^{\frac{1}{2}W(\widetilde{\boldsymbol{f}}_{\alpha}(\lambda)_+,\widetilde{\boldsymbol{f}}_{\alpha}(\lambda)_+)}\,\widehat{S}_t^{-1}\,\Gamma^{\alpha}(t,\lambda). \tag{91}$$

In particular, using the formal λ^{-1}–adic topology we get that the vertex operator $\Gamma^{\alpha}(t,\lambda)$ defines a linear map $\mathbb{C}_{\hbar}[\![\boldsymbol{q}]\!] \to K_{\hbar}[\![\boldsymbol{q}]\!]$, where K is an appropriate field extension of the field $\mathbb{C}((\lambda^{-1}))$.

Let us explain the relation between the phase factors. Recall formula (86), the RHS is interpreted as an element in $\mathbb{C}((\lambda^{-1/\kappa}))((\mu^{-1/\kappa}))$ by taking the Laurent series expansion in λ at $\lambda = \infty$.

Proposition 29 *The following formula holds:*

$$B_{\alpha,\beta}(t,\lambda,\mu) = \widetilde{B}_{\alpha,\beta}(\mu,\lambda)e^{W_t(\widetilde{\boldsymbol{f}}_{\alpha}(\mu)_+,\widetilde{\boldsymbol{f}}_{\beta}(\lambda)_+)}.$$

Proof Conjugating the identity $\widetilde{\Gamma}^{\alpha}(\lambda)\widetilde{\Gamma}^{\beta}(\mu) = \widetilde{B}_{\alpha,\beta}(\lambda,\mu) : \widetilde{\Gamma}^{\alpha}(\lambda)\widetilde{\Gamma}^{\beta}(\mu) :$ by $\widehat{S}_t$ and using formula (91) we get that

$$e^{\frac{1}{2}\left(W_t(\widetilde{\boldsymbol{f}}_{\alpha}(\lambda)_+,\widetilde{\boldsymbol{f}}_{\alpha}(\lambda)_+)+W_t(\widetilde{\boldsymbol{f}}_{\beta}(\mu)_+,\widetilde{\boldsymbol{f}}_{\beta}(\mu)_+)\right)}B_{\alpha,\beta}(t,\lambda,\mu)$$
$$= e^{\frac{1}{2}W_t\left(\widetilde{\boldsymbol{f}}_{\alpha}(\lambda)_+ + \widetilde{\boldsymbol{f}}_{\beta}(\mu)_+,\widetilde{\boldsymbol{f}}_{\alpha}(\lambda)_+ + \widetilde{\boldsymbol{f}}_{\beta}(\mu)_+\right)}\widetilde{B}_{\alpha,\beta}(\lambda,\mu).$$

The quadratic form W is symmetric, so comparing these identities yields the desired formula. □

4.1.2 Vertex operators at a critical value Assume now that λ is near one of the critical values $u_j(t)$ and that β is a cycle vanishing over $\lambda = u_j(t), 1 \le j \le N+1$. According to Lemma 6 we have $\boldsymbol{f}_\beta(t,\lambda;z) = \Psi_t R_t(z) \boldsymbol{f}_{A_1}(u_j,\lambda;z)$. Using formula (123) it is easy to prove (see [23, Section 7]) that

$$\Gamma^\beta(t,\lambda)\,\widehat{\Psi}_t \widehat{R}_t = e^{\frac{1}{2}V_t(\boldsymbol{f}_\beta(t,\lambda)_-,\boldsymbol{f}_\beta(t,\lambda)_-)}\,\widehat{\Psi}_t \widehat{R}_t\,\Gamma^{\pm}_{A_1}(u_j,\lambda), \tag{92}$$

where $\Gamma^{\pm}_{A_1}(u_j,\lambda) =: e^{\pm \widehat{\boldsymbol{f}}_{A_1}(u_j,\lambda)}:$ is the vertex operator of the A_1–singularity, V_t is the second order differential operator defined in formula (123), and

$$V_t(\boldsymbol{f}_\beta(t,\lambda)_-,\boldsymbol{f}_\beta(t,\lambda)_-) = \sum_{\ell,m=0}^{\infty} \big(I^{(-\ell)}_\beta(t,\lambda), V_{\ell m} I^{(-m)}_\beta(t,\lambda)\big).$$

In this case, the action of the vertex operators is well-defined on the subspace spanned by the tame asymptotical functions and it yields a linear map

$$\Gamma^\beta(t,\lambda)\colon \mathbb{C}_\hbar[\![\boldsymbol{q}]\!]_{\mathrm{tame}} \to K_\hbar[\![\boldsymbol{q}]\!],$$

where $K = \mathbb{C}(((\lambda-u_j)^{1/2}))$. Furthermore, the phase factor $B_{\alpha,\beta}(t,\lambda,\mu)$ is well defined if β is a vanishing cycle, since it can be interpreted as an element in

$$\mathbb{C}(((\mu-u_j)^{1/2}))(((\lambda-u_j)^{1/2})).$$

Finally, similarly to Proposition 29, we have

$$B_{\beta,\beta}(t,\lambda,\mu) = B_{A_1}(u_j,\lambda,\mu)e^{-V_t(\boldsymbol{f}_\beta(t,\lambda)_-,\boldsymbol{f}_\beta(t,\mu)_-)}, \tag{93}$$

where $B_{A_1}(u_j,\lambda,\mu)$ is the phase factor of the product $\Gamma^{\pm}_{A_1}(u_j,\lambda)\Gamma^{\pm}_{A_1}(u_j,\mu)$. A straightforward computation gives

$$B_{A_1}(u_j,\lambda,\mu) = \left(\frac{\sqrt{\lambda-u_j}-\sqrt{\mu-u_j}}{\sqrt{\lambda-u_j}+\sqrt{\mu-u_j}}\right)^2, \tag{94}$$

where the RHS should be expanded into a Laurent series with respect to μ at $\mu = u_j$.

4.2 From descendants to ancestors

Following our construction of the HQEs from Section 3.5 we would like to introduce an integrable hierarchy for the ancestor potential $\mathcal{A}_t$. Let us introduce the Heisenberg fields

$$\phi_\beta(t,\lambda) = \partial_\lambda \widehat{\boldsymbol{f}}^\beta(t,\lambda), \quad \beta \in \Delta',$$

and the corresponding Casimir operator

$$\Omega_{\Delta'}(t,\lambda) = \\ -\tfrac{1}{2}\lambda^2\Big(\sum_{m=1}^{N} :\big(\phi_{\beta_m}(t,\lambda)\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\phi_{\beta_m}(t,\lambda)\big)\big(\phi^{\beta_m}(t,\lambda)\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\phi^{\beta_m}(t,\lambda)\big):\Big) \\ + \sum_{\beta\in\Delta'} b_\beta(t,\lambda)\Gamma^{\beta}(t,\lambda)\otimes_{\mathfrak{a}}\Gamma^{-\beta}(t,\lambda) - \tfrac{1}{2}\operatorname{tr}\big(\tfrac{1}{4}+\theta\theta^{\mathrm{T}}\big),$$

where $\{\beta_m\}$ and $\{\beta^m\}$ are chosen as $\{\tilde{\alpha}_m\}$ and $\{\tilde{\alpha}^m\}$ as in Section 3.5, and the coefficients $b_\beta(t,\lambda)$ are defined by

$$b_\beta(t,\lambda)^{-1} = \lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^2 B_{\beta,-\beta}(t,\lambda,\mu). \tag{95}$$

Finally, we need also to discretize the HQEs corresponding to the above Casimir operator to offset the problem of multivaluedness. Note that, for the toroidal cycle φ in Section 2.5.1, according to Proposition 15 the vector $\boldsymbol{f}^{\varphi}(t,\lambda;z)$ has only negative powers of z, so the quantization $\hat{\boldsymbol{f}}^{\varphi}(t,\lambda)$ is a linear function in $\boldsymbol{q}$.

Lemma 30 *Let φ be the toroidal cycle. Then the equation*

$$\hat{\boldsymbol{f}}^{\varphi}(t,\lambda)\otimes 1 - 1\otimes\hat{\boldsymbol{f}}^{\varphi}(t,\lambda) = 2\pi\sqrt{-1}\,n \tag{96}$$

is equivalent to the system of equations

$$[S_t^{-1}\boldsymbol{q}(z)]_{0,00}\otimes 1 - 1\otimes[S_t^{-1}\boldsymbol{q}(z)]_{0,00} = n\sqrt{\hbar}, \tag{97}$$

$$[S_t^{-1}\boldsymbol{q}(z)]_{\ell,00}\otimes 1 - 1\otimes[S_t^{-1}\boldsymbol{q}(z)]_{\ell,00} = 0 \quad \text{for all } \ell\geq 1, \tag{98}$$

where $[S_t^{-1}\boldsymbol{q}(z)]_{\ell,i}$ denotes the coefficient of $S_t^{-1}\boldsymbol{q}(z)$ in front of $\phi_i z^{\ell}$.

Proof Note that

$$\tilde{\boldsymbol{f}}^{\varphi}(\lambda;z) = 2\pi\sqrt{-1}\sum_{\ell=0}^{\infty}\frac{\lambda^{\ell}}{\ell!}\phi_{01}(-z)^{-\ell-1}.$$

The equations (97)–(98) can be written equivalently as

$$\Omega(\tilde{\boldsymbol{f}}^{\varphi}(\lambda;z), S_t^{-1}\boldsymbol{q}(z)) = 2\pi\sqrt{-1}\,n\sqrt{\hbar}.$$

It remains only to recall that S_t is a symplectic transformation and that $\boldsymbol{f}^{\varphi}(t,\lambda;z) = S_t\tilde{\boldsymbol{f}}^{\varphi}(\lambda;z)$. □

We will be interested in the following HQEs: for every integer $n \in \mathbb{Z}$,

$$\text{(99)} \qquad \operatorname{Res}_{\lambda=\infty} \frac{d\lambda}{\lambda}\big(\Omega_{\Delta'}(t,\lambda)(\tau\otimes\tau)\big)\big|_{\hat{f}^{\varphi}(t,\lambda)\otimes 1-1\otimes \hat{f}^{\varphi}(t,\lambda)=2\pi\sqrt{-1}n} = 0,$$

where τ belongs to an appropriate Fock space and we have to require also that the discretization is well defined. For our purposes the HQEs (99) will be on the Fock space $\mathbb{C}_\hbar[\![q_0+t, q_1+1, q_2,\dots]\!]$. On the other hand the operator $\hat{S}_t^{-1}$ gives rise to an isomorphism

$$\hat{S}_t^{-1}\colon \mathbb{C}_\hbar[\![q_0+t, q_1+\mathbf{1}, q_2,\dots]\!] \to \mathbb{C}_\hbar[\![q_0, q_1+1, q_2,\dots]\!],$$

which allows us to identify the HQEs (87) and (99).

