

Spin Hurwitz numbers and topological quantum field theory

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Spin Hurwitz numbers count ramified covers of a spin surface, weighted by the size of their automorphism group (like ordinary Hurwitz numbers), but signed ± 1 according to the parity of the covering surface. These numbers were first defined by Eskin, Okounkov and Pandharipande in order to study the moduli of holomorphic differentials on a Riemann surface. They have also been related to Gromov–Witten invariants of complex 2–folds by work of Lee and Parker and work of Maulik and Pandharipande. In this paper, we construct a (spin) TQFT which computes these numbers, and deduce a formula for any genus in terms of the combinatorics of the Sergeev algebra, generalizing the formula of Eskin, Okounkov and Pandharipande. During the construction, we describe a procedure for averaging any TQFT over finite covering spaces based on the finite path integrals of Freed, Hopkins, Lurie and Teleman.

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Hurwitz numbers count ramified covers of surfaces weighted by the size of the automorphism group of the cover. Combinatorial formulas in terms of the representation theory of the symmetric group go back to Hurwitz [17], Frobenius [14] and Burnside [5].

Suppose Σ is a closed surface equipped with a spin structure. Associated to Σ is an element $p(\Sigma) \in \mathbb{Z}/2\mathbb{Z}$ known as the parity or Atiyah invariant. In this paper, we study a variant of Hurwitz numbers called *spin Hurwitz numbers*, where ramified covers of Σ are counted with a sign according to the parity of the total space of the cover (equipped with the pullback spin structure). One of the main results is the following formula for spin Hurwitz numbers as a sum over the set $\text{SP}(n)$ of *strict partitions* of n (ie partitions into distinct parts).

Theorem 0.1 *The degree n unramified¹ spin Hurwitz numbers of a closed spin surface Σ are given by*

$$2^{-n\chi(\Sigma)/2} \sum_{\mathbf{v} \in \text{SP}(n)} (-1)^{p(\Sigma)\ell(\mathbf{v})} \left(\frac{d(\mathbf{v})}{n!} \right)^{\chi(\Sigma)},$$

¹See [Theorem 1.10](#) for a more general version involving ramified covers.

where

$$d(\mathbf{v}) = 2^{n-\ell(\mathbf{v})/2} \frac{n!}{\nu_1! \cdots \nu_\ell!} \prod_{p < q} \frac{\nu_q - \nu_p}{\nu_p + \nu_q},$$

and $\ell(\mathbf{v})$ denotes the length of the partition \mathbf{v} .

Remark 0.2 The numbers $d(\mathbf{v})$ have a representation theoretic interpretation: namely they are the dimensions of the simple supermodules of the *Sergeev algebra*, up to a factor of $\sqrt{2}$ if $\ell(\mathbf{v})$ is odd (see [Proposition 1.17](#)).

Spin Hurwitz numbers were first introduced by Eskin, Okounkov and Pandharipande [9], who were interested in the volume of strata in the moduli space of differentials on a Riemann surface with prescribed singularities. Kontsevich and Zorich [22] and Eskin and Masur [7] independently explained how these volumes are related to the asymptotics of the enumeration of connected branched coverings of a torus, where the ramification data of the covering matches the singularities of the differential. Eskin and Okounkov [8] were able to use this idea to compute the volumes in the case when the ramification data contained only partitions into even parts; the case when the ramification data consisted of partitions into odd parts required the introduction of spin Hurwitz numbers and was solved by Eskin, Okounkov and Pandharipande [9]. One of the main results of their paper was a group theoretic computation of the parity of a ramified covering of a spin surface. This result was applied in the case of a torus with odd spin structure to give a formula for the spin Hurwitz numbers in that case. In this paper, we will generalize the formula of [9] to any genus ([Theorem 1.10](#)).

Spin Hurwitz numbers have also appeared in the work of Lee and Parker [25] and Maulik and Pandharipande [28] in Gromov–Witten theory. Suppose X is a complex Kähler 2–fold with a smooth canonical divisor $\Sigma \subseteq X$. The normal bundle of Σ in X defines a spin structure on Σ . It was shown in [25] that certain Gromov–Witten invariants of X can be expressed in terms of spin Hurwitz numbers of Σ . More recently, Lee and Parker have proved a recursion formula that allows one to compute higher genus spin Hurwitz numbers in terms of the lower genus numbers using techniques from analysis and PDE (see Lee and Parker [26] and Lee [24]). The recursion formulas of [26] also follow from the techniques of this paper (see [Section 4C](#)).

Remark 0.3 There does not appear to be an obvious connection between the appearance of spin Hurwitz numbers in the Gromov–Witten theory of 2–folds, and in the volume of the moduli spaces of differentials on a Riemann surface.

The main tool of this paper is topological quantum field theory (TQFT). In the 1990s, Dijkgraaf and Witten [6] and Freed and Quinn [13] explained how to construct a

topological quantum field theory from a finite gauge group. Hurwitz numbers are controlled by a 2–dimensional version of Dijkgraaf–Witten theory, with gauge group S_n . The formulas of Hurwitz, Frobenius and Burnside can be recovered by analyzing the structure of this TQFT. To prove [Theorem 0.1](#) we will construct a (spin) TQFT taking values in the category of superalgebras:

Theorem 0.4 *For each positive integer n , there is a fully extended 2d spin TQFT Z_n which assigns invariants as follows:*

- *To a point, Z_n assigns the Sergeev superalgebra, $\mathcal{Y}_n := \mathcal{C}_n \rtimes S_n$, where \mathcal{C}_n denotes the n^{th} complex Clifford algebra.*
- *To the antiperiodic spin circle, Z_n assigns the even super vector space generated by odd partitions of n (ie partitions into odd parts).*
- *To the periodic spin circle, Z_n assigns the super vector space of strict partitions (ie partitions into distinct parts) where the degree of a partition μ is the partition's length mod 2.*
- *To a closed spin surface Σ , Z_n assigns the spin Hurwitz numbers of Σ .*

The spin TQFT Z_n is a convenient way to encode all the information about spin Hurwitz numbers $\mathcal{H}_n(\Sigma)$ for a spin surface Σ . The locality properties of TQFT mean that Z_n is determined by the superalgebra $Z_n(\text{pt}) = \mathcal{Y}_n$, together with a linear map $\mathcal{Z}(\mathcal{Y}_n) \rightarrow \mathbb{C}$. In fact, we can recover Z_n from the following data: the set of simple supermodules $S = \{V^\nu \mid \nu \in \text{SP}(n)\}$ of \mathcal{Y}_n , the decomposition $S = S_0 \sqcup S_1$ into type M and type Q supermodules (see [Section 1D](#)), and a complex number $t(\nu)$ for each $\nu \in \text{SP}(n)$.

Statements about spin Hurwitz numbers often reduce to formal properties of TQFT. For example, the spin Hurwitz formulas in [Theorem 0.1](#) are a special case of a more general formula for spin TQFT ([Proposition 2.13](#)). Recursion formulas, such as those of Lee and Parker [[26](#)], follow directly from the existence of the TQFT Z_n (see [Section 4C](#)).

Outline of the paper

In [Section 1](#), we introduce some of the necessary preliminaries, and state the main results in full generality ([Theorems 1.10](#) and [1.23](#)).

In [Section 2](#), we give a general description of spin TQFTs and explain how to compute the invariants assigned to a closed surface by such a TQFT. The fundamental example of a spin TQFT is the parity theory which assigns ± 1 to a closed spin surface according to its parity. The construction of the parity theory is left to [Section 5](#). In [Section 3](#), we

explain how to construct new TQFTs from old ones by averaging over n -fold covering spaces. The spin Hurwitz theory of [Theorem 1.23](#) is then constructed in [Section 4](#) by averaging the parity theory over n -fold covers. [Theorem 1.10](#) is proved by applying the computation from [Section 2](#) to the spin Hurwitz TQFT.

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1 Preliminaries and main results

1A Ordinary Hurwitz numbers

It will be helpful to recall how ordinary Hurwitz numbers are defined and computed using TQFT. The results in [Sections 1A](#) and [1B](#) are reasonably well known, but we present them here in order make analogies with our own results.

Let $P(n)$ denote the set of partitions of n , ie the set of

$$\boldsymbol{\mu} = (\mu_1 \leq \dots \leq \mu_\ell), \text{ where } \mu_1 + \dots + \mu_\ell = n.$$

We write $\ell(\boldsymbol{\mu}) = \ell$ for the *length* of $\boldsymbol{\mu}$. Note that isomorphism classes of n -fold covering spaces of a circle are in bijection with $P(n)$; we call the partition corresponding to such a cover, the *ramification datum* of the cover.

Fix a closed, oriented surface Σ with marked points p_1, \dots, p_k .

Definition 1.1 An n -fold ramified cover of Σ , with ramification data $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^k$ at p_1, \dots, p_k is a surface $\tilde{\Sigma}$ with a finite, continuous map $\tilde{\Sigma} \rightarrow \Sigma$ which is an n -fold covering space over $\Sigma - \{p_1, \dots, p_k\}$, and such that the restriction to a small circle around p_i has ramification datum $\boldsymbol{\mu}^i \in P(n)$. Define the *Hurwitz numbers* by

$$\hat{\mathcal{H}}_n(\Sigma; \boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^k) = \sum \frac{1}{\#\text{Aut}(\tilde{\Sigma}/\Sigma)},$$

where the sum is taken over isomorphism classes of ramified covers $\tilde{\Sigma} \rightarrow \Sigma$ with the specified ramification data. We will write $\hat{\mathcal{H}}_n(\Sigma, k)$ for the associated linear functional on $\mathbb{C}[\mathbf{P}(n)]^{\otimes k}$.

Recall that the set $\mathbf{P}(n)$ naturally parametrizes both conjugacy classes and irreducible representations of S_n .

Definition 1.2 Let $\hat{V}^{\mathbf{v}}$ denote the irreducible representation of S_n corresponding to $\mathbf{v} \in \mathbf{P}(n)$. The *central character* $\hat{f}_{\boldsymbol{\mu}}^{\mathbf{v}}$ of S_n is defined to be the constant multiple of the identity by which the conjugacy class $\boldsymbol{\mu}$ acts on the representation $\hat{V}^{\mathbf{v}}$.

The following formulas go back to Hurwitz, Frobenius and Burnside (in various forms).

Theorem 1.3 [17; 14; 5] *The numbers $\hat{\mathcal{H}}_n(\Sigma; \boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^k)$ are given by the following expression:*

$$\sum_{\mathbf{v} \in \mathbf{P}(n)} \left(\prod_{i=1}^k \hat{f}_{\boldsymbol{\mu}^i}^{\mathbf{v}} \right) \left(\frac{\dim \hat{V}^{\mathbf{v}}}{n!} \right)^{X(\Sigma)}.$$

Moreover, we have the following expressions for the dimensions and central characters of the symmetric group (see eg Fulton [15]):

$$\dim \hat{V}^{\mathbf{v}} = \frac{n!}{L_1! \cdots L_{\ell}!} \prod_{i < j} (L_i - L_j),$$

where $L_i = \mu_i + \ell - i$, and the $\hat{f}_{\boldsymbol{\mu}}^{\mathbf{v}}$ satisfy

$$S_{\mathbf{v}}(X) = \sum_{\boldsymbol{\mu} \in \mathbf{P}(n)} \frac{\dim \hat{V}^{\mathbf{v}}}{n!} \hat{f}_{\boldsymbol{\mu}}^{\mathbf{v}} p_{\boldsymbol{\mu}}(X)$$

where $S_{\mathbf{v}}(X)$ are the Schur functions, $p_m(X) = \sum_{j \geq 1} x_j^m$ is a power sum function, and $p_{\boldsymbol{\mu}}(X) = p_{\mu_1}(X) \cdots p_{\mu_{\ell}}(X)$.

1B Topological quantum field theory

Let Σ denote a closed, oriented surface and suppose that Σ can be decomposed as $\Sigma_1 \cup_N \Sigma_2$ where the Σ_i are surfaces glued along a common boundary N . The key observation required to prove [Theorem 1.3](#) is that the Hurwitz numbers of Σ can be computed in terms of the Hurwitz numbers of the Σ_i (with extra marked points corresponding to the boundary components N). One convenient way to organize this relationship is via a *topological quantum field theory* (TQFT).

Let \mathbf{Cob}^{or} denote the category whose objects are closed, oriented 1–manifolds, and a morphism $N_1 \rightarrow N_2$ is an oriented surface Σ with an identification $\partial\Sigma \equiv N_1 \sqcup \bar{N}_2$ (up to diffeomorphism). This has a symmetric monoidal structure given by disjoint union of manifolds. Let \mathbf{Vect} denote the category of complex vector spaces with its symmetric monoidal structure given by tensor product.

Definition 1.4 A TQFT is a symmetric monoidal functor $Z: \mathbf{Cob}^{\text{or}} \rightarrow \mathbf{Vect}$.

Given any TQFT Z , the vector space $V = Z(S^1)$ naturally has the structure of a commutative algebra, where the multiplication is given by applying the functor Z to the cobordism $S^1 \sqcup S^1 \rightarrow S^1$ given by a pair of pants (a sphere with three discs removed). Moreover, there is a map $t: V \rightarrow \mathbb{C}$ given by applying Z to the disc, considered as a cobordism $S^1 \rightarrow \emptyset^1$. The map t makes V into a *commutative Frobenius algebra*, ie the composite $V \otimes V \xrightarrow{\text{mult}} V \xrightarrow{t} \mathbb{C}$ is a nondegenerate inner product on V .

If this algebra is *semisimple*, by Wedderburn’s theorem V has a basis $\{e_s \mid s \in S\}$ of orthogonal idempotents.

Proposition 1.5 Let Z be a TQFT with the algebra $V = Z(S^1)$ semisimple, and let $\{e_s \mid s \in S\}$ be the basis of orthogonal idempotents of V . Then for any closed, oriented surface Σ , we have

$$Z(\Sigma) = \sum_{s \in S} t(e_s)^{\chi(\Sigma)/2}.$$

Proposition 1.5 is proved by decomposing Σ into discs and pairs of pants (see Proposition 2.13 for more details). The following theorem gives a precise description of how Hurwitz numbers behave under such decompositions of Σ .

Theorem 1.6 For each positive integer n , there is a TQFT \hat{Z}_n so that:

- To a circle, \hat{Z}_n assigns the vector space $\mathbb{C}[\mathbf{P}(n)]$.
- To a surface Σ with k marked points, considered as a cobordism $(S^1)^{\sqcup k} \rightarrow \emptyset^1$, \hat{Z}_n assigns $\hat{\mathcal{H}}_n(\Sigma, k): \mathbb{C}[\mathbf{P}(n)]^{\otimes k} \rightarrow \mathbb{C}$.

Moreover, the commutative Frobenius algebra structure on $\mathbb{C}[\mathbf{P}(n)]$ is given by identifying it with the algebra of class functions $\mathbb{C}[S_n]^{S_n}$ such that a partition μ maps to the conjugacy class of S_n with cycle type μ . The map t takes a class function f to $(1/n!)f(1)$.

The algebra $\mathbb{C}[S_n]^{S_n}$ is semisimple; the change of basis matrix between the conjugacy classes μ and the orthogonal idempotents e_ν is given by the central character \hat{f}_μ^ν , and $t(e_\nu) = (\dim \hat{V}^\nu/n!)^2$. Combining Theorem 1.6 with Proposition 1.5, we deduce the formula for Hurwitz numbers in Theorem 1.3.

Remark 1.7 There is a natural analogue of this story, where n -fold covers are replaced by G -bundles for any finite group G ; see Freed, Hopkins, Lurie and Teleman [12].

1C Spin Hurwitz numbers

Now suppose Σ is a closed surface equipped with a *spin structure*. Such a structure is equivalent to choosing either of the following pieces of data:

- a quadratic form on $H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ which refines the intersection pairing,
- a square root, $K^{1/2}$, of the canonical bundle (for any complex structure on Σ).

Definition 1.8 In terms of the descriptions above, the *parity* of Σ (with its spin structure) is defined to be

- the Arf invariant of the corresponding quadratic form on $H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$,
- the dimension mod 2 of $H_{\text{hol}}^0(\Sigma, K^{1/2})$.

(See Johnson [18], Atiyah [2] and Mumford [31]).

Recall that there are two spin structures on the circle S^1 : the *antiperiodic* (or *Neveu–Schwarz*) circle S_{ap}^1 which bounds a spin disc, and the *periodic* (or *Ramond*) circle S_{per}^1 which does not [23].

Two subsets of the set $P(n)$ of partitions of n play an important role in this paper:

- the set of *odd partitions*, $\text{OP}(n) = \{\mu \in P(n) : \mu_i \text{ are all odd}\}$,
- the set of *strict partitions*, $\text{SP}(n) = \{\mu \in P(n) : \mu_i \text{ are all distinct}\}$.

We will see that these are analogues of the dual roles played by $P(n)$ for ordinary Hurwitz numbers: indexing conjugacy classes and representations of the symmetric group.²

Suppose we have a cover $\tilde{\Sigma} \rightarrow \Sigma$ ramified at p_1, \dots, p_k . We can canonically lift the spin structure on Σ to $\tilde{\Sigma}$ precisely when the ramification data are all odd partitions (in that case, every component of the covering space over each boundary circle is antiperiodic, thus the spin structure extends over the disc it bounds).

Definition 1.9 The *spin Hurwitz numbers* are defined by

$$\mathcal{H}_n(\Sigma; \mu^1, \dots, \mu^k) = \sum \frac{(-1)^{p(\tilde{\Sigma})}}{\#\text{Aut}(\tilde{\Sigma}/\Sigma)},$$

where the sum is taken over isomorphism classes of branched covers $\tilde{\Sigma}$ of Σ with ramification data $\mu^1, \dots, \mu^k \in \text{OP}(n)$ at p_1, \dots, p_k . We will write $\mathcal{H}_n(\Sigma, k)$ for the associated functional on the vector space $\mathbb{C}[\text{OP}(n)]^{\otimes k}$.

