

Higher torsion and secondary transfer of unipotent bundles

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Given a unipotent bundle of smooth manifolds we construct its secondary transfer map and show that this map determines the higher smooth torsion of the bundle. This approach to higher torsion provides a new perspective on some of its properties. In particular it yields in a natural way a formula for torsion of a composition of two bundles.

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1 Introduction

By a smooth bundle of manifolds we will understand here a smooth submersion $p: E \rightarrow B$ where E and B are smooth compact manifolds. A smooth bundle with the fiber F is unipotent if B is path connected and the graded vector space $H_*(F; \mathbb{Q})$ admits a filtration such that $\pi_1 B$ acts trivially on the filtration quotients. Igusa [10] and J Klein [13] showed using fiberwise Morse theory that to any unipotent bundle one can associate the higher torsion invariant which depends not only on the topological structure of p , but also on its smooth structure. Higher torsion proved to be a useful tool in the study of smooth bundles. In [9] Igusa showed, for example, that it can be used to detect exotic disc bundles constructed by Hatcher.

In [2] and [1] the present authors in collaboration with Klein and B Williams extended ideas of Dwyer, Weiss and Williams [7] to obtain an alternative construction of torsion of unipotent bundles based on the machinery of homotopy theory. This construction can be briefly described as follows. Let $K(\mathbb{Q})$ be the infinite loop space underlying the algebraic K -theory spectrum of the field of rational numbers. Given a smooth bundle $p: E \rightarrow B$ we can construct a map $c_p: B \rightarrow K(\mathbb{Q})$ which, roughly speaking, assigns to each $b \in B$ the point of $K(\mathbb{Q})$ represented by the singular chain complex $C_*(p^{-1}(b); \mathbb{Q})$. The smooth Riemann–Roch theorem of [7] implies that c_p admits a

factorization

$$\begin{array}{ccc}
 & & Q(E_+) \\
 & \nearrow p^! & \downarrow \lambda_E \\
 B & \xrightarrow{c_p} & K(\mathbb{Q})
 \end{array}$$

where $Q(E_+) = \Omega^\infty \Sigma^\infty(E_+)$, the map $p^!$ is the Becker–Gottlieb transfer and λ_E is the linearization map (Section 3.1).

If p is a unipotent bundle then the map c_p is homotopic via a preferred homotopy to a constant map. As a consequence we obtain a lift of $p^!$ to the space $\text{Wh}_s^{\mathbb{Q}}(E)$ which is the homotopy fiber of λ_E :

$$\begin{array}{ccc}
 & & \text{Wh}_s^{\mathbb{Q}}(E) \\
 & \nearrow \tau^s(p) & \downarrow \\
 & & Q(E_+) \\
 & \nearrow p^! & \downarrow \lambda_E \\
 B & \xrightarrow{c_p} & K(\mathbb{Q})
 \end{array}$$

The lift $\tau^s(p)$ is the smooth torsion of the bundle p .

The homotopy class of $\tau^s(p)$ is an invariant of the smooth structure of p in the following sense. If $p': E' \rightarrow B$ is another smooth bundle and $f: E' \rightarrow E$ is a smooth bundle map then f induces a map

$$f_*: \text{Wh}_s^{\mathbb{Q}}(E') \rightarrow \text{Wh}_s^{\mathbb{Q}}(E).$$

The map $f_*\tau^s(p')$ need not be homotopic to $\tau^s(p)$ in general, but this property does hold provided that f is a fiberwise diffeomorphism of bundles.

The map $\tau^s(p)$ gives rise to a certain cohomology class

$$t^s(p) \in \bigoplus_{k>0} H^{4k}(B; \mathbb{R})$$

(see [1, Definition 4.10]), which we will call the cohomological torsion of the bundle p .

In [11, Section 9] Igusa showed that the cohomological torsion of the composition pq can be, in some cases, computed from the torsion of the bundles p and q . Namely, if q is an oriented linear sphere bundle then

$$(1-1) \quad t^s(pq) = \chi(F_q) t^s(p) + \text{tr}_B^E(t^s(q))$$

where $\chi(F_q) \in \mathbb{Z}$ is the Euler characteristic of the fiber of q and

$$\mathrm{tr}_B^E: H^*(E; \mathbb{R}) \rightarrow H^*(B; \mathbb{R})$$

is the transfer map associated to p . In [11] Igusa calls the formula (1-1) the transfer axiom and shows that taken together with a few other properties it uniquely determines the cohomological torsion.

Igusa’s arguments can be used to show that the formula (1-1) holds under more general conditions on p and q , eg if dimensions of fibers of these bundles have the same parity. In [1, Theorem 7.1] we verified that the same is true in the case when p is an arbitrary unipotent bundle and q satisfies the assumptions of the Leray–Hirsch isomorphism theorem. One of our goals in this paper is to show that this formula holds in general:

1.1 Theorem *The formula (1-1) holds for any unipotent bundles $p: E \rightarrow B$ and $q: D \rightarrow E$.*

In order to prove Theorem 1.1 we develop a new construction of smooth torsion based on the notion of the secondary transfer of unipotent bundles. The starting point for this construction is the following fact:

1.2 Theorem *Given a smooth bundle of compact manifolds $p: E \rightarrow B$ with fiber F_p consider the diagram*

$$(1-2) \quad \begin{array}{ccc} Q(B_+) & \xrightarrow{Q(p^!)} & Q(E_+) \\ \lambda_B \downarrow & & \downarrow \lambda_E \\ K(\mathbb{Q}) & \xrightarrow{\chi(F_p)} & K(\mathbb{Q}) \end{array}$$

where the lower horizontal map is given by the multiplication by the Euler characteristic $\chi(F_p) \in \mathbb{Z}$ of F_p and the upper horizontal map is the Becker–Gottlieb transfer of p . If p is a unipotent bundle then this diagram commutes up to a preferred homotopy

$$\eta_p: Q(B_+) \times [0, 1] \rightarrow K(\mathbb{Q}).$$

As an intermediate step in the proof of this result it will be convenient to work in a more general setting of unipotent fibrations, ie fibrations $p: E \rightarrow B$ satisfying some finiteness assumptions and such that the action of $\pi_1(B)$ on homology of the fiber satisfies the same unipotency condition as in the case of unipotent bundles (see Definition 3.4). We show (Theorem 3.5) that for any unipotent fibration an analog of Theorem 1.2 holds, with spaces $Q(B_+)$ and $Q(E_+)$ replaced with Waldhausen’s

algebraic K -theory spaces $A(B)$ and $A(E)$, and with A -theory transfer taken in place of the Becker–Gottlieb transfer.

For a unipotent bundle $p: E \rightarrow B$ the homotopy η_p defines a map of homotopy fibers

$$\mathrm{Wh}_s^{\mathbb{Q}}(p^!): \mathrm{Wh}_s^{\mathbb{Q}}(B) \rightarrow \mathrm{Wh}_s^{\mathbb{Q}}(E).$$

We call this map the smooth secondary transfer of the bundle p . Likewise, for any unipotent fibration p we construct its homotopy secondary transfer

$$\mathrm{Wh}_h^{\mathbb{Q}}(p^!): \mathrm{Wh}_h^{\mathbb{Q}}(B) \rightarrow \mathrm{Wh}_h^{\mathbb{Q}}(E)$$

where $\mathrm{Wh}_h^{\mathbb{Q}}(B) = \mathrm{hofib}(A(B) \rightarrow K(\mathbb{Q}))$.

The smooth secondary transfer shares some of the basic properties of the Becker–Gottlieb transfer. It is additive ([Theorem 7.3](#)) and it preserves composition of bundles:

1.3 Theorem *If $p: E \rightarrow B$ and $q: D \rightarrow E$ are unipotent bundles then*

$$\mathrm{Wh}_s^{\mathbb{Q}}((pq)^!) \simeq \mathrm{Wh}_s^{\mathbb{Q}}(q^!) \circ \mathrm{Wh}_s^{\mathbb{Q}}(p^!).$$

Analogous additivity and composition properties hold for the homotopy secondary transfer ([Theorem 7.1](#), [Theorem 9.1](#)).

The relationship between the smooth secondary transfer and the smooth torsion is as follows. If B is a compact, smooth manifold then the identity map $\mathrm{id}_B: B \rightarrow B$ is a unipotent bundle. We have:

1.4 Theorem *If $p: E \rightarrow B$ is a unipotent bundle then*

$$\tau^s(p) \simeq \mathrm{Wh}_s^{\mathbb{Q}}(p^!) \circ \tau^s(\mathrm{id}_B).$$

This shows that the smooth secondary transfer of a unipotent bundle determines the smooth torsion of the bundle. Since smooth torsion can distinguish bundles that are fiberwise homotopy equivalent, but not fiberwise diffeomorphic, [Theorem 1.4](#) implies that the smooth secondary transfer carries information about the smooth structure of a bundle. In [Proposition 6.1](#) we show that, in contrast, the homotopy secondary transfer is invariant with respect to fiberwise homotopy equivalences of unipotent fibrations.

Combining [Theorems 1.4](#) and [1.3](#) we obtain:

1.5 Corollary *If $p: E \rightarrow B$ and $q: D \rightarrow E$ are unipotent bundles then*

$$\tau^s(pq) \simeq \mathrm{Wh}_s^{\mathbb{Q}}(q^!) \circ \tau^s(p).$$

Theorem 1.1 can be obtained as a direct consequence of this fact. Notice that in this way we exhibit the simple principle underlying the formula (1-1): the torsion of a composition of bundles p and q is a composition of two maps, one depending on p and the other on q .

1.6 Note As it was pointed out to us by the referee, the main results of this paper bear some resemblance to the work of Lott and Bunke on the secondary K -theory pushforward map. In [16] Lott constructed for a smooth manifold B the secondary K -theory group $\bar{K}^0(B)$ which is generated by flat complex vector bundles over B with trivial Borel classes. He also constructed for a smooth bundle $p: E \rightarrow B$ a pushforward map $p_!: \bar{K}^0(E) \rightarrow \bar{K}^0(B)$. Just as the secondary smooth transfer considered in this paper contains information about the smooth torsion of a bundle, the construction of the pushforward map involves higher analytic torsion forms of Bismut and Lott [3]. Lott's pushforward map was studied by Bunke [6] who showed that it preserves composition of bundles: if $q: D \rightarrow E$ and $p: E \rightarrow B$ are smooth bundles then $(pq)_! = p_!q_!$. This parallels our **Theorem 1.3**. Besides the difference in setting between Lott's and Bunke's results and the ones described in this paper the direction of their work is opposite to ours. While Lott's construction of the pushforward map uses explicitly the analytic torsion form, we construct the secondary transfer first and then show that the smooth torsion of a bundle can be recovered from it. Also, while we use the composition property of the secondary transfer to obtain the composition formula for cohomological torsion (1-1), Bunke derives his result using a theorem of Ma [17] which states that an analog of the formula (1-1) holds for higher analytic torsion.

1.7 Organization of the paper **Section 2** contains a brief review of Waldhausen categories which provide the technical setting for the majority of the constructions in this paper. In **Section 3** we take a closer look at the statements of **Theorem 1.2** and its analog for unipotent fibrations, **Theorem 3.5**. The proof of both of these facts uses a theorem of Brown,¹ which states that the singular chain complex of the total space of a fibration is quasi-isomorphic to a twisted tensor product of the chain complexes of the base and the fiber. In **Section 4** we give an overview of this result and describe some properties of Brown's quasi-isomorphism. In **Section 5** we complete proofs of **Theorems 1.2** and **3.5**, which lets us complete the construction of the secondary transfers $\text{Wh}_s^{\mathbb{Q}}(p^!)$ and $\text{Wh}_h^{\mathbb{Q}}(p^!)$. In **Section 6** we show that the homotopy secondary transfer is a fiberwise homotopy invariant. In **Section 7** we obtain additivity formulas for both smooth and homotopy secondary transfers, and in **Sections 8** and **9** we study secondary transfers of compositions of unipotent bundles and fibrations, which leads us

¹ See also Igusa [12] for a nice description of the relationship of Brown's work to the higher torsion of Igusa and Klein.

to the proof of [Theorem 1.3](#). Finally, in [Section 10](#) we prove the relationship between the smooth secondary transfer and the smooth torsion of a bundle p described by [Theorem 1.4](#). We also show that it implies the statement of [Theorem 1.1](#). Several arguments of the paper involve constructions of maps between homotopy fibers and constructions of homotopies of such maps. The [Appendix](#) gives the basic outline of such constructions.

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2 Technical setup

A great majority of constructions described in this paper are set within the realm of Waldhausen categories [\[20\]](#), ie categories with distinguished classes of weak equivalences and cofibrations that satisfy certain axioms. Our basic setup in this respect will be largely the same as that of [\[1, Section 3\]](#), so we summarize it here only briefly. Given a Waldhausen category \mathcal{C} we will denote by $K(\mathcal{C})$ the K -theory of \mathcal{C} . The standard construction of $K(\mathcal{C})$ proceeds using Waldhausen's \mathcal{S}_\bullet -construction. For our purposes it will be more convenient though to use its variant, the \mathcal{S}'_\bullet -construction described by Blumberg and Mandell in [\[4, Section 2\]](#).

2.1 We will work mainly with two specific instances of Waldhausen categories. For a topological space X the category $\mathcal{R}^{\text{fd}}(X)$ has as its objects homotopy finitely dominated retractive spaces over X , while its morphisms are maps of retractive spaces. It is a Waldhausen category with cofibrations given by closed embeddings having the homotopy extension property and weak equivalences defined as homotopy equivalences. The K -theory of $\mathcal{R}^{\text{fd}}(X)$ is the Waldhausen algebraic K -theory of X and it is denoted by $A(X)$.²

Next, by $Ch^{\text{fd}}(\mathbb{Q})$ we will denote the category of homotopy finitely dominated chain complexes of \mathbb{Q} -vector spaces. This is a Waldhausen category with degreewise monomorphisms as cofibrations and quasi-isomorphisms as weak equivalences. We will denote by $K(\mathbb{Q})$ the K -theory of $Ch^{\text{fd}}(\mathbb{Q})$. This space describes the algebraic K -theory of the field of rational numbers: $K(\mathbb{Q}) \simeq \Omega\text{BGL}(\mathbb{Q})^+$.

²If X is a path connected space then by abuse of notation by $\mathcal{R}^{\text{fd}}(X)$ we will understand the Waldhausen category of *path connected* retractive spaces over X . From the perspective of K -theory this change is of little consequence: the functor that embeds this category into the category of all retractive spaces over X induces a homotopy equivalence of the associated K -theory spaces.

In order to obtain a convenient combinatorial model of the space $Q(X_+)$ we will use one more instance of a construction of K -theory which, while it does not come from a Waldhausen category, is very closely related to the S_\bullet -construction. We outline it briefly here and refer to Waldhausen's papers [19; 21] where it was originally developed and [2] for details. Let X be a smooth compact manifold. A partition of $X \times I$ is a codimension-0 submanifold $P \subseteq X \times I$ such that $X \times [0, \frac{1}{3}] \subseteq P$ and that satisfies some further conditions. By $\mathcal{P}_0(X \times I)$ we will denote the poset of partitions ordered by inclusion. Similarly, for $k \geq 0$ we define $\mathcal{P}_k(X \times I)$ to be the poset of locally trivial bundles of partitions over the standard simplex Δ^k . These categories taken together form a simplicial category $\mathcal{P}_\bullet(X \times I)$. In each category $\mathcal{P}_k(X \times I)$ we can introduce an analog of the Waldhausen category structure where, roughly, every morphism is a cofibration and weak equivalences are the identity morphisms. While in general pushouts do not exist in $\mathcal{P}_k(X \times I)$ and so the S_\bullet -construction cannot be performed in this setting, it is possible to use its variant, the \mathcal{T}_\bullet -construction, to obtain a space $|\mathcal{T}_\bullet \mathcal{P}_k(X \times I)|$. Denote by $|\mathcal{T}_\bullet \mathcal{P}_\bullet(X \times I)|$ the space obtained by applying the \mathcal{T}_\bullet -construction to each category $\mathcal{P}_k(X \times I)$ and then taking the geometrical realization of the resulting simplicial space. The final step in the construction of $Q(X_+)$ is stabilization. It is obtained by means of maps $|\mathcal{T}_\bullet \mathcal{P}_\bullet(X \times I^m)| \rightarrow |\mathcal{T}_\bullet \mathcal{P}_\bullet(X \times I^{m+1})|$ that are induced by functors $\mathcal{P}_k(X \times I^m) \rightarrow \mathcal{P}_k(X \times I^{m+1})$ that, roughly, send a partition $P \subseteq X \times I^m$ to $P \times I \subseteq X \times I^{m+1}$. Waldhausen showed that there exists a weak equivalence

$$Q(X_+) \simeq \Omega \operatorname{colim}_m |\mathcal{T}_\bullet \mathcal{P}_\bullet(X \times I^m)|.$$

2.2 A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ of Waldhausen categories is exact if it preserves weak equivalences, cofibrations and pushouts of diagrams

$$(2-1) \quad c' \leftarrow c \rightarrow c''$$

where one of the morphisms is a cofibration. Any such functor induces a map of infinite loop spaces $K(F): K(\mathcal{C}) \rightarrow K(\mathcal{D})$. The advantage of working with the S'_\bullet -construction is that we can obtain the map $K(F)$ under more relaxed assumptions on the functor F . Namely, following the terminology of [1, Definition 3.4] we will say that a functor F is almost exact if it preserves weak equivalences and cofibrations and preserves pushouts of diagrams (2-1) up to a weak equivalence. An almost exact functor induces a functor of simplicial categories $F: S'_\bullet \mathcal{C} \rightarrow S'_\bullet \mathcal{D}$, and so it yields a map $K(F): K(\mathcal{C}) \rightarrow K(\mathcal{D})$.