Proposition 31 *A function τ is a solution to the HQEs (99) if and only if $\hat{S}_t^{-1}\tau$ is a solution to the HQEs (87).*

Proof Using Proposition 29 we get that

$$\widetilde{\Omega}_{\Delta'}(\lambda)\,(\hat{S}_t^{-1}\otimes\hat{S}_t^{-1}) = (\hat{S}_t^{-1}\otimes\hat{S}_t^{-1})\,\Omega_{\Delta'}(t,\lambda).$$

It remains only to notice that the discretization in both HQEs are compatible with the action of $\hat{S}_t$, which follows easily from formula (124) and Lemma 30. □

4.3 The integrable hierarchy for A_1–singularity

It was conjectured by Witten [51] and first proved by Kontsevich [38] that the total descendant potential of a point is a tau-function of the KdV hierarchy. The latter can be written in two different ways: via the Kac–Wakimoto construction and as a reduction of the KP hierarchy. We will need both realizations, so let us recall them.

4.3.1 The Kac–Wakimoto construction of KdV The Casimir operator for the A_1–singularity $f(x)=x^2/2+u$ (see Section 4.2) takes the form

$$\Omega_{A_1}(u,\lambda) = -\tfrac{1}{4}\lambda^2 :\phi_\beta^{V\otimes V}(u,\lambda)\phi_\beta^{V\otimes V}(u,\lambda): \\ + b_\beta(u,\lambda)\big(\Gamma^{\beta}_{A_1}(u,\lambda)\otimes\Gamma^{-\beta}_{A_1}(u,\lambda)+\Gamma^{-\beta}_{A_1}(u,\lambda)\otimes\Gamma^{\beta}_{A_1}(u,\lambda)\big)-\tfrac{1}{8},$$

where the coefficient $b_\beta(u,\lambda)$ is given by

$$b_\beta(u,\lambda) = \lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^{-2} B_{\beta,\beta}(u,\mu,\lambda) = \frac{\lambda^2}{16(\lambda-u)^2}.$$

We denoted by V the Fock space $\mathbb{C}_\hbar[\![\boldsymbol{q}]\!]$, and

$$\phi_\beta^{V\otimes V}(u,\lambda) := \phi_\beta(u,\lambda)\otimes 1 - 1\otimes\phi_\beta(u,\lambda).$$

Witten's conjecture (Kontsevich's theorem) can be stated as follows:

$$\operatorname{Res}_{\lambda=\infty} \Omega_{A_1}(0,\lambda)\,(\mathcal{D}_{\mathrm{pt}} \otimes \mathcal{D}_{\mathrm{pt}}) = 0. \tag{100}$$

To compare the above equation with the principal Kac–Wakimoto hierarchy of type A_1, note that

$$\Gamma^{\beta}_{A_1}(u,\lambda) = \exp\Big(2\sum_{n=0}^{\infty} \frac{(2(\lambda-u))^{n+1/2}}{(2n+1)!!}\,\frac{q_n}{\sqrt{\hbar}}\Big)\exp\Big(-2\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2(\lambda-u))^{n+1/2}}\sqrt{\hbar}\,\partial_n\Big),$$

and that the coefficient in front of λ^{-2} in $\frac{1}{4}:\phi_\beta^{V\otimes V}(0,\lambda)\phi_\beta^{V\otimes V}(0,\lambda):$ is precisely

$$\sum_{n=0}^{\infty}\big(n+\tfrac{1}{2}\big)(q_n\otimes 1 - 1\otimes q_n)(\partial_n\otimes 1 - 1\otimes\partial_n),$$

where $\partial_n := \partial/\partial q_n$. It follows that the above equations coincide with the Kac–Wakimoto form of the KdV hierarchy up to the rescaling $q_n = t_{2n+1}(2n+1)!!$.

On the other hand, the total descendant potential $\mathcal{D}_{\mathrm{pt}}$ satisfies the string equation, which can be stated as follows (see [21]): $e^{(u/z)^\wedge}\mathcal{D}_{\mathrm{pt}} = \mathcal{D}_{\mathrm{pt}}$. Using that

$$\Omega_{A_1}(0,\lambda)\big(e^{(u/z)^\wedge}\otimes e^{(u/z)^\wedge}\big) = \big(e^{(u/z)^\wedge}\otimes e^{(u/z)^\wedge}\big)\Omega_{A_1}(u,\lambda)$$

we get that $\mathcal{D}_{\mathrm{pt}}$ satisfies also the HQEs

$$\operatorname{Res}_{\lambda=\infty} \Omega_{A_1}(u,\lambda)\,(\mathcal{D}_{\mathrm{pt}} \otimes \mathcal{D}_{\mathrm{pt}}) = 0. \tag{101}$$

4.3.2 The KdV hierarchy as a reduction of KP According to Givental [23] the KdV hierarchy (100) can be written also as

$$\operatorname{Res}_{\lambda=0}\Big(\sum_{\pm}\frac{d\lambda}{\pm\sqrt{\lambda}}\,\Gamma^{\pm\beta/2}_{A_1}(0,\lambda)\otimes\Gamma^{\mp\beta/2}_{A_1}(0,\lambda)\Big)(\mathcal{D}_{\mathrm{pt}}\otimes\mathcal{D}_{\mathrm{pt}}) = 0.$$

Using again the string equation and Proposition 29 we get that $\mathcal{D}_{\mathrm{pt}}$ satisfies also the HQEs

$$\operatorname{Res}_{\lambda=u}\Big(\sum_{\pm}\frac{d\lambda}{\pm\sqrt{\lambda-u}}\,\Gamma^{\pm\beta/2}_{A_1}(u,\lambda)\otimes\Gamma^{\mp\beta/2}_{A_1}(u,\lambda)\Big)(\mathcal{D}_{\mathrm{pt}}\otimes\mathcal{D}_{\mathrm{pt}}) = 0. \tag{102}$$

4.4 The phase factors

In this section we will prove Proposition 38, that the phase factors $B_{\alpha,\beta}(t,\lambda,\mu)$ (see (90)) are multivalued analytic function and that the analytic continuation is compatible with the monodromy action on the cycles α and β. To begin with, put

$$B^{\infty}_{\alpha,\beta}(t,\lambda,\mu) = \exp\Omega^{\infty}_{\alpha,\beta}(t,\lambda,\mu),$$

where

$$\Omega^{\infty}_{\alpha,\beta}(t,\lambda,\mu) := \iota_{\lambda^{-1}}\iota_{\mu^{-1}} \sum_{n=0}^{\infty}(-1)^{n+1}(I^{(n)}_{\alpha}(t,\lambda), I^{(-n-1)}_{\beta}(t,\mu)), \tag{103}$$

where $\iota_{\lambda^{-1}}$ (resp. $\iota_{\mu^{-1}}$) is the Laurent series expansion at $\lambda=\infty$ (resp. $\mu=\infty$). The differential of (103) with respect to t is

$$\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu) := I^{(0)}_{\alpha}(t,\lambda)\bullet I^{(0)}_{\beta}(t,\mu) = \sum_{i\in\mathfrak{I}}(I^{(0)}_{\alpha}(t,\lambda), \partial_i \bullet I^{(0)}_{\beta}(t,\mu))\, dt_i,$$

which will be interpreted as a 1–form on M depending on the parameters λ and μ. Furthermore, for each $t\in M$, put $r(t)=\max_j |u_j(t)|$, where $\{u_j(t)\}_{j=1}^{N+1}$ is the set of all critical values of $F(x,t)$. In other words, $r(t)$ is the radius of the smallest disk (with center at 0) that contains all critical values of $F(x,t)$. Let

$$D^{+}_{\infty} = \big\{(t,\lambda,\mu)\in M\times\mathbb{C}^2 : |\lambda-\mu| < |\mu|-r(t) < |\lambda|-r(t)\big\}.$$

Note that since $|\lambda-\mu|\geq 0$ we have $|\lambda|>r(t)$ and $|\mu|>r(t)$ for all $(t,\lambda,\mu)\in D^{+}_{\infty}$, which implies that the Laurent series expansions of $I^{(0)}_{\alpha}(t,\lambda)$ and $I^{(0)}_{\beta}(t,\mu)$ at respectively $\lambda=\infty$ and $\mu=\infty$ are convergent. The first inequality in the definition of D^{+}_{∞} guarantees that the line segment $[\lambda,\mu]$ is outside the disk $|x|\leq r(t)$. In particular, in order to specify a branch of $\mathcal{W}_{\alpha,\beta}(\lambda,\mu)$ it is enough to specify the branches of the period vectors only at the point (t,λ), the branch of the periods at (t,μ) is determined via the line segment $[\lambda,\mu]$.

Proposition 32 *The series (103) is convergent for all $(t,\lambda,\mu)\in D^{+}_{\infty}$.*

Proof Using Proposition 29 we can write (103) as a sum of two formal series

$$\Omega^{\infty}_{\alpha,\beta}(t,\lambda,\mu) = \widetilde{\Omega}^{\infty}_{\alpha,\beta}(\lambda,\mu) + W_t(\widetilde{f}_{\alpha}(\lambda)_+, \widetilde{f}_{\beta}(\mu)_+), \tag{104}$$

where $\widetilde{\Omega}^{\infty}_{\alpha,\beta}$ is the Laurent series expansion of $\log \widetilde{B}_{\alpha,\beta}$ in the domain $|\lambda|>|\mu|$.