²It was shown by Euler that the cardinalities of $\text{SP}(n)$ and $\text{OP}(n)$ are equal.

One of the main results of this paper is a combinatorial expression for the spin Hurwitz numbers. The unramified case was [Theorem 0.1](#); this involved the numbers $d(\mathbf{v})$, for $\mathbf{v} \in \text{SP}(n)$. To state the general case we introduce the matrix $f_{\mu}^{\mathbf{v}}$, for $\mu \in \text{OP}(n)$ and $\mathbf{v} \in \text{SP}(n)$. This matrix has a combinatorial definition in terms of the Schur Q -functions, a certain family of symmetric polynomials $Q_{\mathbf{v}}(X) = Q_{\mathbf{v}}(x_1, x_2, \dots)$ which were introduced by Schur [\[33\]](#) in his study of the spin representations of the symmetric group (the representation theoretic meaning of $f_{\mu}^{\mathbf{v}}$ will be explained in [Section 1F](#)).

The Q -functions were defined by Schur as follows (see also Józefiak [\[20\]](#)). First consider

$$Q(t) = \sum_{n \geq 0} Q_n(X)t^n := \prod_{j \geq 0} \frac{1 + x_j t}{1 - x_j t}.$$

Now we define $Q_{pq}(X)$ by

$$Q(r, s) = \sum_{p, q \geq 0} Q_{pq}(X)r^p s^q := (Q(r)Q(s) - 1) \frac{r - s}{r + s}.$$

Finally, for $\mathbf{v} \in \text{SP}(n)$, we set $Q_{\mathbf{v}}(X) = \text{Pf}(Q_{v_i v_j}(X))$ (if $\ell(\mathbf{v})$ is odd, we set $v_{\ell(\mathbf{v})+1}$ equal to 0 so that $(Q_{v_i v_j}(X))_{ij}$ is an even order antisymmetric matrix and its Pfaffian is well defined).

Theorem 1.10 (spin Hurwitz formulas) *The spin Hurwitz numbers*

$$\mathcal{H}_n(\Sigma; \mu^1, \dots, \mu^k)$$

are given by

$$2^{(\sum_i (\ell(\mu^i) - n) - n\chi(\Sigma))/2} \sum_{\mathbf{v} \in \text{SP}(n)} (-1)^{p(\Sigma)\ell(\mathbf{v})} \left(\prod_i f_{\mu^i}^{\mathbf{v}} \right) \left(\frac{d(\mathbf{v})}{n!} \right)^{\chi(\Sigma)},$$

where

$$d(\mathbf{v}) = 2^{n - \ell(\mathbf{v})/2} \frac{n!}{v_1! \cdots v_{\ell}!} \prod_{p < q} \frac{v_q - v_p}{v_p + v_q},$$

and the numbers $f_{\mu}^{\mathbf{v}}$ satisfy

$$Q_{\mathbf{v}}(X) = 2^{\ell(\mathbf{v})/2} \sum_{\mu \in \text{OP}(n)} \frac{d(\mathbf{v})}{n!} f_{\mu}^{\mathbf{v}} p_{\mu}(X)$$

($p_m(X) = \sum_{j \geq 1} x_j^m$ is a power sum function, and $p_{\mu}(X) = p_{\mu_1}(X) \cdots p_{\mu_{\ell}}(X)$).

1D Semisimple superalgebras

In this section we will recall the basic theory of semisimple superalgebras, after Józefiak [19].

We will use the term *super vector space* to denote a $\mathbb{Z}/2\mathbb{Z}$ graded complex vector space. Let $S\mathbf{Vect}$ denote the category of super vector spaces and (degree preserving) linear maps. This has a natural symmetric monoidal structure, where the symmetric structure incorporates the sign rule: the isomorphism $V \otimes W \rightarrow W \otimes V$ is given by $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ where v and w are homogeneous elements of degree $|v|$ and $|w|$.

A *superalgebra* is an algebra object in $S\mathbf{Vect}$. The notions of supercommutative, opposite superalgebra A^{op} , supercentre $\mathcal{Z}(A)$ and superabelianization $\mathcal{A}b(A)$ of a superalgebra A in this paper are all defined following the sign rule. A supermodule for a superalgebra is just a module with a compatible $\mathbb{Z}/2\mathbb{Z}$ -grading. Two superalgebras A and B are Morita equivalent if there are superbimodules ${}_A M_B$ and ${}_B N_A$ such that $M \otimes_B N \simeq A$ and $N \otimes_A M \simeq B$. Given a superalgebra A , and two supermodules M and N , we denote by $\text{Hom}_A(M, N)$ the space of degree preserving A -linear maps, and by $\text{HOM}_A(M, N)$ the space of *all* A -linear maps with its natural $\mathbb{Z}/2\mathbb{Z}$ -grading.

A superalgebra is called *semisimple* if every supermodule is a direct sum of simple supermodules. Wedderburn theory for semisimple superalgebras states that every simple superalgebra is isomorphic to one of the following:

- $M(r, s) = \text{End}(\mathbb{C}^r \oplus \mathbb{C}^s[1])$, the superalgebra of (non-degree-preserving) endomorphisms,
- $Q(d)$, the subalgebra of $M(d, d)$, consisting of matrices of block form $\begin{pmatrix} C & D \\ D & C \end{pmatrix}$.

Every semisimple superalgebra is isomorphic to a product of such simple superalgebras, with one simple factor for each simple module.

The superalgebras $M(r, s)$ are all Morita equivalent to the trivial algebra $\mathbb{C} = M(1, 0)$, and the superalgebras $Q(d)$ are Morita equivalent to $Q(1) = \mathbb{C}[\eta]$ where η has degree 1 and $\eta^2 = 1$. Simple supermodules for a semisimple algebra will be referred to as type M or type Q depending on which factor they correspond to.

Remark 1.11 The supercentre $\mathcal{Z}(A)$ of every semisimple superalgebra A is in purely even degree, and this has a basis of orthogonal idempotents corresponding to simple supermodules. The superabelianization $\mathcal{A}b(A)$ (the quotient of A by the subspace of all supercommutators) is a vector space of the same dimension. However, the summands corresponding to supermodules of type M are in even degree and those corresponding

to supermodules of type Q are in odd degree (to verify this claim, it is enough to check for the superalgebras \mathbb{C} and $Q(1)$).

The n^{th} Clifford algebra $\mathcal{C}l_n$ is defined as the tensor product $Q(1)^{\otimes n}$. Explicitly, $\mathcal{C}l_n$ is generated by odd elements η_1, \dots, η_n where $\eta_i^2 = 1$ and $\eta_i \eta_j = -\eta_j \eta_i$ whenever $i \neq j$. The Clifford algebra $\mathcal{C}l_n$ is a simple superalgebra. When n is even, $\mathcal{C}l_n$ is Morita equivalent to \mathbb{C} . When n is odd, $\mathcal{C}l_n$ is Morita equivalent to $\mathcal{C}l_1$.

Definition 1.12 The symmetric group S_n acts on $\mathcal{C}l_n$ by permuting the generators η_i . The *Sergeev superalgebra* \mathcal{Y}_n is defined to be the semidirect product $\mathcal{C}l_n \rtimes S_n$.

1E Spin representations and twisted group superalgebras

Suppose G is a finite group, together with a central extension

$$\langle \varepsilon \rangle \hookrightarrow \tilde{G} \xrightarrow{\pi} G$$

where $\varepsilon^2 = 1$. A *spin representation of G* is defined to be a representation of \tilde{G} in which ε acts by -1 . Spin representations of G are the same thing as modules for the *twisted group algebra*

$$\mathcal{T}(G) = \mathbb{C}[\tilde{G}] / (\varepsilon + 1)$$

(we suppress the data of the central extension from the notation).

Suppose in addition that G is equipped with a $\mathbb{Z}/2\mathbb{Z}$ -grading determined by an index 2 subgroup G_0 of G . Then \tilde{G} acquires a grading via \tilde{G}_0 , the preimage of G_0 , and the twisted group algebra is endowed with the structure of a superalgebra. For each supermodule V of $\mathcal{T}(G)$, let $|V|$ denote the spin representation of G obtained by forgetting the grading.

Proposition 1.13 [19] *The superalgebra $\mathcal{T}(G)$ is a semisimple superalgebra. For each simple supermodule V of $\mathcal{T}(G)$,*

- if V is of type M , $|V|$ is also simple, and
- if V is of type Q , $|V|$ splits as a direct sum of two nonisomorphic simple modules $|V|_+$ and $|V|_-$.

Every simple spin representation of G arises as $|V|$, $|V|_+$ or $|V|_-$ for a unique simple supermodule V .

Thus the representation theory of the superalgebra $\mathcal{T}(G)$ encodes the spin representations of G . Here are the main examples that we will consider in this paper:

- (1) Let R_n denote the group $(\mathbb{Z}/2\mathbb{Z})^n$, and label the generators η_1, \dots, η_n . This group has a grading in which each η_i has degree 1, and a central extension \tilde{R}_n where the central element ε satisfies $\eta_i \eta_j = \varepsilon \eta_j \eta_i$ when $i \neq j$. The twisted group superalgebra $\mathcal{T}(R_n)$ is isomorphic to $\mathcal{C}l_n$.
- (2) Let B_n denote the *hyperoctahedral group* $R_n \rtimes S_n$ with a grading induced from the grading on R_n . The group $\tilde{B}_n = \tilde{R}_n \rtimes S_n$ defines a central extension of B_n called the *Sergeev group*. The twisted group algebra $\mathcal{T}(B_n)$ is canonically identified with the Sergeev algebra $\mathcal{Y}_n = \mathcal{C}l_n \rtimes S_n$.
- (3) The symmetric group S_n also has a central extension \tilde{S}_n . To define \tilde{S}_n , let t_1, \dots, t_{n-1} denote the standard Coxeter generators of S_n . Then \tilde{S}_n is generated by the t_i together with the central involution ε , subject to the usual relations $t_i^2 = 1$ and $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$, but now we have that $t_i t_j = \varepsilon t_j t_i$ when $|i - j| > 1$. The symmetric group has a grading given by the parity.

1F Central characters of spin representations

Recall that for a finite group H , each conjugacy class C determines a central element in the group algebra. This acts on a simple representation M by a scalar multiple f_C^M of the identity (equivalently, f_C^M is the change of basis matrix between the basis of conjugacy classes and the basis of orthogonal idempotents in the centre of the group algebra of H). For a given simple representation M , the number f_C^M is called the *central character* of M ; it is related to the *character* (trace) χ_M by the equation

$$\chi_M(C) = \frac{\dim(M)}{\#C} f_C^M.$$

Now let us define the central characters of *spin* representations of a group G (with a grading and central extension as in Section 1E). Let C be a conjugacy class in G , and consider the preimage $\pi^{-1}(C)$ of C in \tilde{G} . There are two possibilities: either $\pi^{-1}(C)$ is a conjugacy class in \tilde{G} , or $\pi^{-1}(C)$ splits as a disjoint union of two conjugacy classes. We will say that the class C *splits in \tilde{G}* if the second case occurs.

If C splits in \tilde{G} , choose a conjugacy class D in \tilde{G} such that $\pi^{-1}(C) = D \sqcup \varepsilon D$. Then the image of D in $\mathcal{T}(G)$ is in the supercentre of $\mathcal{T}(G)$, and the collection of such elements (as C varies over the conjugacy classes in G of the second type) forms a basis for $\mathcal{Z}(\mathcal{T}(G))$. If C does not split in \tilde{G} , then the image of $\pi^{-1}(C)$ in $\mathcal{T}(G)$ is zero, as $\varepsilon \pi^{-1}(C) = \pi^{-1}(C)$. Such conjugacy classes do not contribute to the character, and we will disregard them.

Remark 1.14 This basis is canonical only up to sign as we could have chosen the image of the class εD instead of D . However, in the cases of interest, there will always be a preferred choice of lift D .

Definition 1.15 Given a simple supermodule V of $\mathcal{T}(G)$ its central character f_C^V is the scalar multiple of the identity on which the image of the (chosen) lift D of C acts on V .

Proposition 1.16 [34; 20] *There is a bijection between the set of strict partitions $\lambda \in \text{SP}(n)$ and supermodules V^λ of \mathcal{Y}_n such that V^λ is of type M if $\ell(\lambda)$ is odd, and of type Q if $\ell(\lambda)$ is even.*

Recall that \mathcal{Y}_n is canonically identified with the twisted group algebra $\mathcal{T}(B_n)$ of the hyperoctahedral group. Conjugacy classes C_μ in the hyperoctahedral group which split in the Sergeev group \tilde{B}_n are indexed by odd partitions $\mu \in \text{OP}(n)$. They are precisely the conjugacy classes generated by a permutation of cycle type μ under the embedding $S_n \hookrightarrow B_n$. We also have an embedding $S_n \hookrightarrow \tilde{B}_n$ and the conjugacy class D_μ of a permutation of cycle type μ in \tilde{B}_n defines a lift of C_μ .

Thus we can define the central character $f_{C_\mu}^{V^\nu}$ of the Sergeev algebra \mathcal{Y}_n . Recall that we defined numbers $f_{\mathbf{v}}^\mu$ in Section 1C in terms of the Schur Q -function.

Proposition 1.17 [34; 20] *We have $f_{C_\mu}^{V^\nu} = f_\mu^\nu$. Moreover, the numbers $d(\mathbf{v})$ are equal to $\dim V^\nu$ if $\ell(\mathbf{v})$ is even and equal to $(1/\sqrt{2}) \dim V^\nu$ if $\ell(\mathbf{v})$ is odd.*

Let Δ_μ denote the image of the conjugacy class D_μ in $\mathcal{Z}(\mathcal{Y}_n)$, where $\mu \in \text{OP}(n)$. It will be useful to have an explicit description of the elements Δ_μ . Given an m -cycle $\sigma = (a_1 \dots a_m)$ in S_n where m is odd, define the element $\Delta_\sigma \in \mathcal{Y}_n$ by

$$\sum (\eta_{a_1}^{c_1} \dots \eta_{a_m}^{c_m}) \otimes \sigma,$$

where the sum is over m -tuples $(c_1, \dots, c_m) \in (\mathbb{Z}/2\mathbb{Z})^m$ such that

$$c_1 + \dots + c_m = 0 \in \mathbb{Z}/2\mathbb{Z}.$$

Now, given any element $\sigma \in S_n$ whose cycle type is an odd partition of n , define Δ_σ by $\Delta_{\sigma_1} \dots \Delta_{\sigma_\ell}$ where $\sigma_1 \dots \sigma_\ell$ is a factorization of σ into disjoint cycles.

Proposition 1.18 *The element $\Delta_\mu \in \mathcal{Z}(\mathcal{Y}_n)$ is given by $\sum \Delta_\sigma$, where the sum is over $\sigma \in S_n$ with cycle type μ .*

For example $\Delta_{(3)} \in \mathcal{Z}(\mathcal{Y}_3)$ is given by

$$1 \otimes (123) + \eta_1 \eta_2 \otimes (123) + \eta_2 \eta_3 \otimes (123) + \eta_3 \eta_1 \otimes (123) \\ + 1 \otimes (132) + \eta_1 \eta_3 \otimes (132) + \eta_3 \eta_2 \otimes (132) + \eta_2 \eta_1 \otimes (132).$$

Remark 1.19 Proposition 1.17 above gives a combinatorial description of the spin characters of the hyperoctahedral group in terms of Schur Q -functions. However, Schur's original use of the Q -functions was to describe the spin characters of the symmetric group. The relationship between the spin characters of S_n and B_n is characterized by an isomorphism of superalgebras (see Yameguchi [36], or Kleshchev [21]):

$$\mathcal{Y}_n := \mathcal{C}l_n \rtimes S_n \cong \mathcal{C}l_n \otimes \mathcal{T}(S_n).$$

1G Extended TQFT and 2-categories

The definition of TFQT given in Section 1B assigns a vector space to a closed 1-manifold and a linear map to a 2-dimensional cobordism. A closed 2-manifold is thus assigned a number, thought of as a linear map $\mathbb{C} \rightarrow \mathbb{C}$. There is a natural way to extend this definition which assigns an algebra (or alternatively, a linear category) to a 0-manifold, and a bimodule to a 1-dimensional cobordism. The vector space assigned to a closed 1-manifold in this setting can be thought of as a \mathbb{C} - \mathbb{C} -bimodule. To properly formulate the definition of extended TQFT, we will need to work with 2-categories.

Throughout this paper, the term 2-category will mean a weak 2-category (or bicategory). We refer the reader to Bénabou [4] or Schommer-Pries [32] for a more detailed discussion of 2-categories, but briefly a 2-category \mathcal{C} consists of the following data:

- a set, \mathcal{C}_0 , of objects;
- a 1-category $\text{Hom}_{\mathcal{C}}(x, y)$ of 1-morphisms for each $x, y \in \mathcal{C}_0$ (the morphisms in these 1-categories are called 2-morphisms and composition is referred to as *vertical composition*);
- a functor $\circ: \text{Hom}_{\mathcal{C}}(x, y) \times \text{Hom}_{\mathcal{C}}(y, z) \rightarrow \text{Hom}_{\mathcal{C}}(x, z)$ called *horizontal composition*;
- an identity 1-morphism $1_x \in \text{Hom}(x, x)$ for each object x ;
- natural isomorphisms called *left and right unitors* for each pair of objects, establishing the expected properties of the identity 1-morphism, and a natural transformation called an *associator*, establishing the associativity property of horizontal composition, all of which satisfy some natural coherence axioms.

A symmetric monoidal structure on a 2-category \mathcal{C} is a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ with fixed isomorphisms $x \otimes y \cong y \otimes x$, together with a unit object $1_{\mathcal{C}}$, and various other natural transformations relating these objects (see [32, Section 2.2]). There is a notion of a *symmetric monoidal functor* between two symmetric monoidal 2-categories.