2.3 As we have already mentioned exact and almost exact functors between Waldhausen categories define maps between their associated K -theories. We will frequently need to construct homotopies of such maps. There are two main sources of such constructions. First, if $F, G: \mathcal{C} \rightarrow \mathcal{D}$ are (almost) exact functors then a natural weak

equivalence $\varphi: F \Rightarrow G$ defines a homotopy $K(\varphi)$ between the induced maps $K(F)$ and $K(G)$. Second, if $F_i: \mathcal{C} \rightarrow \mathcal{D}$ are (almost) exact functors for $i = 0, 1, 2$ and $\varphi: F_0 \Rightarrow F_1$ and $\psi: F_1 \Rightarrow F_2$ are natural transformations such that

$$F_0(c) \xrightarrow{\varphi} F_1(c) \xrightarrow{\psi} F_2(c)$$

is a cofibration sequence for each $c \in \mathcal{C}$, then the additivity theorem of Waldhausen [20, Theorem 1.4.2] provides a homotopy $\mathfrak{U}: K(\mathcal{C}) \times I \rightarrow K(\mathcal{D})$ between the maps $K(F_1)$ and $K(F_0 \vee F_2)$. A convenient combinatorial construction of this homotopy was given by Grayson in [8].

2.4 Our work will require us to go a step further beyond homotopies and consider homotopies of homotopies. If $f_0, f_1: X \rightarrow Y$ are two maps between topological spaces and $h_0, h_1: X \times I \rightarrow Y$ are homotopies between f_0 and f_1 then by a homotopy of homotopies we will understand a map $H: X \times I \times I \rightarrow Y$ such that for each $t \in I$ the map $H_t = H(-, -, t)$ is a homotopy between the maps f_0 and f_1 with $H_0 = h_0$ and $H_1 = h_1$. In our constructions of homotopies of homotopies we will almost always have $Y = K(\mathbb{Q})$. The homotopies of homotopies we will consider will be obtained as an application of one of the following lemmas:

2.5 Lemma *Let \mathcal{C} be a Waldhausen category, let $F_1, F_2: \mathcal{C} \rightarrow Ch^{fd}(\mathbb{Q})$ be almost exact functors, and let $\varphi_1, \varphi_2: F_1 \Rightarrow F_2$ be natural weak equivalences. If Φ is a natural chain homotopy between φ_1 and φ_2 then Φ defines a homotopy of homotopies $K(\Phi)$ between $K(\varphi_1)$ and $K(\varphi_2)$.*

Proof We start with the following observation. Assume that we are given three almost exact functors $G_i: \mathcal{C} \rightarrow Ch^{fd}(\mathbb{Q})$, $i = 0, 1, 2$ and two natural weak equivalences $\psi_0: G_0 \Rightarrow G_1$ and $\psi_1: G_1 \Rightarrow G_2$. In such a situation we obtain a map

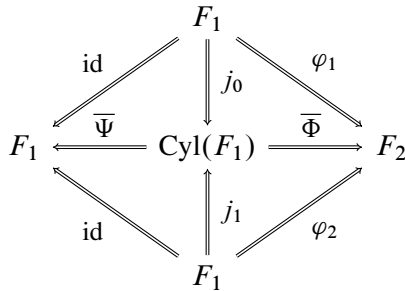
$$K(\mathcal{C}) \times \Delta^2 \rightarrow K(\mathbb{Q}).$$

If we consider it as a family of maps $K(\mathcal{C}) \rightarrow K(\mathbb{Q})$ parametrized by Δ^2 then vertices of Δ^2 correspond to the functions $K(G_i)$ and edges of Δ^2 correspond to the homotopies $K(\psi_0)$, $K(\psi_1)$ and $K(\psi_1\psi_0)$.

For a chain complex C , let $Cyl(C)$ denote the mapping cylinder of the identity function $id: C \rightarrow C$ [22, Section 1.5.5]. We have $Cyl(C)_n = C_n \oplus C_{n-1} \oplus C_n$. Let the maps $j_0, j_1: C \rightarrow Cyl(C)$ be the chain maps given by $j_0(c) = (c, 0, 0)$ and $j_1(c) = (0, 0, c)$. Recall that two chain maps $f_0, f_1: C \rightarrow D$ are chain homotopic if and only if there exists a chain map $h: Cyl(C) \rightarrow D$ such that $hj_i = f_i$ for $i = 0, 1$.

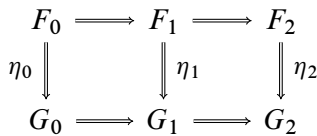
Going back to the setting of the lemma the natural chain homotopy Φ gives a natural weak equivalence of functors $\overline{\Phi}: Cyl(F_1) \Rightarrow F_2$. Let $\overline{\Psi}: Cyl(F_1) \Rightarrow F_1$ denote the

natural weak equivalence corresponding to the natural chain homotopy from the identity natural transformation $\text{id}: F_1 \Rightarrow F_1$ to itself. We obtain the following commutative diagram:



Each vertex of this diagram corresponds to a functor $\mathcal{C} \rightarrow \mathcal{C}h^{\text{fd}}(\mathbb{Q})$ and edges are natural weak equivalences of such functors. By the observation above each commutative triangle in this diagram induces a map $K(\mathcal{C}) \times \Delta^2 \rightarrow K(\mathbb{Q})$. Taken together these maps define a map $H: K(\mathcal{C}) \times I^2 \rightarrow K(\mathbb{Q})$. Considering H as a family of functions $K(\mathcal{C}) \rightarrow K(\mathbb{Q})$ parametrized by I^2 we obtain that two adjacent edges of the square I^2 parametrize the homotopies $K(\varphi_1)$ and $K(\varphi_2)$ and each of remaining two edges parametrizes the homotopy defined by the identity natural transformation $\text{id}: F_1 \Rightarrow F_1$. Using the identity $\varphi_i \text{id} = \varphi_i$ we can further modify H to a homotopy of homotopies between $K(\varphi_1)$ and $K(\varphi_2)$. □

2.6 Lemma *Assume that $F_i, G_i: \mathcal{C} \rightarrow \mathcal{C}h^{\text{fd}}(\mathbb{Q})$, for $i = 0, 1, 2$, are almost exact functors, and that we have a commutative diagram of natural transformations*



where both rows are cofibration sequences and vertical arrows are natural weak equivalences. This yields a diagram:

$$(2-2) \quad \begin{array}{ccc}
 K(F_1) & \xrightarrow{\mathcal{U}} & K(F_0 \vee F_2) \\
 K(\eta_1) \downarrow & & \downarrow K(\eta_0 \vee \eta_2) \\
 K(G_1) & \xrightarrow{\mathcal{U}} & K(G_0 \vee G_2)
 \end{array}$$

In this diagram every vertex represents a map $K(\mathcal{C}) \rightarrow K(\mathbb{Q})$ and each edge is a homotopy of such maps. In this setting there exists a homotopy of homotopies that

fills this diagram, ie a homotopy of homotopies between the concatenation of \mathfrak{U} with $K(\eta_0 \vee \eta_2)$ and the concatenation of $K(\eta_1)$ with \mathfrak{U} .

Proof Let $\mathcal{ECh}^{\text{fd}}(\mathbb{Q})$ be the Waldhausen category of short exact sequences in $\mathcal{Ch}^{\text{fd}}(\mathbb{Q})$. The functors F_i and G_i define functors $F, G: \mathcal{C} \rightarrow \mathcal{ECh}^{\text{fd}}(\mathbb{Q})$, and the natural weak equivalences η_i define a natural weak equivalence $\eta: F \Rightarrow G$. On the level of K -theory this yields a homotopy

$$K(\eta): K(\mathcal{C}) \times I \rightarrow K(\mathcal{ECh}^{\text{fd}}(\mathbb{Q})).$$

For $i = 0, 1, 2$, let $\text{ev}_i: \mathcal{ECh}^{\text{fd}}(\mathbb{Q}) \rightarrow \mathcal{Ch}^{\text{fd}}(\mathbb{Q})$ denote the functor given by

$$\text{ev}_i(C_0 \rightarrow C_1 \rightarrow C_2) = C_i.$$

The Waldhausen additivity theorem can be equivalently stated by saying that there exists a homotopy

$$\mathcal{E}\mathfrak{U}: K(\mathcal{ECh}^{\text{fd}}(\mathbb{Q})) \times I \rightarrow K(\mathbb{Q})$$

between the maps $K(\text{ev}_1)$ and $K(\text{ev}_0 \vee \text{ev}_2)$. The homotopy of homotopies in the diagram (2-2) is obtained by composing the map

$$K(\eta) \times \text{id}_I: K(\mathcal{C}) \times I \times I \rightarrow K(\mathcal{ECh}^{\text{fd}}(\mathbb{Q})) \times I$$

with $\mathcal{E}\mathfrak{U}$. □

2.7 Lemma Consider the following commutative diagram of almost exact functors $\mathcal{C} \rightarrow \mathcal{Ch}^{\text{fd}}(\mathbb{Q})$ and their natural transformations:

$$(2-3) \quad \begin{array}{ccccc} A_0 & \longrightarrow & B_0 & \longrightarrow & C_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 \end{array}$$

Assume that each row and each column of this diagram is a short exact sequence of functors and that the canonical natural transformation $A_1 \cup_{A_0} B_0 \Rightarrow B_1$ is a cofibration. In this situation we obtain two homotopies between the map $K(B_1)$ and the map $K(A_0) + K(C_0) + K(A_2) + K(C_2)$:

- (1) the homotopy \mathfrak{U}_{rc} , which is obtained by applying the additivity theorem to the middle row, which produces a homotopy $K(B_1) \simeq K(A_1) + K(C_1)$, and then applying the additivity theorem to the left and right columns, which gives a homotopy $K(A_1) + K(C_1) \simeq K(A_0) + K(A_2) + K(C_0) + K(C_2)$;

- (2) the homotopy \mathfrak{U}_{cr} , which is obtained in the same manner, but applying additivity to the middle column first, and then to the top and bottom rows.

There exists a homotopy of homotopies between \mathfrak{U}_{rc} and \mathfrak{U}_{cr} .

Proof Let $\mathcal{E}Ch^{fd}(\mathbb{Q})$ be the Waldhausen category of short exact sequences in $Ch^{fd}(\mathbb{Q})$ and let $\mathcal{E}_2Ch^{fd}(\mathbb{Q})$ be the Waldhausen category of short exact sequences in $\mathcal{E}Ch^{fd}(\mathbb{Q})$. The diagram (2-3) can be interpreted as a functor $F: \mathcal{C} \rightarrow \mathcal{E}_2Ch^{fd}(\mathbb{Q})$ while columns of this diagram define functors $A, B, C: \mathcal{C} \rightarrow \mathcal{E}Ch^{fd}(\mathbb{Q})$. The additivity theorem applied to $\mathcal{E}_2(Ch^{fd}(\mathbb{Q}))$ gives a homotopy $\mathcal{E}_2\mathfrak{U}: K(\mathcal{E}_2Ch^{fd}(\mathbb{Q})) \rightarrow K(\mathcal{E}Ch^{fd}(\mathbb{Q}))$ such that the composition

$$K(\mathcal{C}) \times I \xrightarrow{K(F) \times id_I} K(\mathcal{E}_2Ch^{fd}(\mathbb{Q})) \times I \xrightarrow{\mathcal{E}_2\mathfrak{U}} K(\mathcal{E}Ch^{fd}(\mathbb{Q}))$$

is a homotopy between the maps $K(B)$ and $K(A) + K(C)$. Let

$$\mathcal{E}\mathfrak{U}: K(\mathcal{E}Ch^{fd}(\mathbb{Q})) \times I \rightarrow K(\mathcal{E}Ch^{fd}(\mathbb{Q}))$$

denote the additivity homotopy described in the proof of Lemma 2.6. The homotopy of homotopies between \mathfrak{U}_{rc} and \mathfrak{U}_{cr} is then given by the composition

$$K(\mathcal{C}) \times I^2 \xrightarrow{K(F) \times id_{I^2}} K(\mathcal{E}_2Ch^{fd}(\mathbb{Q})) \times I^2 \xrightarrow{\mathcal{E}_2\mathfrak{U} \times id_I} K(\mathcal{E}Ch^{fd}(\mathbb{Q})) \times I \xrightarrow{\mathcal{E}\mathfrak{U}} K(\mathbb{Q}). \quad \square$$

3 The linearization and transfer maps

3.1 In preparation for the proof of Theorem 1.2 we start this section by reviewing briefly the construction of the linearization map $\lambda_B: Q(B_+) \rightarrow K(\mathbb{Q})$. For an arbitrary space B we have the assembly map $a_B: Q(B_+) \rightarrow A(B)$. The combinatorial construction of this map can be outlined as follows. Recall (Section 2.1) that we are working with a model of $Q(B_+)$ built using the category of partitions, and that the space $A(B)$ is constructed using the category $\mathcal{R}^{fd}(B)$. By definition a partition $P \subseteq B \times I$ contains $B \times \{0\}$ as a subspace which gives P the structure of a retractive space over B . In this way we can regard P as an object of $\mathcal{R}^{fd}(B)$. It is possible to extend this assignment to all parametrized and stabilized partitions (using parametrized and stabilized retractive spaces over B as an intermediate step). The map a_B is induced by this assignment. We refer to [2, Section 3] for details.

Next, recall that $K(\mathbb{Q})$ was constructed from the Waldhausen K category of chain complexes $Ch^{fd}(\mathbb{Q})$. Consider the functor

$$\Lambda_B: \mathcal{R}^{fd}(B) \rightarrow Ch^{fd}(\mathbb{Q})$$

that assigns to a retractive space X the relative singular chain complex $C_*(X, B)$. This functor is almost exact, so it induces a map $\lambda_B^h: A(B) \rightarrow K(\mathbb{Q})$. We will call λ_B^h the A -theory linearization. The linearization map $\lambda_B: Q(B_+) \rightarrow K(\mathbb{Q})$ is given by the composition

$$\lambda_B = \lambda_B^h a_B.$$

3.2 Assume that we have a fibration $p: E \rightarrow B$. For a retractive space $X \in \mathcal{R}^{\text{fd}}(B)$ let p^*X denote the pullback

$$p^*X := \lim(X \rightarrow B \xleftarrow{p} E).$$

The assignment $X \mapsto p^*X$ defines an exact functor $\mathcal{R}^{\text{fd}}(B) \rightarrow \mathcal{R}^{\text{fd}}(E)$. We will call the induced map $A(p^!): A(B) \rightarrow A(E)$ the A -theory transfer of p .

If p is a smooth bundle of manifolds then in a similar way we can define a map

$$Q(p^!): Q(B_+) \rightarrow Q(E_+).$$

This map is induced by functors of categories of partitions $\mathcal{P}_k(B \times I^m) \rightarrow \mathcal{P}_k(E \times I^m)$ that, roughly, associate to a partition $P \subseteq B \times I^m$ the partition $(p \times \text{id})^{-1}(P) \subseteq E \times I^m$. The map $Q(p^!)$ obtained in this way coincides with the Becker–Gottlieb transfer [2, Section 4].

3.3 Let $p: E \rightarrow B$ be a fibration with a homotopy finitely dominated fiber F_p . The maps described above can be assembled into a diagram:

$$(3-1) \quad \begin{array}{ccc} Q(B_+) & \xrightarrow{Q(p^!)} & Q(E_+) \\ a_B \downarrow & & \downarrow a_E \\ A(B) & \xrightarrow{A(p^!)} & A(E) \\ \lambda_B^h \downarrow & & \downarrow \lambda_E^h \\ K(\mathbb{Q}) & \xrightarrow{\chi(F_p)} & K(\mathbb{Q}) \end{array}$$

The outer rectangle in this diagram coincides with the diagram (1-2). If $p: E \rightarrow B$ is a smooth bundle then directly from the constructions described above it follows that the upper square commutes up to a homotopy induced by the natural transformation that for a partition $P \subseteq B \times I$ is given by the isomorphism $(p \times \text{id})^{-1}(P) \rightarrow p^*P$

of retractive spaces over E . We will denote this homotopy by μ_p .³ In order to obtain [Theorem 1.2](#) it is then enough to show that the lower square of (3-1) is homotopy commutative. We will show that this fact holds for any unipotent fibration p :

3.4 Definition A fibration $p: E \rightarrow B$ is *unipotent* if B is a path connected space, both B and the fiber F_p of p have the homotopy type of a finite CW-complex, and $H_*(F_p)$ admits a filtration by $\pi_1 B$ -modules such that the action of $\pi_1 B$ on the filtration quotients is trivial.

3.5 Theorem Let $p: E \rightarrow B$ be a unipotent fibration with fiber F_p . The diagram

$$(3-2) \quad \begin{array}{ccc} A(B) & \xrightarrow{A(p^!)} & A(E) \\ \lambda_B^h \downarrow & & \downarrow \lambda_E^h \\ K(\mathbb{Q}) & \xrightarrow{\chi(F_p)} & K(\mathbb{Q}) \end{array}$$

commutes up to a preferred homotopy $\eta_p^h: A(B) \times I \rightarrow K(\mathbb{Q})$.

In the next section we describe some technical tools that we will use in the proof of this fact. The proof itself is given in [Section 5](#).