Since the series $\widetilde{\Omega}^{\infty}_{\alpha,\beta}$ is convergent for $|\lambda|>|\mu|>|\lambda-\mu|$, it is enough to prove the proposition for the second series on the RHS of (104). Recalling the definition of W_t and using the fact that modulo Q the series $S_t(z)=e^{(1/z)t\,\cup}$, where $t\,\cup$ means the classical orbifold cup product multiplication by t, we get that

$$\lim_{\mathrm{Re}(t_{01})\to-\infty}\ \lim_{t\to(0,\dots,0,t_{01})}(W_t - t_{01}P) = 0,$$

On the other hand, since

$$dW_t(\widetilde{f}_{\alpha}(\lambda)_+, \widetilde{f}_{\beta}(\mu)_+) = d\Omega_{\alpha,\beta}(t,\lambda,\mu) = I^{(0)}_{\alpha}(t,\lambda)\bullet I^{(0)}_{\beta}(t,\mu),$$

the series

$$W_t(\widetilde{f}_\alpha(\lambda)_+,\widetilde{f}_\beta(\mu)_+)-t_{01}(\alpha_0|\beta_0)/\chi, \tag{105}$$

viewed as a formal Laurent series in λ^{-1} and μ^{-1} can be identified with the improper integral

$$\lim_{\varepsilon\to\infty}\int_\varepsilon^t\big(I_\alpha^{(0)}(t',\lambda)\bullet I_\beta^{(0)}(t',\mu)-dt'_{01}(\alpha_0|\beta_0)/\chi\big), \tag{106}$$

where $\varepsilon\in M$ and the limit is taken along a straight segment such that $\varepsilon_i\to 0$ for $i\neq 01$ and $\mathrm{Re}(\varepsilon_{01})\to-\infty$. More precisely, if we take the Laurent series expansion of the integrand at $\lambda=\infty$ and $\mu=\infty$ and integrate termwise, we get (105). It remains only to notice that the integrand extends holomorphically at the limiting point $\varepsilon=\infty$ (because we removed the singularity), so the termwise integration preserves the convergence. □

The proof of Proposition 32 yields slightly more. Namely, we proved that the second summand on the RHS of (104) is a convergent Laurent series in λ^{-1} and μ^{-1} and that the corresponding limit is a multivalued analytic function on

$$D_\infty:=\big\{(t,\lambda,\mu)\in M\times\mathbb{C}^2:|\lambda-\mu|<\min(|\lambda|-r(t),|\mu|-r(t))\big\}.$$

On the other hand, the phase factor $\widetilde{B}_{\alpha,\beta}(\lambda,\mu)$ is also a multivalued analytic function on D_∞ except for a possible pole along $\lambda=\mu$. Hence we have the following corollary (of the proof).

Corollary 33 *The series $B^\infty_{\alpha,\beta}(t,\lambda,\mu)$ extends analytically to a multivalued analytic function on D_∞ except for a possible pole along the diagonal $\lambda=\mu$.*

Using the analytic extension of $B^\infty_{\alpha,\beta}(t,\lambda,\mu)$ we define a multivalued function with values in the space $\mathbb{C}\{\{\xi\}\}$ of convergent Laurent series at $\xi=0$ in the following way:

$$B_{\alpha,\beta}\colon (M\times\mathbb{C})_\infty\to\mathbb{C}\{\{\xi\}\},\qquad (t,\lambda)\mapsto \iota_{\mu-\lambda}B^\infty_{\alpha,\beta}(t,\lambda,\mu),$$

where $\xi=\mu-\lambda$, $\iota_{\mu-\lambda}$ is the Laurent series expansion at $\mu=\lambda$, and

$$(M\times\mathbb{C})_\infty:=\{(t,\lambda)\in M\times\mathbb{C}:|\lambda|>r(t)\}.$$

It is convenient to introduce the 1–form $\mathcal{W}_{\alpha,\beta}(\xi):=\widetilde{\mathcal{W}}_{\alpha,\beta}(0,\xi)$. Following [23] we call $\mathcal{W}_{\alpha,\beta}(\xi)$ the *phase form*. Note that if $C\subset(M\times\mathbb{C})_\infty$ is a path from (t,λ) to (t',λ'), then

$$B_{\alpha,\beta}(t',\lambda')=B_{\alpha,\beta}(t,\lambda)\,e^{\int_C\mathcal{W}_{\alpha,\beta}(\xi)}. \tag{107}$$

Therefore we can uniquely extend the function $B_{\alpha,\beta}$ to a function on $(M\times\mathbb{C})'$, so that formula (107) holds for all paths $C\subset (M\times\mathbb{C})'$. Finally, for every $(t,\lambda)\in (M\times\mathbb{C})'$ and μ sufficiently close to λ we define

$$B_{\alpha,\beta}(t,\lambda,\mu)=B_{\alpha,\beta}(t,\lambda)|_{\xi=\mu-\lambda},\quad \Omega_{\alpha,\beta}(t,\lambda,\mu):=\log B_{\alpha,\beta}(t,\lambda,\mu).$$

Note that $B_{\alpha,\beta}(t,\lambda,\mu)=B^{\infty}_{\alpha,\beta}(t,\lambda,\mu)$ if $(t,\lambda,\mu)\in D^{+}_{\infty}$.

Let $t_0\in M$ be a generic point, so that all critical points of $F(x,t_0)$ are of type A_1 and the absolute values of the corresponding critical values are pairwise distinct. Let $u_j(t_0)$ be a critical value of $F(x,t_0)$ with a maximal absolute value, ie $|u_j(t_0)|=r(t_0)$. There exists a real number $\epsilon_0>0$ such that if $|x|<\epsilon_0$, then $r(t_0+x\mathbf{1})=|u_j(t_0)+x|$. We fix $t=t_0+x_0\mathbf{1}$, λ, and μ such that the line segment $[\mu-u_j(t_0),x_0]$ is contained inside the disk $\{|x|<\epsilon_0\}\subset\mathbb{C}$ and the line segment $[t_0,t]\times\{(\lambda,\mu)\}\subset D^{+}_{\infty}$. For example, fix μ such that $|\mu|>u_j(t_0)$ and $|\mu-u_j(t_0)|<\varepsilon_0$ and put $x_0=\frac{1}{2}(\mu-u_j(t_0))$. Then we can find λ such that all requirements are fulfilled.

Lemma 34 *If $\beta\in H_2(X_{t,\mu};\mathbb{Z})$ is a cycle vanishing over $t=t_0+(\mu-u_j(t_0))\mathbf{1}$, then*

$$\Omega_{\alpha,\beta}(t,\lambda,\mu)=\lim_{\varepsilon\to 0}\int_{t_0+(\varepsilon+\mu-u_j(t_0))\mathbf{1}}^{t}\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu),\tag{108}$$

where the integration is along a straight segment.

Proof By definition $\Omega_{\alpha,\beta}(t,\lambda,\mu)$ is the Laurent series expansion near $\lambda=\infty$ of the series

$$\sum_{n=0}^{\infty}(-1)^{n+1}\big(I^{(n)}_{\alpha}(t,\lambda),I^{(-n-1)}_{\beta}(t,\mu)\big),\tag{109}$$

while the RHS of (108) is

$$\lim_{\varepsilon\to 0}\int_{\varepsilon+\mu-u_j}^{x_0}\big(I^{(0)}_{\alpha}(t_0,\lambda-x),I^{(0)}_{\beta}(t_0,\mu-x)\big)\,dx.\tag{110}$$

Using integration by parts $(n+1)$ times and the fact that the periods $I^{(-p-1)}_{\beta}(t_0,\mu-x)$ vanish at $x=\mu-u_j$, we get that the integral (110) coincides with

$$\sum_{p=0}^{n}(-1)^{p+1}\big(I^{(p)}_{\alpha}(t_0',\lambda),I^{(-p-1)}_{\beta}(t_0',\mu)\big)\\ +\lim_{\varepsilon\to 0}(-1)^{n+1}\int_{\varepsilon+\mu-u_j}^{x_0}\big(I^{(n+1)}_{\alpha}(t_0,\lambda-x),I^{(-n-1)}_{\beta}(t_0,\mu-x)\big)\,dx.\tag{111}$$

The Laurent series expansion of $I^{(n+1)}_\alpha(t_0,\lambda-x)=I^{(n+1)}_\alpha(t_0+x\mathbf{1},\lambda)$ in λ^{-1} has radius of convergence $r(t_0+x\mathbf{1})$. Hence, it is uniformly convergent for all x that vary along a compact subset of the open subset in $\mathbb{C}$ defined by the inequality

$$\{x\in\mathbb{C}:|\lambda|>r(t_0+x\mathbf{1})\}.$$

On the other hand, according to our choice of x_0,λ and μ, the point $(t_0+x\mathbf{1},\lambda,\mu)$ lies in D^+_∞ for all x on the integration path. In particular, $|\lambda|>|\mu|>r(t_0+x\mathbf{1})$, which means that the integration path is entirely contained in the above open subset. Hence the integral in (111) has a convergent Laurent series in λ^{-1}. Moreover, the leading order term of the expansion is λ^{-e} for some rational number $e>n$. This proves that the Laurent series expansions (in λ^{-1}) of the integral (110) and of the series (109) coincide. □

Our next goal is to prove that the analytic continuation of the phase factor $B_{\alpha,\beta}(t,\lambda,\mu)$ is compatible with the monodromy representation in the following sense. Recall the monodromy representation (see Section 2.2)

$$\rho\colon \pi_1((M\times\mathbb{C})')\to \mathrm{GL}(\mathfrak{h}).$$

Let $U\subset(M\times\mathbb{C})'$ be an open subdomain and $f_{\alpha,\beta}(t,\lambda)$ be a (vector-valued) function depending bilinearly on $(\alpha,\beta)\in\mathfrak{h}\times\mathfrak{h}$ and analytic in a neighborhood of some point $(t_0,\lambda_0)\in U$. We say that $f_{\alpha,\beta}$ is *multivalued analytic* on U if it can be extended analytically along any path in U. Furthermore, we say that $f_{\alpha,\beta}$ is *compatible* with the monodromy representation ρ, if for every closed loop C in U, the analytic continuation of $f_{\alpha,\beta}(t,\lambda)$ along C coincides with $f_{w(\alpha),w(\beta)}(t,\lambda)$, where $w=\rho(C)$ is the corresponding monodromy transformation.