Our primary examples of symmetric monoidal 2–categories are 2–categories of cobordisms. For example, the 2–category $\mathbf{Bord}^{\text{or}}$ of oriented cobordisms can be described as follows (see [32] for a more careful definition):

- Objects of $\mathbf{Bord}^{\text{or}}$ are oriented 0–manifolds, ie disjoint unions of points equipped with an orientation.
- A 1–morphism between objects N_0 and N_1 is a cobordism between N_0 and N_1 , ie an oriented 1–manifold with boundary, M , together with an oriented diffeomorphism $\partial M \cong \bar{N}_0 \sqcup N_1$, where \bar{N} denotes the opposite orientation on N .
- If M_0 and M_1 are both 1–cobordisms between N_0 and N_1 , then a 2–morphism between M_0 and M_1 is a cobordism Σ between M_0 and M_1 which is trivial on the boundary; more precisely, Σ is an oriented 2–manifold with corners together with an isomorphism

$$\partial \Sigma \simeq \bar{M}_0 \sqcup_{N_0 \sqcup \bar{N}_1} ((\bar{N}_0 \sqcup N_1) \times [0, 1]) \sqcup_{\bar{N}_0 \sqcup N_1} M_1.$$

- The horizontal and vertical compositions are both described by gluing cobordisms along the relevant parts of the boundary (this requires some care to make precise).
- The symmetric monoidal structure is given by disjoint union of manifolds.

Note that the 1–category $\text{Hom}_{\mathbf{Bord}^{\text{or}}}(\emptyset^0, \emptyset^0)$ is equivalent to \mathbf{Cob}^{or} . We would like to extend the notion of TQFT from Section 1B to a symmetric monoidal functor from $\mathbf{Bord}_2^{\text{or}}$ to a suitable target symmetric monoidal 2–category \mathcal{C} , with the property that $\text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ is equivalent to \mathbf{Vect} . A common choice for \mathcal{C} is the 2–category \mathbf{Alg} , which can be described as follows:

- Objects of \mathbf{Alg} are \mathbb{C} –algebras.
- A 1–morphism between objects A and B is an A – B –bimodule, M .
- If M and M' are A – B –bimodules, then a 2–morphism between M and M' is a homomorphism of bimodules.
- Horizontal composition is given by relative tensor product, and vertical composition is given by composition of homomorphisms.
- The symmetric monoidal structure is given by tensor product of algebras.

Definition 1.20 A fully extended, oriented, 2d TQFT is a symmetric monoidal functor $\mathbf{Bord}^{\text{or}} \rightarrow \mathbf{Alg}$.

Example 1.21 The Hurwitz TQFT from Theorem 1.6 extends to a fully extended TQFT which assigns the group algebra $\mathbb{C}[S_n]$ to a point.

1H Spin TQFT

Just as ordinary Hurwitz numbers are controlled by a TQFT, spin Hurwitz numbers are controlled by a *spin* TQFT. The 2–category $\mathbf{Bord}^{\text{spin}}$ is defined similarly to $\mathbf{Bord}^{\text{or}}$, but all manifolds are now endowed with a spin structure (see [32, Section 3.4] for more details). To allow for interesting spin TQFTs, we will also change the target 2–category from \mathbf{Alg} to $S\mathbf{Alg}$, in which the algebras and bimodules are equipped with a compatible $\mathbb{Z}/2\mathbb{Z}$ –grading, and the bimodule homomorphisms are compatible with the grading.

Definition 1.22 A *2d spin topological quantum field theory* (TQFT) is a symmetric monoidal functor $\mathbf{Bord}^{\text{spin}} \rightarrow S\mathbf{Alg}$.

The following theorem is analogous to [Theorem 1.6](#) for ordinary Hurwitz numbers.

Theorem 1.23 (The spin Hurwitz theory) *For each positive integer n , there is a fully extended 2d spin TQFT Z_n which assigns invariants as follows:*

- To a point, Z_n assigns the Sergeev algebra, \mathcal{Y}_n .
- To the antiperiodic spin circle, Z_n assigns the even super vector space $\mathbb{C}[\text{OP}(n)]$. This can be identified with the supercentre $\mathcal{Z}(\mathcal{Y}_n)$.
- To the periodic spin circle, Z_n assigns the super vector space $\mathbb{C}[\text{SP}(n)]$ where the degree of $\mu \in \text{SP}(n)$ is $\ell(\mu) \bmod 2$. This can be identified with the super-abelianization $Ab(\mathcal{Y}_n)$.
- To a closed spin surface Σ with k punctures, considered as a cobordism $(S_{\text{ap}}^1)^{\sqcup k} \rightarrow \emptyset^1$, Z_n assigns $\mathcal{H}_n(\Sigma, k): \mathbb{C}[\text{OP}(n)]^{\otimes k} \rightarrow \mathbb{C}$.

Remark 1.24 Our construction gives a canonical basis for $Z_n(S_{\text{ap}}^1)$ indexed by $\text{OP}(n)$, which is necessary to identify the value of Z_n on a punctured surface with spin Hurwitz numbers. Strictly speaking, this basis is not an invariant of the functor Z_n (it is not a Morita invariant of \mathcal{Y}_n), rather it is a feature of our construction of it. It is also important to note that the basis of $Z_n(S_{\text{per}}^1)$ given above is actually only well defined up to sign.

2 2d spin TQFTs

2A Dualizable and fully dualizable objects

Here we recall the notions of dualizability and full dualizability (see Lurie [27] and Ben-Zvi and Nadler [3, Section 2] for more details).

Definition 2.1 Let (\mathcal{C}, \otimes) be a symmetric monoidal 2–category. An object x in \mathcal{C} is *dualizable* if there is another object x^\vee and morphisms $\text{ev}: x \otimes x^\vee \rightarrow 1_{\mathcal{C}}$ and $\text{coev}: 1_{\mathcal{C}} \rightarrow x^\vee \otimes x$ such that

$$x \xrightarrow{1_x \otimes \text{coev}} x \otimes x^\vee \otimes x \xrightarrow{\text{ev} \otimes 1_x} x$$

is isomorphic to 1_x , and

$$x^\vee \xrightarrow{\text{coev} \otimes 1_{x^\vee}} x^\vee \otimes x \otimes x^\vee \xrightarrow{1_{x^\vee} \otimes \text{ev}} x^\vee$$

is isomorphic to 1_{x^\vee} .

Example 2.2

- An object V in **Vect** (considered as a 2–category with only identity 2–morphisms) is dualizable if and only if it is finite-dimensional.
- Every object A in **Alg** is dualizable. The dual is given by the opposite algebra A^{op} . The 1–morphisms $\text{ev}: \mathbb{C} \rightarrow A \otimes A^{\text{op}}$ and $\text{coev}: A^{\text{op}} \otimes A \rightarrow \mathbb{C}$ in **Alg** are both given by the bimodule A .

Definition 2.3 An object x of \mathcal{C} is *fully dualizable* if it is dualizable, and the morphism ev admits a left and a right adjoint (ie we have morphisms $\text{ev}^L, \text{ev}^R: 1_{\mathcal{C}} \rightarrow x \otimes x^\vee$, satisfying the usual adjunction properties).

Definition 2.4 Given a fully dualizable object x in \mathcal{C} , there are canonical maps

$$S: x \rightarrow x \quad \text{and} \quad T: x \rightarrow x$$

such that

$$\text{ev}^R \simeq (S \otimes \text{id}_{x^\vee}) \circ \text{coev} \quad \text{and} \quad \text{ev}^L \simeq (T \otimes \text{id}_{x^\vee}) \circ \text{coev}.$$

(Here, and for the remainder of the paper, we freely use the symmetric structure to identify $x \otimes x^\vee$ with $x^\vee \otimes x$.) The map S is called the *Serre automorphism*, and T is inverse to S .

Example 2.5 [12, Example 2.8; 32, A.3] Suppose A is an object of **Alg**; we denote by A^e the algebra $A \otimes A^{\text{op}}$ (which can be canonically identified with $A^{\text{op}} \otimes A$ using the symmetric monoidal structure). We will denote by A^* the linear dual $\text{Hom}_{\mathbb{C}}(A, \mathbb{C})$ and by $A^!$ the bimodule dual $\text{Hom}_{A^e}(A, A^e)$. Both are A – A –bimodules. An object A of **Alg** is fully dualizable if and only if it is finite-dimensional and semisimple. If this is the case, then ev^L is given by $A^!$ and ev^R is given by A^* (both considered as \mathbb{C} – A^e –bimodules). The analogous statement also holds for **SAlg**. Alternatively, one can view A^* and $A^!$ as morphisms $A \rightarrow A$; as such A^* is identified with the Serre automorphism S and $A^!$ is identified with the inverse Serre automorphism T (see Definition 2.4). An analogous statement holds for superalgebras, where we write $A^* := \text{HOM}_{\mathbb{C}}(A, \mathbb{C})$, and $A^! := \text{HOM}_{A^e}(A, A^e)$.

2B Oriented TQFTs and Frobenius algebras

The following result indicates a fundamental relationship between TQFT and full dualizability.

Proposition 2.6 *The object pt_+ represented by a single point in the oriented bordism category \mathbf{Bord}^{or} is fully dualizable with dual pt_- .*

Proof (sketch) The evaluation map $ev: pt_- \sqcup pt_+ \rightarrow \emptyset^0$ and coevaluation map $coev: \emptyset^0 \rightarrow pt_+ \sqcup pt_-$ are both represented by the line interval $[0, 1]$. The proof of the duality statement is sometimes referred to as Zorro's Lemma; see [Figure 1](#).

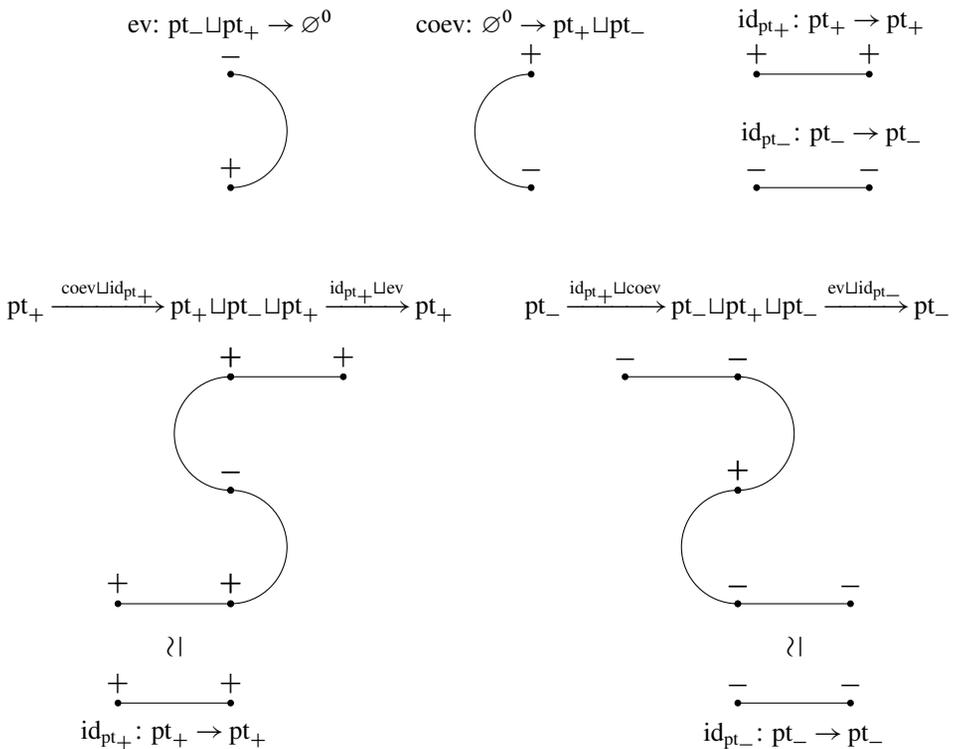


Figure 1: Duality data for pt_+ in \mathbf{Bord}^{or}

The morphism $coev$ can be identified with the left and right adjoint to ev in \mathbf{Bord}^{or} . The 2-dimensional disc, when considered as a cobordism **Cap**: $ev \circ coev \rightarrow \emptyset^1$, is the counit and, when considered as a cobordism **Saddle**: $pt_+ \sqcup pt_- \rightarrow coev \circ ev$, is the unit in the adjunction realizing $coev$ as right adjoint to ev (reading these cobordisms the other way gives the unit and counit of the other adjunction); see [Figure 2](#). \square

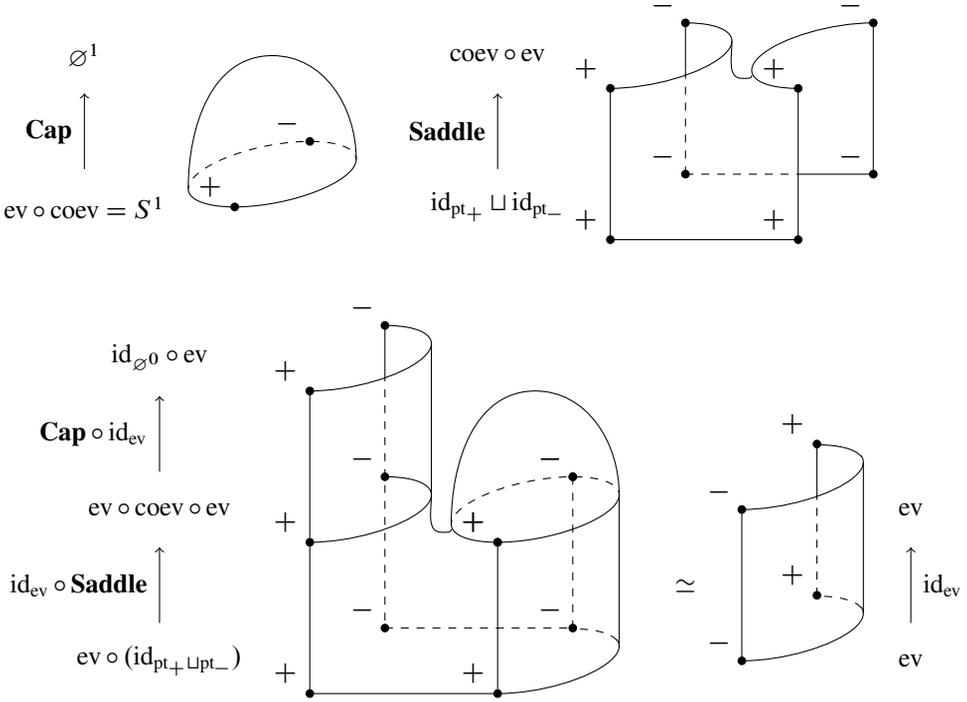


Figure 2: Full dualizability data for pt_+ in \mathbf{Bord}^{or}

It follows from Proposition 2.6 that if $Z: \mathbf{Bord}^{or} \rightarrow \mathbf{Alg}$ is an oriented TQFT, then the object $A = Z(pt_+) \in \mathbf{Alg}$ is fully dualizable, and thus is finite-dimensional and semisimple. The vector space $Z(S^1)$ can be identified with the abelianization

$$Ab(A) = A \otimes_{A \otimes A^{op}} A = A/[A, A].$$

Moreover, as $coev$ is both left and right adjoint to ev in \mathbf{Bord}^{or} , there is an isomorphism of bimodules $A^* \simeq A$; this isomorphism can be encoded by a nondegenerate inner product on A , which factors as

$$A \otimes A \rightarrow A \rightarrow A/[A, A] \xrightarrow{t} \mathbb{C},$$

where t is the linear map given by applying Z to the disc cobordism $S^1 \rightarrow \emptyset^1$. Thus A has the structure of a Frobenius algebra, and we have isomorphisms

$$Z(S^1) \simeq Ab(A) \simeq \mathcal{Z}(A)$$

giving $Z(S^1)$ the structure of a commutative Frobenius algebra. In fact, more is true:

Theorem 2.7 (The cobordism hypothesis [32; 27]) *Given a finite-dimensional, semisimple Frobenius algebra A , there is a unique oriented TQFT*

$$Z_A: \mathbf{Bord}^{\text{or}} \rightarrow \mathbf{Alg}$$

with $Z(\text{pt}_+) = A$, and trace map $Z(\mathbf{Cap}): Ab(A) \rightarrow \mathbb{C}$.

2C 1–dimensional spin cobordisms

Just as in the oriented case, the 0–manifold given by a single point has two nonequivalent spin structures. We shall denote these two objects of $\mathbf{Bord}^{\text{spin}}$ by pt_+ and pt_- as before.

Note that there is a canonical involution α_M of every spin manifold M , which fixes each point of M but switches the sheets of the spin structure. We can consider this as a spin cobordism with total space $M \times [0, 1]$, by using id_M to identify $\{0\} \times M$ with M , but α_M to identify $\{1\} \times M$ with M .