4 Twisted tensor products

Consider the diagram (3-2). The lower horizontal map in this diagram can be described combinatorially as follows. Let $H_*(F_p)$ be the chain complex of rational homology groups of F_p with trivial differentials. The functor

$$- \otimes H_*(F_p): Ch^{\text{fd}}(\mathbb{Q}) \rightarrow Ch^{\text{fd}}(\mathbb{Q})$$

is exact and $\chi(F_p): K(\mathbb{Q}) \rightarrow K(\mathbb{Q})$ is the map induced by this functor. As a consequence the map $\chi(F_p)\lambda_B^h$ comes from the functor $\mathcal{R}^{\text{fd}}(B) \rightarrow Ch^{\text{fd}}(\mathbb{Q})$ that associates to a retractive space X the chain complex $C_*(X, B) \otimes H_*(F_p)$. On the other hand the composition $\lambda_E^h A(p^!)$ is induced by the functor $\mathcal{R}^{\text{fd}}(B) \rightarrow Ch^{\text{fd}}(\mathbb{Q})$ that assigns to a space X the chain complex $C_*(p^*X, E)$. The main ingredient of the proof of [Theorem 3.5](#) is the fact that under the assumption that $p: E \rightarrow B$ is a unipotent fibration

³While all maps in the upper square of (3-1), ie the Becker–Gottlieb transfer, the A -theory transfer and assembly maps, are defined for any fibration, smoothness of p is essential for commutativity. This diagram does not commute, in general, when p is a fibration. See eg the proof of [[14](#), Theorem F].

for any $X \in \mathcal{R}^{\text{fd}}(B)$ we can construct a path in $K(\mathbb{Q})$ joining the points represented by these two chain complexes. This path is natural to the extent that it gives rise to a homotopy filling the diagram (3-2). Our main tool in the construction of this path will be the theorem of Brown [5] which shows that the chain complex of the total space of a fibration is quasi-isomorphic to a twisted tensor product of the chain complex of the base and the homology of the fiber. We begin this section by reviewing the relevant notions in homological algebra. Subsequently we describe Brown’s result and develop some of its properties that we will need later on.

4.1 Twisting cochains and twisted tensor products Let A be a differential graded \mathbb{Q} -algebra with multiplication $\mu: A \otimes A \rightarrow A$, and let K be a dg \mathbb{Q} -coalgebra with comultiplication $\nabla: K \rightarrow K \otimes K$. Given homomorphisms of graded vector spaces $\varphi, \psi: K \rightarrow A$, the cup product $\varphi \cup \psi: K \rightarrow A$ is given by the formula

$$\varphi \cup \psi := \mu(\varphi \otimes \psi)\nabla.$$

If M is a left A -module with multiplication $\nu: A \otimes M \rightarrow M$ then for φ as above and $c \in K \otimes M$ the cap product $\varphi \cap c \in K \otimes M$ is given by

$$\varphi \cap c := (\text{id}_K \otimes \nu)(\text{id}_K \otimes \varphi \otimes \text{id}_M)(\nabla \otimes \text{id}_M)(c).$$

For a fixed φ the map

$$\varphi \cap -: K \otimes M \rightarrow K \otimes M$$

is a homomorphism of graded vector spaces.

4.2 Definition Let A , K and M be respectively a dg \mathbb{Q} -algebra, a dg \mathbb{Q} -coalgebra and a left A -module as above.

(i) A twisting cochain is a homomorphism of graded vector spaces $\varphi: K \rightarrow A$ that lowers grading by 1 and satisfies the identity

$$\partial\varphi - \varphi\partial + \varphi \cup \varphi = 0.$$

(ii) If $\varphi: K \rightarrow A$ is a twisting cochain then the twisted tensor product $K \otimes_{\varphi} M$ is a chain complex such that $K \otimes_{\varphi} M = K \otimes M$ as a graded vector space, and the differential in $K \otimes_{\varphi} M$ is given by

$$(4-1) \quad \partial_{\varphi} := \partial \otimes \text{id} + \text{id} \otimes \partial + \varphi \cap -.$$

4.3 Twisted chain complex of a fibration Let X be a topological space with a basepoint x_0 . By $C'_*(X)$ we will denote the subcomplex of the singular chain complex $C_*(X)$ with coefficients in \mathbb{Q} generated by all singular simplices $\sigma: \Delta^n \rightarrow X$ that send all vertices of Δ^n into x_0 . If X is a path connected space then the inclusion

$C'_*(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence. The chain complex $C'_*(X)$ can be equipped with the usual dg coalgebra structure with comultiplication

$$\nabla: C'_*(X) \rightarrow C'_*(X) \otimes C'_*(X)$$

defined by

$$\nabla(\sigma) = \sum_{i=0}^n f_i(\sigma) \otimes b_{n-i}(\sigma)$$

where $\sigma \in C'_n(X)$ is a singular n -simplex and $f_i(\sigma)$ and $b_{n-i}(\sigma)$ denote, respectively, the front i^{th} face and the back $(n-i)^{\text{th}}$ face of σ .

For a space X we can also consider its associated dg homology algebra $\text{End}(H_*(X))$ defined as follows. Let $\text{End}_n(H_*(X))$ denote the vector space of all maps of graded vector spaces $H_*(X) \rightarrow H_*(X)$ that increase the grading by n , and let

$$\text{End}(H_*(X)) = \bigoplus_{n \geq 0} \text{End}_n(H_*(X)).$$

We view $\text{End}(H_*(X))$ as a chain complex with trivial differentials. The dg algebra structure on $\text{End}(H_*(X))$ comes from composition of maps. Naturally $H_*(X)$ is a module over this dg algebra.

The main result of [5] says that given a fibration $p: E \rightarrow B$ with a path connected base space and a fiber F_p we can find a twisting cochain $\varphi_p: C'_*(B) \rightarrow \text{End}(H_*(F_p))$ such that the twisted tensor product $C'_*(B) \otimes_{\varphi_p} H_*(F_p)$ is naturally quasi-isomorphic to $C'_*(E)$. For our purposes it will be convenient to state this fact in the following form. Let \mathcal{S}_* denote the category of path connected, pointed spaces, and $\mathcal{S}_* \downarrow B$ be the over category of \mathcal{S}_* over a space B . Given a fibration $p: E \rightarrow B$ and an object $X \in \mathcal{S}_* \downarrow B$ denote by $p_X: p^*X \rightarrow X$ the fibration induced from p .

4.4 Theorem *Let $p: E \rightarrow B$ be a fibration with a path connected, pointed base space B and a fiber F_p .*

- (1) *For every $X \in \mathcal{S}_* \downarrow B$ there exists a twisting cochain*

$$\varphi_{p_X}: C'_*(X) \rightarrow \text{End}(H_*(F_p))$$

and a quasi-isomorphism

$$\beta_{p_X}: C_*(p^*X) \xrightarrow{\cong} C'_*(X) \otimes_{\varphi_{p_X}} H_*(F_p).$$

- (2) *On $C'_1(X)$ the twisting cochain $\varphi_{p_X}: C'_1(X) \rightarrow \text{End}_0(H_*(F_p))$ is given as follows. If σ is a singular simplex in $C'_1(X)$ then σ is a loop in X , and so it*

represents an element $[\sigma] \in \pi_1 X$. For $z \in H_*(F_p)$ we have

$$\varphi_{p_X}(\sigma)(z) = [\sigma]z - z$$

where the product $[\sigma]z$ is defined by the action of $\pi_1 X$ on $H_*(F_p)$.

(3) The assignment $X \mapsto C'_*(X) \otimes_{\varphi_{p_X}} H_*(F_p)$ defines a functor

$$F: \mathcal{S}_* \downarrow B \rightarrow \text{Ch}(\mathbb{Q}),$$

where $\text{Ch}(\mathbb{Q})$ is the category of chain complexes over \mathbb{Q} . If $f: X \rightarrow Y$ is a morphism in $\mathcal{S}_* \downarrow B$ then $F(f) = f_* \otimes \text{id}_{H_*(F_p)}$.

(4) The quasi-isomorphisms β_{p_X} define a natural transformation of functors.

4.5 Notation For simplicity from now on we will write $C'_*(X) \otimes_{p_X} H_*(F_p)$ to denote the complex $C'_*(X) \otimes_{\varphi_{p_X}} H_*(F_p)$.

4.6 While we refer to Brown’s paper [5] for the proof of **Theorem 4.4**, a few comments will be useful later on. Brown constructs the quasi-isomorphisms β_{p_X} in two stages. First, he shows that given a path connected space X one can construct a twisting cochain $\psi_X: C'_*(X) \rightarrow C_*(\Omega X)$ [5, Theorem 4.1], where the dg algebra structure on $C_*(\Omega X)$ is obtained by composing the Eilenberg–Zilber map and the map $C_*(\Omega X \times \Omega X) \rightarrow C_*(\Omega X)$ induced by the loop multiplication. The twisting cochain ψ_X depends on the space X only, not on a fibration over X . Moreover, the construction of ψ_X is natural on the category of path connected spaces. The action of ΩX on the fiber F_p of p_X defines a $C_*(\Omega X)$ –module structure on $C_*(F_p)$. Brown shows [5, Theorem 4.2] that the twisted tensor product $C'_*(X) \otimes_{\psi_X} C_*(F_p)$ is chain homotopy equivalent to $C_*(p^* X)$ via a chain homotopy equivalence natural in X . A minor technical point here is that in order to get a suitable action of ΩX on F_p one needs to specify a weakly transitive lifting function for the fibration p . This can be taken care of by first replacing the fibration $p: E \rightarrow B$ by the homotopy equivalent fibration $\tilde{p}: PB \times_B E \rightarrow B$ where PB is the space of Moore paths in B . The fibration \tilde{p} admits a canonical lifting function [5, page 225] which can be used to get a chain homotopy equivalence

$$C_*(p^* X) \xrightarrow{\cong} C_*(\tilde{p}^* X) \xrightarrow{\cong} C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}})$$

where $F_{\tilde{p}}$ is the fiber of \tilde{p} . Since $F_{\tilde{p}} \simeq F_p$ we have $C_*(F_{\tilde{p}}) \simeq C_*(F_p)$.⁴

⁴In [5] Brown gives a quasi-isomorphism $C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}) \xrightarrow{\cong} C_*(\tilde{p}^* X)$ going in the opposite direction. However, since his argument relies on the method of acyclic models it also produces a natural homotopy inverse of that map, and we work here with this inverse for convenience.

To complete the construction of β_{p_X} it suffices to show that we have quasi-isomorphisms

$$C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}) \xrightarrow{\cong} C'_*(X) \otimes_{p_X} H_*(F_p).$$

One can proceed as follows. Using the fact that we deal here with chain complexes over a field we can find chain maps

$$j_{F_p}: H_*(F_p) \rightleftarrows C_*(F_{\tilde{p}}) : r_{F_p}$$

such that for $z \in H_*(F_p) \cong H_*(F_{\tilde{p}})$ the element $j_{F_p}(z) \in C_*(F_{\tilde{p}})$ is a chain representing z , the composition $r_{F_p} j_{F_p}$ is equal to $\text{id}_{H_*(F_p)}$, and $j_{F_p} r_{F_p} \simeq \text{id}_{C_*(F_{\tilde{p}})}$. The maps j_{F_p} and r_{F_p} define a strong deformation retraction of (untwisted) tensor products

$$\text{id} \otimes j_{F_p}: C'_*(X) \otimes H_*(F_p) \rightleftarrows C'_*(X) \otimes C_*(F_{\tilde{p}}) : \text{id} \otimes r_{F_p}.$$

The basic perturbation lemma (see eg [15, Lemma 2.6]) shows that in such a situation there is a twisting cochain $\varphi_{p_X}: C'_*(X) \rightarrow \text{End}(H_*(F_p))$ and a strong deformation retraction of twisted tensor products

$$(\text{id} \otimes j_{F_p})^\infty: C'_*(X) \otimes_{p_X} H_*(F_p) \rightleftarrows C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}) : (\text{id} \otimes r_{F_p})^\infty.$$

Following our convention (Notation 4.5) by $C'_*(X) \otimes_{p_X} H_*(F_p)$ we denote here the complex $C'_*(X) \otimes_{\varphi_{p_X}} H_*(F_p)$. We define β_{p_X} as the composition

$$\beta_{p_X}: C_*(p^*X) \longrightarrow C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}) \xrightarrow{(\text{id} \otimes r_{F_p})^\infty} C'_*(X) \otimes_{p_X} H_*(F_p).$$

4.7 Homological filtration Let $\varphi: K \rightarrow A$ be a twisting cochain and let M be an A -module. Directly from the definition of a twisted tensor product it follows that the chain complex $K \otimes_\varphi M$ admits an increasing filtration

$$U_0 \subset U_1 \subset \dots \subset K \otimes_\varphi M$$

where $U_n := (\bigoplus_{q \leq n} K_q) \otimes_\varphi M$. In the case where M has trivial differentials we have also a decreasing filtration

$$K \otimes_\varphi M = L_0 \supset L_1 \supset \dots$$

given by $L_n := K \otimes_\varphi (\bigoplus_{q \geq n} M_q)$. Since we will consider this filtration in the situation where M is the homology of some chain complex we will call it the homological filtration of $K \otimes_\varphi M$.

Let $p: E \rightarrow B$ be a fibration with fiber F_p , let $X \in \mathcal{S}_* \downarrow B$, and let $\{L_n(p_X)\}$ denote the homological filtration of the chain complex $C'_*(X) \otimes_{p_X} H_*(F_p)$. We will need

an explicit description of the quotients $L_n(pX)/L_{n+1}(pX)$. On the level of graded vector spaces we have isomorphisms

$$L_n(pX)/L_{n+1}(pX) \cong C'_*(X) \otimes H_n(F_p).$$

In order to describe the differential in the filtration quotients notice that the differential in $C'_*(X) \otimes_{pX} H_*(F_p)$ is given by

$$\partial(\sigma \otimes z) = \partial\sigma \otimes z + \sum_{i=0}^n f_i(\sigma) \otimes \varphi_{pX}(b_{n-i}(\sigma))(z)$$

where σ is a singular simplex in $C'_*(X)$, $z \in H_*(F_p)$, and $f_i(\sigma)$ and $b_{n-i}(\sigma)$ are, respectively, the i^{th} front face and the $(n-i)^{\text{th}}$ back face of σ . Using part 2) of [Theorem 4.4](#) we get from here

$$\partial(\sigma \otimes z) = \partial\sigma \otimes z + f_{n-1}(\sigma) \otimes ([b_1(\sigma)]z - z) \pmod{L_{n+1}(pX)}.$$

As a consequence we obtain:

4.8 Proposition *Let $p: E \rightarrow B$ be a fibration with a fiber F_p . For $X \in \mathcal{S}_* \downarrow B$ let $C'_*(X) \otimes_{\pi_1 X} H_n(F_p)$ denote the chain complex such that*

$$(C'_*(X) \otimes_{\pi_1 X} H_n(F_p))_k = C'_{k-n}(X) \otimes H_n(F_p)$$

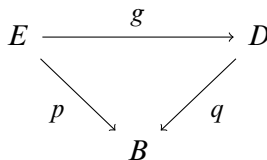
with differential given by

$$\partial(\sigma \otimes z) = \partial\sigma \otimes z + f_{n-1}(\sigma) \otimes ([b_1(\sigma)]z - z)$$

for a singular simplex $\sigma \in C'_*(X)$ and $z \in H_n(F_p)$. We have a canonical isomorphism

$$C'_*(X) \otimes_{\pi_1 X} H_n(F_p) \cong L_n(pX)/L_{n+1}(pX).$$

4.9 Maps of fibrations Assume that we have a map of fibrations over B :



For $X \in \mathcal{S}_* \downarrow B$ let $g_X: p^*X \rightarrow q^*X$ be the map of the induced fibrations over X . Consider the diagram

$$\begin{array}{ccc}
 C_*(p^*X) & \xrightarrow{g_X^*} & C_*(q^*X) \\
 \beta_{p_X} \downarrow & & \downarrow \beta_{q_X} \\
 C'_*(X) \otimes_{p_X} H_*(F_p) & \dashrightarrow & C'_*(X) \otimes_{q_X} H_*(F_q)
 \end{array}$$

where F_p and F_q denote, respectively, the fibers of p and q . We would like to construct a natural lower horizontal map such that the resulting diagram commutes up to a homotopy. The obvious candidate for such map is $\text{id} \otimes (g|_{F_p})_*$, where the homomorphism $(g|_{F_p})_*: H_*(F_p) \rightarrow H_*(F_q)$ is induced by restriction of g to the fibers, but this map is not a chain map in general. We can, however, proceed as follows. By the construction of quasi-isomorphisms β_{p_X} and β_{q_X} (Section 4.6) we have a diagram:

$$\begin{array}{ccc}
 C_*(p^*X) & \xrightarrow{g_X^*} & C_*(q^*X) \\
 \simeq \downarrow & & \downarrow \simeq \\
 C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}) & \xrightarrow{\text{id} \otimes (g|_{\tilde{F}_p})_*} & C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{q}}) \\
 (\text{id} \otimes r_{F_p})^\infty \downarrow & & \downarrow (\text{id} \otimes r_{F_q})^\infty \\
 C'_*(X) \otimes_{p_X} H_*(F_p) & \dashrightarrow^{g_X^\infty} & C'_*(X) \otimes_{q_X} H_*(F_q)
 \end{array}
 \tag{4-2}$$

The compositions of the vertical maps give β_{p_X} and β_{q_X} . The upper square commutes by the naturality properties of Brown’s theorem [5, Theorem 4.2]. Recall that the map $(\text{id} \otimes r_{F_p})^\infty$ is a part of the strong deformation retraction data

$$(\text{id} \otimes j_{F_p})^\infty: C'_*(X) \otimes_{p_X} H_*(F_p) \rightleftarrows C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}) : (\text{id} \otimes r_{F_p})^\infty.$$

Define a map $g_X^\infty: C'_*(X) \otimes_{p_X} H_*(F_p) \rightarrow C'_*(X) \otimes_{q_X} H_*(F_q)$ by

$$g_X^\infty := (\text{id} \otimes r_{F_q})^\infty \circ (\text{id} \otimes (g|_{\tilde{F}_p})_*) \circ (\text{id} \otimes j_{F_p})^\infty.$$

The lower square in the diagram (4-2) commutes then up to a chain homotopy. The basic perturbation lemma gives explicit formulas for this chain homotopy and for the maps $(\text{id} \otimes r_{F_q})^\infty$ and $(\text{id} \otimes j_{F_p})^\infty$. Direct computations involving these formulas yield the following fact:

4.10 Proposition *Let B be a pointed, path connected space. Let $p: E \rightarrow B$ and $q: D \rightarrow B$ be fibrations, and let $g: E \rightarrow D$ be a map of fibrations.*

- (1) *The maps g_X^∞ define a natural transformation of functors $\mathcal{S}_* \downarrow B \rightarrow Ch(\mathbb{Q})$.*
- (2) *The diagram*

$$\begin{array}{ccc}
 C_*(p^*X) & \xrightarrow{g_X^*} & C_*(q^*X) \\
 \beta_{pX} \downarrow \simeq & & \simeq \downarrow \beta_{qX} \\
 C'_*(X) \otimes_{pX} H_*(F_p) & \xrightarrow{g_X^\infty} & C'_*(X) \otimes_{qX} H_*(F_q)
 \end{array}$$

commutes up to a chain homotopy that is natural in X .