Recall that (see Corollary 33) the Laurent series $\Omega^\infty_{\beta,\alpha}(t,\lambda,\mu)$ extends analytically to a multivalued analytic function $\Omega_{\beta,\alpha}(t,\lambda,\mu)$ defined for all $(t,\lambda,\mu)\in D_\infty$ such that $\lambda\neq\mu$.

Lemma 35 *Let α and β be cycles in the vanishing cohomology such that $(\alpha|\beta)=0$. Then*

$$\Omega_{\alpha,\beta}(t,\lambda,\mu)-\Omega_{\beta,\alpha}(t,\mu,\lambda)=2\pi\sqrt{-1}\,\mathrm{SF}(\alpha,\beta)\quad \textit{for all } (t,\lambda,\mu)\in D^+_\infty,$$

where SF *is the bilinear form* (56).

Proof Since the difference

$$\Omega_{\beta,\alpha}(t,\lambda,\mu)-\widetilde{\Omega}_{\beta,\alpha}(\lambda,\mu),\quad \text{where}\quad \widetilde{\Omega}_{\beta,\alpha}(\lambda,\mu):=\log\widetilde{B}_{\beta,\alpha}(\lambda,\mu),$$

has a convergent Laurent series expansion in D_∞ and it is invariant under switching $(\beta,\lambda)\leftrightarrow(\alpha,\mu)$, it is enough to prove the statement for $\widetilde{\Omega}_{\alpha,\beta}(\lambda,\mu)$ where (λ,μ) is a point in the open subset

$$\{|\lambda-\mu|<\min(|\lambda|,|\mu|)\}\subset\mathbb{C}^2.$$

Recalling formula (86), the rest of the proof is a straightforward computation (see also the proof of Lemma 24, where some of the computations were already done). $\square$

Remark 36 If we omit the condition $(\alpha|\beta)=0$ in Lemma 35, then the identity is true only up to an integer multiple of $2\pi\sqrt{-1}(\alpha|\beta)$. The ambiguity comes from the fact that the phase factor $\widetilde{\Omega}_{\alpha,\beta}(\lambda,\mu)$ has a logarithmic singularity along $\lambda=\mu$ of the type $(\alpha|\beta)\log(\lambda-\mu)$.

Proposition 37 *The phase factor $B_{\alpha,\beta}(t,\lambda)$ is compatible with the monodromy representation in the domain $(M\times\mathbb{C})'$.*

Proof By definition we have to prove that if $C'\subset(M\times\mathbb{C})'$ is an arbitrary loop based at (t,λ) and μ is sufficiently close to λ, then

$$B_{w(\alpha),w(\beta)}(t,\lambda,\mu)=B_{\alpha,\beta}(t,\lambda,\mu)\,e^{\int_C\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)},$$

where $w=\rho(C')$ and $C\subset M$ is the path parametrized by

$$t'+(\lambda-\lambda')\mathbf{1},\quad (t',\lambda')\in C'.$$

We may assume that $(t,\lambda,\mu)\in D_\infty^+$, because by definition the value of $B_{\alpha,\beta}$ at any other point differs by an integral along the path of the phase form $\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)$. Under this assumption the above equality is equivalent to

$$\Omega^\infty_{w(\alpha),w(\beta)}(t,\lambda,\mu)=\Omega^\infty_{\alpha,\beta}(t,\lambda,\mu)+\int_C\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)\pmod{2\pi\sqrt{-1}\mathbb{Z}}. \tag{112}$$

We first prove a special case of the above formula. Namely, let us choose a generic point $t_0\in M$ such that the absolute values of the critical values of $F(x,t_0)$ are pairwise distinct and let $u_j(t_0)$ be the critical value with maximal absolute value (here the notation is the same as in Lemma 34). We will assume that $t=t_0+x_0\mathbf{1}$ is sufficiently close to $t_0+(\mu-u_j(t_0))\mathbf{1}$ and that C is a closed loop of the type $t_0+x\mathbf{1}$, where the parameter x varies along a small closed loop based at $x_0\in\mathbb{C}$ going around $\mu-u_j(t_0)$, so that the line segment $[\lambda-x,\mu-x]$ moves around u_j. Let us denote by $\gamma\in H_2(X_{t,\lambda};\mathbb{Z})$ the vanishing cycle vanishing over $(t_0,u_j(t_0))$. Then we have the following decompositions:

$$\alpha=\alpha'+\tfrac{1}{2}(\alpha|\gamma)\gamma,\quad \beta=\beta'+\tfrac{1}{2}(\beta|\gamma)\gamma,$$

where α' and β' are cycles invariant with respect to the local monodromy around the point $(t_0, u_j(t_0))$. After a straightforward computation we get

$$\Omega_{w(\alpha),w(\beta)}(t,\lambda,\mu) - \Omega_{\alpha,\beta}(t,\lambda,\mu) = -(\alpha|\gamma)\Omega_{\gamma,\beta'}(t,\lambda,\mu) - (\beta|\gamma)\Omega_{\alpha',\gamma}(t,\lambda,\mu),$$

while $\int_C \widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)$ is

$$\tag{113} \tfrac{1}{2}(\beta|\gamma)\int_C \widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu) + \tfrac{1}{2}(\alpha|\gamma)\int_C \widetilde{\mathcal{W}}_{\gamma,\beta'}(\lambda,\mu) + \tfrac{1}{4}(\alpha|\gamma)(\beta|\gamma)\int_C \widetilde{\mathcal{W}}_{\gamma,\gamma}(\lambda,\mu),$$

where we used that $\int_C \widetilde{\mathcal{W}}_{\alpha',\beta'}(\lambda,\mu) = 0$, because the periods $I^{(0)}_{\alpha'}(t_0, \lambda - x)$ and $I^{(0)}_{\beta'}(t_0, \mu - x)$ are holomorphic respectively at $x = \lambda - u_j$ and $x = \mu - u_j$, which means that the phase form is holomorphic inside the loop C. The last integral in the above formula is easy to compute because only the singular terms of $I^{(0)}_{\gamma}(t_0, \lambda - x)$ and $I^{(0)}_{\gamma}(t_0, \mu - x)$ contribute, ie

$$\int_C \widetilde{\mathcal{W}}_{\gamma,\gamma}(\lambda,\mu) = 2\oint \frac{dx}{\sqrt{(\lambda - u_j(t_0) - x)(\mu - u_j(t_0) - x)}} = 4\pi\sqrt{-1}.$$

According to Lemma 34,

$$\Omega_{\alpha',\gamma}(t,\lambda,\mu) = \int_{t_0 + (\mu - u_j(t_0))\mathbf{1}}^{t} \widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu),$$

and the integral on the RHS has a convergent Laurent series expansion in $\lambda - u_j(t)$ and $(\mu - u_j(t))^{1/2}$, which allows us to evaluate the integral

$$\begin{aligned}\int_C \widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu) &= -2\int_{t_0 + (\mu - u_j(t_0))\mathbf{1}}^{t} \widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu) \\ &= -2\Omega^{\infty}_{\alpha',\gamma}(t,\lambda,\mu) = -2\Omega_{\alpha',\gamma}(t,\lambda,\mu).\end{aligned}$$

It remains only to evaluate the second integral in (113). We have

$$\int_C \widetilde{\mathcal{W}}_{\gamma,\beta'}(\lambda,\mu) = \int_C \widetilde{\mathcal{W}}_{\beta',\gamma}(\mu,\lambda) = -2\Omega_{\beta',\gamma}(t,\mu,\lambda),$$

where the second identity is derived just like above when $|\mu| > |\lambda|$, and then we use analytic continuation to extend the formula for $|\mu| < |\lambda|$ as well. Recalling Lemma 35, we get

$$\Omega_{\beta',\gamma}(t,\mu,\lambda) = \Omega_{\gamma,\beta'}(t,\lambda,\mu) + 2\pi\sqrt{-1}\,\mathrm{SF}(\beta',\gamma).$$

Using that $\beta' = \beta - \frac{1}{2}(\beta|\gamma)\gamma$ and that $\mathrm{SF}(\gamma,\gamma) = 1$, we finally get

$$\int_C \widetilde{\mathcal{W}}_{\gamma,\beta'}(\lambda,\mu) = -2\Omega_{\gamma,\beta'}(t,\lambda,\mu) - 4\pi\sqrt{-1}\,\mathrm{SF}(\beta,\gamma) + 2\pi\sqrt{-1}(\beta|\gamma).$$

Since $\mathrm{SF}(\beta,\gamma) \in \mathbb{Z}$, the proof of formula (112) in the special case is complete.

The general case follows easily, because the fundamental group $\pi_1((M \times \mathbb{C})')$ is generated by loops like the above one. Indeed, we already know that the affine cusp polynomial $f(x)$ has a real Morsification $F(x,t_0')$, ie all critical points of $F(x,t_0')$ are real and the corresponding critical values are real as well. In particular, we can find a small deformation $F(x,t_0)$ of the real Morsification such that the critical values u_j are vertices of a convex polygon. The fundamental group $\pi_1((M\times\mathbb{C})')$ is generated by simple loops in $\{t_0\}\times\mathbb{C}$ that go around the vertices of the polygon. Let us pick one of these loops and let $(t_0,u_j(t_0))$ be the corresponding vertex of the polygon. Since the translations of the type $t_0 \mapsto t_0 + c\mathbf{1}$, $c\in\mathbb{C}$, do not change the homotopy class of the loop, we can find a representative (namely, pick c such that the $|u_j(t_0)+c| > |u_j(t_0)+c|$ for all other vertices $(t_0,u_j(t_0))$) of the homotopy class, which has the special form from above. □

Proposition 38 *There exists a generic point $t_0 \in M$ (ie $F(x,t_0)$ is a Morse function) and a critical value $u_j(t_0)$ such that*

$$B_{\alpha,\beta}(t,\lambda,\mu) = \lim_{\varepsilon\to 0} \exp\biggl(-\int_t^{t_0+(\varepsilon+\mu-u_j(t_0))\mathbf{1}} \widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)\biggr), \tag{114}$$

where the integration is along any path avoiding the poles of the 1–form $\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)$ such that the cycle $\beta\in H_2(X_{t,\mu},\mathbb{Z})$ vanishes along it.