Thus we have 1–dimensional cobordisms

$$\text{id}_{\text{pt}_{\pm}}: \text{pt}_{\pm} \rightarrow \text{pt}_{\pm} \quad \text{and} \quad \alpha_{\text{pt}_{\pm}}: \text{pt}_{\pm} \rightarrow \text{pt}_{\pm}.$$

Unlike in the oriented case, there is no canonical spin cobordism $\text{pt}_- \sqcup \text{pt}_+ \rightarrow \emptyset^0$ which is the evaluation morphism for a duality; rather, there are two choices which can be interchanged by precomposing with $\alpha_{\text{pt}_-} \sqcup \text{id}_{\text{pt}_+}$. However, we will arbitrarily pick one such cobordism:

$$\text{ev}: \text{pt}_- \sqcup \text{pt}_+ \rightarrow \emptyset^0.$$

Then the cobordism

$$\text{coev}: \emptyset^0 \rightarrow \text{pt}_+ \sqcup \text{pt}_-,$$

is defined so that (ev, coev) identify pt_- as dual to pt_+ in $\mathbf{Bord}^{\text{spin}}$. We will denote

$$\overline{\text{ev}} := \text{ev} \circ (\text{id}_{\text{pt}_-} \sqcup \alpha_{\text{pt}_+}) \simeq \text{ev} \circ (\alpha_{\text{pt}_-} \sqcup \text{id}_{\text{pt}_+}).$$

and

$$\overline{\text{coev}} := (\text{id}_{\text{pt}_+} \sqcup \alpha_{\text{pt}_-}) \circ \text{coev} \simeq (\alpha_{\text{pt}_+} \sqcup \text{id}_{\text{pt}_-}) \circ \text{coev}.$$

We have identifications of closed 1–manifolds:

$$\text{ev} \circ \text{coev} \simeq \overline{\text{ev}} \circ \overline{\text{coev}} \simeq S_{\text{per}}^1$$

and

$$\text{ev} \circ \overline{\text{coev}} \simeq \overline{\text{ev}} \circ \text{coev} \simeq S_{\text{ap}}^1.$$

Figure 3 illustrates the various spin cobordisms and their interactions.

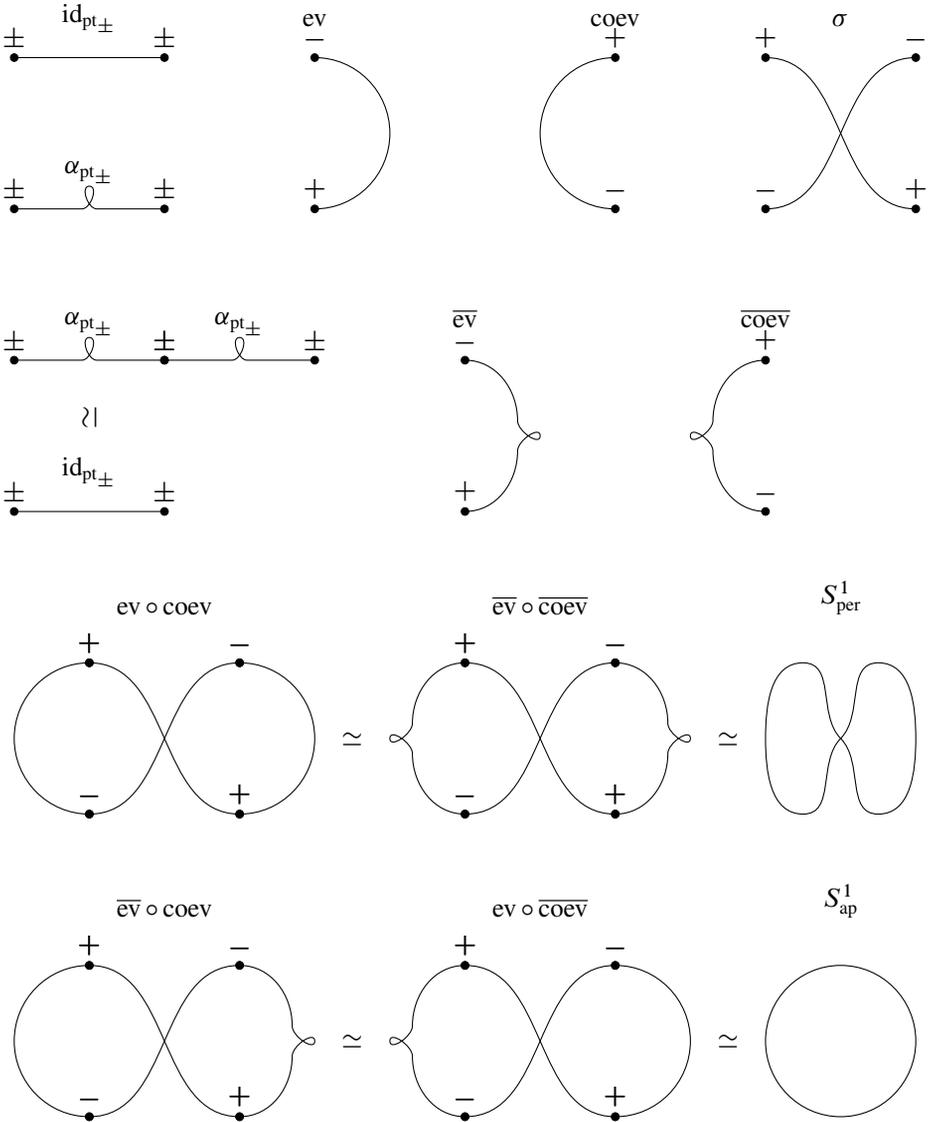


Figure 3: 1-dimensional spin cobordisms. The spin structures are inherited from the immersion in the plane as illustrated.

The following result can be seen in the same way as Proposition 2.6, noting that $\text{ev} \circ \text{coev} = S_{\text{ap}}^1$ is the boundary of a spin disc.

Proposition 2.8 *The object pt_+ is fully dualizable in $\mathbf{Bord}^{\text{spin}}$. The morphism $\overline{\text{coev}}$ is both left and right adjoint to ev .*

Proposition 2.9 Let $Z: \mathbf{Bord}^{\text{spin}} \rightarrow \mathbf{SAlg}$ be a spin TQFT, and let $A = Z(\text{pt}_+)$. Then

- $Z(S_{\text{per}}^1) = \mathcal{A}b(A)$,
- $Z(S_{\text{ap}}^1) = \mathcal{Z}(A)$.

Proof Note that the Serre automorphism α_{pt_+} in $\mathbf{Bord}^{\text{spin}}$ is equivalent to its inverse. This defines an isomorphism of bimodules $Z(\alpha_{\text{pt}_+}) \simeq A^! \simeq A^*$ in \mathbf{SAlg} (see [Example 2.5](#)). We have

$$\begin{aligned} Z(S_{\text{per}}^1) &= Z(\text{ev} \circ \text{coev}) = A \otimes_{A^e} A = \mathcal{A}b(A), \\ Z(S_{\text{ap}}^1) &= Z(\text{ev} \circ \overline{\text{coev}}) = A^! \otimes_{A^e} A = \mathcal{Z}(A), \end{aligned}$$

as required. □

2D 2–dimensional spin cobordisms

The 2–disc carries a unique spin structure with gives rise to cobordisms

$$\mathbf{Cap}: S_{\text{ap}}^1 \rightarrow \emptyset^1 \quad \text{and} \quad \mathbf{Cup}: \emptyset^1 \rightarrow S_{\text{ap}}^1.$$

Recall that the surface S^2 carries a unique (even) spin structure. Removing 3 discs from S^2 , we obtain spin cobordisms

$$\text{ap,ap}\mathbf{Pants}_{\text{ap}}: S_{\text{ap}}^1 \sqcup S_{\text{ap}}^1 \rightarrow S_{\text{ap}}^1 \quad \text{and} \quad \text{ap}\mathbf{Pants}_{\text{ap,ap}}: S_{\text{ap}}^1 \sqcup S_{\text{ap}}^1 \rightarrow S_{\text{ap}}^1.$$

We can now build higher genus cobordisms by gluing copies of the cobordisms $\text{ap,ap}\mathbf{Pants}_{\text{ap}}$, $\text{ap}\mathbf{Pants}_{\text{ap,ap}}$, \mathbf{Cup} and \mathbf{Cap} . The punctured spin surfaces obtained in this way will always have even parity.

There are two nonisomorphic spin cobordisms:

$$\text{id}_{S_{\text{per}}^1}, \alpha_{S_{\text{per}}^1}: S_{\text{per}}^1 \rightarrow S_{\text{per}}^1,$$

(note that the involution $\alpha_{S_{\text{per}}^1}$ is isotopic to the identity). Now let us remove a disc from the cylinder $\text{id}_{S_{\text{per}}^1}$. If we consider the resulting boundary circle as incoming, we obtain a cobordism:

$$\text{per,ap}\mathbf{Pants}_{\text{per}}: S_{\text{per}}^1 \sqcup S_{\text{ap}}^1 \rightarrow S_{\text{per}}^1.$$

If we consider the resulting boundary circle as outgoing, we obtain a cobordism:

$$\text{per}\mathbf{Pants}_{\text{ap,per}}: S_{\text{per}}^1 \rightarrow S_{\text{ap}}^1 \sqcup S_{\text{per}}^1.$$

In a similar manner, we can remove a disc from the cylinder $\alpha_{S_{\text{per}}^1}$ to obtain cobordisms $\text{per,ap}\overline{\mathbf{Pants}}_{\text{per}}$ and $\text{per}\overline{\mathbf{Pants}}_{\text{per,ap}}$.

The two cylinders $\text{id}_{S_{\text{per}}^1}$ and $\alpha_{S_{\text{per}}^1}$ can also be considered as cobordisms

$$\text{ev}_{S_{\text{per}}^1}, \overline{\text{ev}}_{S_{\text{per}}^1} : S_{\text{per}}^1 \sqcup S_{\text{per}}^1 \rightarrow \emptyset^1$$

and

$$\text{coev}_{S_{\text{per}}^1}, \overline{\text{coev}}_{S_{\text{per}}^1} : \emptyset^1 \rightarrow S_{\text{per}}^1 \sqcup S_{\text{per}}^1.$$

As in Section 2C, neither one of $\text{ev}_{S_{\text{per}}^1}$ or $\overline{\text{ev}}_{S_{\text{per}}^1}$ is distinguished—they are exchanged by precomposing with $\alpha_{S_{\text{per}}^1} \sqcup \text{id}_{S_{\text{per}}^1}$. An analogous statement holds for $\text{coev}_{S_{\text{per}}^1}$. By removing a disc from $\text{ev}_{S_{\text{per}}^1}$ we obtain a cobordism

$$\text{per,per}\mathbf{Pants}_{\text{ap}} : S_{\text{per}}^1 \sqcup S_{\text{per}}^1 \rightarrow S_{\text{ap}}^1.$$

Similarly we define cobordisms

$$\text{per,per}\overline{\mathbf{Pants}}_{\text{ap}} \quad \text{and} \quad \text{ap}\mathbf{Pants}_{\text{per,per}} \quad \text{and} \quad \text{ap}\overline{\mathbf{Pants}}_{\text{per,per}}.$$

The notation is chosen such that the pairs

$$(\text{ev}_{S_{\text{per}}^1}, \text{coev}_{S_{\text{per}}^1}) \quad \text{and} \quad (\overline{\text{ev}}_{S_{\text{per}}^1}, \overline{\text{coev}}_{S_{\text{per}}^1})$$

both define evaluation and coevaluation maps which exhibit S_{per}^1 as its own dual.

Let $E_{\text{even}} : S_{\text{ap}}^1 \rightarrow S_{\text{ap}}^1$ denote the cobordism obtained by removing two discs from a surface of genus 1 with an even spin structure. Similarly, let E_{odd} denote the analogous cobordism where the genus 1 surface has an odd spin structure. The following lemmas may easily be checked using (for example) Johnson’s computation of the parity [18].

Lemma 2.10 *We have*

$$\begin{aligned} \text{per,per}\mathbf{Pants}_{\text{ap}} \circ \text{ap}\mathbf{Pants}_{\text{per,per}} &\simeq \text{per,per}\overline{\mathbf{Pants}}_{\text{ap}} \circ \text{ap}\overline{\mathbf{Pants}}_{\text{per,per}} \simeq E_{\text{even}}, \\ \text{per,per}\overline{\mathbf{Pants}}_{\text{ap}} \circ \text{ap}\mathbf{Pants}_{\text{per,per}} &\simeq \text{per,per}\mathbf{Pants}_{\text{ap}} \circ \text{ap}\overline{\mathbf{Pants}}_{\text{per,per}} \simeq E_{\text{odd}}. \end{aligned}$$

Lemma 2.11 *The composite*

$$\mathbf{Cap} \circ E_{\text{even}}^{\text{og}} \circ \mathbf{Cup}$$

is an even spin surface of genus g . The composite

$$\mathbf{Cap} \circ E_{\text{odd}} \circ E_{\text{even}}^{\text{o}(g-1)} \circ \mathbf{Cup}$$

is an odd spin surface of genus g .

2E Computing invariants of spin TQFT

Let $Z : \mathbf{Bord}^{\text{spin}} \rightarrow \mathbf{SAlg}$ be a spin TQFT and denote the superalgebra $Z(\text{pt}_+)$ by A .

By Wedderburn theory (see Section 1D), we may assume that A takes the form

$$A = \prod_{S_0} \mathbb{C} \times \prod_{S_1} \mathcal{C}l_1,$$

where $S = S_0 \sqcup S_1$ is the set of simple supermodules of A decomposed into type M and type Q parts. Write $\deg(s) = i$ whenever $s \in S_i$. Combining Proposition 2.9 with the results of Section 1D, we obtain:

Proposition 2.12 *A spin TQFT Z as above assigns the following invariants to a spin circle:*

- $Z(S_{\text{ap}}^1) = \mathcal{Z}(A) \cong \mathbb{C}^{|S|}$, which has a basis of orthogonal idempotents e_s (all in degree 0),
- $Z(S_{\text{per}}^1) = \mathcal{A}b(A) \cong \mathbb{C}^{|S_0|} \oplus \mathbb{C}^{|S_1|}[1]$.

Consider the linear map

$$t := Z(\mathbf{Cap}): Z(S_{\text{ap}}^1) \rightarrow \mathbb{C},$$

and write $t(s) = t(e_s) \in \mathbb{C}$ for each $s \in S$. The goal of this section is to derive a formula for the value of Z on surfaces in terms of the data of $S = S_0 \sqcup S_1$ and the numbers $t(s)$.

Proposition 2.13 *Let Σ be a closed spin surface with k marked points considered as a cobordism $(S_{\text{ap}}^1)^{\sqcup k} \rightarrow \emptyset^1$. Then we have*

$$Z(\Sigma): e_{s_1} \otimes \cdots \otimes e_{s_k} \mapsto \begin{cases} (-1)^{\deg(s_1)p(\Sigma)} t(s_1)^{\chi(\Sigma)/2} & \text{if } s_1 = \cdots = s_k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, Z assigns $\sum_{s \in S} (-1)^{\deg(s)p(\Sigma)} t(s)^{\chi(\Sigma)/2}$ to a closed spin surface.

Proof The result for even surfaces follows from the usual techniques of oriented 2d TQFT. Explicitly, the map $Z_{(\text{ap}, \text{ap})\mathbf{Pants}_{\text{ap}}}$ recovers the multiplication map on $Z(S_{\text{ap}}^1) = \mathcal{Z}(A)$. The cobordism \mathbf{Cap} defines a nondegenerate trace, giving $\mathcal{Z}(A)$ the structure of a commutative Frobenius algebra (in degree 0). It is an easy exercise in the theory of semisimple Frobenius algebras that the comultiplication map $Z_{(\text{ap})\mathbf{Pants}_{\text{ap}, \text{ap}}}$ is given by $e_s \mapsto t(s)^{-1} e_s \otimes e_s$. It follows that

$$Z(E_{\text{even}}) = Z_{(\text{ap}, \text{ap})\mathbf{Pants}_{\text{ap}}} \circ_{\text{ap}} \mathbf{Pants}_{\text{ap}, \text{ap}}: e_s \mapsto t(s)^{-1} e_s.$$

The even spin surface of genus g with k punctures can be factorized as

$$\mathbf{Cap} \circ E_{\text{even}}^g \circ_{\text{ap}} k\mathbf{Pants}_{\text{ap}},$$

where ${}_{\text{ap}}k\mathbf{Pants}_{\text{ap}}$ is the pair of pants with k incoming boundary circles. This gives the result in the even case.

In order to complete the computation for odd surfaces, it remains to compute $Z(E_{\text{odd}})$. First we observe the following:

Lemma 2.14 *The linear map $Z(\alpha_{S^1_{\text{per}}}) : \mathcal{A}b(A) \rightarrow \mathcal{A}b(A)$ is given by the grading involution.*

Proof It is sufficient to prove the lemma in the cases $A = \mathbb{C}$ and $A = \mathcal{C}l_1$. Note that the cylinder $\alpha_{S^1_{\text{per}}}$ can be thought of as a product $S^1_{\text{per}} \times \alpha_{\text{pt}_+}$. Thus the map $Z(\alpha_{S^1_{\text{per}}})$ is given by the automorphism of $\mathcal{A}b(A)$ induced from the A - A -bimodule $Z(\alpha_{\text{pt}_+})$ (in general, any Morita equivalence of algebras induces a trace map on abelianizations). This bimodule is necessarily the Serre automorphism of A . In the case $A = \mathbb{C}$, the Serre automorphism and grading involution are trivial. In the case $A = \mathcal{C}l_1$, the Serre automorphism is the shift bimodule $A[1]$; the trace of the shift bimodule is precisely the grading automorphism, as required. \square

Remark 2.15 Often it is included in the axioms of spin TQFT that the canonical spin automorphism of a 1-manifold gives rise to the grading automorphism on the corresponding super vector space. For us this follows from the fact that the spin automorphism of a point is also the Serre automorphism in $\mathbf{Bord}^{\text{spin}}$ as shown in Lemma 2.14.

The map

$$Z({}_{\text{per,ap}}\mathbf{Pants}_{\text{per}}) : Z(S^1_{\text{ap}}) \otimes Z(S^1_{\text{per}}) \rightarrow Z(S^1_{\text{per}})$$

recovers the canonical action of $Z(A)$ on $\mathcal{A}b(A)$. The elements $e_s \in Z(A)$ give a family of orthogonal idempotent operators on $\mathcal{A}b(A)$. The eigenspaces I_s of the operators e_s are 1-dimensional and generate $\mathcal{A}b(A)$.