- (3) *For $X \in \mathcal{S}_* \downarrow B$ consider the homological filtrations $\{L_n(p_X)\}$ and $\{L_n(q_X)\}$ of the complexes $C'_*(X) \otimes_{pX} H_*(F_p)$ and $C'_*(X) \otimes_{qX} H_*(F_q)$, respectively (Section 4.7). The map g_X^∞ preserves these filtrations. Moreover, for every n the following diagram commutes:*

$$\begin{array}{ccc}
 L_n(p_X)/L_{n-1}(p_X) & \xrightarrow{g_X^\infty} & L_n(q_X)/L_{n-1}(q_X) \\
 \cong \downarrow & & \downarrow \cong \\
 C'_*(X) \otimes_{\pi_1 X} H_n(F_p) & \xrightarrow{\text{id} \otimes (g|_{F_p})_*} & C'_*(X) \otimes_{\pi_1 X} H_n(F_q)
 \end{array}$$

The vertical isomorphisms in this diagram come from Proposition 4.8.

5 The secondary transfer

We are now ready to give:

Proof of Theorem 3.5 Consider diagram (3-2). We want to construct a homotopy η_p^h between the maps $\lambda_E^h A(p^!)$ and $\chi(F_p)\lambda_B^h$. Recall that the map $\lambda_E^h A(p^!)$ is induced by the functor

$$\Phi: \mathcal{R}^{\text{fd}}(B) \rightarrow Ch^{\text{fd}}(\mathbb{Q})$$

that assigns to a retractive space X the relative chain complex $C_*(p^*X, E)$, while the map $\chi(F_p)\lambda_B^h$ is induced by the functor

$$\Psi: \mathcal{R}^{\text{fd}}(B) \rightarrow Ch^{\text{fd}}(\mathbb{Q})$$

given by $\Psi(X) = C_*(X, B) \otimes H_*(F_p)$. We will build a sequence of intermediate functors between Φ and Ψ and show that maps induced by these functors can be connected by homotopies.

First, for $X \in \mathcal{R}^{\text{fd}}(B)$ the inclusion $i_X: B \hookrightarrow X$ induces an inclusion $\tilde{i}_X: E \hookrightarrow p^*X$. The naturality of the quasi-isomorphisms β_{pX} described in [Theorem 4.4](#) implies that we have a commutative diagram:

$$\begin{array}{ccc}
 C_*(E) & \xrightarrow{\tilde{i}_{X*}} & C_*(p^*X) \\
 \beta_p \downarrow & & \downarrow \beta_{pX} \\
 C'_*(B) \otimes_p H_*(F_p) & \xrightarrow{i_{X*} \otimes \text{id}} & C'_*(X) \otimes_{pX} H_*(F_p)
 \end{array}$$

Define

$$C'_*(X, B) \otimes_{pX} H_*(F_p) := \text{coker}(i_{X*} \otimes \text{id}).$$

The assignment $X \mapsto C'_*(X, B) \otimes_{pX} H_*(F_p)$ defines an almost exact functor

$$\Phi_1: \mathcal{R}^{\text{fd}}(B) \rightarrow \text{Ch}^{\text{fd}}(\mathbb{Q})$$

and the natural quasi-isomorphisms

$$\beta_{pX}: C_*(p^*X, E) \xrightarrow{\cong} C'_*(X, B) \otimes_{pX} H_*(F_p)$$

define a natural weak equivalence $\beta: \Phi \Rightarrow \Phi_1$. Denote by $K(\Phi_1): A(B) \rightarrow K(\mathbb{Q})$ the map induced by Φ_1 . The natural weak equivalence β defines a homotopy

$$(5-1) \quad \lambda_E^h A(p^!) \simeq K(\Phi_1).$$

Next, since the map $i_{X*} \otimes \text{id}$ preserves the homological filtration of twisted tensor products we can define

$$L_n(pX, p) := \text{coker}(L_n(p) \xrightarrow{i_{X*} \otimes \text{id}} L_n(pX)).$$

The chain complexes $L_n(pX, p)$ form a decreasing filtration of $C'_*(X, B) \otimes_{pX} H_*(F_p)$. [Proposition 4.8](#) shows that the filtration quotient $L_n(pX, p)/L_{n+1}(pX, p)$ can be identified with the chain complex

$$C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p) := \text{coker}(C'_*(B) \otimes_{\pi_1 B} H_n(F_p) \rightarrow C'_*(X) \otimes_{\pi_1 X} H_n(F_p)).$$

Since F_p is a homotopy finite space we have $H_q(F_p) = 0$ for q large enough, and so $\{L_n(pX, p)\}$ is in fact a finite filtration. The assignments $X \mapsto L_n(pX, p)$ and

$X \mapsto C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p)$ define almost exact functors $\mathcal{R}^{\text{fd}}(B) \rightarrow \mathcal{C}h^{\text{fd}}(\mathbb{Q})$. These functors are connected by natural short exact sequences

$$(5-2) \quad 0 \rightarrow L_{n+1}(pX, p) \rightarrow L_n(pX, p) \rightarrow C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p) \rightarrow 0.$$

Let $\Phi_2: \mathcal{R}^{\text{fd}}(B) \rightarrow \mathcal{C}h^{\text{fd}}(\mathbb{Q})$ denote the almost exact functor given by

$$\Phi_2(X) := \bigoplus_n C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p)$$

and let $K(\Phi_2): A(B) \rightarrow K(\mathbb{Q})$ be the map induced by Φ_2 . Applying repeatedly Waldhausen’s additivity theorem to the sequences (5-2) we obtain a homotopy

$$(5-3) \quad K(\Phi_1) \simeq K(\Phi_2).$$

Assume now for a moment that $p: E \rightarrow B$ is a fibration with the trivial action of $\pi_1 B$ on $H_*(F)$. In this case the action of $\pi_1 X$ on $H_*(F_p)$ is trivial as well, so we have isomorphisms

$$\Phi_2(X) \cong C'_*(X, B) \otimes H_*(F_p).$$

Since by assumption X and B are path connected spaces we also have natural quasi-isomorphisms

$$C'_*(X, B) \otimes H_*(F_p) \simeq C_*(X, B) \otimes H_*(F_p) = \Psi(X).$$

As a consequence, for every $X \in \mathcal{R}^{\text{fd}}(B)$ we obtain $\Phi_2(X) \simeq \Psi(X)$, which induces a homotopy

$$(5-4) \quad K(\Phi_2) \simeq \chi(F_p)\lambda_B^h.$$

Concatenating the homotopies (5-1), (5-3) and (5-4) we get the desired homotopy η_p^h .

If p is an arbitrary unipotent fibration we need an additional step to pass between the maps $K(\Phi_2)$ and $\chi(F_p)\lambda_B^h$. In this case the action of $\pi_1 B$ need not be trivial, but $H_*(F_p)$ admits a decreasing filtration $\{V^i\}$ such that V^i is a $\pi_1 B$ -module and the action of $\pi_1 B$ on the quotients V^{i+1}/V^i is trivial. This defines a filtration $\{C'_*(X, B) \otimes_{\pi_1 X} V^i\}$ of the complex $\Phi_2(X)$. The quotients of this filtration are the (untwisted) tensor products $C'_*(X, B) \otimes (V^i/V^{i-1})$. Define a functor

$$\Phi_3: \mathcal{R}^{\text{fd}}(B) \rightarrow \mathcal{C}h^{\text{fd}}(\mathbb{Q})$$

by

$$\Phi_3(X) := \bigoplus_i C'_*(X, B) \otimes (V^i/V^{i-1}).$$

Naturally we also have a filtration $\{C'_*(X, B) \otimes V^i\}$ of the untwisted tensor product $C'_*(X, B) \otimes H_*(F_p)$ and $\Phi_3(X)$ is the direct sum of the quotients of this filtration.

This means that using Waldhausen’s additivity theorem (and the quasi-isomorphisms $C'_*(X, B) \otimes H_*(F_p) \simeq C_*(X, B) \otimes H_*(F_p)$) we get homotopies

$$(5-5) \quad K(\Phi_2) \simeq K(\Phi_3) \simeq \chi(F_p)\lambda_B^h.$$

The homotopy η_p^h is then obtained as a concatenation of the homotopies (5-1), (5-3) and (5-5). □

Proof of Theorem 1.2 The homotopy η_p is obtained by concatenating the above homotopy η_p^h and the homotopy μ_p described in Section 3.3. □

Let $C \in Ch^{fd}(\mathbb{Q})$ be a chain complex. Notice that our construction of $K(\mathbb{Q})$ lets us identify C with a point of $K(\mathbb{Q})$.

5.1 Definition Let B be a path connected space and let $C \in Ch^{fd}(\mathbb{Q})$. Denote the homotopy fiber of the linearization map taken over the point $C \in K(\mathbb{Q})$ by $Wh_s^{\mathbb{Q}}(B)_C$, that is,

$$Wh_s^{\mathbb{Q}}(B)_C := \text{hofib}(\lambda_B: Q(B_+) \rightarrow K(\mathbb{Q}))_C.$$

For the zero chain complex $0 \in Ch^{fd}(\mathbb{Q})$ we will write $Wh_s^{\mathbb{Q}}(B)$ to denote $Wh_s^{\mathbb{Q}}(B)_0$.

Let $p: E \rightarrow B$ be a unipotent bundle with a fiber F_p . For any $C \in Ch^{fd}(\mathbb{Q})$ we have by Theorem 1.2 a map

$$Wh_s^{\mathbb{Q}}(B)_C \longrightarrow Wh_s^{\mathbb{Q}}(E)_{C \otimes H_*(F_p)}.$$

This gives rise to the following:

5.2 Definition The smooth secondary transfer of a unipotent bundle $p: E \rightarrow B$ is the map

$$Wh_s^{\mathbb{Q}}(p^!): Wh_s^{\mathbb{Q}}(B) \longrightarrow Wh_s^{\mathbb{Q}}(E)$$

determined by the Becker–Gottlieb transfer $Q(p^!)$ and the homotopy η_p given by Theorem 1.2.

It will be convenient to consider a variant of this definition in the setting of unipotent fibrations:

5.3 Definition For a path connected space B let

$$Wh_h^{\mathbb{Q}}(B) := \text{hofib}(\lambda_B^h: A(B) \rightarrow K(\mathbb{Q}))_0.$$

The homotopy secondary transfer of a unipotent fibration $p: E \rightarrow B$ is the map

$$\text{Wh}_h^{\mathbb{Q}}(p^!): \text{Wh}_h^{\mathbb{Q}}(B) \longrightarrow \text{Wh}_h^{\mathbb{Q}}(E)$$

determined by the transfer $A(p^!)$ and the homotopy η_p^h given by [Theorem 3.5](#).

5.4 Note Let $p: E \rightarrow B$ be a unipotent fibration with a fiber F_p . The construction of the homotopy η_p^h described in the proof of [Theorem 3.5](#) makes use of a choice of a strong deformation retraction $H_*(F_p) \rightleftarrows C_*(F_p)$ and a choice of a unipotent filtration $\{V^i\}$ of $H_*(F_p)$. One can check though that the homotopy class of the map $\text{Wh}_h^{\mathbb{Q}}(p^!)$ is independent of these choices, and so it depends on the fibration p only. Likewise, if p is a unipotent bundle then the homotopy class of the smooth secondary transfer $\text{Wh}_s^{\mathbb{Q}}(p^!)$ depends only on the bundle p .

6 Homotopy invariance of $\text{Wh}_h^{\mathbb{Q}}(p^!)$

Let $f: E_1 \rightarrow E_2$ be a map of topological spaces. Such a map defines an exact functor of Waldhausen categories

$$f_*: \mathcal{R}^{\text{fd}}(E_1) \rightarrow \mathcal{R}^{\text{fd}}(E_2)$$

given by $f_*(X) = X \cup_{E_1} E_2$ for $X \in \mathcal{R}^{\text{fd}}(E_1)$. This functor in turn induces a map $f_*: A(E_1) \rightarrow A(E_2)$. Consider the diagram:

$$\begin{array}{ccc} A(E_1) & \xrightarrow{f_*} & A(E_2) \\ & \searrow \lambda_{E_1}^h & \swarrow \lambda_{E_2}^h \\ & K(\mathbb{Q}) & \end{array}$$

Recall $\lambda_{E_1}^h$ is induced by the functor $\mathcal{R}^{\text{fd}}(E_1) \rightarrow \text{Ch}^{\text{fd}}(\mathbb{Q})$ given by $X \mapsto C_*(X, E_1)$. Similarly, $\lambda_{E_2}^h f_*$ comes from the functor defined by $X \mapsto C_*(X \cup_{E_1} E_2, E_2)$. The natural quasi-isomorphisms $C_*(X, E_1) \rightarrow C_*(X \cup_{E_1} E_2, E_2)$ define a homotopy h_f between $\lambda_{E_1}^h$ and $\lambda_{E_2}^h f_*$. As a result we obtain a map

$$f_*: \text{Wh}_h^{\mathbb{Q}}(E_1) \longrightarrow \text{Wh}_h^{\mathbb{Q}}(E_2).$$

Our goal in this section is to prove the following:

6.1 Proposition For $i = 1, 2$, let $p_i: E_i \rightarrow B$ be a unipotent fibration, and let $f: E_1 \rightarrow E_2$ be a fiberwise homotopy equivalence. There is a homotopy

$$f_* \text{Wh}_h^{\mathbb{Q}}(p_1^!) \simeq \text{Wh}_h^{\mathbb{Q}}(p_2^!).$$

Proof Let F_{p_i} denote the fiber of p_i . Using Section A.2 we see that in order to obtain the desired homotopy it is enough to construct the following data:

- (1) a homotopy $H_f^A: A(B) \times I \rightarrow A(E_2)$ between $f_*A(p_1^!)$ and $A(p_2^!)$;
- (2) a homotopy $H_f^K: K(\mathbb{Q}) \times I \rightarrow K(\mathbb{Q})$ between the maps $\chi(F_{p_1})$ and $\chi(F_{p_2})$;
- (3) a homotopy of homotopies that fills the diagram

$$(6-1) \quad \begin{array}{ccc} \lambda_{E_2}^h f_* A(p_1^!) & \xrightarrow{\lambda_{E_2}^h H_f^A} & \lambda_{E_2}^h A(p_2^!) \\ \downarrow h_f \circ (A(p_1^!) \times \text{id}_I) & & \downarrow \eta_{p_2}^h \\ \lambda_{E_1}^h A(p_1^!) & & \\ \downarrow \eta_{p_1}^h & & \\ \chi(F_{p_1}) \lambda_B^h & \xrightarrow{H_f^K(\lambda_B^h \times \text{id}_I)} & \chi(F_{p_2}) \lambda_B^h \end{array}$$

where each vertex represents a map $A(B) \rightarrow K(\mathbb{Q})$ and edges represent homotopies of such maps.

(1) **Construction of H_f^A** Note that the map $f_*A(p_1^!)$ is induced by the functor $\mathcal{R}^{\text{fd}}(B) \rightarrow \mathcal{R}^{\text{fd}}(E_2)$ given by $X \mapsto f_*p_1^*X$ while $A(p_2^!)$ comes from the functor given by $X \mapsto p_2^*X$. Since f is a fiberwise homotopy equivalence the natural maps $f_*p_1^*X \rightarrow p_2^*X$ induced by f are weak equivalences, and so they define the homotopy H_f^A .

(2) **Construction of H_f^K** Recall for $i = 1, 2$ the map $\chi(F_i)$ is induced by the functor $\mathcal{C}h^{\text{fd}}(\mathbb{Q}) \rightarrow \mathcal{C}h^{\text{fd}}(\mathbb{Q})$ given by $C \mapsto C \otimes H_*(F_i)$. Since the map $f|_{F_1}: F_1 \rightarrow F_2$ is a homotopy equivalence it induces an isomorphism of homology groups of the fibers

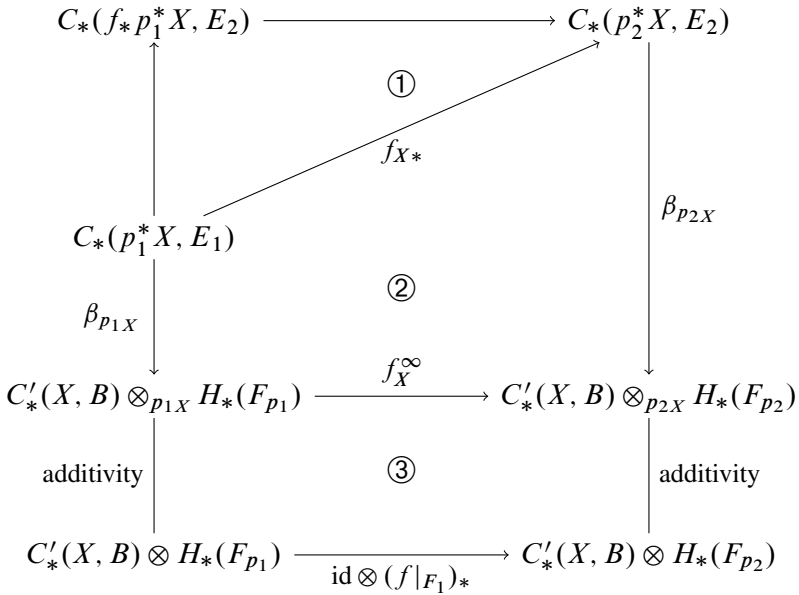
$$(f|_{F_1})_*: H_*(F_1) \xrightarrow{\cong} H_*(F_2).$$

This gives a natural isomorphism of functors

$$- \otimes H_*(F_1) \Rightarrow - \otimes H_*(F_2).$$

The homotopy H_f^K is defined by this natural isomorphism.