Proof Let us assume that t_0 is a generic point and that $u_j(t_0)$ is the critical value with maximal absolute value. It is enough to prove the statement for an arbitrary point $(t,\lambda,\mu)\in D_\infty^+$, because by definition the value of $B_{\alpha,\beta}(t',\lambda,\mu)$ at any other point (t',λ,μ) differs by an integral of $\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)$ along a path connecting t and t', while the RHS of (114) clearly has the same property. Let $(t,\lambda,\mu)\in D_\infty^+$ be a point such that Lemma 34 holds and let C_ε'' be the straight segment $[t, t_0+(\varepsilon+\mu-u_j(t_0))\mathbf{1}]$. Put $C' = (C_\varepsilon'')^{-1}\circ C_\varepsilon$ and $w=\rho(C')$, where C_ε is the integration path (from t to $t_0+(\varepsilon+\mu-u_j(t_0))\mathbf{1}$). Then by definition the cycle $w(\beta)\in H_2(X_{t,\mu};\mathbb{Z})$ is the vanishing cycle along the line segment $[t,t_0+(\mu-u_j(t_0))\mathbf{1}]$. According to Lemma 34, formula (114) holds for C'' and $B_{w(\alpha),w(\beta)}$. Therefore, we need to prove that

$$-\int_{C'} \widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu) = \Omega_{\alpha,\beta}(t,\lambda,\mu) - \Omega_{w(\alpha),w(\beta)}(t,\lambda,\mu) \pmod{2\pi\sqrt{-1}\mathbb{Z}}, \tag{115}$$

which follows from Proposition 37. □

4.5 The ancestor solution

Theorem 39 *The total ancestor potential $\mathcal{A}_t(\hbar;\boldsymbol{q})$ is a solution to the HQEs (99).*

We do some preparatory work in Sections 4.5.1–4.5.3 and prove Theorem 39 in Section 4.5.4.

To begin with, put $\boldsymbol{q}' = \boldsymbol{q} \otimes 1$, $\boldsymbol{q}'' = 1 \otimes \boldsymbol{q}$, and let us assume that the discretization condition (96) is satisfied for some integer n. The tameness of $\mathcal{A}(\hbar;\boldsymbol{q})$ implies that the LHS of (99) (for $\tau = \mathcal{A}(\hbar;\boldsymbol{q})$) is a formal series in $\boldsymbol{q}'$ and $\boldsymbol{q}''$ with coefficients formal Laurent series in $\sqrt{\hbar}$, whose coefficients are polynomial expressions of the period vectors $I^{(n)}_\alpha(t,\lambda)$. In particular, the residue in (99) can be computed via the residue theorem, ie we have to compute the residues at all critical points and at $\lambda = 0$ and prove that their sum is 0.

Let $u_j(t)$ be one of the critical points of F, where $t \in M$ is a generic point such that all critical values are pairwise different. Furthermore, we assume that λ is near $u_j(t)$ and that a path in $(M \times \mathbb{C})'$ from the reference point $(0,1)$ to (t,λ) is fixed in such a way that the vanishing cycle β, vanishing over $\lambda = u_j(t)$, belongs to the subset Δ' of affine roots defined in Section 3.5.

4.5.1 The Virasoro term Let us compute

$$-\operatorname{Res}_{\lambda=u_j(t)} \tfrac{1}{2}\lambda\, d\lambda \sum_{m=1}^{N} :\phi^{V\otimes V}_{\beta_m}(t,\lambda)\phi^{V\otimes V}_{\beta^m}(t,\lambda): \mathcal{A}_t^{\otimes 2}, \tag{116}$$

where $\phi^{V\otimes V}_\alpha := \phi_\alpha \otimes 1 - 1 \otimes \phi_\alpha$. Put

$$\beta_m = \alpha_m + \tfrac{1}{2}(\beta_m|\beta)\beta \quad \text{and} \quad \beta^m = \alpha^m + \tfrac{1}{2}(\beta^m|\beta)\beta,$$

where $(\alpha_m|\beta) = (\alpha^m|\beta) = 0$. The above operator can be written as the sum of

$$\sum_{m=1}^{N} :\phi^{V\otimes V}_{\alpha_m}(t,\lambda)\phi^{V\otimes V}_{\alpha^m}(t,\lambda): + \frac{1}{4}\left(\sum_{m=1}^{N}(\beta_m|\beta)(\beta^m|\beta)\right) :\phi^{V\otimes V}_{\beta}(t,\lambda)\phi^{V\otimes V}_{\beta}(t,\lambda):$$

and

$$\frac{1}{2}\sum_{m=1}^{N}\Big((\beta_m|\beta) :\phi^{V\otimes V}_{\beta}(t,\lambda)\phi^{V\otimes V}_{\alpha^m}(t,\lambda): + (\beta^m|\beta) :\phi^{V\otimes V}_{\beta}(t,\lambda)\phi^{V\otimes V}_{\alpha_m}(t,\lambda):\Big) \tag{117}$$

The Picard–Lefschetz formula implies that the periods $I^{(n)}_{\alpha_m}(t,\lambda)$ and $I^{(n)}_{\alpha^m}(t,\lambda)$ are invariant with respect to the local monodromy around $\lambda = u_j(t)$, so they must be holomorphic in a neighborhood of $\lambda = u_j(t)$. The operator $\phi^{V\otimes V}_{\varphi}(t,\lambda)$, where φ is the toroidal cycle, vanishes after we impose the discretization condition (96). On the other hand, since $\sum_m (\beta_m|\beta)(\beta^m|\alpha) = (\beta|\alpha)$, the cycles

$$-\beta + \sum_{m=1}^{N} (\beta_m|\beta)\beta^m \quad \text{and} \quad -\beta + \sum_{m=1}^{N} (\beta^m|\beta)\beta_m$$

are in the kernel of the intersection form, so they must be proportional to φ. Hence the operator (117) vanishes after the discretization condition (96) is imposed. The residue (116) turns into

$$-\operatorname{Res}_{\lambda=u_j(t)} \tfrac{1}{4}\lambda\, d\lambda : \phi_\beta^{V\otimes V}(t,\lambda)\phi_\beta^{V\otimes V}(t,\lambda) : \mathcal{A}_t(\hbar;\boldsymbol{q}')\mathcal{A}_t(\hbar;\boldsymbol{q}'').$$

To compute the above residue, note that the expression

$$:\phi_\beta^{V\otimes V}(t,\lambda)\phi_\beta^{V\otimes V}(t,\lambda) : (\widehat{\Psi}_t \widehat{R}_t)^{\otimes 2}$$

can be written as

$$(\widehat{\Psi}_t \widehat{R}_t)^{\otimes 2} : \phi_{A_1}^{V\otimes V}(u_j,\lambda)\phi_{A_1}^{V\otimes V}(u_j,\lambda) : +2V_t(\phi_\beta(t,\lambda)_-, \phi_\beta(t,\lambda)_-).$$

Let us compute

$$\begin{aligned}-\operatorname{Res}_{\lambda=u_j(t)} \tfrac{1}{4}\lambda\, d\lambda\, 2V_t(\phi_\beta(t,\lambda)_-, \phi_\beta(t,\lambda)_-)& \\ = -\operatorname{Res}_{\lambda=u_j(t)} \tfrac{1}{2}\lambda\, d\lambda\, (V_{00}(t) I_\beta^{(0)}(t,\lambda), I_\beta^{(0)}(t,\lambda)),&\end{aligned}$$

where we used the fact that only the leading term (with respect to z) of

$$\phi_\beta(t,\lambda;z)_- = -I_\beta^{(0)}(t,\lambda)z^{-1} + \cdots$$

will contribute because the remaining ones have a zero at $\lambda = u_j(t)$ of order at least $\frac{1}{2}$. Furthermore, the Laurent series expansion of $I_\beta^{(0)}$ at $\lambda = u_j(t)$ has the form

$$I_\beta^{(0)}(t,\lambda) = 2(2(\lambda - u_j))^{-1/2} e_j + \cdots, \quad e_j = du_j/\sqrt{\Delta_j},$$

where the dots stand for terms that have at $\lambda = u_j$ a zero of order at least $\frac{1}{2}$. These terms do not contribute to the residue, so we get

$$-\operatorname{Res}_{\lambda=u_j(t)} \tfrac{1}{2}\lambda\, d\lambda\, (V_{00}(t)e_j, e_j)\frac{2}{\lambda - u_j(t)} = u_j(t)\,(R_1(t)e_j, e_j).$$

We get the following formula for the residue (116):

$$(\widehat{\Psi}_t \widehat{R}_t)^{\otimes 2}\big(u_j R_1^{jj} - \operatorname{Res}_{\lambda=u_j} \tfrac{1}{4}\lambda\, d\lambda : \phi_{A_1}^{V\otimes V}(u_j,\lambda)\phi_{A_1}^{V\otimes V}(u_j,\lambda) :\big) \prod_{m=1}^{N+1} \mathcal{D}_{\mathrm{pt}}(\hbar\Delta_m; {}^m\boldsymbol{q})^{\otimes 2},$$

where $R_1^{jj} = (R_1 e_j, e_j)$ is the j^{th} diagonal entry of R_1.