The super vector space $\mathcal{A}b(A)$ can be equipped with the inner product given by $Z(\text{ev}_{S^1_{\text{per}}})$. The eigenspaces I_s are mutually orthogonal with respect to this inner product, and we can choose a basis $\{f_s \mid s \in S\}$ of $\mathcal{A}b(A)$ which is orthonormal with respect to the inner product (since we could have picked the inner product given by $Z(\overline{\text{ev}}_{S^1_{\text{per}}})$, such a basis is only canonical up to sign). Note that $\text{deg}(f_s) = \text{deg}(s)$.

Now, by construction,

$$Z({}_{\text{per,per}}\mathbf{Pants}_{\text{ap}}) : f_{s_1} \otimes f_{s_2} \mapsto \begin{cases} e_s & \text{if } s_1 = s_2 = s, \\ 0 & \text{if } s_1 \neq s_2. \end{cases}$$

and

$$Z({}_{\text{ap}}\mathbf{Pants}_{\text{per,per}}) : e_s \mapsto f_s \otimes f_s.$$

As $\text{per,per}\overline{\mathbf{Pants}}_{\text{ap}}$ is given by the composition $\text{per,per}\overline{\mathbf{Pants}}_{\text{ap}} \circ (\text{id}_{S^1_{\text{per}}} \sqcup \alpha_{S^1_{\text{per}}})$, we also have

$$\text{per,per}\overline{\mathbf{Pants}}_{\text{ap}}: f_S \otimes f_S \mapsto (-1)^{\deg(s)} e_S.$$

Finally, E_{odd} is given by the composition

$$Z(\text{per,per}\overline{\mathbf{Pants}}_{\text{ap}}) \circ Z(\text{ap}\mathbf{Pants}_{\text{per,per}}),$$

so $Z(E_{\text{odd}})$ maps e_S to $(-1)^{\deg(s)} e_S$. This completes the proof of [Proposition 2.13](#). \square

2F Classification of 2d spin TQFTs

[Proposition 2.13](#) computes the invariants of a TQFT Z such that $Z(\text{pt})$ is a finite-dimensional semisimple algebra A , starting from the linear map $t: \mathcal{Z}(A) \rightarrow \mathbb{C}$ given by applying Z to a spin disc. In fact, the cobordism hypothesis [\[27\]](#) implies that any TQFT is completely determined by such data, and that, given the data of a semisimple algebra A and a map t satisfying certain conditions, there is a TQFT which assigns those data to a point and a disc.³

Proposition 2.16 *A 2d spin TQFT is given by a semisimple superalgebra A , together with a linear map $t: \mathcal{Z}(A) \rightarrow \mathbb{C}$ such that the composite*

$$A^! \otimes A \rightarrow A^! \otimes_{A^e} A = \mathcal{Z}(A) \xrightarrow{t} \mathbb{C}$$

identifies $A^!$ with the linear dual of A . This is equivalent to the following data: a finite set $S = S_0 \sqcup S_1$ (the spectrum of $\mathcal{Z}(A)$ decomposed into types M and Q), together with a function $t: S = S_0 \sqcup S_1 \rightarrow \mathbb{C}^\times$ (the trace map on $\mathcal{Z}(A)$ restricted to the orthogonal idempotents).

Proof (sketch) According to the cobordism hypothesis, a spin TQFT should be given by a fully dualizable object of \mathbf{SAlg} , equipped with the structure of a $\text{Spin}(2)$ homotopy fixed point. The fully dualizable object determines a *framed* theory, and the fixed point data allows an extension to spin manifolds.

An algebra A is fully dualizable if and only if it is finite-dimensional and semisimple (see [Example 2.5](#)), so by Wedderburn theory A is Morita equivalent to a product $\prod_{S_0} \mathbb{C} \times \prod_{S_1} \mathcal{C}_1$.

The action of $\text{SO}(2)$ on the space of fully dualizable objects gives an automorphism of each object (the Serre automorphism), which in the case of algebras is the bimodule dual $A^!$ of A (or its inverse A^*). Similarly, the action of $\text{Spin}(2)$ on this space gives the bimodule $A^! \otimes_A A^!$.

³A classification of $\text{Spin}(2, 1)$ TQFTs (ie nonextended TQFTs) was described in [\[1\]](#).

Let us first identify the space of $SO(2)$ fixed points (corresponding to oriented TQFTs). Giving the structure of a homotopy fixed point for $SO(2)$ is to give an isomorphism of bimodules $A \cong A^!$. Unwinding the definition of dual bimodule gives that this is the same as a linear map $\mathcal{A}b(A) = A \otimes_{A \otimes_{A^{op}}} A \rightarrow \mathbb{C}$ such that the composite

$$A \otimes A \xrightarrow{m} A \otimes_{A^e} A \xrightarrow{\text{tr}} \mathbb{C}$$

identifies A with its linear dual A^* . This is the structure of a symmetric Frobenius algebra on A [27; 32]. The centre and abelianization of A are both identified with the vector space with basis $\{e_s \mid s \in S\}$. A map tr as above is exactly given by a function $t: S \rightarrow \mathbb{C}^*$.

Now, to give a $Spin(2)$ fixed point is to give a bimodule isomorphism $A^! \cong A^*$. Again, unwinding the dualities gives us a map $\mathcal{Z}(A) = A^! \otimes_{A^e} A \rightarrow \mathbb{C}$, such that the composite $A^! \otimes A \rightarrow A^! \otimes_{A^e} A \rightarrow \mathbb{C}$ identifies $A^!$ with the linear dual of A . This is equivalent to a function $S \rightarrow \mathbb{C}^*$ as required. \square

Remark 2.17 If the 2–category $S\mathbf{Alg}$ was replaced by an $(\infty, 2)$ –category with nontrivial higher morphisms (for example, the category of differential graded algebras, bimodules, maps of bimodules, homotopies between maps, etc), there would be higher coherence data to consider. In particular, specifying a $SO(2)$ or $Spin(2)$ (homotopy) fixed point would involve more data.

Remark 2.18 If A contains only factors of type M (ie $S_1 = \emptyset$), then the corresponding TQFT cannot distinguish spin structures. It follows that a spin TQFT valued in ordinary (ungraded) algebras necessarily comes from an oriented theory.

Example 2.19 (parity TQFT) Consider the spin TQFT Z_1 for which $S_0 = \emptyset$ and $S_1 = \{*\}$ and $t(*) = 1$. This assigns $\mathcal{C}l_1$ to a point, the vector space \mathbb{C} to S_{per}^1 , $\mathbb{C}[1]$ to S_{per}^1 , and the number $(-1)^{p(\Sigma)}$ to every spin surface Σ . A direct construction of Z_1 is given in Section 5.

3 Averaging TQFTs over finite covers

The aim of this section is to give a procedure for taking a TQFT Z_1 and producing a new TQFT Z_n for each positive integer n which averages Z_1 over n –fold covers. This procedure is based on the machinery of finite path integrals as defined in Freed, Hopkins, Lurie and Teleman [12] (based on ideas from Freed [10]). See also the recent paper of Morton [30].

For example, applying this procedure to the trivial TQFT recovers the theory which assigns the ordinary Hurwitz numbers to a closed 2–manifold. Applying this to the parity theory (see [Example 2.19](#)) will produce the spin Hurwitz theory of [Theorem 1.23](#).

Notation 3.1 In this section, we will write **Bord** instead of $\mathbf{Bord}^{\text{spin}}$, **Alg** instead of **SAlg** and **Vect** instead of **SVect**. The results of this section apply much more generally. For example **Bord** could be the oriented or framed bordism category (or $(\infty, 2)$ –category), and **Alg** could be the Morita category of ordinary algebras, or the $(\infty, 2)$ –category of differential graded algebras.

If \mathcal{C} is a symmetric monoidal 2–category, write $\Omega\mathcal{C}$ for the symmetric monoidal 1–category of endomorphisms of the identity object $1_{\mathcal{C}}$. Similarly, write $\Omega^2\mathcal{C}$ for the monoid of endomorphisms of $1_{\Omega\mathcal{C}}$. Objects of \mathcal{C} we call 0–objects, objects of $\Omega\mathcal{C}$ we call 1–objects, and elements of $\Omega^2\mathcal{C}$ we call 2–objects. For example, if $\mathcal{C} = \mathbf{Bord}$ then n –objects are closed n –manifolds.

3A The averaging theory Z_n

Let $Z_1: \mathbf{Bord} \rightarrow \mathbf{Alg}$ be a TQFT. The goal of this section is to prove the following:

Proposition 3.2 *Given Z_1 as above, there is a family of TQFTs $Z_n: \mathbf{Bord} \rightarrow \mathbf{Alg}$ for $n = 2, 3, \dots$, such that:*

- $Z_n(\text{pt}) = Z_1(\text{pt})^{\otimes n} \rtimes S_n$.
- If N is a closed 1–manifold, $Z_n(N) = \bigoplus Z_1(\tilde{N})^{\text{Aut}(\tilde{N}/N)}$, where the sum is over isomorphism classes of n –fold covers \tilde{N} of N .
- If Σ is a cobordism between closed 1–manifolds N_0 and N_1 , $Z_n(\Sigma)$ is given by the following linear map:

$$\bigoplus Z_1(\tilde{N}_0)^{\text{Aut}(\tilde{N}_0/N_0)} \rightarrow \bigoplus Z_1(\tilde{N}_1)^{\text{Aut}(\tilde{N}_1/N_1)},$$

$$u = (u_{\tilde{N}_0}) \mapsto v = (v_{\tilde{N}_1}),$$

where

$$v_{\tilde{N}_1} = \sum \frac{Z_1(\tilde{\Sigma})(u)}{\#\text{Aut}(\tilde{\Sigma}/\Sigma)}$$

and the sum is over isomorphism classes of covers $\tilde{\Sigma}/\Sigma$.

The field theory Z_n will be defined as a composite

$$\mathbf{Bord} \xrightarrow{\text{Cov}_n} \mathbf{Fam}_2(\mathbf{Bord}) \xrightarrow{\mathbf{Fam}_2(Z)} \mathbf{Fam}_2(\mathbf{Alg}) \xrightarrow{\text{Sum}_2} \mathbf{Alg}.$$

Very loosely, the map Cov_n takes a manifold to its groupoid of n -fold covers (whilst remembering the spin structure on the total space of each cover), the functor $\mathbf{Fam}_2(Z)$ applies the TQFT Z_1 to the total space of the cover, and then Sum_2 takes the average. Most of the categories and functors appearing above are defined in [12, Section 3] (see also [27]). Here, we will briefly review the definitions (note that we only need the case of 2-categories, rather than the general m -categories discussed in [12]).

3B Toy example: 1d TQFTs

Before we delve into the details of the construction, let us examine what it gives us in the case of a 1d TQFT. Recall that a 1d (say, oriented) TFT Z valued in vector spaces is determined by a finite-dimensional vector space V which is the value at a (positively oriented) point. The linear maps $\mathbb{C} \rightarrow V \otimes V^*$ and $V^* \otimes V \rightarrow \mathbb{C}$ which Z assigns to semicircles are necessarily the unit and trace maps which identify V^* as the dual of V . Hence the value of Z on a circle is given by the integer $d := \dim V = \text{tr}(1_V)$.

The analogue of Proposition 3.2 says that we can define a new TFT Z_n whose value at a point is $(V^{\otimes n})^{S_n}$ and whose value on a circle is $\sum_{\mu} (d^{\ell(\mu)} / |C(\mu)|)$. Here, the sum is taken over partitions μ which index $\pi_0(Cov(S^1))$, and $C(\mu)$ is the centralizer of the corresponding conjugacy class in S_n .

By comparing the results from Proposition 3.2 with what we know by duality, we recover the formula

$$\dim(V^{\otimes n})^{S_n} = \sum_{\mu} \frac{d^{\ell(\mu)}}{\#C(\mu)}.$$

The formulas in Theorem 1.10 will be proved using a very similar idea.

3C The category $\mathbf{Fam}_2(\mathcal{C})$

Suppose \mathcal{C} is a symmetric monoidal 2-category. The symmetric monoidal 2-category $\mathbf{Fam}_2(\mathcal{C})$ has objects essentially finite groupoids X (ie $\pi_0(X)$ and $\pi_1(X)$ are finite) with a functor $f: X \rightarrow \mathcal{C}$. The 1-morphisms are spans of groupoids over \mathcal{C} , ie each is a groupoid W , with maps p_1 to X and p_2 to Y and a natural transformation $\alpha: f_1 \circ p_1 \Rightarrow f_2 \circ p_2$:

$$(1) \quad \begin{array}{ccc} & W & \\ p_1 \swarrow & & \searrow p_2 \\ X & \xrightarrow{\alpha} & Y \\ f_1 \searrow & & \swarrow f_2 \\ & \mathcal{C} & \end{array}$$

Composition of morphisms is given by the homotopy fibre product of groupoids. Similarly, the 2-morphisms are given by spans of such diagrams (up to equivalence), and the symmetric monoidal structure is given by products of groupoids. The details of how 2-morphisms compose etc quickly become quite complicated and we will not attempt to flesh them out here. We refer the reader to [30] for more details.

Remark 3.3 Note that the 2-category of 2-vector spaces defined in [30] is equivalent to the full subcategory of **Alg** given by finite-dimensional semisimple algebras: to such an algebra A , one assigns its category of finite-dimensional modules, which is a 2-vector space.

However, the category $\Omega\mathbf{Fam}_2(\mathcal{C}) = \mathbf{Fam}_1(\Omega\mathcal{C})$ is easier to describe: its objects are essentially finite groupoids X with a map $f: X \rightarrow \Omega\mathcal{C}$, and morphisms are spans of groupoids as in the diagram (1), except that \mathcal{C} is replaced by $\Omega\mathcal{C}$. Similarly, $\Omega^2\mathbf{Fam}_2(\mathcal{C}) = \mathbf{Fam}_0(\Omega^2\mathcal{C})$ is the monoid of equivalence classes of groupoids X with a map $\pi_0(X) \rightarrow \Omega^2\mathcal{C}$.

Note that if we have a symmetric monoidal functor of 2-categories $F: \mathcal{C} \rightarrow \mathcal{D}$, then we get a symmetric monoidal functor $\mathbf{Fam}_2(F): \mathbf{Fam}_2(\mathcal{C}) \rightarrow \mathbf{Fam}_2(\mathcal{D})$ by composing all the maps with F .

3D The functor Cov_n

If M is a topological space, let $Cov_n(M)$ denote the groupoid whose objects are n -fold covers of M and whose morphisms are deck transformations. The functor $Cov_n: \mathbf{Bord} \rightarrow \mathbf{Fam}_2(\mathbf{Bord})$ takes a closed manifold M to $Cov_n(M)$ equipped with the map $Cov_n(M) \rightarrow \mathbf{Bord}$ which takes a covering space to the its total space thought of as an object of **Bord**. To a cobordism $M_0 \xrightarrow{N} M_1$ between closed manifolds, we associate the span

$$Cov_n(M_0) \longleftarrow Cov_n(N) \longrightarrow Cov_n(M_1)$$

equipped with the maps $Cov_n(M_i) \rightarrow \mathbf{Bord}$.

3E The functor Sum_2

Let us first describe what $\text{Sum}_2: \mathbf{Fam}_2(\mathbf{Alg}) \rightarrow \mathbf{Alg}$ does on i -objects, $i = 0, 1, 2$:

- To a 2-object $(X, f: \pi_0(X) \rightarrow \mathbb{C})$ of $\mathbf{Fam}_2(\mathbf{Alg})$, Sum_2 assigns the number

$$\sum_{x \in \pi_0(X)} \frac{f(x)}{\#\text{Aut}(x)}.$$

- To a 1-object $(X, f: X \rightarrow \mathbf{Vect})$ of $\mathbf{Fam}_2(\mathbf{Alg})$, Sum_2 assigns the limit of the diagram f , which can be identified with

$$\bigoplus_{x \in \pi_0(X)} f(x)^{\text{Aut}(x)}.$$

- To a 0-object $(X, f: X \rightarrow \mathbf{Alg})$, Sum_2 assigns the (homotopy) limit of the diagram f , which can be identified with⁴

$$\bigoplus_{x \in \pi_0(X)} f(x) \rtimes \text{Aut}(x).$$

We would like to describe how Sum_2 acts on more general morphisms in $\mathbf{Fam}_2(\mathbf{Alg})$. This will be given by an integral transform formula as follows, which we will describe in the case of a morphism in $\Omega\mathbf{Fam}_2(\mathbf{Alg}) = \mathbf{Fam}_1(\mathbf{Vect})$.

First let us note that if $(X, f: X \rightarrow \mathbf{Vect})$ is a 1-object, then the natural map

$$\lim f = \bigoplus_{x \in \pi_0(X)} f(x)^{\text{Aut}(x)} \longrightarrow \bigoplus_{x \in \pi_0(X)} f(x)_{\text{Aut}(x)} = \text{colim } f$$

is an isomorphism (where the subscripts denote coinvariants). The inverse is given by lifting $v \in f(x)_{\text{Aut}(x)}$ to $\tilde{v} \in f(x)$ and then mapping to

$$\sum_{g \in \text{Aut}(x)} g \cdot \tilde{v} \in f(x)^{\text{Aut}(x)}.$$

Now, given a sequence $W \xrightarrow{p} X \xrightarrow{f} \mathbf{Vect}$, as well as the natural pullback map $f^*: \lim(f) \rightarrow \lim(fp)$, we have the pushforward $f_*: \lim(fp) \rightarrow \lim(f)$, defined by first identifying limits with colimits, then using the natural pushforward map for colimits.