(3) **Construction of the homotopy of homotopies** Consider the following diagram:



Each vertex of this diagram represents a functor $\mathcal{R}^{\text{fd}}(B) \rightarrow \mathcal{C}h^{\text{fd}}(\mathbb{Q})$. The edges represent natural weak equivalences, with the exception of the lowest vertical edges where the passage between functors is obtained using additivity. The maps β_{p_1X} and β_{p_2X} are the Brown quasi-isomorphisms (Theorem 4.4) and the maps f_X^∞ come from Proposition 4.10. On the level of K -theory each vertex of this diagram represents a map $A(B) \rightarrow K(\mathbb{Q})$ and the edges represent homotopies of such maps. The outer rectangle coincides with diagram (6-1).

In order to show that the diagram (6-1) can be filled by a homotopy of homotopies it is enough to show that each of the subdiagrams ① through ③ in the above diagram of functors can be filled by a homotopy of homotopies. In the case of subdiagram ① such a homotopy of homotopies exists since this subdiagram commutes. By Proposition 4.10 subdiagram ② commutes up to a natural chain homotopy, so it again can be filled by a homotopy of homotopies. Proposition 4.10 says also that the maps f_X^∞ preserve the homological filtration of the twisted tensor products and that they induce the map $\text{id} \otimes (f|_{F_1})_*$ on the filtration quotients. This, together with the fact that the map $(f|_{F_1})_*: H_*(F_1) \rightarrow H_*(F_2)$ is an isomorphism of $\pi_1 B$ -modules, implies that we also have a homotopy of homotopies filling subdiagram ③. \square

7 Additivity of the secondary transfer

Our goal of this section is to prove that secondary transfer maps have additivity properties that are analogous to additivity of the Becker–Gottlieb transfer and the A -theory transfer. We start by considering additivity of the homotopy secondary transfer:

7.1 Theorem For $i = 0, 1, 2$, let $p_i: E_i \rightarrow B$ be a unipotent fibration. Assume that we have maps of fibrations

$$\begin{array}{ccccc}
 E_1 & \longleftarrow & E_0 & \xrightarrow{j} & E_2 \\
 & \searrow p_1 & \downarrow p_0 & \swarrow p_2 & \\
 & & B & &
 \end{array}$$

where j is a cofibration over B . Let $E := E_1 \cup_{E_0} E_2$ and let $p: E \rightarrow B$ be the fibration given by $p := p_1 \cup_{p_0} p_2$. Then p is a unipotent fibration and we have

$$(7-1) \quad [\text{Wh}_h^{\mathbb{Q}}(p^!)] = [k_{1*} \text{Wh}_h^{\mathbb{Q}}(p_1)] + [k_{2*} \text{Wh}_h^{\mathbb{Q}}(p_2)] - [k_{0*} \text{Wh}_h^{\mathbb{Q}}(p_0)].$$

Here $k_{i*}: \text{Wh}_h^{\mathbb{Q}}(E_i) \rightarrow \text{Wh}_h^{\mathbb{Q}}(E)$ is induced by the map $k_i: E_i \rightarrow E$.

7.2 Lemma Consider a diagram of chain complexes

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 g \downarrow & & \downarrow g' \\
 A' & \xrightarrow{f'} & B'
 \end{array}$$

that commutes up to a chain homotopy h . There exists a map $\tilde{g}': \text{Cyl}(f) \rightarrow B'$ such that the diagram

$$\begin{array}{ccc}
 A & \longrightarrow & \text{Cyl}(f) \\
 g \downarrow & & \downarrow \tilde{g}' \\
 A' & \xrightarrow{f'} & B'
 \end{array}$$

commutes. Moreover \tilde{g}' is chain homotopic to the composition $\text{Cyl}(f) \rightarrow B \xrightarrow{g'} B'$.

Proof Recall that $\text{Cyl}(f)_n = A_n \oplus A_{n-1} \oplus B_n$. The map $\tilde{g}'_n: \text{Cyl}(f)_n \rightarrow B'_n$ is given by

$$\tilde{g}'_n(a_1, a_2, b) = f'_n g(a_1) + h(a_2) + g'_n(b).$$

The second statement of the lemma is easy to verify. □

Proof of Theorem 7.1 Let F denote the fiber of the fibration p and for $i = 0, 1, 2$, let F_i be the fiber of p_i . The fact that the action of $\pi_1(B)$ on $F = F_1 \cup_{F_0} F_2$ is unipotent can be obtained using the Mayer–Vietoris sequence for homology of the fibers.

The strategy of the proof of additivity of the secondary transfer is as follows. We will construct maps

$$f_1^h: \text{Wh}_h^{\mathbb{Q}}(B) \rightarrow \text{Wh}_h^{\mathbb{Q}}(E) \quad \text{and} \quad f_2^h: \text{Wh}_h^{\mathbb{Q}}(B) \rightarrow \text{Wh}_h^{\mathbb{Q}}(E_2)$$

such that $[f_1^h] = [k_{2*} f_2^h]$. We will also show that

$$[\text{Wh}_h^{\mathbb{Q}}(p^!)] = [k_{1*} \text{Wh}_h^{\mathbb{Q}}(p_1^!)] + [f_1^h] \quad \text{and} \quad [\text{Wh}_h^{\mathbb{Q}}(p_2^!)] = [j_* \text{Wh}_h^{\mathbb{Q}}(p_0^!)] + [f_2^h].$$

Since $k_0 = k_2 j$ the second of these equations will give

$$[k_{2*} \text{Wh}_h^{\mathbb{Q}}(p_2^!)] = [k_{0*} \text{Wh}_h^{\mathbb{Q}}(p_0^!)] + [k_{2*} f_2^h] = [k_{0*} \text{Wh}_h^{\mathbb{Q}}(p_0^!)] + [f_1^h],$$

which, combined with the first equation, will yield the formula (7-1).

The construction of the map f_1^h will proceed following the scheme outlined in Section A.1. First, we will construct a map $f_1^A: A(B) \rightarrow A(E)$. Subsequently we will consider the diagram:

$$(7-2) \quad \begin{array}{ccc} A(B) & \xrightarrow{f_1^A} & A(E) \\ \lambda_B^h \downarrow & & \downarrow \lambda_E^h \\ K(\mathbb{Q}) & \xrightarrow{\chi(\text{Cone}(k_1|_{F_1*}))} & K(\mathbb{Q}) \end{array}$$

Here $\text{Cone}(k_1|_{F_1*})$ denotes the mapping cone of the map $k_1|_{F_1*}: H_*(F_1) \rightarrow H_*(F)$, and the map

$$\chi(\text{Cone}(k_1|_{F_1*})): K(\mathbb{Q}) \rightarrow K(\mathbb{Q})$$

is induced by the functor $\mathcal{C}h^{\text{fd}}(\mathbb{Q}) \rightarrow \mathcal{C}h^{\text{fd}}(\mathbb{Q})$ given by tensoring by $\text{Cone}(k_1|_{F_1*})$. We will show that the diagram (7-2) commutes up to a preferred homotopy h_1 . This homotopy together with the map f_1^A will define the map f_1^h .

In order to obtain the map f_1^A recall that the map $k_{1*} A(p_1^!)$ is induced by the functor that assigns to a space $X \in \mathcal{R}^{\text{fd}}(B)$ the space $k_{1*} p_1^* X \in \mathcal{R}^{\text{fd}}(E)$, and that the map $A(p^!)$ is induced by the functor $X \mapsto p^* X$. For $X \in \mathcal{R}^{\text{fd}}(B)$ we have a cofibration $k_{1*} p_1^* X \rightarrow p^* X$. Let $M_X \in \mathcal{R}^{\text{fd}}(E)$ denote the cofiber of this map. The

assignment $X \mapsto M_X$ defines an exact functor $\mathcal{R}^{\text{fd}}(B) \rightarrow \mathcal{R}^{\text{fd}}(E)$. The map f_1^A is induced by this functor. The above constructions give a short exact sequence of functors

$$(7-3) \quad k_{1*}p_1^*X \rightarrow p^*X \rightarrow M_X.$$

Applying Waldhausen’s additivity theorem we obtain a homotopy

$$(7-4) \quad A(p^!) \simeq k_{1*}A(p_1^!) + f_1^A.$$

Next, in order to describe a homotopy that fills the diagram (7-2) consider the following diagram of functors:

$$(7-5) \quad \begin{array}{ccccc} & C_*(k_{1*}p_1^*X, E) & & & C_*(M_X, E) \\ & \uparrow \simeq & \searrow & & \uparrow \simeq \\ & C_*(p_1^*X, E_1) & \xrightarrow{k_{X*}} & C_*(p^*X, E) & \xrightarrow{\quad} & \text{coker}(k_{X*}) \\ & \uparrow = & & \uparrow & & \uparrow \\ & C_*(p_1^*X, E_1) & \xrightarrow{\quad} & \text{Cyl}(k_{X*}) & \xrightarrow{\quad} & \text{Cone}(k_{X*}) \\ & \downarrow \beta_{p_1X} & & \downarrow g_X & & \downarrow \\ C'_*(X, B) \otimes_{p_1X} H_*(F_1) & \xrightarrow{\quad} & \text{Cyl}(k_X^\infty) & \xrightarrow{\quad} & \text{Cone}(k_X^\infty) \\ \text{additivity} & & \downarrow \text{additivity} & & \downarrow \text{additivity} \\ C_*(X, B) \otimes H_*(F_1) & \longrightarrow & C_*(X, B) \otimes \text{Cyl}(k_1|_{F_1*}) & \longrightarrow & C_*(X, B) \otimes \text{Cone}(k_1|_{F_1*}) \end{array}$$

The map k_{X*} in this diagram is induced by the map of fibrations $k_X: p_1^*X \rightarrow p^*X$. The complexes $\text{Cyl}(k_{X*})$ and $\text{Cone}(k_{X*})$ are, respectively, the mapping cylinder and the mapping cone of k_{X*} . Similarly $\text{Cyl}(k_X^\infty)$ and $\text{Cone}(k_X^\infty)$ are the mapping cylinder and the mapping cone of the map

$$k_X^\infty: C'_*(X, B) \otimes_{p_1X} H_*(F_1) \rightarrow C'_*(X, B) \otimes_{p_X} H_*(F)$$

given by Proposition 4.10. Finally, $\text{Cyl}(k_1|_{F_1*})$ and $\text{Cone}(k_1|_{F_1*})$ are the mapping cylinder and the mapping cone of the map $k_1|_{F_1*}: H_*(F_1) \rightarrow H_*(F)$. The horizontal maps are defined in the obvious way so that each row of the diagram forms a short exact sequence.

All vertical maps are quasi-isomorphisms. They are defined in the obvious way with the exception of the map g_X , which is given as follows. By Proposition 4.10 we have

a diagram

$$\begin{array}{ccc}
 C_*(p_1^*X, E_1) & \xrightarrow{k_{X*}} & C_*(p^*X, E) \\
 \beta_{p_{1X}} \downarrow & & \downarrow \beta_{p_X} \\
 C'_*(X, B) \otimes_{p_{1X}} H_*(F_1) & \xrightarrow{k_X^\infty} & C'_*(X, B) \otimes_{p_X} H_*(F)
 \end{array}$$

that commutes up to a chain homotopy. Using Lemma 7.2 we obtain from here a commutative diagram:

$$\begin{array}{ccc}
 C_*(p_1^*X, E_1) & \xrightarrow{\quad\quad\quad} & \text{Cyl}(k_{X*}) \\
 \beta_{p_{1X}} \downarrow & & \downarrow \bar{\beta}_{p_X} \\
 C'_*(X, B) \otimes_{p_{1X}} H_*(F_1) & \xrightarrow{k_X^\infty} & C'_*(X, B) \otimes_{p_X} H_*(F)
 \end{array}$$

The map g_X is the composition of $\bar{\beta}_{p_X}$ and the inclusion

$$C'_*(X, B) \otimes_{p_X} H_*(F) \hookrightarrow \text{Cyl}(k_X^\infty).$$

The lowest vertical edges in the diagram indicate a passage between chain complexes using additivity. This means the following construction. Since the map k_X^∞ preserves the homological filtrations on $C'_*(X, B) \otimes_{p_{1X}} H_*(F_1)$ and $C'_*(X, B) \otimes_{p_X} H_*(F)$, the complexes $\text{Cyl}(k_X^\infty)$ and $\text{Cone}(k_X^\infty)$ are endowed with induced filtrations. Each lowest vertical edge indicates a passage from the filtered chain complex to the direct sum of the filtration quotients. As usual, after passage to the induced maps $A(B) \rightarrow K(\mathbb{Q})$ each additivity edge gives a homotopy obtained using Waldhausen’s additivity theorem. We note here that the additivity edges are related to one another as follows. The maps in the short exact sequence

$$C'_*(X, B) \otimes_{p_{1X}} H_*(F_1) \longrightarrow \text{Cyl}(k_X^\infty) \longrightarrow \text{Cone}(k_X^\infty)$$

preserve filtrations. Moreover, their restrictions to the filtration subcomplexes also form short exact sequences, and so do the induced maps of filtrations quotients. The bottom row of the diagram is the direct sum of the short exact sequences of these filtration quotients.

Each vertex of the diagram (7-5) induces a map $A(B) \rightarrow K(\mathbb{Q})$. All vertical edges define homotopies of such maps. Concatenation of homotopies defined by rightmost vertical edges gives a homotopy filling the diagram (7-2). This homotopy, together with the map f_1^A , defines the map $f_1^h: \text{Wh}_h^{\mathbb{Q}}(B) \rightarrow \text{Wh}_h^{\mathbb{Q}}(E)$.

The existence of a homotopy between the maps $\text{Wh}_h^{\mathbb{Q}}(p^!)$ and $k_{1*}\text{Wh}_h^{\mathbb{Q}}(p_1^!) + f_1^h$ follows directly from the above construction. The map $k_{1*}\text{Wh}_h^{\mathbb{Q}}(p_1^!)$ is defined by the map $k_{1*}A(p_1^!)$ and the homotopy induced by the leftmost vertical edges in the diagram (7-5). The map $\text{Wh}_h^{\mathbb{Q}}(p^!)$ is homotopic to the map defined by $A(p^!)$ and the homotopy induced by the middle vertical edges in (7-5). As we have already noticed applying Waldhausen’s additivity theorem to the short exact sequence of functors (7-3) defines a homotopy between $A(p^!)$ and $k_{1*}A(p_1^!) + f_1^A$. In order to lift this homotopy to a homotopy between the maps $\text{Wh}_h^{\mathbb{Q}}(p^!)$ and $k_{1*}\text{Wh}_h^{\mathbb{Q}}(p_1^!) + f_1^h$ it is enough to apply the additivity theorem to the horizontal short exact sequences in the diagram (7-5).

Construction of the map $f_2^h: \text{Wh}_h^{\mathbb{Q}}(B) \rightarrow \text{Wh}_h^{\mathbb{Q}}(E_2)$ proceeds in the same way as the construction of f_1^h , with the cofibration $j: E_0 \rightarrow E_2$ used in place of k_1 . By the same argument as above we obtain a homotopy $\text{Wh}_h^{\mathbb{Q}}(p_2^!) \simeq j_*\text{Wh}_h^{\mathbb{Q}}(p_0^!) + f_2^h$. Finally, that the maps f_1^h and $k_{2*}f_2^h$ are homotopic can be verified directly by inspecting the construction of f_1^h and f_2^h . □

A statement analogous to [Theorem 7.1](#) holds for the smooth secondary transfer. Given a smooth bundle $p: E \rightarrow B$ whose fibers are manifolds with boundary, by the vertical boundary of p we will understand the smooth bundle $\partial^v p: \partial^v E \rightarrow B$ obtained by restricting p to the union of boundaries of its fibers. We have:

7.3 Theorem *Let $p: E \rightarrow B$ be a smooth bundle with closed fibers, and for $i = 0, 1, 2$, let $p_i: E_i \rightarrow B$ be unipotent subbundles of p such that p_0 is the vertical boundary of both p_1 and p_2 , and that $E = E_1 \cup_{E_0} E_2$. Then p is a unipotent bundle and we have*

$$[\text{Wh}_s^{\mathbb{Q}}(p^!)] = [k_{1*}\text{Wh}_s^{\mathbb{Q}}(p_1)] + [k_{2*}\text{Wh}_s^{\mathbb{Q}}(p_2)] - [k_{0*}\text{Wh}_s^{\mathbb{Q}}(p_0)].$$

Here $k_{i*}: \text{Wh}_s^{\mathbb{Q}}(E_i) \rightarrow \text{Wh}_s^{\mathbb{Q}}(E)$ is induced by the map $k_i: E_i \rightarrow E$.