4.5.2 The A_1–subroot system The vanishing cycles $\{-\beta,\beta\}$ form a subroot system of type A_1. Let us compute the residue of the corresponding vertex operator terms, ie

$$\operatorname{Res}_{\lambda=u_j(t)} \frac{d\lambda}{\lambda}\Big(\sum_{\pm} b_{\pm\beta}(t,\lambda)\Gamma^{\pm\beta}(t,\lambda)\otimes\Gamma^{\mp\beta}(t,\lambda)\Big)\mathcal{A}_t^{\otimes 2}. \tag{118}$$

We have $b_\beta(t,\lambda)=b_{-\beta}(t,\lambda)$ and

$$b_\beta(t,\lambda)\Gamma^{\pm\beta}(t,\lambda)\otimes\Gamma^{\mp\beta}(t,\lambda)(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2} = (\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2} b_{A_1}(u_j,\lambda)\Gamma_{A_1}^{\pm\beta}(u_j,\lambda)\otimes\Gamma_{A_1}^{\mp\beta}(u_j,\lambda),$$

where we used formula (92) together with the identity

$$b_\beta(t,\lambda)e^{V_t(\boldsymbol{f}_\beta(t,\lambda)_-,\boldsymbol{f}_\beta(t,\lambda)_-)}=b_{A_1}(u_j,\lambda),$$

which follows immediately from (93). Using that $\mathcal{A}_t=\widehat{\Psi}_t\widehat{R}_t\prod_j\mathcal{D}_{\mathrm{pt}}^{(j)}$, where the factors $\mathcal{D}_{\mathrm{pt}}^{(j)}=\mathcal{D}_{\mathrm{pt}}(\hbar\Delta_j;{}^j\boldsymbol{q})$ are solutions to KdV, we can compute the residue (118) via the Kac–Wakimoto form of the KdV hierarchy (101). After a short computation we get that the residue (118) is

$$(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}\big(\tfrac{1}{8}+\operatorname{Res}_{\lambda=u_j}\tfrac{1}{4}\lambda\,d\lambda\,:\phi_{A_1}^{V\otimes V}(u_j,\lambda)\phi_{A_1}^{V\otimes V}(u_j,\lambda):\big)\prod_{m=1}^{N+1}\mathcal{D}_{\mathrm{pt}}(\hbar\Delta_m;{}^m\boldsymbol{q})^{\otimes 2}.$$

4.5.3 The A_2–subroot subsystem Let $\alpha\in\Delta'$ be a cycle such that $(\alpha|\beta)=1$. We claim that the expression

$$\big(b_\alpha(t,\lambda)\Gamma^{\alpha}(t,\lambda)\otimes\Gamma^{-\alpha}(t,\lambda)+b_{\alpha-\beta}(t,\lambda)\Gamma^{\alpha-\beta}(t,\lambda)\otimes\Gamma^{-\alpha+\beta}(t,\lambda)\big)\mathcal{A}_t^{\otimes 2} \tag{119}$$

is analytic near $\lambda=u_j$. Using the decompositions

$$\alpha=\alpha'+\tfrac{1}{2}\beta,\quad \alpha-\beta=\alpha'-\tfrac{1}{2}\beta,$$

where $(\alpha'|\beta)=0$, the above expression can be written as

$$\Gamma^{\alpha'}\otimes\Gamma^{-\alpha'}\big(a'\Gamma^{\beta/2}\otimes\Gamma^{-\beta/2}+a''\Gamma^{-\beta/2}\otimes\Gamma^{\beta/2}\big)\mathcal{A}_t^{\otimes 2},$$

where the coefficients a' and a'' are given by

$$a'(t,\lambda)=\lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^{-2}B_{\alpha,\alpha}(t,\lambda,\mu)B^{u_j}_{\alpha',-\beta}(t,\lambda,\mu),$$

$$a''(t,\lambda)=\lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^{-2}B_{\alpha-\beta,\alpha-\beta}(t,\lambda,\mu)B^{u_j}_{\alpha',\beta}(t,\lambda,\mu),$$

where the phase factor is given by

$$B^{u_j}_{\alpha',\beta} = \exp \Omega^{u_j}_{\alpha',\beta}$$

with

$$\Omega^{u_j}_{\alpha',\beta}(t,\lambda,\mu) = \iota_{\lambda-u_j}\iota_{\mu-u_j}\sum_{n=0}^{\infty}(-1)^{n+1}(I^{(n)}_{\alpha'}(t,\lambda), I^{(-n-1)}_{\beta}(t,\mu)),$$

where $\iota_{\lambda-u_j}$ (resp. $\iota_{\mu-u_j}$) is the Laurent series expansion at $\lambda = u_j$ (resp. $\mu = u_j$). Since the Laurent series expansions converge for λ and μ sufficiently close to u_j, integration by parts yields

$$\Omega^{u_j}_{\alpha',\beta}(t,\lambda,\mu) = \lim_{\varepsilon\to 0}\int_{L_\varepsilon} \mathcal{W}_{\alpha',\beta}(\mu-\lambda),$$

where L_ε is the straight segment $[t+(\varepsilon+\mu-\lambda-u_j)\mathbf{1}, t-\lambda\mathbf{1}]$. On the other hand we have

$$\Gamma^{\pm\beta/2}\otimes\Gamma^{\mp\beta/2}(\hat{\Psi}_t\hat{R}_t)^{\otimes 2} = (\hat{\Psi}_t\hat{R}_t)^{\otimes 2}\, e^{V_t(\mathbf{f}_{\beta/2}(t,\lambda)_-,\mathbf{f}_{\beta/2}(t,\lambda)_-)}\Gamma^{\pm\beta/2}_{A_1}\otimes\Gamma^{\mp\beta/2}_{A_1}.$$

The exponential factor can be expressed in terms of the phase factors as follows (see Section 4.1.2):

$$e^{V_t(\mathbf{f}_{\beta/2}(t,\lambda)_-,\mathbf{f}_{\beta/2}(t,\lambda)_-)} = \frac{1}{2\sqrt{\lambda-u_j}}\lim_{\mu\to\lambda}(\lambda-\mu)^{1/2}B^{u_j}_{\beta/2,-\beta/2}(t,\lambda,\mu),$$

where the limit is taken in the region $|\lambda|>|\mu|$. Recalling the KP-reduction HQEs of KdV (102) we get that if the coefficients

$$c'(t,\lambda) = \lambda^2\lim_{\mu\to\lambda}(\lambda-\mu)^{-3/2}B_{\alpha,\alpha}(t,\lambda,\mu)B^{u_j}_{\alpha',-\beta}(t,\lambda,\mu)B^{u_j}_{\beta/2,-\beta/2}(t,\lambda,\mu),$$

$$c''(t,\lambda) = \lambda^2\lim_{\mu\to\lambda}(\lambda-\mu)^{-3/2}B_{\alpha-\beta,\alpha-\beta}(t,\lambda,\mu)B^{u_j}_{\alpha',\beta}(t,\lambda,\mu)B^{u_j}_{\beta/2,-\beta/2}(t,\lambda,\mu)$$

are analytic near $\lambda = u_j$, and $c'/c'' = -1$, then the expression (119) is analytic near $\lambda = u_j$.

Let us prove the analyticity of c'. The argument for c'' is similar. Let us choose a small $\varepsilon\in\mathbb{C}$ and a generic point $t_0\in M$ on the discriminant, so that Proposition 38 holds. Furthermore, we fix two paths C'_ε, and C''_ε in $M' = M\setminus\{\text{discr}\}$ from $t_0+(\mu-\lambda+\varepsilon)\mathbf{1}$ to $t-\lambda\mathbf{1}$ such that the parallel transport transforms the cycle φ vanishing over t_0 respectively into α, and $\alpha-\beta$. The phase factors in the definition of c' can be written in terms of integrals along the path as follows:

$$B_{\alpha,\alpha}(t,\lambda,\mu) = \lim_{\varepsilon\to 0} \exp\Big(\int_{C'_\varepsilon} \mathcal{W}_{\alpha,\alpha}(\mu-\lambda)\Big),$$
$$B_{\alpha',-\beta}(t,\lambda,\mu) = \lim_{\varepsilon\to 0} \exp\Big(\int_{L_\varepsilon} \mathcal{W}_{\alpha',-\beta}(\mu-\lambda)\Big),$$
$$B_{\beta/2,-\beta/2}(t,\lambda,\mu) = \lim_{\varepsilon\to 0} \exp\Big(\int_{L_\varepsilon} \mathcal{W}_{\beta/2,-\beta/2}(\mu-\lambda)\Big).$$

Using these formulas, we can express the coefficient $c'(t,\lambda)$ as the limit $\varepsilon\to 0$ of the following expression:

$$\lambda^2 \lim_{\mu\to\lambda} (\lambda-\mu)^{-3/2} \exp\Big(\int_{C'_\varepsilon} \mathcal{W}_{\alpha,\alpha}(\mu-\lambda) - \int_{L_\varepsilon} \mathcal{W}_{\alpha,\alpha}(\mu-\lambda) + \int_{L_\varepsilon} \mathcal{W}_{\alpha',\alpha'}(\mu-\lambda)\Big).$$

Let us examine the dependence on the parameters t,λ, and $\xi := \mu-\lambda$. The difference of the first two integrals in the above formula does not depend on λ, because the paths C'_ε and L_ε have the same ending point, while the starting points are independent of λ. After passing to the limit the difference contributes a constant independent of λ, and μ. The last integral is analytic near $\lambda = u_j$, because the cycle α' is invariant with respect to the local monodromy, which means that the period vector $I^{(0)}_{\alpha'}(t',\xi)$ and respectively the phase form $\mathcal{W}_{\alpha',\alpha'}(\xi)$ are analytic for t' sufficiently close to $t-u_j\mathbf{1}$ and $|\xi|\ll 1$. This proves the analyticity of c'.

It remains only to prove that $c'/c'' = -1$. Using the above path integrals, we can write $\log(c'/c'')$ as

$$\int_{C'_\varepsilon} \mathcal{W}_{\alpha,\alpha} - \int_{L_\varepsilon} \mathcal{W}_{\alpha,\alpha} - \int_{C''_\varepsilon} \mathcal{W}_{\alpha-\beta,\alpha-\beta} + \int_{L_\varepsilon} \mathcal{W}_{\alpha-\beta,\alpha-\beta} + \int_{\gamma_\varepsilon} \mathcal{W}_{\alpha,\alpha} - \int_{\gamma_\varepsilon} \mathcal{W}_{\alpha,\alpha},$$

where γ_ε is a small loop in M' based at the starting point of L_ε (ie $t+(\varepsilon+\mu-\lambda-u_j)\mathbf{1}$) that goes counterclockwise around the discriminant and the branch of the phase form is determined by its value at the point $t-\lambda\mathbf{1}$ (which belongs to the integration paths of the first four integrals and it is connected via the line segment L_ε to the contour of the last two ones). The above expression coincides with

$$\oint_{(C''_\varepsilon)^{-1}\circ L_\varepsilon\circ\gamma_\varepsilon\circ L_\varepsilon^{-1}\circ C'_\varepsilon} \mathcal{W}_{\alpha,\alpha} - \oint_{\gamma_\varepsilon} \mathcal{W}_{\alpha,\alpha}.$$

By definition the cycle α is invariant along the integration contour of the first integral, so the first integral is an integer multiple of $2\pi\sqrt{-1}$. We get

$$c'/c'' = \lim_{\xi\to 0}\lim_{\varepsilon\to 0} \exp\Big(-\oint_{\gamma_\varepsilon} \mathcal{W}_{\alpha,\alpha}(\xi)\Big).$$

The limit here is easy to compute because the integral involves only local information. Using again the decomposition $\alpha = \alpha' + \frac{1}{2}\beta$ and Lemma 6 we get

$$I^{(0)}_{\beta}(t',\xi) = 2(2(\xi-u))^{-1/2}\frac{du}{\sqrt{\Delta}} + (\text{higher order terms}).$$

On the other hand, the period vector $I^{(0)}_{\alpha'}(t',\xi)$ is analytic for (t',ξ) sufficiently close to (t,u_j). Expanding the phase form into a Laurent series about $\xi = u$ we get

$$\lim_{\varepsilon\to 0}\oint_{\gamma_\varepsilon}\mathcal{W}_{\alpha,\alpha}(\xi) = \frac{1}{4}\oint_{\gamma_\varepsilon}\mathcal{W}_{\beta,\beta}(\xi) = \frac{1}{4}\oint\frac{2du}{\sqrt{(-u)(\xi-u)}} = \pi\sqrt{-1},$$

ie $c'/c'' = -1$.

