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$\mathrm{GL}^+(2,\mathbb{R})$–orbits in Prym eigenform loci

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This paper is devoted to the classification of $\mathrm{GL}^+(2,\mathbb{R})$–orbit closures of surfaces in the intersection of the Prym eigenform locus with various strata of abelian differentials. We show that the following dichotomy holds: an orbit is either closed or dense in a connected component of the Prym eigenform locus.

The proof uses several topological properties of Prym eigenforms. In particular, the tools and the proof are independent of the recent results of Eskin and Mirzakhani and Eskin, Mirzakhani and Mohammadi.

As an application we obtain a finiteness result for the number of closed $\mathrm{GL}^+(2,\mathbb{R})$–orbits (not necessarily primitive) in the Prym eigenform locus $\Omega E_D(2,2)$ for any fixed D that is not a square.

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1 Introduction

For any $g \geq 1$ and any integer partition $\kappa = (\kappa_1, \dots, \kappa_r)$ of $2g-2$ we denote by $\mathcal{H}(\kappa)$ a stratum of the moduli space of marked abelian differentials of type κ, ie of pairs (X,ω), where X is a Riemann surface of genus g and ω is a holomorphic 1–form having r zeros with prescribed multiplicities $\kappa_1, \dots, \kappa_r$. Analogously, one defines the strata of the moduli space of *marked quadratic differentials* $\mathcal{Q}(\kappa')$ having zeros and simple poles of multiplicities $\kappa'_1, \dots, \kappa'_s$ with $\sum_{i=1}^{s} \kappa'_s = 4g-4$ (simple poles correspond to "zeros of multiplicity -1").

The 1–form ω defines a canonical flat metric on S (the underlying topological surface) with conical singularities at Σ, the zeros of ω. Therefore we will refer to points of $\mathcal{H}(\kappa)$ as flat surfaces or *translation surfaces* (two translation surfaces are equivalent if they differ by precomposition by a homeomorphism of S which fixes Σ and is isotopic to the identity rel Σ). The strata admit a natural action of the group $\mathrm{GL}^+(2,\mathbb{R})$ that can be viewed as a generalization of the $\mathrm{GL}^+(2,\mathbb{R})$–action on the space $\mathrm{GL}^+(2,\mathbb{R})/\,\mathrm{SL}(2,\mathbb{Z})$ of flat tori. For an introduction to this subject, we refer to the excellent surveys by Masur and Tabachnikov [17] and Zorich [31].

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It has been discovered that many topological and dynamical properties of a translation surface can be revealed by its $\mathrm{GL}^+(2,\mathbb{R})$–orbit closure. The most spectacular example of this phenomenon is the case of *Veech surfaces*, or *lattice surfaces*, that is, surfaces whose $\mathrm{GL}^+(2,\mathbb{R})$–orbit is a closed subset in its stratum; for such surfaces, the famous *Veech dichotomy* holds: the linear flow in any direction is either periodic or uniquely ergodic.

It follows from the foundational results of Masur and Veech that most $\mathrm{GL}^+(2,\mathbb{R})$ orbits are dense in their stratum. However, in any stratum there always exist surfaces whose orbits are closed: for example, coverings of the standard flat torus, which are commonly known as *square-tiled surfaces*.

During the past three decades, much effort has been made in order to obtain the list of possible $\mathrm{GL}^+(2,\mathbb{R})$–orbit closures and to understand their structure as subsets of strata. So far, such a list is only known in genus two by the work of McMullen [26], but the problem is widely open in higher genus, even though some breakthroughs have been achieved recently (see below).

In genus two the complex dimensions of the connected strata $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$ are, respectively, 4 and 5. In this situation, McMullen proved that if a $\mathrm{GL}^+(2,\mathbb{R})$–orbit is not dense, then it belongs to a *Prym eigenform locus*, which is a submanifold of complex dimension 3. In this case, the orbit is either closed or dense in the whole Prym eigenform locus. These (closed) invariant submanifolds, which we denote by ΩE_D, where D is a *discriminant* (that is $D \in \mathbb{N}$, $D \equiv 0, 1 \bmod 4$), are characterized by the following properties:

(1) Every surface $(X,\omega) \in \Omega E_D$ has a holomorphic involution $\tau\colon X \to X$.