Recall that a morphism of 1-objects is given by a span of groupoids as in diagram (1) (where $\mathcal{C} = \mathbf{Vect}$). In particular, for each w in $\pi_0(W)$, we have a linear map $\alpha(w): f_1(p_1(w)) \rightarrow f_2(p_2(w))$ which is $\text{Aut}(w)$ equivariant. Then we have that $\text{Sum}_2(W, p_1, p_2, \alpha)$ is given by the integral transform formula:

$$\lim p_1 \xrightarrow{p_1^*} \lim f_1 p_1 \xrightarrow{\alpha} \lim f_2 p_2 \xrightarrow{p_2^*} \lim p_2.$$

Explicitly, this is given by the linear map

$$\bigoplus_{x \in \pi_0(X)} f_1(x)^{\text{Aut}(x)} \longrightarrow \bigoplus_{y \in \pi_0(Y)} f_2(y)^{\text{Aut}(y)}$$

⁴The computation of this limit is given by Morton [30] in the context of 2-vector spaces (see Remark 3.3).

where $u = (u_x)$ gets mapped to $v = (v_y)$ and

$$v_y = \sum_{w \in \pi_0(W_y)} \frac{\alpha(w)(u)}{|\text{Aut}(w)|}.$$

The fact that this construction is functorial follows from a base change formula for the pullback and pushforward: $p_2^* p_{1*} \cong \tilde{p}_{1*} \tilde{p}_2^*$ whenever $p_1, p_2, \tilde{p}_1, \tilde{p}_2$ form a cartesian square.

The same integral transform idea allows us to define Sum_2 on 1–morphisms; the description for general 2–morphisms is somewhat more complicated (again, see [30] for a careful construction of the functor Sum_2).

[Proposition 3.2](#) now follows from the description of the functor Sum_2 in [Section 3E](#).

4 Spin Hurwitz Numbers

In this section we will combine the results from [Sections 2 and 3](#) to construct the spin Hurwitz theory of [Theorem 1.23](#) and prove the spin Hurwitz formula of [Theorem 1.10](#).

4A The spin Hurwitz theory

The TQFT Z_n is obtained by applying [Proposition 3.2](#) in the case when Z_1 is the parity theory (see [Example 2.19](#)). It is clear from the proposition that the value of Z_n on a closed spin surface Σ is the spin Hurwitz number $\mathcal{H}_n(\Sigma)$.

To complete the proof of [Theorem 1.23](#) we should also identify $Z_n(S_{\text{ap}}^1)$ with $\mathbb{C}[\text{OP}(n)]$. Here we want more than just an identification as abstract vector spaces, rather, we want to identify a particular basis $\{\delta_{\mu} \mid \mu \in \text{OP}(n)\}$ such that

$$(2) \quad \mathcal{H}_n(\Sigma; \mu^1, \dots, \mu^k) = Z_n(\Sigma^*)(\delta_{\mu^1} \otimes \dots \otimes \delta_{\mu^k}),$$

where Σ^* is the cobordism from $(S_{\text{ap}}^1)^{\sqcup k}$ to \emptyset^1 obtained by puncturing Σ at k points.

Let $\mu \in P(n)$ be a partition corresponding to a covering $\bigsqcup_i S_{\mu_i} \rightarrow S_{\text{ap}}^1$ (so that each $S_{\mu_i} \rightarrow S^1$ is a connected cover of degree μ_i); the automorphism group of this covering is the centralizer $C(\mu)$ in S_n .

If one of the μ_i is even, then the corresponding component of the cover will be periodic. There is a unique nontrivial order 2 automorphism of the cover fixing all the components except S_{μ_i} , on which it acts by switching the sheets of the spin structure. Thus it acts by -1 on the odd vector space $Z_1(S_{\mu_i})$, so that the invariants of the action of

$C(\mu)$ on $Z_1(\mathcal{S}_\mu)$ vanish. On the other hand, if μ is an odd partition, then the action on $Z_1(\mathcal{S}_\mu)$ is trivial. Thus

$$(3) \quad Z_n(\mathcal{S}_{\text{ap}}^1) = \bigoplus_{\mu \in \text{OP}(n)} Z_1(\mathcal{S}_{\mu_1}) \otimes \cdots \otimes Z_1(\mathcal{S}_{\mu_{\ell(\mu)}}).$$

As $\mathcal{S}_{\mu_i} \simeq \mathcal{S}_{\text{ap}}^1$, there is a spin disc bounding \mathcal{S}_{μ_i} , and thus $Z_1(\mathcal{S}_{\mu_i})$ contains a canonical element, 1_{μ_i} . The element δ_μ is defined to be

$$1_{\mu_1} \otimes \cdots \otimes 1_{\mu_{\ell(\mu)}} \in Z_1(\mathcal{S}_{\mu_1}) \otimes \cdots \otimes Z_1(\mathcal{S}_{\mu_{\ell(\mu)}}) = Z_1(\mathcal{S}_\mu) \subseteq Z_n(\mathcal{S}_{\text{ap}}^1).$$

The property in (2) follows from [Proposition 3.2](#).

Remark 4.1 The statement that only odd partitions appear as boundary conditions in our TQFT can be thought of more concretely. Suppose you have a decomposition of a spin surface Σ into two halves along a boundary circle. We wish to construct covers of the surface by gluing together covers along the two halves. If we have a cover on one of the halves with ramification datum that is not odd at the boundary, one of the components of the boundary of the cover will be periodic. There are always two ways to glue this circle to a corresponding one over the other half, which lead to spin surfaces covering our original surface *which have opposite parity*. Thus, when we sum over all covers, these will cancel and will not contribute to the sum.

Now let us investigate $Z_n(\mathcal{S}_{\text{per}}^1)$ (this will not be necessary for the spin Hurwitz formulas). In this case, all components of the covering \mathcal{S}_μ are periodic. Given a partition μ for which $\mu_i = \mu_j$, we have an automorphism which switches the two components. As in the antiperiodic case, this acts by -1 on $Z_1(\mathcal{S}_\mu)$ so is killed in the space of invariants. On the other hand if μ is a strict partition, then $Z_1(\mathcal{S}_\mu)$ is given by

$$Z_1(\mathcal{S}_{\mu_1}) \otimes \cdots \otimes Z_1(\mathcal{S}_{\mu_{\ell(\mu)}})$$

which is a super vector space of degree $\ell(\mu) \bmod 2$. This completes the proof of [Theorem 1.23](#).

Remark 4.2 The TQFT point of view allows us to generalize the notion of spin Hurwitz number to allow periodic boundary components. One has to be careful though, as the odd line $L = Z_1(\mathcal{S}_{\text{per}}^1)$ is not trivialized; rather, it has a canonical element defined up to sign. This means that we should interpret such invariants as linear maps between linear combinations of copies of L rather than numbers.

4B The spin Hurwitz formulas

Theorem 1.23 states that the spin Hurwitz numbers $\mathcal{H}_n(\Sigma; \mu^1, \dots, \mu^k)$ are given by evaluating $Z_n(\Sigma^*)$ on the element $\delta_{\mu^1} \otimes \dots \otimes \delta_{\mu^k} \in \mathcal{Z}(\mathcal{Y}_n)$. On the other hand, **Proposition 2.13** computes the invariant obtained by evaluating $Z(\Sigma^*)$ on the elements of the form $e_{\mathbf{v}^1} \otimes \dots \otimes e_{\mathbf{v}^k}$ in terms of the numbers $t_{\mathbf{v}} = Z_n(\mathbf{Cap})(e_{\mathbf{v}})$. Note that

$$(4) \quad Z_n(\mathbf{Cap})(\delta_{\mu}) = \begin{cases} 1/n! & \text{if } \mu = 1^n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, to prove **Theorem 1.10** (the spin Hurwitz formulas), it remains to identify the change of basis matrix between the bases $\{\delta_{\mu} \mid \mu \in \text{OP}(n)\}$ and $\{e_{\mathbf{v}} \mid \mathbf{v} \in \text{SP}(n)\}$.

Recall that \mathcal{Y}_n is canonically identified with the twisted group algebra of the hyperoctahedral group B_n . Thus there is another basis $\{\Delta_{\mu} \mid \mu \in \text{OP}(n)\}$ given by the conjugacy classes in B_n (see **Section 1F**). The following lemma is the final technical step needed to prove **Theorems 0.1** and **1.10**. Its proof will be left to **Section 4F**.

Lemma 4.3

$$\delta_{\mu} = 2^{(\ell(\mu-n))/2} \Delta_{\mu}.$$

Assuming **Lemma 4.3**, let us complete the proof of **Theorem 1.10** (and thus also **Theorem 0.1** which is a special case). According to **Proposition 1.17**, the change of basis matrix between $\{\Delta_{\mu} \mid \mu \in \text{OP}(n)\}$ and $\{e_{\mathbf{v}} \mid \mathbf{v} \in \text{SP}(n)\}$ is $f_{\mu}^{\mathbf{v}}$ (indeed, the central character of a group may be defined as the change of basis matrix between the orthogonal idempotents and the conjugacy classes).

Lemma 4.4

$$t(\mathbf{v}) = (d(\mathbf{v})/n!)^2.$$

Proof The linear map $t: \mathcal{Z}(\mathcal{Y}_n) \rightarrow \mathbb{C}$ takes the value $1/n!$ on $\delta_{1^n} = \Delta_{1^n}$ and zero on all other Δ_{μ} . Thus to prove the lemma we should express the idempotents $e_{\mathbf{v}}$ in terms of the basis of conjugacy classes $\{\Delta_{\mu}\}$. This is a straightforward exercise in the representation theory of finite groups, complicated only slightly by the fact that we are dealing with spin representations rather than linear representations (see **Section 1F** for details). The result follows by noting the interpretation of $d(\mathbf{v})$ in terms of the dimension of the module $V^{\mathbf{v}}$ in **Proposition 1.17**. \square

The spin Hurwitz formulas of **Theorem 1.10** now follow from **Proposition 2.13**.

4C Recursion formulas of Lee and Parker

Let Σ_g^p denote a closed spin surface of genus g and parity $p \in \mathbb{Z}/2\mathbb{Z}$. The following theorem was proved by Lee and Parker in [26]. Let $C(\mu)$ denote the centralizer of the conjugacy class μ in S_n . Note that $\#C(\mu) = \prod_i \mu_i(\mu_i!)$.

Theorem 4.5 *Let $\mu^1, \dots, \mu^k \in \text{OP}(n)$.*

(1) *If $g = g_1 + g_2$, and $p = p_1 + p_2$, then for $0 \leq k_0 \leq k$,*

$$\mathcal{H}_n(\Sigma_g; \mu^1, \dots, \mu^k) = \sum_{\mu \in \text{OP}(n)} \#C(\mu) \mathcal{H}_n(\Sigma_{g_1}; \mu^1, \mu^{k_0}, \mu) \mathcal{H}_n(\Sigma_{g_2}; \mu, \mu^{k_0+1}, \dots, \mu^k).$$

(2) *If $g \geq 2$, or if $(g, p) = (1, 0)$, then*

$$\mathcal{H}_n(\Sigma_g^p; \mu^1, \dots, \mu^k) = \sum_{\mu \in \text{OP}(n)} \#C(\mu) \mathcal{H}_n(\Sigma_{g-1}^p; \mu, \mu, \mu^1, \dots, \mu^k).$$

Proof Considered as a linear map $\mathbb{C} \rightarrow \mathbb{C}$, the number $Z_n(\Sigma_g^p)$ factors as

$$\mathbb{C} \xrightarrow{Z_n(\Sigma_{g_1}^{*p_1})} Z_n(S_{\text{ap}}^1) \xrightarrow{Z_n(\Sigma_{g_2}^{p_2})} \mathbb{C},$$

where $\Sigma_{g_1}^{*p_1}$ means $\Sigma_{g_1}^{p_1}$ punctured at one point, considered as a cobordism $\emptyset^1 \rightarrow S_{\text{ap}}^1$, and $\Sigma_{g_2}^{p_2}$ means $\Sigma_{g_2}^{p_2}$ punctured at one point, considered as a cobordism $S_{\text{ap}}^1 \rightarrow \emptyset^1$. Let us expand the element

$$Z_n(\Sigma_{g_1}^{*p_1})(1) \in Z_n(S_{\text{ap}}^1)$$

in the δ_μ basis. As $\langle Z_n(\Sigma_{g_1}^{*p_1})(1), \delta_\mu \rangle = \mathcal{H}_n(\Sigma_{g_1}^{p_1}; \mu)$, the coefficient of δ_μ in this expansion is

$$\mathcal{H}_n(\Sigma_{g_1}^{p_1}; \mu) \langle \delta_\mu, \delta_\mu \rangle^{-1}.$$

The number $\langle \delta_\mu, \delta_\mu \rangle$ is a genus zero spin Hurwitz number with two ramification points; it is easy to compute that this is equal to $\#C(\mu)^{-1}$. Thus

$$\begin{aligned} \mathcal{H}_n(\Sigma) &= Z_n(\Sigma_{g_2}^{*p_2}) \left(\sum_{\mu \in \text{OP}(n)} \mathcal{H}_n(\Sigma_{g_1}^{p_1}; \mu) \#C(\mu) \delta_\mu \right) \\ &= \sum_{\mu \in \text{OP}(n)} \#C(\mu) \mathcal{H}_n(\Sigma_{g_1}^{p_1}; \mu) \mathcal{H}_n(\Sigma_{g_2}^{p_2}; \mu). \end{aligned}$$

This proves part (1) in the case $k = 0$. The ramified case is proved similarly. To prove part (2), note that we can factor $Z_n(\Sigma_g^p)$ as

$$\mathbb{C} \xrightarrow{Z_n(\Sigma_{g-1}^{*p})} Z_n(S_{\text{ap}}^1) \otimes Z_n(S_{\text{ap}}^1) \xrightarrow{\langle -, - \rangle} \mathbb{C},$$

where Σ_{g-1}^{**p} is Σ_{g-1}^p with two punctures, considered as a cobordism $\emptyset^1 \rightarrow S_{\text{ap}}^1 \sqcup S_{\text{ap}}^1$. Note that the inner product $\langle -, - \rangle$ is the value of Z_n on a 2–sphere with two punctures, considered as a cobordism $S_{\text{ap}}^1 \sqcup S_{\text{ap}}^1 \rightarrow \emptyset^1$. The proof now proceeds in the same way as part (1). \square

4D Boundary conditions for extended TQFT

To prove Lemma 4.3, we need to compare the element $\delta_\mu \in Z_n(S_{\text{ap}}^1)$ with $\Delta_\mu \in \mathcal{Z}(\mathcal{Y}_n)$. To describe the embedding $Z_n(S_{\text{ap}}^1) \hookrightarrow \mathcal{Y}_n$ explicitly, it is necessary to make use of a cobordism which does not appear naturally in the category $\mathbf{Bord}^{\text{spin}}$, namely, the whistle or closed-to-open string transition. We will briefly explain what this means below.

Let us first describe the category $\mathbf{Bord}^{\text{or,oc}}$ of *oriented* cobordisms with boundary condition (the theory of cobordism categories with boundary conditions is described in Lurie [27, Section 4.3.22]; we will need only a small part of the theory). In general, the 2–category $\mathbf{Bord}^{\text{or,oc}}$ is defined in the same way as $\mathbf{Bord}^{\text{or}}$, except that the word “manifold” is replaced by “manifold with (marked) boundary”. Thus, 0–objects of $\mathbf{Bord}^{\text{or,oc}}$ are the same as in $\mathbf{Bord}^{\text{or}}$, but there is an additional 1–object, given by the line interval $I = [0, 1]$, where the boundary $\{0, 1\}$ is considered “marked” (thus I is a morphism $\emptyset^0 \rightarrow \emptyset^0$ rather than $\text{pt}_+ \rightarrow \text{pt}_+$, say). If M_0 and M_1 are both 1–objects in $\mathbf{Bord}^{\text{or,oc}}$ (thus, a disjoint union of copies of S^1 and I), then a 2–morphism between M_0 and M_1 is a 2–manifold with corners Σ , where $\partial\Sigma$ is equipped with an isomorphism

$$\partial\Sigma \simeq \overline{M}_0 \sqcup_{\partial_{\text{marked}} \overline{M}_0} \sqcup_{\partial_{\text{marked}} M_1} M_1.$$

Remark 4.6 The 1–category $\Omega\mathbf{Bord}^{\text{or,oc}} = \text{Hom}_{\mathbf{Bord}^{\text{or,oc}}}(\emptyset^0, \emptyset^0)$ is equivalent to the category of *open-closed cobordisms* as described eg in Moore and Segal [29].

Given an oriented 2d TQFT $Z: \mathbf{Bord}^{\text{or}} \rightarrow \mathbf{Alg}$ with $Z(\text{pt}_+) = A$, an extension of Z to a symmetric monoidal functor $\tilde{Z}: \mathbf{Bord}^{\text{or,oc}} \rightarrow \mathbf{Alg}$ is called a *boundary condition* for Z . Boundary conditions for Z are in one-to-one correspondence with A –modules; given an A –module M , there is an extension \tilde{Z}_M with $\tilde{Z}_M(I) = \text{End}(M)$. (More generally, we could consider categories of cobordisms where we allow marked boundary of various colours. An extension of Z to such a category corresponds to a choice of A –module M_i for each colour i . Such an extension assigns the vector space $\text{Hom}_A(M_i, M_j)$ to the line interval in which one endpoint has colour i and the other has colour j .)

Thus, any oriented TQFT Z as above, has a canonical boundary condition \tilde{Z} given by the A –module A itself. Thus $\tilde{Z}(I) = A$. There are various 2–morphisms given

by marking intervals along the boundary of a disc. For example, marking 3 intervals along the boundary of a disc defines the *pair of chaps* cobordism

$$\mathbf{Chaps}: I \sqcup I \rightarrow I.$$

Applying \tilde{Z} to **Chaps** recovers the multiplication map for the algebra A (respectively, comultiplication if read in the other direction). Similarly, marking one interval on the boundary of the disc gives the trace map defining the Frobenius algebra structure on A (or the unit map, if read in the other direction).