4.5.4 Proof of Theorem 39 The 1–form

$$\frac{d\lambda}{\lambda}\Omega_{\Delta'}(t,\lambda)\mathcal{A}_t(\hbar;\boldsymbol{q}')\mathcal{A}_t(\hbar;\boldsymbol{q}'')$$

has poles only at $\lambda = 0,\infty$, and the critical values u_j, $1 \le j \le N+1$. Let u_j be one of the critical values and β be the cycle vanishing over $\lambda = u_j$. Note that nontrivial contributions to the residue at $\lambda = u_j$ come only from vertex operator terms corresponding to vanishing cycles that have nonzero intersection with β. Recalling our computations in Sections 4.5.1, 4.5.2 and 4.5.3, we get that the residue at $\lambda = u_j$ is $\left(\frac{1}{8} + u_j R_1^{jj}\right)\mathcal{A}_t^{\otimes 2}$, while the residue at $\lambda = 0$ is $-\frac{1}{2}\operatorname{tr}\left(\frac{1}{4} + \theta\theta^T\right)\mathcal{A}_t^{\otimes 2}$. In order to prove that the residue at $\lambda = \infty$ is 0, we just need to check that

$$\sum_{j=1}^{N+1} u_j R_1^{jj} = \tfrac{1}{2}\operatorname{tr}(\theta\theta^T).$$

The above identity is well-known from the theory of Frobenius manifolds (see [24; 29]). Hence the ancestor potential $\mathcal{A}_t(\hbar;\boldsymbol{q})$ is a solution to the HQEs (99). Theorem 39 is thus proved. □

Proof of Theorem 1 Given Theorem 39, Proposition 31 implies that the total descendant potential $\mathcal{D}_{\boldsymbol{a}}(\hbar;\boldsymbol{q})$ is a solution to the HQEs (87). Theorem 1 then follows from Theorem 28. □

5 An example: $\mathbb{P}^1_{2,2,2}$

In this section we consider the example $\boldsymbol{a} = \{2,2,2\}$, namely $\mathbb{P}^1_{\boldsymbol{a}} = \mathbb{P}^1_{2,2,2}$. In this case $\Delta^{(0)}$ is the root system of type D_4. It is convenient to denote the indices in the index set $\mathfrak{I}_{\mathrm{tw}} = \{(1,1),(2,1),(3,1)\}$ simply by $1,2,3$. There are 12 positive roots

$$\gamma_i\ (1\le i\le 3),\quad \gamma_b,\quad \gamma_b+\gamma_i\ \ (1\le i\le 3),\quad \gamma_b+\gamma_i+\gamma_j\ \ (1\le i<j\le 3),$$
$$\gamma_b+\gamma_1+\gamma_2+\gamma_3,\quad 2\gamma_b+\gamma_1+\gamma_2+\gamma_3,$$

where γ_b is the simple root corresponding to the branching node of the Dynkin diagram and γ_i $(1\le i\le 3)$ are the remaining simple roots. The fundamental weight is $\omega_b = 2\gamma_b+\gamma_1+\gamma_2+\gamma_3$. The eigenbasis for σ_b used in our construction is

$$H_i := -(\kappa/2)^{1/2}\gamma_i \quad (1\le i\le 3),\qquad H_0 := (\kappa/2)^{1/2}\omega_b,$$

and we have $m_i = \frac{1}{2}\kappa$, $d_i = \frac{1}{2}$, $1\le i\le 3$, where $\kappa = 4$.

Let us write the HQEs for $\tau = (\tau_n(y))_{n\in\mathbb{Z}}$. We have

$$a_\alpha(\zeta) = \tfrac{1}{4}2^{(\sigma_b(\alpha)|\alpha)}\zeta^{\kappa|\alpha_0|^2}e^{2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)},$$
$$(E_\alpha(\zeta)\tau)_0 = \zeta^{-\kappa|\alpha_0|^2}e^{-2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}E^*_\alpha(\zeta)\tau_{-(\omega_b|\alpha)},$$

where the subscript 0 on the LHS means the 0^{th} component of the corresponding vector in our Fock space. The HQEs give rise to a system of PDEs as follows. First we make a substitution

$$\boldsymbol{y}' := \boldsymbol{y}\otimes 1 = \boldsymbol{x}+\boldsymbol{t},\qquad \boldsymbol{y}'' := 1\otimes\boldsymbol{y} = \boldsymbol{x}-\boldsymbol{t},$$

which implies that

$$\boldsymbol{y}'-\boldsymbol{y}'' = 2\boldsymbol{t},\qquad \frac{\partial}{\partial\boldsymbol{y}'}-\frac{\partial}{\partial\boldsymbol{y}''} = \frac{\partial}{\partial\boldsymbol{t}},$$

and that

$$\operatorname{Res}_{\zeta=0}\big(a_\alpha(\zeta)E_\alpha(\zeta)\tau\otimes E_{-\alpha}(\zeta)\tau\big)_{0,0}$$

is the coefficient in front of ζ^0 in the expression

$$2^{(\sigma_b(\alpha)|\alpha)-2}e^{-2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}$$
$$\times\big(\zeta^{-\kappa|\alpha_0|^2}e^{\sum_{i,\ell}2(\alpha|H_i)\zeta^{m_i+\ell\kappa}t_{i,\ell}}\big)\big(e^{-\sum_{i,\ell}(\alpha|H_{i^*})\frac{\zeta^{-m_i-\ell\kappa}}{m_i+\ell\kappa}\partial_{x_{i,\ell}}}\tau_{-(\omega_b|\alpha)}(\boldsymbol{x}+\boldsymbol{t})\big)$$
$$\times\big(e^{\sum_{i,\ell}(\alpha|H_{i^*})\frac{\zeta^{-m_i-\ell\kappa}}{m_i+\ell\kappa}\partial_{x_{i,\ell}}}\tau_{(\omega_b|\alpha)}(\boldsymbol{x}-\boldsymbol{t})\big).$$

By definition the HQEs are

$$\operatorname{Res}_{\zeta=0}\sum_{\alpha\in\Delta^{(0)}}\big(a_\alpha(\zeta)E_\alpha(\zeta)\tau\otimes E_{-\alpha}(\zeta)\tau\big)_{m,n}$$
$$= \Big(\tfrac{3}{8}+\tfrac{1}{4}(m-n)^2+2\sum_{i,\ell}(d_{i^*}+\ell)t_{i,\ell}\partial_{t_{i,\ell}}\Big)\tau_m(\boldsymbol{x}+\boldsymbol{t})\tau_n(\boldsymbol{x}-\boldsymbol{t}).$$

Comparing the coefficients in front of the various monomials in $\boldsymbol{t}$ we obtain a system of PDEs whose equations are some quadratic polynomials in the partial derivatives of τ. Let us specialize to the case $m = n = 0$. In order to get nontrivial equations we have to compare coefficients in front of monomials that are invariant under the involution $\boldsymbol{t} \mapsto -\boldsymbol{t}$. The simplest case is $\boldsymbol{t}^0$, which corresponds to the identity

$$\sum_{\substack{\alpha \in \Delta^{(0)} \\ (\omega_b|\alpha)=0}} 2^{(\sigma_b(\alpha)|\alpha)-2} = \tfrac{3}{8}.$$

Comparing the coefficients in front of the monomial $t_{01,0}^2$, we get

$$4\frac{\partial^2}{\partial x_{01,0}^2} \log \tau(\boldsymbol{x}) = 8\kappa \frac{\tau_{-2}(\boldsymbol{x})\tau_2(\boldsymbol{x})}{\tau^2(\boldsymbol{x})} - 4(2/\kappa)^{1/2} \frac{\partial^3}{\partial t_{1,0}\partial t_{2,0}\partial t_{3,0}} \left(\frac{\tau_{-1}(\boldsymbol{x}+\boldsymbol{t})\tau_1(\boldsymbol{x}-\boldsymbol{t})}{\tau^2(\boldsymbol{x})} \right)\bigg|_{\boldsymbol{t}=0}.$$

Recalling the substitution (78)–(79), which in this case is

$$y_{01,0} = \frac{1}{\sqrt{\hbar}} \frac{\sqrt{2}}{\kappa\sqrt{\kappa}} q_0^{01}, \quad y_{i,0} = \frac{1}{\sqrt{\hbar}} \frac{\sqrt{2}}{\kappa} q_0^i, \quad 1 \le i \le 3,$$

we get

$$\hbar \frac{\partial^2}{\partial (q_0^{01})^2} \log \tau(\boldsymbol{q}) = \frac{4}{\kappa^2} \frac{\tau_{-2}(\boldsymbol{q})\tau_2(\boldsymbol{q})}{\tau^2(\boldsymbol{q})} - \frac{\hbar^{3/2}}{\kappa^{1/2}} \partial_1\partial_2\partial_3 \left(\frac{\tau_{-1}(\boldsymbol{q}+\boldsymbol{t})\tau_1(\boldsymbol{q}-\boldsymbol{t})}{\tau^2(\boldsymbol{q})} \right)\bigg|_{\boldsymbol{t}=0},$$

where for brevity we put $\partial_i := \partial/\partial t_0^i$. To get a differential equation for the total descendant potential we just have to substitute

$$\tau_{\pm 2}(\boldsymbol{q}) = C^2 \mathcal{D}(\hbar; \boldsymbol{q} \pm 2\sqrt{\hbar}), \quad \tau_{\pm 1}(\boldsymbol{q}) = C^{1/2} \mathcal{D}(\hbar; \boldsymbol{q} \pm \sqrt{\hbar}), \quad C = \kappa^{1/2} Q.$$