7.4 Note Recall that the construction of the smooth secondary transfer we are working with uses the combinatorial model of $Q(E_+)$ built using partitions ([Section 2.1](#)). This model is functorial with respect to embeddings of submanifolds of codimension 0. As a consequence, in the notation of [Theorem 7.3](#), for $i = 1, 2$ the inclusion maps $k_i: E_i \hookrightarrow E$ induce maps $k_{i*}: Q(E_{i+}) \rightarrow Q(E)$, which then lift to maps $k_{i*}: \text{Wh}_s^{\mathbb{Q}}(E_i) \rightarrow \text{Wh}_s^{\mathbb{Q}}(E)$. The map $k_{0*}: Q(E_{0+}) \rightarrow Q(E_+)$ is constructed as follows. Let $b: E_0 \times [-1, 1] \rightarrow E$ be a fiberwise bicollar neighborhood of E_0 . Thus, b is a smooth embedding of bundles over B such that $b(E_0 \times \{0\}) = E_0$, $b(E_0 \times [-1, 0]) \subseteq E_1$ and $b(E_0 \times [0, 1]) \subseteq E_2$. The inclusion $k_0: E_0 \hookrightarrow E$ coincides with the composition

$$E_0 \rightarrow E_0 \times \{0\} \subseteq E_0 \times [-1, 1] \xrightarrow{b} E.$$

For a partition $P \subseteq E_0 \times I$ the submanifold $P \times [-1, 1] \subseteq E_0 \times [-1, 1]$ defines (modulo permutation of coordinates) a partition of $(E_0 \times [-1, 1]) \times I$. This assignment induces a map $Q(E_{0+}) \rightarrow Q(E_0 \times [-1, 1]_+)$. Furthermore, since b is an embedding of codimension 0 it gives a map $b_*: Q(E_0 \times [-1, 1]_+) \rightarrow Q(E_+)$. Composing these two maps we obtain a map $k_{0*}: Q(E_{0+}) \rightarrow Q(E_+)$ which lifts to

$$k_{0*}: \text{Wh}_s^{\mathbb{Q}}(E_0) \rightarrow \text{Wh}_s^{\mathbb{Q}}(E).$$

Proof of Theorem 7.3 The basic scheme of the proof is the same as that of the proof of Theorem 7.1: it suffices to show that there exist maps $f_1^s: \text{Wh}_s^{\mathbb{Q}}(B) \rightarrow \text{Wh}_s^{\mathbb{Q}}(E)$ and $f_2^s: \text{Wh}_s^{\mathbb{Q}}(B) \rightarrow \text{Wh}_s^{\mathbb{Q}}(E_2)$ such that

$$[\text{Wh}_s^{\mathbb{Q}}(p^!)] = [k_{1*} \text{Wh}_s^{\mathbb{Q}}(p_1^!)] + [f_1^s] \quad \text{and} \quad [\text{Wh}_s^{\mathbb{Q}}(p_2^!)] = [j_* \text{Wh}_s^{\mathbb{Q}}(p_0^!)] + [f_2^s],$$

where $j: E_0 \hookrightarrow E_2$ is the inclusion map, and that $[f_1^s] = [k_{2*} f_2^s]$. Moreover, the construction of the maps f_1^s, f_2^s and verification that they satisfy the above identities also mimics the arguments we used in the proof of Theorem 7.1. Recall that the map f_1^h in that proof was defined using a map $f_1^A: A(B) \rightarrow A(E)$ and a preferred homotopy in the diagram (7-2). Similarly, to define the map f_1^s we need to construct a map $f_1^Q: Q(B_+) \rightarrow Q(E_+)$ together with a preferred homotopy $\lambda_E f_1^Q \simeq \chi(\text{Cone}(k_1|F_{1*}))\lambda_B$. To obtain a homotopy $\text{Wh}_s^{\mathbb{Q}}(p^!) \simeq k_{1*} \text{Wh}_s^{\mathbb{Q}}(p_1^!) + f_1^s$ it suffices to construct a homotopy

$$(7-6) \quad Q(p^!) \simeq k_{1*} Q(p_1^!) + f_1^Q$$

together with an appropriate homotopy of homotopies.

The construction of the map f_1^Q can be simplified in two ways. First, let $\mathcal{S}(B)$ denote the simplicial set of smooth singular simplices of B . It will suffice to construct a map $f_1^Q: |\mathcal{S}(B)| \rightarrow Q(E_+)$. Since $B \simeq |\mathcal{S}(B)|$ this map will extend to a map of infinite loop spaces $f_1^Q: Q(B_+) \rightarrow Q(E_+)$. Second, the construction of the map $f_1^Q: |\mathcal{S}(B)| \rightarrow Q(E_+)$ can be reduced to the following combinatorial construction. Let $\mathcal{T}_1\mathcal{P}_\bullet(E \times I)$ be the simplicial category given by the first stage of the \mathcal{T}_\bullet -construction. Thus for $k \geq 0$ the objects of $\mathcal{T}_1\mathcal{P}_k(E \times I)$ are pairs (P_0, P_1) where P_1 is a bundle of partitions of $E \times I$ over the standard simplex Δ^k , and P_0 is a subbundle of P_1 . Considering $\mathcal{S}(B)$ as a (discrete) simplicial category, it will suffice to give a functor $F_1^Q: \mathcal{S}(B) \rightarrow \mathcal{T}_1\mathcal{P}_\bullet(E \times I)$. Such a functor will determine the map $f_1^Q: |\mathcal{S}(B)| \rightarrow Q(E_+)$ (see [2, Remark 2.5]).

In order to describe the functor F_1^Q we will use the setup of [2, Section 4.3]. Using the notation introduced there for a k -simplex $\sigma \in \mathcal{S}(B)$, by σ^*TB_ε we denote the

disc bundle over Δ^k induced by σ from the disc bundle of the tangent bundle of B . The exponential map defines a map of bundles

$$\begin{array}{ccc} \sigma^*TB_\varepsilon & \xrightarrow{\text{exp}} & B \times \Delta^k \\ & \searrow & \swarrow \\ & \Delta^k & \end{array}$$

which is a fiberwise embedding. Let $\frac{1}{3} < a < b < 1$. Define

$$P_\sigma := \sigma^*TB_\varepsilon \times [a, b] \cup (B \times \Delta^k) \times [0, \frac{1}{3}].$$

This space admits a map $P_\sigma \rightarrow B \times \Delta^k \times I$ which lets us consider it (modulo permutation of factors) as a bundle of partitions of $B \times I$ over Δ^k . For $\sigma \in \mathcal{S}(B)$ we set

$$F_1^Q(\sigma) := ((p_1 \times \text{id})^{-1}(P_\sigma) \cup E \times \Delta^k \times [0, \frac{1}{3}], (p \times \text{id})^{-1}(P_\sigma)).$$

To describe the homotopy (7-6) we will use Waldhausen’s preadditivity theorem [1, Section 3.6]. Let $\mathcal{T}_2\mathcal{P}_\bullet(E \times I)$ be the second stage of the \mathcal{T}_\bullet -construction on $\mathcal{P}_\bullet(E \times I)$. This is a simplicial category with objects in the k^{th} simplicial dimension given by triples (P_0, P_1, P_2) where P_i is a bundle of partitions of $E \times I$ over Δ^k , and P_i is a subbundle of P_{i+1} . For $0 \leq i < j \leq 2$, let $D_{ij}: \mathcal{T}_2\mathcal{P}_\bullet(E \times I) \rightarrow \mathcal{T}_1\mathcal{P}_\bullet(E \times I)$ denote the functor given by $D_{ij}(P_0, P_1, P_2) = (P_i, P_j)$. For a functor $H: \mathcal{S}(B) \rightarrow \mathcal{T}_2\mathcal{P}_\bullet(E \times I)$ the preadditivity theorem gives a homotopy between the map $|\mathcal{S}(B)| \rightarrow Q(E_+)$ induced by the functor $D_{02}H$ and the sum of maps induced by the functors $D_{01}H$ and $D_{12}H$. Consider the functor H given by

$$H(\sigma) := (E \times \Delta^k \times [0, \frac{1}{3}], (p_1 \times \text{id})^{-1}(P_\sigma) \cup E \times \Delta^k \times [0, \frac{1}{3}], (p \times \text{id})^{-1}(P_\sigma)).$$

Notice that $D_{12}H = F_1^Q$. It remains to notice that by [2, Section 4] the maps induced by the functors $D_{01}H$ and $D_{02}H$ are, respectively, $k_{1*}Q(p_1^!)\eta$ and $Q(p^!)\eta$ where $\eta: |\mathcal{S}(B)| \rightarrow Q(B_+)$ is the coaugmentation map.

The above constructions of the map f_1^Q and the homotopy $Q(p^!) \simeq Q(p_1^!) + f_1^Q$ are obtained by replicating the constructions from the proof of Theorem 7.1 of the map f_1^h and the homotopy $A(p^!) \simeq A(p_1^!) + f_1^A$ using partitions and the \mathcal{T}_\bullet -construction in place of retractive spaces and the \mathcal{S}_\bullet -construction. The arguments that show that f_1^Q admits a lift to $f_1^s: \text{Wh}_s^Q(B) \rightarrow \text{Wh}_s^Q(E)$ and that the homotopy (7-6) lifts to a homotopy $\text{Wh}_s^Q(p^!) \simeq k_{1*}\text{Wh}_s^Q(p_1^!) + f_1^s$ involve constructions on the level of chain complexes that duplicate the constructions from the proof of Theorem 7.1. The map f_2^s and the homotopy $\text{Wh}_s^Q(p_2^!) = j_*\text{Wh}_s^Q(p_0^!) + f_2^s$ are obtained in an analogous way. \square

8 Composition of unipotent fibrations

Our next goal is to give a proof of [Theorem 1.3](#). Recall that this theorem states that the homotopy secondary transfer preserves compositions of unipotent fibrations, and the smooth secondary transfer preserves compositions of unipotent bundles.

The statement of [Theorem 1.3](#) relies on the fact that the composition of unipotent bundles (or unipotent fibrations) is again unipotent. While this property is implicitly present in the work of Igusa [\[11\]](#) we give its proof below for completeness, and also because its main ingredient, [Lemma 8.1](#), will be needed later on.

8.1 Lemma [\[11, Lemma 8.9\]](#) *Let*

$$F_p \rightarrow E \xrightarrow{p} B$$

be a unipotent fibration. There is a finite sequence of unipotent fibrations

$$\begin{array}{ccccccc}
 E_0 & \longrightarrow & E_1 & \longrightarrow & \dots & \longrightarrow & E_k \\
 & \searrow p_0 & \downarrow p_1 & & & \swarrow p_k & \\
 & & B & & & &
 \end{array}$$

such that

- (i) $E_0 = \Sigma_B^n E$ for some $n \geq 0$;
- (ii) $p_k: E_k \rightarrow B$ is a rational homotopy equivalence;
- (iii) for every i we have a cofibration sequence

$$B \times S^{n_i} \xrightarrow{\alpha_i} E_i \longrightarrow E_{i+1}$$

over B (ie $E_{i+1} = E_i \cup_{B \times S^{n_i}} B \times D^{n_i+1}$).

We denote here by $\Sigma_B^n E \rightarrow B$ the n -fold fiberwise suspension of the fibration p , while $B \times S^{n_i} \rightarrow B$ and $B \times D^{n_i+1} \rightarrow B$ are, respectively, a product sphere bundle and a product disc bundle.

8.2 Theorem (Igusa) *If $q: D \rightarrow E$ and $p: E \rightarrow B$ are unipotent bundles (resp. unipotent fibrations) then the composition $pq: D \rightarrow B$ is also a unipotent bundle (resp. a unipotent fibration).*

Proof Let F_p, F_q, F_{pq} denote the fibers of p, q, pq , respectively. Since the only nontrivial property we need to verify is that the action of $\pi_1 B$ on $H_*(F_{pq})$ is unipotent, it is enough to show that the statement of the theorem holds for unipotent fibrations. We will split our argument into a few steps.

Step 1 The fibration pq is unipotent for an arbitrary unipotent fibration p and any product fibration $q: E \times F_q \rightarrow E$.

Indeed, in this case we have an isomorphism of $\pi_1 B$ -modules

$$H_*(F_{pq}) \cong H_*(F_p) \otimes H_*(F_q)$$

where the action of $\pi_1 B$ on the right-hand side is given by $\alpha(x \otimes y) = \alpha x \otimes y$.

Step 2 The fibration pq is unipotent for an arbitrary unipotent fibration $p: E \rightarrow B$ and any fibration $q: D \rightarrow E$ with fiber F_q such that $\tilde{H}_*(F_q) = 0$.

This holds since the map $(q|_{F_{pq}})_*: H_*(F_{pq}) \rightarrow H_*(F_p)$ is in this case an isomorphism of $\pi_1 B$ -modules.

Step 3 For $i = 0, 1, 2$, let $p_i: E_i \rightarrow B$ be a fibration with a path connected base space B and fiber F_i of a finite homotopy type. Assume that we have maps of fibrations

$$E_1 \xleftarrow{f} E_0 \xrightarrow{i} E_2$$

where i is a cofibration over B . Let $p: E_1 \cup_{E_0} E_2 \rightarrow B$ be the pushout map. If three of the fibrations p_0, p_1, p_2, p are unipotent then so is the fourth.

This follows from the Mayer-Vietoris sequence for the homology of the fibers and the fact that unipotent $\pi_1 B$ -modules form a Serre category.

Step 4 As an application of Step 3 we obtain that if p and q are unipotent fibrations and $\Sigma_E q: \Sigma_E D \rightarrow E$ is a fiberwise suspension of q then pq is a unipotent fibration if and only if $p\Sigma_E q$ is unipotent.

Step 5 Assume now that p and q are arbitrary unipotent fibrations. Applying Lemma 8.1 to q we obtain a sequence of fibrations:

$$\begin{array}{ccccccc}
 D_0 & \longrightarrow & D_1 & \longrightarrow & \dots & \longrightarrow & D_k \\
 & \searrow & \downarrow & & & \swarrow & \\
 & & q_0 & & & q_1 & & q_k \\
 & & & & & & & \\
 & & & & & & & E
 \end{array}$$

We will show that pq_i is a unipotent fibration for all i . In the case of $i = k$ this is a consequence of Step 2. Arguing inductively, assume that pq_{i+1} is unipotent for

some i . Since pq_{i+1} is the pushout in the diagram of fibrations over B

$$E \times D^{n_i+1} \leftarrow E \times S^{n_i} \rightarrow D_i$$

and the fibrations $E \times D^{n_i+1} \rightarrow B$ and $E \times S^{n_i} \rightarrow B$ are unipotent by Step 1, we obtain using Step 3 that pq_i is also a unipotent fibration. As a consequence we get that pq_0 is unipotent. Since q_0 is an iterated fiberwise suspension of q by Step 4 we obtain that pq is unipotent as well. \square

9 Composition of secondary transfers

In this section we prove [Theorem 1.3](#) as well as its analog for the homotopy secondary transfer:

9.1 Theorem *If $p: E \rightarrow B$ and $q: D \rightarrow E$ are unipotent fibrations then*

$$\text{Wh}_h^{\mathbb{Q}}((pq)^\dagger) \simeq \text{Wh}_h^{\mathbb{Q}}(q^\dagger) \circ \text{Wh}_h^{\mathbb{Q}}(p^\dagger).$$

Our strategy will be as follows. First, in [Lemmas 9.2](#) and [9.3](#) we show that [Theorem 9.1](#) holds in two special cases, and then we will use an argument of Igusa to show that its general statement follows from these special cases. Finally, we will show that essentially the same reasoning can be used to obtain a proof of [Theorem 1.3](#).

9.2 Lemma *Let $p: E \rightarrow B$ and $q: D \rightarrow E$ be unipotent fibrations with fibers F_p and F_q respectively. If $\tilde{H}_*(F_q) = 0$ then*

$$\text{Wh}_h^{\mathbb{Q}}((pq)^\dagger) \simeq \text{Wh}_h^{\mathbb{Q}}(q^\dagger) \circ \text{Wh}_h^{\mathbb{Q}}(p^\dagger).$$

Proof Let F_{pq} denote the fiber of pq . Recall ([Definition 5.3](#)) the map $\text{Wh}_h^{\mathbb{Q}}((pq)^\dagger)$ is defined by the diagram

$$\begin{array}{ccc} A(B) & \xrightarrow{A((pq)^\dagger)} & A(D) \\ \lambda_B^h \downarrow & & \downarrow \lambda_D^h \\ K(\mathbb{Q}) & \xrightarrow{\chi(F_{pq})} & K(\mathbb{Q}) \end{array}$$

which commutes up to the homotopy η_{pq}^h . On the other hand, the composition

$\text{Wh}_h^{\mathbb{Q}}(q^!) \text{Wh}_h^{\mathbb{Q}}(p^!)$ is induced by the diagram:

$$\begin{array}{ccccc}
 A(B) & \xrightarrow{A(p^!)} & A(E) & \xrightarrow{A(q^!)} & A(D) \\
 \lambda_B^h \downarrow & & \downarrow \lambda_E^h & & \downarrow \lambda_D^h \\
 K(\mathbb{Q}) & \xrightarrow{\chi(F_p)} & K(\mathbb{Q}) & \xrightarrow{\chi(F_q)} & K(\mathbb{Q})
 \end{array}$$

The left square in this diagram commutes up to the homotopy η_p^h , and the right square commutes up to the homotopy η_q^h . It follows that the outer square, defining the map $\text{Wh}_h^{\mathbb{Q}}(q^!) \text{Wh}_h^{\mathbb{Q}}(p^!)$, commutes up to the homotopy obtained by concatenating $\eta_q \circ (A(p^!) \times \text{id}_I)$ with $\chi(F_q)\eta_p$.

By Section A.2 in order to obtain a homotopy between the maps $\text{Wh}_h^{\mathbb{Q}}(q^!) \text{Wh}_h^{\mathbb{Q}}(p^!)$ and $\text{Wh}_h^{\mathbb{Q}}((pq)^!)$ it is enough to construct the following data:

- (1) a homotopy $H_A(p, q): A(B) \times I \rightarrow A(D)$ between $A((pq)^!)$ and $A(q^!)A(p^!)$;
- (2) a homotopy $H_K: K(\mathbb{Q}) \times I \rightarrow K(\mathbb{Q})$ between the maps $\chi(F_{pq})$ and $\chi(F_q)\chi(F_p)$;
- (3) a homotopy of homotopies that fills the following diagram:

$$\begin{array}{ccc}
 \lambda_D^h A(q^!)A(p^!) & \xrightarrow{\lambda_D^h H_A(p, q)} & \lambda_D^h A((pq)^!) \\
 \eta_q^h \circ (A(p^!) \times \text{id}_I) \downarrow & & \downarrow \eta_{pq}^h \\
 \chi(F_q)\lambda_E^h A(p^!) & & \\
 \chi(F_q)\eta_p^h \downarrow & & \\
 \chi(F_q)\chi(F_p)\lambda_B^h & \xrightarrow{H_K(\lambda_B^h \times \text{id}_I)} & \chi(F_{pq})\lambda_B^h
 \end{array}$$

(9-1)

Each vertex of this diagram represents a map $A(B) \rightarrow K(\mathbb{Q})$ and edges represent homotopies of such maps.