The main example that we will need is the *whistle* cobordism (also known as the closed-to-open string transition),

$$\mathbf{Whistle}: S^1 \rightarrow I.$$

The whistle has a cylinder $S^1 \times [0, 1]$ as its underlying manifold, where the marked portion of the boundary is a line segment embedded in $S^1 \times \{1\}$. Applying \tilde{Z} to the whistle cobordism recovers the inclusion of the centre $\mathcal{Z}(A) \hookrightarrow A$ (or the projection $A \rightarrow \mathcal{A}b(A)$ if read in the other direction).

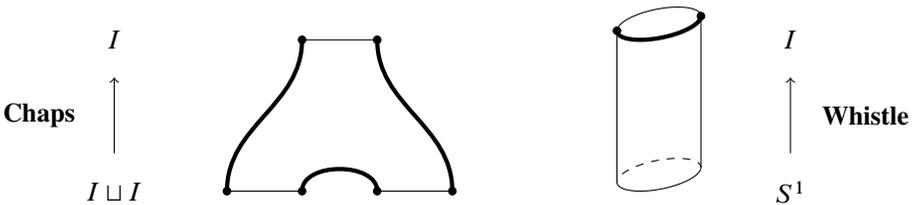


Figure 4: The chaps and whistle cobordisms. The marked portion of the boundary is shown in bold.

The theory of boundary conditions for TQFTs carries over to the spin setting: there is a 2–category $\mathbf{Bord}^{\text{spin,oc}}$, having the property that an extension \tilde{Z} of a TQFT $Z: \mathbf{Bord}^{\text{spin}} \rightarrow \mathbf{SAlg}$ corresponds to a supermodule for the superalgebra $A = Z(\text{pt}_+)$. Naturally, there are some new features in the spin case: There are now two 1–objects I and \bar{I} whose underlying manifold is the interval $[0, 1]$. The canonical boundary condition \tilde{Z} for Z assigns $\tilde{Z}(I) = A$ and $\tilde{Z}(\bar{I}) = A^1$, and the various spin structures on the pair of chaps cobordisms recover the multiplication on A , the A –module structure on A^1 , and the map $A^1 \otimes A^1 \rightarrow A$ expressing the triviality of the square of the Serre automorphism. We will only require the use of a single cobordism in $\mathbf{Bord}^{\text{spin,oc}}$, namely the whistle cobordism $S_{\text{ap}}^1 \rightarrow I$.

4E The map $\tilde{Z}_n(\text{Whistle})$

Consider the parity theory Z_1 with its canonical boundary condition \tilde{Z}_1 and let A denote $\mathcal{C}l_1 = Z_1(\text{pt}_+)$. The construction of the covering theories Z_n as in Section 3 can be extended to cobordisms with boundary conditions. For example, to compute $\tilde{Z}_n(I)$, we look at the groupoid $\text{Cov}_n(I)$. Note that the boundary of I is marked, and thus we should consider n -fold covers of I equipped with trivializations at $\partial I = \{0, 1\}$. Thus $\text{Cov}_n(I)$ is isomorphic to the discrete groupoid S_n , and $\tilde{Z}_n(I) = \bigoplus_{S_n} A^{\otimes n}$. Applying \tilde{Z}_n to the pair of chaps cobordism identifies the algebra structure on $\bigoplus_{S_n} A^{\otimes n}$ as the semidirect product $A^{\otimes n} \rtimes S_n$.

Now let us apply the covering theory construction of Section 3 to the whistle cobordism. Consider the following correspondence of groupoids:

$$(5) \quad \text{Cov}_n(S_{\text{ap}}^1) \leftarrow \text{Cov}_n(\text{Whistle}) \rightarrow \text{Cov}_n(I).$$

Objects of the groupoid $\text{Cov}_n(S_{\text{ap}}^1)$ are given by covers of the circle S_{ap}^1 . Equivalence classes of such covers are indexed by $P(n)$. For each $\mu \in P(n)$, we have a cover \mathcal{S}_μ which is a disjoint union of circles \mathcal{S}_{μ_i} each of which are connected covers of S_{ap}^1 of degree μ_i .

Objects of $\text{Cov}_n(\text{Whistle})$ are given by covers of (S_{ap}^1, p) where p is a choice of basepoint on S_{ap}^1 , together with a framing of the cover at p (ie an identification between the fibre over p and the set $\{1, \dots, n\}$). Such covers are indexed by S_n . For each $\sigma \in S_n$, we have such a cover \mathcal{S}_σ which is a disjoint union of circles \mathcal{S}_{σ_i} where $\sigma = \sigma_1 \cdots \sigma_\ell$ is a disjoint cycle decomposition.

Finally, objects of $\text{Cov}_n(I)$ are covers of $(I = [0, 1], \{0, 1\})$ with framings at $\{0, 1\}$. Such covers are indexed by S_n . For each $\sigma \in S_n$, we have I_σ which is a disjoint union of n line intervals.

Following the construction of Section 3, \tilde{Z}_n assigns to the whistle cobordism a linear map, obtained by averaging the theory Z_1 over the correspondence of groupoids (5):

$$(6) \quad \tilde{Z}_n(S_{\text{ap}}^1) = \bigoplus_{\mu \in P(n)} Z_1(\mathcal{S}_\mu)^{C(\mu)} \xrightarrow{\tilde{Z}_n(\text{Whistle})} \bigoplus_{\sigma \in S_n} Z_1(I_\sigma) = Z_n(S_{\text{ap}}^1).$$

4F Proof of Lemma 4.3

Let us describe $\tilde{Z}_n(\text{Whistle})$ more explicitly. To begin with we have the natural inclusion

$$\bigoplus_{\mu \in P(n)} Z_1(\mathcal{S}_\mu)^{C(\mu)} \simeq \left(\bigoplus_{\sigma \in S_n} Z_1(\mathcal{S}_\sigma) \right)^{S_n} \hookrightarrow \bigoplus_{\sigma \in S_n} Z_1(\mathcal{S}_\sigma).$$

As explained in Section 4A, the action of $C(\mu)$ on the 1-dimensional super vector space $Z_1(\mathcal{S}_\mu)$ is trivial if $\mu \in \text{OP}(n)$ and nontrivial otherwise. Thus we need only consider $\mu \in \text{OP}(n)$.

Fix an element $\sigma \in S_n$ of cycle type $\mu \in \text{OP}(n)$, with a corresponding disjoint cycle decomposition $\sigma = \sigma_1 \cdots \sigma_\ell$. We have $Z_1(\mathcal{S}_\sigma) = Z_1(\mathcal{S}_{\sigma_1}) \otimes \cdots \otimes Z_1(\mathcal{S}_{\sigma_\ell})$. Each \mathcal{S}_{σ_i} is a μ_i -fold cover of the base S_{ap}^1 , thus we compute $Z_1(\mathcal{S}_{\mu_i})$ as a tensor product μ_i copies of $A^!$ over A , arranged in a circle (see Figure 5). We can write this circular tensor product linearly as

$$(7) \quad Z_1(\mathcal{S}_{\sigma_i}) = \underbrace{(A^! \otimes_A \cdots \otimes_A A^!)}_{\mu_i} \otimes_{A^e} A.$$

To describe the map $Z_1(\mathcal{S}_{\sigma_i}) \rightarrow Z_1(I_{\sigma_i})$, note that each factor $A^!$ canonically embeds in A^e . Replacing each instance of $A^!$ by A^e in the tensor product (7), we arrive at $A^{\otimes \mu_i}$ which is precisely $Z_1(I_{\sigma_i})$.

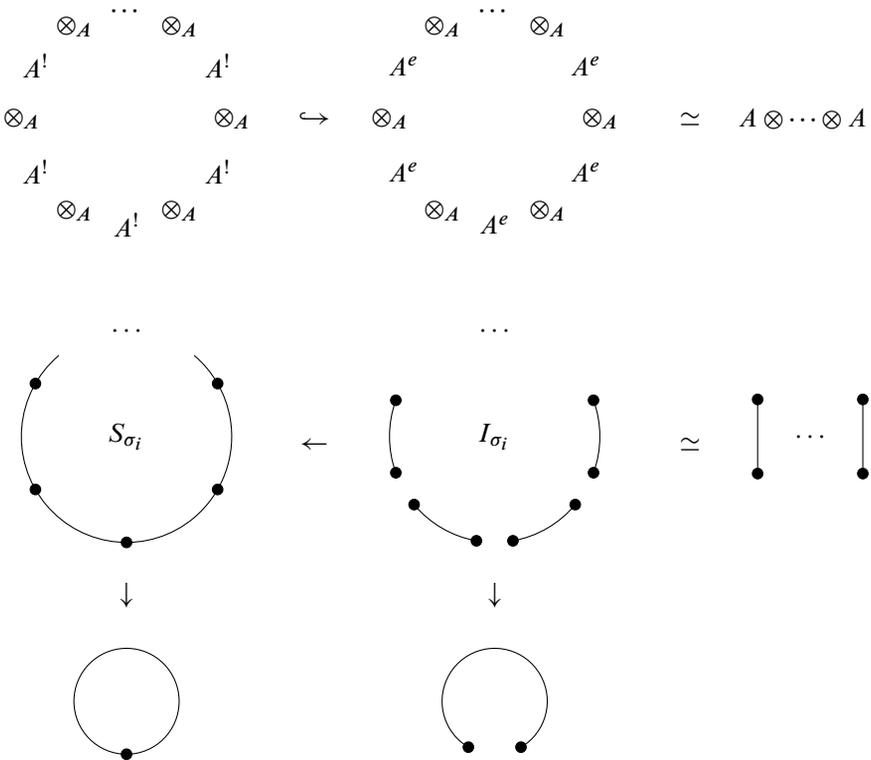


Figure 5: Computing $\tilde{Z}_n(\text{Whistle})$

On the other hand, S_{σ_i} is isomorphic to S_{ap}^1 , and thus $Z_1(S_{\mu}) \simeq \mathbb{C}$. To go between the two descriptions of $Z_1(S_{\sigma_i})$, we can use the trivialization of the Serre automorphism $A^! \otimes_A A^! \simeq A$ to contract pairs of $A^!$'s in the tensor product, eventually arriving at $A^! \otimes_{A^e} A \simeq \mathbb{C}$.

Lemma 4.7 Under the map $\mathbb{C} \simeq Z_1(S_{\sigma_i}) \rightarrow Z_1(I_{\sigma_i}) \simeq A^{\otimes \mu_i}$, the element 1 maps to

$$\sum 2^{-(\mu_i-1)/2} \eta^{c_1} \otimes \dots \otimes \eta^{c_{\mu_i}}$$

where the sum is over $(c_1, \dots, c_{\mu_i}) \in (\mathbb{Z}/2\mathbb{Z})^{\mu_i}$ such that $\sum_j c_j = 0 \in \mathbb{Z}/2\mathbb{Z}$.

Proof Note that $A^!$ can be identified with the sub-bimodule of A^e generated by the element $\varepsilon = 1 \otimes 1 + \eta \otimes \eta$. Unwinding the duality data for $A^!$, we obtain that under the trivialization of the square of the Serre automorphism, the element $1 \in A$ corresponds to the tensor $\frac{1}{2}\varepsilon \otimes \varepsilon$ in $A^! \otimes_A A^!$. Also, the element $1 \in \mathbb{C}$ corresponds to $\frac{1}{2}\varepsilon \otimes 1$ in $A^! \otimes_{A^e} A$.

It follows that the element $1 \in \mathbb{C}$ maps to $2^{-(\mu_i+1)/2} \varepsilon \otimes \dots \otimes \varepsilon \otimes 1$ in the tensor product (7). Lemma 4.7 follows by taking canonical embedding $A^! \hookrightarrow A^e$ for each $A^!$ in the tensor product, and simplifying to $A^{\otimes \mu_i}$. □

To complete the proof of Lemma 4.3, consider the tensor product over $i = 1, \dots, \ell$ of the maps $Z_1(S_{\sigma_i}) \rightarrow Z_1(I_{\sigma_i})$. By Lemma 4.7, the element $\delta_{\mu} \in Z_1(S_{\sigma})$ maps to

$$\bigotimes_{i=1}^{\ell} \left(\sum 2^{-(\mu_i-1)/2} \eta^{c_1} \otimes \dots \otimes \eta^{c_{\mu_i}} \right) = 2^{(\ell-n)/2} \Delta_{\sigma} \in A^{\otimes n},$$

where the sum is over $(c_1, \dots, c_{\mu_i}) \in (\mathbb{Z}/2\mathbb{Z})^{\mu_i}$ such that $\sum_j c_j = 0 \in \mathbb{Z}/2\mathbb{Z}$. Here $\bigotimes_{i=1}^{\ell} A^{\mu_i} = A^{\otimes n}$ is identified with the subspace $A^{\otimes n} \otimes \sigma$ of \mathcal{Y}_n , and Δ_{σ} is defined as in Proposition 1.18. Thus $\tilde{Z}_n(\text{Whistle})$ maps δ_{μ} to $2^{(\ell-n)/2} \Delta_{\mu}$ as required.

5 The parity TQFT

In this section, we explain how the parity of a closed spin surface Σ can be upgraded to a TQFT

$$Z_p: \mathbf{Bord}^{\text{spin}} \rightarrow \mathbf{SAlg},$$

such that $Z_p(\Sigma) = (-1)^{p(\Sigma)}$. We will give a direct but somewhat ad hoc construction, then in Section 5C we explain how this theory can be constructed via homotopical methods using the spin orientation of KO.

Remark 5.1 The TQFT Z_p is *invertible* in the sense that for each object N , the object $Z_p(N)$ is (weakly) invertible for the monoidal structure, and for each 1– or 2–morphism M , $Z_p(M)$ is a (weakly) invertible 1– or 2–morphism. The parity TQFT and more general invertible TQFTs are discussed in Freed [11].

5A An aside on conformal spin structures and metrics

For the construction of the TQFT, it will be helpful to make use of metrics on the various vector bundles associated to the spin structure. Rather than making choices and then proving the invariance of those choices it will be more convenient to work with a particular model of the spin bordism category for which canonical metrics are built in. The definitions and conventions in this subsection are drawn from Stolz and Teichner [35].

Let V be a finite-dimensional (real or complex) vector space with a quadratic form q . The Clifford algebra $\mathcal{C}\ell(V)$ is the quotient of the tensor algebra on V by the relation $v^2 = -q(v)$. We write $\mathcal{C}\ell_d(\mathbb{R})$ for the Clifford algebra of \mathbb{R}^d with its canonical positive definite quadratic form and $\mathcal{C}\ell_{-d}(\mathbb{R})$ for the Clifford algebra of \mathbb{R}^d with its negative definite form. We can extend the involution $v \mapsto -v$ on V uniquely to an algebra involution α on $\mathcal{C}\ell(V)$. This gives it the structure of a superalgebra (ie a $\mathbb{Z}/2\mathbb{Z}$ –graded algebra), and we can speak of supermodules, etc. Note that the Clifford algebra $\mathcal{C}\ell(V)$ inherits an inner product from that on V .

A *spin structure* on V is a choice of an irreducible $\mathcal{C}\ell(V)$ – $\mathcal{C}\ell_d(\mathbb{R})$ –superbimodule S_V with a compatible inner product (ie multiplication by elements of unit length in V and \mathbb{R}^n are isometries). Given two spin vector spaces V and W , the space $\text{Spin}(V, W)$ of spin isometries consists of an isometry $f: V \rightarrow W$, together with a homomorphism of bimodules $\phi: f^*S_W \rightarrow S_V$ which is additionally an isometry. The map $(f, \phi) \mapsto f$ exhibits $\text{Spin}(V, W)$ as a double cover of $\text{SO}(V, W)$ (the connected component of $O(V, W)$ for which *there exists* such a ϕ).

If $E \rightarrow X$ is a real vector bundle with a positive definite metric, let $\mathcal{C}\ell(E) \rightarrow X$ denote the bundle of algebras whose fibre over $x \in X$ is the Clifford algebra $\mathcal{C}\ell(E_x)$. A spin structure on E is a bundle S_E of irreducible $\mathcal{C}\ell(E)$ – $\mathcal{C}\ell_d(\mathbb{R})$ –superbimodules (ie a compatible family of spin structures on the quadratic vector spaces E_x).

Given a smooth manifold M of dimension d , we write L^k for the (trivializable) line bundle of k –densities on M (in our convention a d –density can be integrated over M to get a real number). The *weightless cotangent bundle* T_0^*M is defined to be $L^{-1} \otimes T^*M$. A *conformal spin structure* on a conformal manifold M is defined to be a spin structure on T_0^*M .

Note that a choice of conformal structure equips the weightless cotangent bundle with a *canonical* metric, and thus the Clifford bundles, spinor bundles, etc all come equipped with canonical metrics. Moreover, if M has boundary ∂M , then $T_0^*M|_{\partial M}$ is canonically equivalent to $T_0^*\partial M \oplus \mathbb{R}$, thus a conformal spin structure on M canonically defines a conformal spin structure on the boundary.

We can define a 2–category of (topological) conformal spin cobordisms $c\mathbf{Bord}^{\text{spin}}$ in a similar way to Section 1G. Note that 0– and 1–dimensional manifolds carry a unique conformal structure. Moreover, we will consider two conformal spin 2–cobordisms to give rise to the same 2–morphism if there is a spin diffeomorphism between them, fixing the given isomorphisms on the boundary (such a diffeomorphism is not required to preserve the conformal structure).

One can check that the symmetric monoidal 2–category $c\mathbf{Bord}^{\text{spin}}$ is equivalent to $\mathbf{Bord}^{\text{spin}}$ as defined, for example in Schommer-Pries [32]. We will not attempt to give a rigorous proof of this statement,⁵ but intuitively this should be clear: a choice of Riemannian metric on a conformal spin manifold trivializes the density bundle and defines a spin structure in the usual sense. Moreover, the space of such Riemannian metrics is contractible.