Let us use the above equation to compute the genus-0 primary potential F. Put $q_k^i = 0$ for all $k > 0$, and compare the leading terms of the genus expansion. We get the following PDE for F:

$$F_{01,01} = 4Q^4 e^{4F_{00,00}} + Q e^{F_{00,00}} \big(8F_{00,1}F_{00,2}F_{00,3} + 4(F_{00,1}F_{2,3} + F_{00,2}F_{1,3} + F_{00,3}F_{1,2}) \big),$$

where $F_{i,j} := \partial^2 F/\partial q_0^i \partial q_0^j$. To simplify the notation, let us put $t_i := q_0^i$. String equation gives

$$F_{00,00} = t_{01}, \quad F_{00,i} = \tfrac{1}{2} t_i,$$

so from the above equation we get the relation

$$F_{01,01} = 4Q^4 e^{4t_{01}} + Qe^{t_{01}}\big(t_1t_2t_3 + 2(t_1F_{2,3} + t_2F_{1,3} + t_3F_{1,2})\big). \tag{120}$$

Equation (120) allows us to compute the potential F recursively, by the degree of the Novikov variable Q. Indeed, it is easy to see that up to degree-1 terms, F is given by

$$\tfrac{1}{2}t_{00}^2t_{01} + \tfrac{1}{4}t_{00}(t_1^2+t_2^2+t_3^2) + \tfrac{1}{96}(t_1^4+t_2^4+t_3^4) + Qe^{t_{01}}t_1t_2t_3.$$

Comparing the degree-2 terms in (120) we get that the degree-2 term of F must be $\frac{1}{2}(t_1^2+t_2^2+t_3^2)Q^2e^{2t_{01}}$. Arguing in the same way we get that F does not have degree-3 terms, while the degree-4 term must be $\frac{1}{4}Q^4e^{4t_{01}}$. The potential F takes the form

$$\begin{aligned} F(t) = \tfrac{1}{2}t_{00}^2t_{01} + \tfrac{1}{4}t_{00}(t_1^2+t_2^2+t_3^2) + \tfrac{1}{96}(t_1^4+t_2^4+t_3^4) \\ + Qe^{t_{01}}t_1t_2t_3 + \tfrac{1}{2}Q^2e^{2t_{01}}(t_1^2+t_2^2+t_3^2) + \tfrac{1}{4}Q^4e^{4t_{01}}. \end{aligned}$$

The above formula agrees with the computation of P Rossi [41, Example 3.2] based on symplectic field theory.

Appendix: Givental's formalism

Canonical quantization Equip the space

$$\mathcal{H} := H((z^{-1}))$$

of formal Laurent series in z^{-1} with coefficients in H with the following *symplectic form*:

$$\Omega(\phi_1(z),\phi_2(z)) := \mathrm{Res}_z(\phi_1(-z),\phi_2(z)), \quad \phi_1(z),\phi_2(z)\in\mathcal{H},$$

where, as before, $(\cdot,\cdot)$ denotes the residue pairing on H and the formal residue Res_z gives the coefficient in front of z^{-1}.

Let $\{\phi_i\}_{i\in\mathfrak{I}}$ and $\{\phi^i\}_{i\in\mathfrak{I}}$ be dual bases of H with respect to the residue pairing. Then

$$\Omega(\phi^i(-z)^{-\ell-1},\phi_jz^m) = \delta_{ij}\delta_{\ell m}.$$

Hence, a Darboux coordinate system is provided by the linear functions q_ℓ^i, $p_{\ell,i}$ on $\mathcal{H}$ given by:

$$q_\ell^i = \Omega(\phi^i(-z)^{-\ell-1},\cdot), \quad p_{\ell,i} = \Omega(\cdot,\phi_iz^\ell).$$

In other words,

$$\phi(z) = \sum_{\ell=0}^{\infty}\sum_{i\in\mathfrak{I}} q_\ell^i(\phi(z))\phi_iz^\ell + \sum_{\ell=0}^{\infty}\sum_{i\in\mathfrak{I}} p_{\ell,i}(\phi(z))\phi^i(-z)^{-\ell-1}, \quad \phi(z)\in\mathcal{H}.$$

The *quantization* of linear functions on $\mathcal{H}$ is given by the rules

$$\widehat{q}^i_\ell = \hbar^{-1/2} q^i_\ell, \quad \widehat{p}_{\ell,i} = \hbar^{1/2} \frac{\partial}{\partial q^i_\ell},$$

where the RHSs of the above definitions are operators acting on the Fock space

(121) $\mathbb{C}_\hbar[\![\boldsymbol{q}]\!] := \mathbb{C}_\hbar[\![q_0, q_1 + \mathbf{1}, q_2, \dots]\!], \quad \text{where} \quad \mathbb{C}_\hbar := \mathbb{C}((\hbar)), \quad q_\ell := (q^i_\ell)_{i\in\mathfrak{I}}.$

For any $\phi(z) \in \mathcal{H}$, the function $\Omega(\phi(z), \cdot)$ on $\mathcal{H}$ is linear, so we can define the quantization $\widehat{\phi(z)}$ by

(122) $$(\phi_i z^\ell)^\wedge = -\hbar^{1/2} \frac{\partial}{\partial q^i_\ell}, \quad (\phi^i (-z)^{-\ell-1})^\wedge = \hbar^{-1/2} q^i_\ell.$$

The quantization also makes sense for $\phi(z) \in H[\![z, z^{-1}]\!]$ if we interpret $\widehat{\phi(z)}$ as a formal differential operator in the variables q^i_ℓ with coefficients in $\mathbb{C}_\hbar$. For all $\phi_1(z), \phi_2(z) \in \mathcal{H}$,

$$[\widehat{\phi_1(z)}, \widehat{\phi_2(z)}] = \Omega(\phi_1(z), \phi_2(z)).$$

Quantization of symplectic transformations It is known that both series $S_t(z)$ and $R_t(z)$ described in Sections 2.2.1 and 2.2.2 are symplectic transformations on $(\mathcal{H}, \Omega)$. Moreover, they both have the form $e^{A(z)}$, where $A(z)$ is an infinitesimal symplectic transformation.

A linear operator $A(z)$ on $\mathcal{H} := H((z^{-1}))$ is infinitesimal symplectic if and only if the map $\phi(z) \mapsto A(\phi(z))$ is a Hamiltonian vector field with a Hamiltonian given by the quadratic function

$$h_A(\phi(z)) = \tfrac{1}{2}\Omega(A(\phi(z)), \phi(z)).$$

By definition, the *quantization* of $e^{A(z)}$ is given by the differential operator $e^{\widehat{h}_A}$, where the quadratic Hamiltonians are quantized according to the following rules:

$$(p_{\ell,i} p_{m,j})^\wedge = \hbar \frac{\partial^2}{\partial q^i_\ell \partial q^j_m}, \quad (p_{\ell,i} q^j_m)^\wedge = (q^j_m p_{\ell,i})^\wedge = q^j_m \frac{\partial}{\partial q^i_\ell}, \quad (q^i_\ell q^j_m)^\wedge = \frac{1}{\hbar} q^i_\ell q^j_m.$$

The action of the asymptotical operator The operator $\widehat{U_t/z}$ is known to annihilate $\mathcal{D}_{\mathrm{pt}}$. Therefore, its exponential is redundant and it can be dropped from the formula. The action of the operator $\widehat{R}_t$ on formal functions, whenever it makes sense, is given by (see [22])

(123) $$\widehat{R}_t^{-1}\, F(\boldsymbol{q}) = \left(e^{\frac{1}{2}\hbar V_t(\partial,\partial)} F(\boldsymbol{q})\right)\Big|_{\boldsymbol{q}\mapsto R_t \boldsymbol{q}},$$

where $V_t(\partial,\partial)$ is the quadratic differential operator

$$V_t(\partial,\partial)=\sum_{\ell,m=0}^{\infty}\sum_{i,j\in\mathfrak{I}}(\phi^i,V_{\ell m}(t)\phi^j)\,\frac{\partial^2}{\partial q_\ell^i\partial q_m^j}$$

whose coefficients $V_{\ell m}(t)$ are given by

$$\sum_{\ell,m=0}^{\infty}V_{\ell m}(t)z^\ell w^m=\frac{1-R_t(z)({}^TR_t(w))}{z+w}$$

and ${}^TR_t(w)$ denotes the transpose of $R_t(w)$ with respect to the Poincaré pairing. The substitution $\boldsymbol{q}\mapsto R_t\boldsymbol{q}$ can be written more explicitly as follows:

$$q_0\mapsto q_0,\quad q_1\mapsto R_1(t)q_0+q_1,\quad q_2\mapsto R_2(t)q_0+R_1(t)q_1+q_2,\quad\ldots.$$

The above substitution is well-defined for *tame* formal functions, including $\mathcal{A}_t$ (see [21]).

The action of the calibration The quantized symplectic transformation $\widehat{S}_t^{-1}$ acts by (see [22])

$$\widehat{S}_t^{-1}\,F(\boldsymbol{q})=e^{\frac{1}{2\hbar}W_t(\boldsymbol{q},\boldsymbol{q})}F((S_t\boldsymbol{q})_+),\tag{124}$$

where the subscript $+$ in (124) means truncation of all negative powers of z, and $W_t(\boldsymbol{q},\boldsymbol{q})$ is the quadratic form

$$W_t(\boldsymbol{q},\boldsymbol{q})=\sum_{\ell,m=0}^{\infty}(W_{\ell m}(t)q_m,q_\ell)$$

whose coefficients are defined by

$$\sum_{\ell,m=0}^{\infty}W_{\ell m}(t)z^{-\ell}w^{-m}=\frac{{}^TS_t(z)S_t(w)-1}{z^{-1}+w^{-1}}.$$

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