(1) **Construction of $H_A(p, q)$** The map $A((pq)^!): A(B) \rightarrow A(D)$ comes from the functor $\mathcal{R}^{\text{fd}}(B) \rightarrow \mathcal{R}^{\text{fd}}(D)$ that assigns to a retractive space X the space $(pq)^* X$, while the composition $A(q^!)A(p^!)$ comes from the functor that sends X to $q^* p^* X$. The canonical isomorphisms

$$(pq)^* X \xrightarrow{\cong} q^* p^* X$$

define a natural transformation of functors, and so they induce a homotopy between $A((pq)^!)$ and $A(q^!)A(p^!)$. This is the homotopy $H_A(p, q)$.

(2) **Construction of H_K** Recall that the maps $\chi(F_p)$, $\chi(F_q)$ and $\chi(F_{pq})$ are induced by functors $Ch^{fd}(\mathbb{Q}) \rightarrow Ch^{fd}(\mathbb{Q})$ that tensor a chain complex C by, respectively, $H_*(F_p)$, $H_*(F_q)$ and $H_*(F_{pq})$. As a consequence the map $\chi(F_q)\chi(F_p)$ is induced by the functor that tensors $C \in Ch^{fd}(\mathbb{Q})$ by the chain complex $H_*(F_p) \otimes H_*(F_q)$. Since $\tilde{H}_*(F_q) = 0$ we have an isomorphism

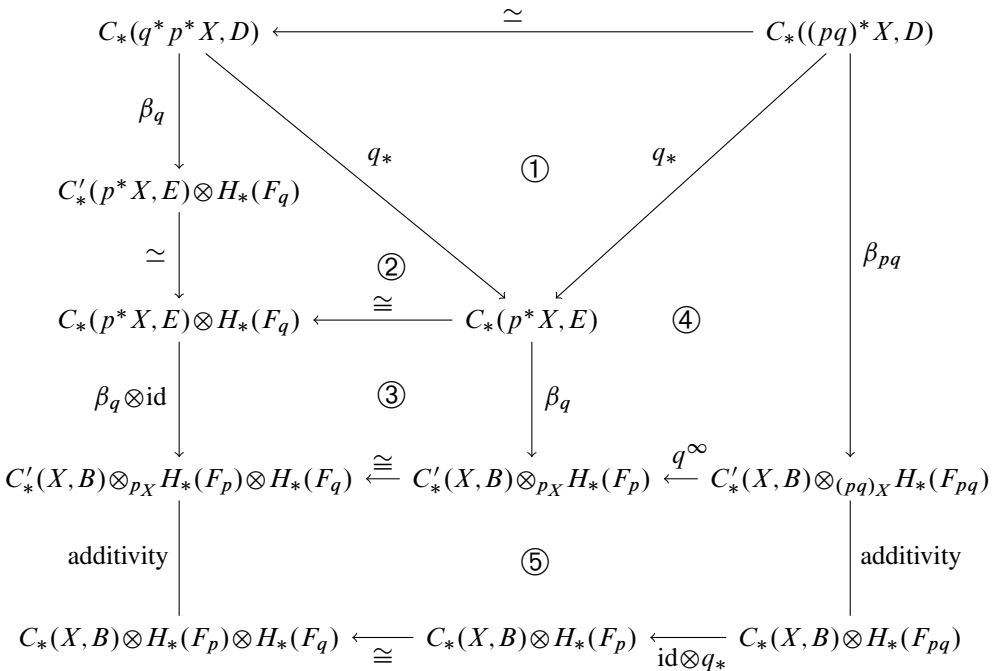
$$H_*(F_{pq}) \xrightarrow{(q|_{F_q})^*} H_*(F_p) \xrightarrow{\cong} H_*(F_p) \otimes H_*(F_q).$$

This induces a natural isomorphism of functors

$$- \otimes H_*(F_{pq}) \Rightarrow - \otimes (H_*(F_p) \otimes H_*(F_q))$$

which, in turn, defines the homotopy H_K .

(3) **Construction of the homotopy of homotopies** In order to show that the diagram (9-1) can be filled by a homotopy of homotopies we will first replace it by the underlying diagram of functors:



Each vertex of this diagram represents a functor $\mathcal{R}^{fd}(B) \rightarrow Ch^{fd}(\mathbb{Q})$. The edges represent natural weak equivalences, with the exception of the lowest vertical edges where the passage between functors is obtained using additivity. The outer edges of this diagram correspond to the homotopies in the diagram (9-1). Since $\tilde{H}_*(F_q) = 0$ the twisted tensor product of the fibration $q^* p^* X \rightarrow p^* X$ is just the untwisted tensor

product $C_*(p^*X, E) \otimes H_*(F_q)$. In effect the homotopy $\eta_q^h \circ (A(p^!) \times \text{id}_I)$ in (9-1) is induced simply by the quasi-isomorphisms

$$C_*(q^*p^*X, D) \xrightarrow{\beta_q} C'_*(p^*X, E) \otimes H_*(F_q) \xrightarrow{\cong} C_*(p^*X, E) \otimes H_*(F_q)$$

without using additivity.

In order to show that the diagram (9-1) can be filled by a homotopy of homotopies it is enough to show that each of the subdiagrams in the above diagram of functors can be filled by a homotopy of homotopies. In the case of the subdiagrams ① through ④ such homotopies of homotopies exist since subdiagrams ① and ③ commute strictly and subdiagrams ② and ④ commute up to natural chain homotopies. Homotopy commutativity of the subdiagram ② follows from [5, (7.4)]. The subdiagram ④ is homotopy commutative since it is obtained by applying Proposition 4.10 to the map of fibrations:

$$\begin{array}{ccc} D & \xrightarrow{q} & E \\ & \searrow pq & \swarrow p \\ & & B \end{array}$$

Finally, the subdiagram ⑤ can be filled by a homotopy of homotopies since the maps $C_*(X, B) \otimes_{(pq)_X} H_*(F_{pq}) \xrightarrow{q^\infty} C_*(X, B) \otimes_{p_X} H_*(F_p) \cong C_*(X, B) \otimes_{p_X} H_*(F_p) \otimes H_*(F_q)$ preserve the homological filtrations and induce isomorphisms on the filtration quotients. □

9.3 Lemma *Let $p: E \rightarrow B$ be a unipotent fibration and let $q: E \times S^0 \rightarrow E$ be the product fibration with fiber S^0 . Then*

$$\text{Wh}_h^{\mathbb{Q}}((pq)^!) \simeq \text{Wh}_h^{\mathbb{Q}}(q^!) \circ \text{Wh}_h^{\mathbb{Q}}(p^!).$$

Proof The basic outline of our argument is the same as in the proof of Lemma 9.2. The same construction as there gives a homotopy $H_A(p, q)$ between the maps $A((pq)^!)$ and $A(p^!)A(q^!)$. Next, we need a homotopy H_K between the maps $\chi(F_p \times S^0)$ and $\chi(S^0)\chi(F_p)$. The first map comes from the functor $C \mapsto C \otimes H_*(F_p \times S^0)$ while the second map is induced by the functor $C \mapsto C \otimes H_*(F_p) \otimes H_*(S^0)$. The isomorphism $H(F_p) \otimes H_*(S^0) \cong H_*(F_p \times S^0)$ induces a natural isomorphism of the above functors, which in turn defines the homotopy H_K .

As the result of these constructions we obtain a diagram of homotopies (9-1) (with $F_q = S^0$ and $F_{pq} = F_p \times S^0$). The remaining step is to show that this diagram can be filled by a homotopy of homotopies. Existence of such a homotopy of homotopies can

be verified in a straightforward manner using the fact that for $X \in \mathcal{R}^{\text{fd}}(B)$ we have a commutative diagram:

$$\begin{array}{ccc}
 C_*(q^* p^* X, E \times S^0) & \xrightarrow{\cong} & C(p^* X, E) \otimes H_*(S^0) \\
 \beta_{pq} \downarrow & & \downarrow \beta_p \otimes \text{id} \\
 C'_*(X, B) \otimes_{(pq)_X} H(E \times S^0) & \xrightarrow{\cong} & (C(X, B) \otimes_{p_X} H_*(F_p)) \otimes H_*(S^0) \quad \square
 \end{array}$$

9.4 Igusa’s argument In [11], Igusa used Lemma 8.1 to show that two higher torsion invariants of unipotent bundles coincide. We will adapt this argument to prove Theorems 1.3 and 9.1. The main idea of Igusa’s proof is to define a “difference torsion”, which is a new invariant that measures the difference between the given two invariants of bundles, and then to show that this difference torsion vanishes for all unipotent bundles. The essential properties of Igusa’s difference torsion are encapsulated in Definition 9.5. Proposition 9.6 spells out the conditions that guarantee its vanishing.

9.5 Definition Let B be a space of the homotopy type of a finite CW-complex, and let Λ be an abelian group. An additive homotopy B -invariant of unipotent fibrations with values in Λ is an assignment Φ that associates to each unipotent fibration $p: E \rightarrow B$ an element $\Phi(p) \in \Lambda$ and that satisfies the following conditions:

Additivity Given maps of fibrations over B

$$\begin{array}{ccccc}
 & & j & & \\
 & & \longleftarrow & & \longrightarrow \\
 E_1 & & E_0 & & E_2 \\
 & \searrow p_1 & \downarrow p_0 & \swarrow p_2 & \\
 & & B & &
 \end{array}$$

where p_i is a unipotent fibration for $i = 0, 1, 2$ and j is a cofibration, we have

$$\Phi(p_1 \cup_{p_0} p_2) = \Phi(p_1) + \Phi(p_2) - \Phi(p_0).$$

Homotopy invariance If unipotent fibrations $p_i: E_i \rightarrow B$ ($i = 1, 2$) are fiberwise homotopy equivalent then $\Phi(p_1) = \Phi(p_2)$.

9.6 Proposition Let Φ be an additive homotopy B -invariant with values in Λ . Assume that

- $\Phi(p) = 0$ for the product fibration $p: B \times S^0 \rightarrow B$;
- $\Phi(p) = 0$ if the map $p: E \rightarrow B$ is a rational homotopy equivalence.

Then $\Phi(p) = 0$ for all unipotent fibrations $p: E \rightarrow B$.

Proof First notice that $\Phi(p) = 0$ if $p: B \times F \rightarrow B$ is a product fibration where F is either a disc or a sphere. Indeed, in the first case p is a rational homotopy equivalence. If F is a sphere we can argue inductively starting with $F = S^0$ and using additivity.

Next, let $p: E \rightarrow B$ be an arbitrary unipotent fibration. Applying Lemma 8.1 to p we obtain a sequence of fibrations:

$$(9-2) \quad \begin{array}{ccccccc} E_0 & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_k \\ & \searrow p_0 & \downarrow p_1 & & & \swarrow p_k & \\ & & B & & & & \end{array}$$

Since p_k is a rational homotopy equivalence we have $\Phi(p_k) = 0$. Next, since p_i is obtained as a pushout of p_{i-1} and product fibrations with disc and sphere fibers we can use additivity of Φ to get that $\Phi(p_i) = \Phi(p_{i-1})$. This gives

$$0 = \Phi(p_k) = \Phi(p_{k-1}) = \cdots = \Phi(p_0).$$

Finally, since p_0 is an n -fold fiberwise suspension of p we can use additivity of Φ again to get

$$\Phi(p) = (-1)^n \Phi(p_0) = 0. \quad \square$$

9.7 Note Igusa uses a variant of Proposition 9.6 that will be also useful to us later on. Namely, for a smooth compact manifold B consider an assignment Φ that satisfies additivity and homotopy invariance properties as in Definition 9.5, but is defined only for unipotent bundles over B . Then the statement of Proposition 9.6 still holds: if Φ vanishes on the product bundle $B \times S^0 \rightarrow B$ and on bundles that are given by a rational homotopy equivalence then it vanishes on all unipotent bundles. The proof of this fact is essentially the same as the proof of Proposition 9.6 with two additional observations:

- The homotopy invariance of Φ lets us define this invariant on all smoothable unipotent fibrations over B , ie all fibrations that are fiberwise homotopy equivalent to a unipotent bundle.
- If $p: E \rightarrow B$ is a unipotent bundle then all fibrations appearing in the diagram (9-2) are smoothable [11, Lemma 8.6].

We are now ready to give:

Proof of Theorem 9.1 Let $p: E \rightarrow B$ and $q: D \rightarrow E$ be unipotent fibrations. We have commutative diagrams:

$$\begin{array}{ccc}
 \mathrm{Wh}_h^{\mathbb{Q}}(B) & \xrightarrow{\mathrm{Wh}_h^{\mathbb{Q}}((pq)^{\dagger})} & \mathrm{Wh}_h^{\mathbb{Q}}(D) \\
 i_B \downarrow & & \downarrow i_D \\
 A(B) & \xrightarrow{A((pq)^{\dagger})} & A(D)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathrm{Wh}_h^{\mathbb{Q}}(B) & \xrightarrow{\mathrm{Wh}_h^{\mathbb{Q}}(q^{\dagger})\mathrm{Wh}_h^{\mathbb{Q}}(p^{\dagger})} & \mathrm{Wh}_h^{\mathbb{Q}}(D) \\
 i_B \downarrow & & \downarrow i_D \\
 A(B) & \xrightarrow{A(q^{\dagger})A(p^{\dagger})} & A(D)
 \end{array}$$

In the same way as in the proof of Lemma 9.2 we construct a homotopy $H_A(p, q)$ between the maps $A((pq)^{\dagger})$ and $A(q^{\dagger})A(p^{\dagger})$. By Section A.3 this data defines a map $\varphi(p, q): \mathrm{Wh}_h^{\mathbb{Q}}(B) \rightarrow \Omega K(\mathbb{Q})$ such that $[\varphi(p, q)] = 0$ if and only if H_A admits a lift to a homotopy between the maps $\mathrm{Wh}_h^{\mathbb{Q}}((pq)^{\dagger})$ and $\mathrm{Wh}_h^{\mathbb{Q}}(q^{\dagger})\mathrm{Wh}_h^{\mathbb{Q}}(p^{\dagger})$.

Fix a fibration $p: E \rightarrow B$. Let Φ_p be the assignment that associates to a unipotent fibration $q: D \rightarrow E$ the homotopy class $[\varphi(p, q)]$. We claim that Φ_p is an additive homotopy E -invariant with values in the group $[\mathrm{Wh}_h^{\mathbb{Q}}(B), \Omega K(\mathbb{Q})]$; see Definition 9.5. In order to verify homotopy invariance of Φ_p assume that we have a fiberwise homotopy equivalence:

$$\begin{array}{ccc}
 D_1 & \xrightarrow{f} & D_2 \\
 & \searrow q_1 & \swarrow q_2 \\
 & E &
 \end{array}$$

We need to check that $\varphi(p, q_1)$ and $\varphi(p, q_2)$ are homotopic. By Proposition 6.1 we can construct a homotopy

$$H_f^{\mathrm{Wh}(E)}: \mathrm{Wh}_h^{\mathbb{Q}}(E) \times I \rightarrow \mathrm{Wh}_h^{\mathbb{Q}}(D_2)$$

between the maps $f_*\mathrm{Wh}_h^{\mathbb{Q}}(q_1^{\dagger})$ and $\mathrm{Wh}_h^{\mathbb{Q}}(q_2^{\dagger})$. As a consequence the map

$$\bar{H}_f^{\mathrm{Wh}(E)} := H_f^{\mathrm{Wh}(E)} \circ (\mathrm{Wh}_h^{\mathbb{Q}}(p^{\dagger}) \times \mathrm{id}_I)$$

is a homotopy between $f_*\mathrm{Wh}_h^{\mathbb{Q}}(q_1^{\dagger})\mathrm{Wh}_h^{\mathbb{Q}}(p^{\dagger})$ and $\mathrm{Wh}_h^{\mathbb{Q}}(q_2^{\dagger})\mathrm{Wh}_h^{\mathbb{Q}}(p^{\dagger})$. On the other hand f is also a fiberwise homotopy equivalence of the fibrations pq_1 and pq_2 , so we have a homotopy

$$H_f^{\mathrm{Wh}(B)}: \mathrm{Wh}_h^{\mathbb{Q}}(B) \times I \rightarrow \mathrm{Wh}_h^{\mathbb{Q}}(D_2)$$

between the maps $f_*\mathrm{Wh}_h^{\mathbb{Q}}((pq_1)^{\dagger})$ and $\mathrm{Wh}_h^{\mathbb{Q}}((pq_2)^{\dagger})$. By the proof of Proposition 6.1 we also have a homotopy

$$H_f^{A(B)}: A(B) \times I \rightarrow A(D_2)$$

between the maps $f_*A((pq_1)^!)$ and $A((pq_2)^!)$ as well as a homotopy

$$\bar{H}_f^{A(E)} := H_f^{A(E)} \circ (A(p^!) \times \text{id}_I)$$

between $f_*A(q_1^!)A(p^!)$ and $A(q_2^!)A(p^!)$. All these homotopies fit into commutative diagrams:

$$\begin{array}{ccc} \text{Wh}_h^{\mathbb{Q}}(B) \times I & \xrightarrow{H_f^{\text{Wh}(B)}} & \text{Wh}_h^{\mathbb{Q}}(D_2) \\ i_B \times \text{id}_I \downarrow & & \downarrow i_{D_2} \\ A(B) \times I & \xrightarrow{H_f^{A(B)}} & A(D_2) \end{array} \qquad \begin{array}{ccc} \text{Wh}_h^{\mathbb{Q}}(B) \times I & \xrightarrow{\bar{H}_f^{\text{Wh}(E)}} & \text{Wh}_h^{\mathbb{Q}}(D) \\ i_B \times \text{id}_I \downarrow & & \downarrow i_{D_2} \\ A(B) \times I & \xrightarrow{\bar{H}_f^{A(E)}} & A(D) \end{array}$$

Consider the diagram:

$$(9-3) \quad \begin{array}{ccc} f_*A((pq_1)^!) & \xrightarrow{f_*H_A(p, q_1)} & f_*A(q_1^!)A(p^!) \\ H_f^{A(B)} \downarrow & & \downarrow \bar{H}_f^{A(E)} \\ A((pq_2)^!) & \xrightarrow{H_A(p, q_2)} & A(q_2^!)A(p^!) \end{array}$$

Each vertex of the above diagram represents a map $A(B) \rightarrow A(D_2)$, and its edges represent homotopies of such maps. The diagram is induced by a diagram of functors $\mathcal{R}^{\text{fd}}(B) \rightarrow \mathcal{R}^{\text{fd}}(D_2)$ and natural equivalences of such functors. It is straightforward to check that this underlying diagram of functors commutes. This implies that the diagram (9-3) can be filled by a homotopy of homotopies. This homotopy of homotopies can be interpreted as a homotopy between $H_f^{A(B)}$ and $\bar{H}_f^{A(E)}$. By Section A.3 this homotopy defines a map $\text{Wh}_h^{\mathbb{Q}}(B) \times I \rightarrow \Omega K(\mathbb{Q})$. One can check that this map determines a homotopy between $\varphi(p, q_1)$ and $\varphi(p, q_2)$. Additivity of Φ_p can be verified in a similar way, using additivity of secondary transfers.