5B The direct construction

The construction of the 0, 1–dimensional part of the theory is taken from [35, Section 2.3]. Given a (conformal) spin 0–manifold N , $\mathcal{S}(N)$ is a finite-dimensional super vector space equipped with an inner product. We define a symmetric bilinear form b_N by $b_N(v, w) = \langle v, i(w) \rangle$, where i is the grading involution and $\langle -, - \rangle$ is the inner product. We set

$$Z_p(N) = \mathcal{C}\ell(\mathcal{S}(N), b_N).$$

Given a spin 1–cobordism M with $\partial N = \bar{N}_0 \sqcup N_1$, the even spinor bundle S_M on M is a metric line bundle and thus is equipped with a canonical flat connection. The space of flat sections $\mathcal{S}^h(M)$ of S_M over M is called the space of harmonic spinors. The space $\mathcal{S}_0^h(M)$ of even harmonic spinors is finite-dimensional, and comes equipped with a map

$$L: \mathcal{S}_0^h(M) \rightarrow \mathcal{S}(\partial N),$$

given by restriction to the boundary. The morphism L is a *generalized Lagrangian* in the sense of [35, Definition 2.2.9]. As explained in [35, Definition 2.2.4], the *Fock*

⁵Indeed to give such a rigorous proof, we would first need to give a more rigorous definition of the bordism bicategory, taking care of gluing using collars eg as in [32].

module of L is defined to be

$$F(L) := \ker(L)^* \otimes \Lambda^*(\text{Im}(L)).$$

The Fock module $F(L)$ carries a canonical action of $\mathcal{C}l(\mathcal{S}(\partial N))$ or, equivalently, is a $\mathcal{C}l(\mathcal{S}(N_0))$ – $\mathcal{C}l(\mathcal{S}(N_1))$ –bimodule. We set $Z_p(M) = F(L)$. Note that if M is closed, then $F(L) = \bigwedge^{\text{top}}(\mathcal{S}^h(M))$ is the Pfaffian line of the Dirac operator on T_0^*M . For example, if $M \simeq S_{\text{ap}}^1$, there are no harmonic spinors and thus $Z_p(S_{\text{ap}}^1)$ is canonically isometric to \mathbb{R} . On the other hand, if $M \simeq S_{\text{per}}^1$, the space of harmonic spinors is 1–dimensional and thus $Z_p(M)$ is a 1–dimensional odd metric line.

Lemma 5.2 [35, Gluing Lemma 2.2.8]⁶ *Given conformal spin 1–cobordisms*

$$N_0 \xrightarrow{M} N_1 \quad \text{and} \quad N_1 \xrightarrow{M'} N_2$$

there is a canonical isomorphism

$$Z_p(M \sqcup_{N_1} M') \simeq Z_p(M) \otimes_{Z_p(N_1)} Z_p(M').$$

It remains to explain what Z_p assigns to a 2–morphism in $\mathbf{Bord}^{\text{spin}}$. Let us first consider the case of a spin surface Σ with boundary $M := \partial\Sigma$. We would like to define an element $Z_p(\Sigma) \in Z_p(M)$. There are various approaches to this; for example one could use a refinement of the mod 2 index of the Dirac operator to a spin conformal 2–manifold with boundary which takes values in the Pfaffian line of the boundary. However, we will just pursue a more down-to-earth method in terms of the parity of closed spin manifolds obtained by gluing in discs and cylinders to Σ .

Let D be the unit disc in \mathbb{R}^2 equipped with its canonical conformal spin structure, and $C = S_{\text{per}}^1 \times [0, 1]$ similarly equipped.

Definition 5.3 Suppose M is a closed conformal spin 1–manifold with n_D periodic components and $2n_C$ antiperiodic components (where $n_D, n_C \in \mathbb{Z}_{\geq 0}$). A capping of M is defined to be a spin isomorphism

$$c: M \simeq \partial \left(\bigsqcup_{n=1}^{n_D} D \sqcup \bigsqcup_{n=1}^{n_C} C \right).$$

The set of cappings of M is denoted $\text{Cappings}(M)$.

Note that a capping of M only exists if M has an even number of periodic components. This condition is satisfied if M is the boundary of a spin surface.

⁶Note that in this case, all Hilbert spaces and Clifford algebras are finite-dimensional.

Lemma 5.4 *Each capping of M canonically determines an element of $Z_p(M)$ of unit length.*

Proof First consider the case where M has only antiperiodic components; the space of harmonic spinors is 0, and thus its determinant line $Z_p(M)$ is canonically trivial. In particular $Z_p(M)$ carries the canonical element 1. Now let us consider the case when M has precisely two periodic components M_0 and M_1 . A choice of capping allows us to define a parallel transport isometry $\phi: \mathcal{S}^h(\overline{M}_0) \simeq \mathcal{S}^h(M_1)$ which in turn defines an element $\det(\phi) \in Z_p(M)$. The general case follows. \square

Given a spin surface Σ with boundary $\partial\Sigma = M$, a capping c of M canonically determines a closed spin surface Σ_c by gluing along the map c .

Lemma 5.5 *The assignment*

$$\begin{aligned} \text{Cappings}(M) &\rightarrow \{\pm 1\} \subseteq \mathbb{R}, \\ c &\mapsto (-1)^{p(\Sigma_c)}, \end{aligned}$$

factors through the map $\text{Cappings}(M) \rightarrow Z_p(M)$ given by Lemma 5.4, defining an isometry of graded metric lines

$$Z_p(\Sigma): Z_p(M) \rightarrow \mathbb{R}.$$

Proof We prove this using the definition of parity via the Arf invariant of the quadratic form $\varphi: H_1(\Sigma_c; \mathbb{F}_2) \rightarrow \mathbb{F}_2$ associated to the spin structure. Recall that if M is a closed loop in Σ , then $\varphi([M]) = 0$ (respectively, 1) if M has an antiperiodic (respectively, periodic) spin structure.

Changing the choice of capping on antiperiodic components does not affect the corresponding quadratic form, as one can choose representatives of cycles in $H_1(\Sigma_c; \mathbb{F}_2)$ which do not intersect the antiperiodic components. Thus we may assume M has only periodic components. Consider a particular component M_0 of M ; recall that there are two choices of spin isomorphism $S_{\text{per}}^1 \simeq M_0$ (up to spin isotopy). Given a capping c , we can modify c by changing the choice of $S_{\text{per}}^1 \simeq M_0$ to obtain a new capping c' . The effect of changing c to c' has precisely the effect of changing the value of the quadratic form on the Poincaré dual cycle to M_0 , which in turn changes the value of the Arf invariant; simultaneously, the corresponding value of $Z_p(M)$ gets multiplied by -1 . The result follows readily from this observation. \square

Finally, given an arbitrary 2–morphism $\Sigma: M_0 \rightarrow M_1$, where $M_0, M_1: N_0 \rightarrow N_1$ are 1–cobordisms, we can now define a map,

$$Z_p(\Sigma): Z_p(M_0) \rightarrow Z_p(M_1),$$

of $Z_p(N_0)$ – $Z_p(N_1)$ –bimodules as follows. Note that, topologically, the manifold with corners underlying Σ may be thought of as a spin manifold Σ' with closed boundary M' (ie we can smooth the corners of Σ). To identify the spin structure on the boundary M' , let us consider the underlying spin 1–manifold with boundary of M_0 , as a cobordism

$$M'_0: N_0 \sqcup \bar{N}_1 \rightarrow \emptyset^0.$$

Taking the opposite spin structure, we have

$$\bar{M}'_0: \bar{N}_0 \sqcup N_1 \rightarrow \emptyset^0.$$

Similarly, we can consider

$$M'_1: \emptyset^0 \rightarrow \bar{N}_0 \sqcup N_1.$$

Thus, we can compose \bar{M}'_0 and M'_1 in the bordism category to obtain the closed spin 1–manifold M' . The general principles of preservations of duality under symmetric monoidal functors tell us that

$$Z_p(M') \simeq \text{Hom}_{Z_p(N_0)^{\text{op}} \otimes Z_p(N_1)}(Z_p(\bar{M}'_0), Z_p(M'_1)).$$

We now define $Z_p(\Sigma) \in \text{Hom}_{Z_p(N_0)^{\text{op}} \otimes Z_p(N_1)}(Z_p(M_0), Z_p(M_1))$ to correspond to $Z_p(\Sigma') \in Z_p(M')$ under the above isomorphism.

It remains to check that Z_p , as defined above, defines a TQFT, ie a symmetric monoidal functor

$$Z_p: \mathbf{Bord}^{\text{spin}} \rightarrow \mathbf{SAlg}_{\mathbb{R},m}^{\times},$$

where $\mathbf{SAlg}_{\mathbb{R},m}^{\times}$ is the 2–category whose objects are Clifford algebras of quadratic vector spaces, morphisms are Fock modules associated to Lagrangian subspaces, and 2–morphisms are isomorphisms of bimodules which are also isometries for the given inner products (see [35, Definition 2.2.4]). (We obtain a TQFT with target \mathbf{SAlg} by just applying $\otimes_{\mathbb{R}} \mathbb{C}$ and forgetting the inner products.) This amounts to checking the compatibility of the above definition under various kinds of gluing. We will not spell out all the necessary compatibilities here; rather, we will prove the following gluing law from which all necessary compatibilities may be deduced.

Proposition 5.6 *Let Σ be a closed spin surface, and M an embedded disjoint union of circles in Σ with an even number of periodic components. Define $\Sigma_{\text{cut},M}$ by cutting along M to obtain a surface with boundary $M \sqcup \bar{M}$. Choose a capping c of M , and let \bar{c} denote the corresponding capping of \bar{M} . We define the closed surface $\Sigma_{c \sqcup \bar{c},M}$ by capping the boundary of $\Sigma_{\text{cut},M}$ according to $c \sqcup \bar{c}$. Then,*

$$p(\Sigma) = p(\Sigma_{c \sqcup \bar{c},M}).$$

Proof We use the definition of parity via the Arf invariant of the quadratic form on H_1 . The proof is an elementary check based on the properties of embedded spin circles under cutting and gluing. We will give the argument in the case where M has two periodic components (the case when M has a single antiperiodic component is proved similarly, and all other cases can be proved by induction from these). For simplicity, we assume further that Σ is connected.

Note that the genus of $\Sigma_{c \sqcup \bar{c}, M}$ is always one greater than the genus of Σ . One can choose a symplectic basis for $H_1(\Sigma; \mathbb{F}_2)$ which contains the class κ of either component of M (the two components necessarily define the same class in $H_1(\Sigma; \mathbb{F}_2)$) and its Poincaré dual κ^\vee . The \mathbb{F}_2 -vector space $H_1(\Sigma_{c \sqcup \bar{c}}; \mathbb{F}_2)$ has a similar basis but with two copies of κ , κ_1 and κ_2 and their respective Poincaré duals. Here, there are two possible cases: either κ^\vee is periodic or antiperiodic. In the first case, precisely one of κ_1^\vee or κ_2^\vee will be periodic (which one depends on the choice of c). In the second case, either both κ_1^\vee and κ_2^\vee will be periodic, or both antiperiodic. Thus in all cases, the Arf invariant is unchanged as required. \square

Remark 5.7 One could also prove [Proposition 5.6](#) by noting that the surfaces Σ and $\Sigma_{c \sqcup \bar{c}, M}$ are spin cobordant, and the parity is a spin cobordism invariant (see Atiyah [\[2\]](#)).

5C Homotopical construction

One way of thinking of the parity of a closed spin surface is in terms of the Gysin map in KO-theory afforded by the spin structure. Namely, we have a map

$$\int_{\Sigma}: \mathrm{KO}(\Sigma) \rightarrow \mathrm{KO}^{-2}(\mathrm{pt}) \simeq \mathbb{Z}/2\mathbb{Z},$$

and $\int_{\Sigma} 1 = p(\Sigma)$.

Similarly, one has invariants $\int_M 1 \in \mathrm{KO}^{-1}(\mathrm{pt}) \simeq \mathbb{Z}/2\mathbb{Z}$, and $\int_N 1 \in \mathrm{KO}(\mathrm{pt}) = \mathbb{Z}$ given a spin 1-manifold M and a spin 0-manifold. There is a very general procedure by which one can upgrade the assignment $\Sigma \mapsto \int_{\Sigma} 1$ to a TQFT.

First note that the spin orientation of KO may be thought of as a map of E_{∞} -ring spectra

$$\mathrm{ABS}: \mathrm{MSpin} \rightarrow \mathrm{ko}.$$

Here, MSpin is the spectrum representing spin cobordism, and ko represents connective KO. Thus a closed spin d -manifold M represents a class $[M] \in \Omega_d^{\mathrm{Spin}} = \pi_d(\mathrm{MSpin})$ and the element $\int_M 1$ is equal to the image of $[M]$ in $\pi_d(\mathrm{KO}) = \mathrm{KO}^{-d}(\mathrm{pt})$ under the

map ABS . Given a connective spectrum E (or equivalently an infinite loop space), its fundamental 2–groupoid $\pi_{\leq 2}(E)$ is naturally symmetric monoidal. The morphism ABS induces a functor of symmetric monoidal 2–groupoids on the level of fundamental 2–groupoids:

$$\pi_{\leq 2}(\text{MSpin}) \rightarrow \pi_{\leq 2}(\text{ko}).$$

It remains to describe the 2–groupoids $\pi_{\leq 2}(\text{MSpin})$ and $\pi_{\leq 2}(\text{ko})$ in terms of bordism categories and superalgebras respectively.

Define a 2–category $\mathbf{Bord}_{\infty}^{\text{spin}}$ as follows: the objects and morphisms of $\mathbf{Bord}_{\infty}^{\text{spin}}$ are the same as those of $\mathbf{Bord}^{\text{spin}}$, but the 2–morphisms are given by 2–cobordisms *up to cobordism* (recall that in the 2–category $\mathbf{Bord}^{\text{spin}}$, the 2–morphisms are 2–cobordisms up to diffeomorphism). Note there is a canonical quotient functor $\mathbf{Bord}^{\text{spin}} \rightarrow \mathbf{Bord}_{\infty}^{\text{spin}}$. The following proposition is surely known, but I have been unable to track down a precise reference; it should follow from an appropriate version of the results of Galatius, Madsen, Tillmann and Weiss [16].

Proposition 5.8 *The fundamental 2–groupoid of MSpin is equivalent as symmetric monoidal 2–categories to $\mathbf{Bord}_{\infty}^{\text{spin}}$.*

On the other hand, we have the groupoid $\text{SAlg}_{\mathbb{R},m}^{\times}$ of *invertible* real metric superalgebras, invertible metric superbimodules, and invertible isometries of bimodules. Again, the following result is surely known; we give a proof below.

Lemma 5.9 *There is a canonical symmetric monoidal functor*

$$\text{Cliff}: \pi_{\leq 2}(\text{ko}) \rightarrow \text{SAlg}_{\mathbb{R},m}^{\times}.$$

Remark 5.10 The functor Cliff is not quite an equivalence: it induces an isomorphism on π_1 and π_2 , but on π_0 it induces the quotient map $\mathbb{Z} \rightarrow \mathbb{Z}/8\mathbb{Z}$.

Finally, the TQFT Z_p is given by the composite:

$$\mathbf{Bord}^{\text{spin}} \rightarrow \mathbf{Bord}_{\infty}^{\text{spin}} \simeq \pi_{\leq 2}(\text{MSpin}) \xrightarrow{\pi_{\leq 2}(\text{ABS})} \pi_{\leq 2}(\text{ko}) \xrightarrow{\text{Cliff}} \text{SAlg}_{\mathbb{R},m}.$$

Proof of Lemma 5.9 Let $\mathbf{Vect}_{\mathbb{R},m}$ be the *topological* category of real vector spaces equipped with an inner product, with isometries as morphisms. Recall that the space ko is obtained by taking the group completion of this category. We will define a symmetric monoidal functor from $\mathbf{Vect}_{\mathbb{R},m}$ (with symmetric monoidal structure given by orthogonal direct sum) to the 2–category $\text{SAlg}_{\mathbb{R},m}$, which by the universal property of group completion must factor through ko . To define such a functor, we must

assign a superalgebra to each quadratic vector space V , a bimodule to each isometry $f: V \rightarrow W$, and a map of bimodules to each homotopy class of paths of isometries $f_t: V \rightarrow W$.

The map Cliff is defined by taking a vector space V as described above to its Clifford algebra $\mathcal{Cl}(V)$. An isometry $f: V \rightarrow W$ naturally defines an invertible $\mathcal{Cl}(V)$ – $\mathcal{Cl}(W)$ –bimodule M_f . Recall that a *spin structure* on W means a choice of an irreducible $\mathcal{Cl}(W)$ – $\mathcal{Cl}_d(\mathbb{R})$ –bimodule. Note that for any choice of spin structure S_W on W , the $\mathcal{Cl}(V)$ – $\mathcal{Cl}(W)$ –bimodule M_f can be identified with $f^*S_W \otimes_{\mathcal{Cl}_d(\mathbb{R})} S_W^*$, where $S_W^* = \text{Hom}_{\mathbb{R}}(S_W, \mathbb{R})$ is the $\mathcal{Cl}_d(\mathbb{R})$ – $\mathcal{Cl}(W)$ –bimodule dual to S_W . Moreover, picking a compatible spin structure S_V on V and a lift of f to a spin isometry (f, ϕ) induces an isomorphism of bimodules $\hat{\phi}: M_f \rightarrow S_V \otimes_{\mathcal{Cl}_d(\mathbb{R})} S_W$.

A path in $O(V, W)$ is given by a family f_t of isometries where $t \in [0, 1]$. As the map $\text{Spin}(V, W) \rightarrow \text{SO}(V, W)$ is a covering map, if we pick a lift ϕ_0 of f_0 as above, then we obtain a lift ϕ_t of f_t (depending only on the homotopy class of the path). This gives the required isomorphism of bimodules

$$\hat{\phi}_1^{-1} \circ \hat{\phi}_0: M_{f_0} \rightarrow M_{f_1}$$

which doesn't depend on the choices of lift above. □

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