By Lemma 9.2 and Lemma 9.3 we have $\Phi_p(q) = 0$ if q is a product fibration or a rational homotopy equivalence. Proposition 9.6 implies then that $[\varphi(p, q)] = \Phi_p(q) = 0$ for any unipotent fibration $q: D \rightarrow E$. □

Proof of Theorem 1.3 Let $p: E \rightarrow B$ and $q: D \rightarrow E$ be unipotent bundles. Recall (Section 3.2) that the map $Q(p^!)$ is induced by functors between categories of partitions that assign to a partition $P \subseteq B \times I$ the partition $(p \times \text{id})^{-1}(P) \subseteq E \times I$. Since $(pq \times \text{id})^{-1}(P) = (q \times \text{id})^{-1}(p \times \text{id})^{-1}(P)$ we obtain a homotopy $H_Q(p, q)$ between the maps $Q((pq)^!)$ and $Q(q^!)Q(p^!)$. By Section A.3 the maps $\text{Wh}_s^{\mathbb{Q}}(q^!)\text{Wh}_s^{\mathbb{Q}}(p^!)$,

$\text{Wh}_s^{\mathbb{Q}}((pq)^!)$ and the homotopy $H_Q(p, q)$ define a map $\psi(p, q): \text{Wh}_s^{\mathbb{Q}}(B) \rightarrow \Omega K(\mathbb{Q})$ such that $\text{Wh}_s^{\mathbb{Q}}(q^!)\text{Wh}_s^{\mathbb{Q}}(p^!) \simeq \text{Wh}_h^{\mathbb{Q}}((pq)^!)$ if $[\psi(p, q)] = 0$.

It remains to show that $[\psi(p, q)] = 0$ for all unipotent bundles p and q . Recall (Section 3.3) that by μ_p we denoted the homotopy between $A(p^!)a_B$ and $a_E Q(p^!)$. Consider the diagram:

$$(9-4) \quad \begin{array}{ccc} a_D Q((pq)^!) & \xrightarrow{a_D H_Q(p, q)} & a_D Q(q^!)Q(p^!) \\ \downarrow \mu_{pq} & & \downarrow \mu_q \circ (Q(p^!) \times \text{id}_I) \\ A((pq)^!)a_B & \xrightarrow{H_A(p, q) \circ (a_B \times \text{id}_I)} & A(q^!)A(p^!)a_B \end{array}$$

Vertices of this diagram corresponds to a map $Q(B_+) \rightarrow A(D)$ and edges correspond to homotopies of such maps. Each of these homotopies is induced by natural isomorphisms between retractive spaces over D obtained from partitions $P \subseteq B \times I$:

$$\begin{array}{ccc} (pq \times \text{id})^{-1}(P) & \longrightarrow & (q \times \text{id})^{-1}(p \times \text{id})^{-1}(P) \\ \downarrow & & \downarrow \\ (pq)^* P & \longrightarrow & q^* p^* P \end{array}$$

Since this diagram commutes we obtain a homotopy of homotopies filling diagram (9-4).

Let $b_B: \text{Wh}_s^{\mathbb{Q}}(B) \rightarrow \text{Wh}_h^{\mathbb{Q}}(B)$ be the map induced by the map of fibrations:

$$\begin{array}{ccc} \text{Wh}_s^{\mathbb{Q}}(B) & \xrightarrow{b_B} & \text{Wh}_h^{\mathbb{Q}}(B) \\ \downarrow & & \downarrow \\ Q(B_+) & \xrightarrow{a_B} & A(B) \\ \searrow \lambda_B & & \swarrow \lambda_B^h \\ & K(\mathbb{Q}) & \end{array}$$

The construction of the map $\psi(p, q)$ described in Section A.3 combined with existence of a homotopy of homotopies in the diagram (9-4) gives

$$\psi(p, q) \simeq \varphi(p, q)b_B$$

where $\varphi(p, q)$ is the map defined in the proof of Theorem 9.1. In that proof we showed that $[\varphi(p, q)] = 0$, so also $[\psi(p, q)] = 0$. \square

10 Secondary transfer and smooth torsion

In [2] and [1] (joint with Williams and Klein) we described a homotopy theoretical construction of the smooth torsion of unipotent bundles and showed that it defines characteristic classes which coincide with the higher torsion invariants of Igusa and Klein. The construction of the smooth torsion of a bundle $p: B \rightarrow E$ proceeds as follows. Let $\eta_B: B \rightarrow Q(B_+)$ denote the coaugmentation map. By [2, Theorem 6.7] the map

$$\lambda_E Q(p^!) \eta_B: B \rightarrow K(\mathbb{Q})$$

is homotopic via a preferred homotopy ω_p to a constant map. This defines a map $\tau^s(p): B \rightarrow \text{Wh}_s^{\mathbb{Q}}(E)$ which is a lift of $Q(p^!) \eta_B$. The map $\tau^s(p)$ is the smooth torsion of the bundle p .

The secondary transfer of unipotent bundles described in this paper can be used to construct another map $\bar{\tau}^s(p): B \rightarrow \text{Wh}_s^{\mathbb{Q}}(E)$. Namely, since the identity map $\text{id}_B: B \rightarrow B$ can be considered as a unipotent bundle, it defines the smooth torsion $\tau^s(\text{id}_B): B \rightarrow \text{Wh}_s^{\mathbb{Q}}(B)$. We set

$$\bar{\tau}^s(p) := \text{Wh}_s^{\mathbb{Q}}(p^!) \tau^s(\text{id}_B).$$

Our final goal in this paper is to prove Theorem 1.4, which says that for any unipotent bundle p the maps $\tau^s(p)$ and $\bar{\tau}^s(p)$ are homotopic. We will also show that as a consequence the statement of Theorem 1.1 holds: for any pair of composable unipotent bundles p and q the higher torsion cohomology classes of p , q and pq are related by the formula (1-1).

The proof of Theorem 1.4 will use the same scheme as the proof of the composition formula of unipotent fibrations (Theorem 9.1). We will first show that this theorem holds when p is either a product bundle or it is a rational homotopy equivalence, and then we will use Igusa's argument (Section 9.4) to extend this result to all unipotent bundles.

10.1 Lemma *Let $p: E \rightarrow B$ be a unipotent bundle. If p is either the product bundle with fiber S^0 or a rational homotopy equivalence then $\tau^s(p) \simeq \bar{\tau}^s(p)$.*

Proof This follows essentially from [1, Section 6]. We proved there that if $p: E \rightarrow B$ is a unipotent bundle that satisfies the assumptions of the Leray–Hirsch theorem then the quasi-isomorphisms

$$C_*(p^* X) \xrightarrow{\simeq} C_*(X) \otimes H_*(F_p)$$

given for $X \in \mathcal{R}^{\text{fd}}(B)$ by that theorem define a map

$$\text{Wh}_{LH}^{\mathbb{Q}}(p^!): \text{Wh}_s^{\mathbb{Q}}(B) \rightarrow \text{Wh}_s^{\mathbb{Q}}(E)$$

and that $\tau^s(p) \simeq \text{Wh}_{LH}^{\mathbb{Q}}(p^!)\tau^s(\text{id}_B)$. It is straightforward to check that if p is a product bundle $B \times S^0 \rightarrow B$ or a rational homotopy equivalence then $\text{Wh}_{LH}^{\mathbb{Q}}(p^!) \simeq \text{Wh}_s^{\mathbb{Q}}(p^!)$. □

Proof of Theorem 1.4 Both $\tau^s(p)$ and $\bar{\tau}^s(p)$ are defined as lifts of the map $\varrho(p^!)\eta_B$. As a consequence they define a map $\varrho(p): B \rightarrow \Omega K(\mathbb{Q})$ such that the homotopy class of the composition

$$B \xrightarrow{\varrho(p)} \Omega K(\mathbb{Q}) \longrightarrow \text{Wh}_s^{\mathbb{Q}}(E)$$

coincides with the element $[\tau^s(p)] - [\bar{\tau}^s(p)] \in [B, \text{Wh}_s^{\mathbb{Q}}(E)]$. It suffices to show that $[\varrho(p)] = 0$ in $[B, \Omega K(\mathbb{Q})]$.

We claim that the assignment $p \mapsto [\varrho(p)]$ is an additive homotopy B -invariant of unipotent bundles with values in $[B, \Omega K(\mathbb{Q})]$; see Note 9.7. Indeed, additivity of this assignment follows essentially from the additivity of the secondary transfer (Theorem 7.3) and the additivity of the smooth torsion [1, Theorem 5.1]. Homotopy invariance can be verified using the fact that the construction of $\varrho(p)$ involves only chain complexes associated to p . Using Lemma 10.1 we obtain that $[\varrho(p)] = 0$ if p is either a product bundle or a rational homotopy equivalence. By Proposition 9.6 and Note 9.7 we get then that $[\varrho(p)] = 0$ for all unipotent bundles p . □

Proof of Theorem 1.1 Combining Theorems 1.4 and 1.3 we obtain the statement of Corollary 1.5: for any unipotent bundles $p: E \rightarrow B$ and $q: D \rightarrow E$ there is a homotopy

$$\tau^s(pq) \simeq \text{Wh}_s^{\mathbb{Q}}(q^!)\tau^s(p).$$

In [1, Theorem 7.1] an analogous decomposition of the smooth torsion of pq (in the case where q is a Leray–Hirsch bundle) was the main ingredient in the proof of the fact that the cohomological torsion of p and q satisfies the formula (1-1). The same argument can be now used to obtain the formula (1-1) for arbitrary unipotent bundles p and q . □

Appendix: Maps of homotopy fibers

Multiple arguments in this paper involve constructions of maps between homotopy fibers as well as constructions of homotopies between such maps. We summarize here the basic scheme of such constructions.

A.1 Maps of homotopy fibers For a space X with a basepoint x_0 let $P_{x_0}X$ denote the space of paths in X that start at x_0 . By the homotopy fiber of a map $p: Y \rightarrow X$ over x_0 we understand the standard construction

$$\text{hofib}(p)_{x_0} := \{(\omega, y) \in P_{x_0}X \times Y \mid \omega(1) = p(y)\}.$$

We will denote by $i_Y: \text{hofib}(p)_{x_0} \rightarrow Y$ the map given by $i_Y(\omega, y) = y$.

Assume that we have a diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{f}} & Y' \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{f} & X' \end{array}$$

such that $f(x_0) = x'_0$. Given a homotopy h from fp to $p'\tilde{f}$ we obtain a map $\tilde{f}: \text{hofib}(p)_{x_0} \rightarrow \text{hofib}(p')_{x'_0}$ given by

$$\tilde{f}(\omega, y) := (f\omega * h_y, \tilde{f}(y)).$$

Here h_y denotes the path in X' defined by $h_y(t) = h(y, t)$, and $*$ indicates concatenation of paths. We have

$$i_{Y'} \tilde{f} = \tilde{f} i_Y.$$

A.2 Homotopies of maps of homotopy fibers Assume that we have two diagrams

$$(A-1) \quad \begin{array}{ccc} Y & \xrightarrow{\tilde{f}_0} & Y' \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{f_0} & X' \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{\tilde{f}_1} & Y' \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{f_1} & X' \end{array}$$

that commute up to homotopies h_0 and h_1 , respectively, with $f_0(x_0) = f_1(x_0) = x'_0$. This gives two maps of the homotopy fibers:

$$\tilde{f}_0, \tilde{f}_1: \text{hofib}(p)_{x_0} \rightarrow \text{hofib}(p')_{x'_0}.$$

In order to obtain a homotopy between these maps it suffices to construct the following data:

- (1) a homotopy $\tilde{H}: Y \times I \rightarrow Y'$ between \tilde{f}_0 and \tilde{f}_1 ;
- (2) a basepoint preserving homotopy $H: X \times I \rightarrow X'$ between f_0 and f_1 ;
- (3) a homotopy of homotopies between the two homotopies $p'\tilde{f}_0 \simeq f_1p$: the one given as a concatenation of $p'\tilde{H}$ with h_1 , and the one obtained by concatenating h_0 with $H(p \times \text{id}_I)$:

$$\begin{array}{ccc}
 f_0p & \xrightarrow{H(p \times \text{id}_I)} & f_1p \\
 h_0 \uparrow & & \uparrow h_1 \\
 p'\tilde{f}_0 & \xrightarrow{p'\tilde{H}} & p'\tilde{f}_1
 \end{array}$$

Giving such a homotopy of homotopies is equivalent to giving a map

$$\Theta: Y \times I \times I \rightarrow X'$$

such that $\Theta|_{Y \times \{i\} \times I} = h_i$ for $i = 0, 1$, $\Theta_{Y \times I \times \{0\}} = p'\tilde{H}$ and $\Theta_{Y \times I \times \{1\}} = H(p \times \text{id}_I)$. The homotopy \tilde{H} between \tilde{f}_1 and \tilde{f}_2 is defined by

$$\tilde{H}((\omega, y), t) := (H_t\omega * \Theta_{y,t}, \tilde{H}(y, t))$$

where $H_t: X \rightarrow X'$ is given by $H_t(x) = H(t, x)$ and $\Theta_{y,t}$ is the path in X' given by $\Theta_{y,t}(s) = \Theta(y, t, s)$.

A.3 An obstruction to lifting a homotopy Assume again that we have two homotopy commutative squares (A-1) with \tilde{f}_0 and \tilde{f}_1 the maps of homotopy fibers defined by these squares. Assume also that we have a homotopy \tilde{H} between \tilde{f}_0 and \tilde{f}_1 , and that we want to determine whether there exists a homotopy \tilde{H} between \tilde{f}_1 and \tilde{f}_2 that fits into a commutative diagram:

$$\begin{array}{ccc}
 \text{hofib}(p)_{x_0} \times I & \overset{\tilde{H}}{\dashrightarrow} & \text{hofib}(p')_{x'_0} \\
 i_Y \times \text{id}_I \downarrow & & \downarrow i_{Y'} \\
 Y \times I & \xrightarrow{\tilde{H}} & Y'
 \end{array}$$

In Section A.2 we described data that suffice to construct such \tilde{H} , but here we are interested in a condition that is equivalent to the existence of this homotopy. We can describe such a condition as follows.⁵

⁵ See also [18, Lemma 4.1].

Let $C_p: \text{hofib}(p)_{x_0} \times I \rightarrow X$ be the map given by $C_p((\omega, y), t) = \omega(t)$. This is a homotopy between the constant map into x_0 and the map pi_Y . Consider the following diagram:

$$\begin{array}{ccc}
 p' \tilde{f}_0 i_Y & \xrightarrow{p' \tilde{H}(i_Y \times id_I)} & p' \tilde{f}_1 i_Y \\
 h_0(i_Y \times id_I) \Big| & & \Big| h_1(i_{Y'} \times id_I) \\
 f_0 pi_Y & & f_1 pi_{Y'} \\
 f_0 C_p \searrow & & \swarrow f_1 C_p \\
 & x'_0 &
 \end{array}$$

Each vertex of this diagram represent a map $\text{hofib}(p)_{x_0} \rightarrow X'$ and edges represent homotopies of such maps. Concatenating all homotopies appearing here we obtain a homotopy from the constant map into x'_0 to itself, or equivalently a map

$$\varphi: \text{hofib}(p)_{x_0} \rightarrow \Omega X'.$$

It is straightforward to verify that the map φ is contractible if and only if there exists a homotopy \tilde{H} such that the diagram (A-2) commutes. In other words the homotopy class of φ is an obstruction to lifting the homotopy \tilde{H} to a homotopy defined on the level of the homotopy fibers.

Note Let $p': (Y', y'_0) \rightarrow (X', x'_0)$ be a map of infinite loop spaces where x'_0 and y'_0 are the trivial elements in X' and Y' . In this case the map φ has a simpler interpretation. Namely, let $j_{Y'}: \Omega X' \rightarrow \text{hofib}(p')_{x'_0}$ be the map given by $j_{Y'}(\omega) = (\omega, y_0)$. The set of homotopy classes $[\text{hofib}(p)_{x_0}, \text{hofib}(p')_{x'_0}]$ has a structure of an abelian group and we have

$$j_{Y'*}[\varphi] = [\tilde{f}_1] - [\tilde{f}_2].$$

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