(2) The *Prym variety* $\mathrm{Prym}(X,\tau) = (\Omega^-(X,\tau))^*/H_1(X,\mathbb{Z})^-$ admits a real multiplication by some quadratic order $\mathcal{O}_D := \mathbb{Z}[x]/(x^2+bx+c)$, $b,c \in \mathbb{Z}$, $b^2-4c = D$.

(Here $\Omega^-(X,\tau) = \{\eta \in \Omega(X) \mid \tau^*\eta = -\eta\}$).

Later, these properties were extended to higher genera (up to genus five); see McMullen [20; 24] and Lanneau and Nguyen [15] for more details.

Recently, Eskin, Mirzakhani and Mohammadi [7; 8] have announced a proof of the conjecture that any $\mathrm{GL}^+(2,\mathbb{R})$–orbit closure is an affine invariant submanifold of $\mathcal{H}(\kappa)$. This result is of great importance in view of the classification of orbit closures as it provides some very important characterizations of such subsets. However *a priori* this result does not allow us to construct explicitly such invariant submanifolds.

So far, most $\mathrm{GL}^+(2,\mathbb{R})$–invariant submanifolds of a stratum are obtained from coverings of translation surfaces of lower genera. The only known examples of invariant submanifolds *not* arising from this construction belong to one of the following families:

(1) Primitive Teichmüller curves (closed orbits).

(2) Prym eigenforms.

This paper is concerned with the classification of $\mathrm{GL}^+(2,\mathbb{R})$–orbit closures in the space of Prym eigenforms. To be more precise, for any nonempty stratum $\mathcal{Q}(\kappa')$, there is a (local) affine map $\phi\colon \mathcal{Q}_{g'}(\kappa') \to \mathcal{H}_g(\kappa)$ given by the orientation double covering (the indices g and g' are the genera of the corresponding Riemann surfaces). When $g-g'=2$, following McMullen [24] we call the image of ϕ a Prym locus and denote it by $\mathrm{Prym}(\kappa)$. Those Prym loci contain $\mathrm{GL}^+(2,\mathbb{R})$–invariant suborbifolds denoted by $\Omega E_D(\kappa)$ (see Section 2 for more precise definitions). We will investigate the $\mathrm{GL}^+(2,\mathbb{R})$–orbit closures in $\Omega E_D(\kappa)$. The first main theorem of this paper is the following:

Theorem 1.1 *Let $(X,\omega)\in\Omega E_D(\kappa)$ be a Prym eigenform, where $\Omega E_D(\kappa)$ has complex dimension 3 (ie $\Omega E_D(\kappa)$ is contained in one of the Prym loci in Table 1). We denote by $\mathcal{O}$ its orbit under $\mathrm{GL}^+(2,\mathbb{R})$. Then:*

(1) *Either $\mathcal{O}$ is closed (ie (X,ω) is a Veech surface), or*

(2) *$\overline{\mathcal{O}}$ is a connected component of $\Omega E_D(\kappa)$.*

$\mathcal{Q}(\kappa')$	$\mathrm{Prym}(\kappa)$	$g(X)$
$\mathcal{Q}_0(-1^6,2)$	$\mathrm{Prym}(1,1)\simeq\mathcal{H}(1,1)$	2
$\mathcal{Q}_1(-1^3,1,2)$	$\mathrm{Prym}(1,1,2)$	3
$\mathcal{Q}_1(-1^4,4)$	$\mathrm{Prym}(2,2)^{\mathrm{odd}}$	3
$\mathcal{Q}_2(-1^2,6)$	$\mathrm{Prym}(3,3)\simeq\mathcal{H}(1,1)$	4
$\mathcal{Q}_2(1^2,2)$	$\mathrm{Prym}(1^2,2^2)\simeq\mathcal{H}(0^2,2)$	4
$\mathcal{Q}_2(-1,2,3)$	$\mathrm{Prym}(1,1,4)$	4
$\mathcal{Q}_2(-1,1,4)$	$\mathrm{Prym}(2,2,2)^{\mathrm{even}}$	4
$\mathcal{Q}_3(8)$	$\mathrm{Prym}(4,4)^{\mathrm{even}}$	5

Table 1: Prym loci for which the corresponding stratum of quadratic differentials has (complex) dimension 5. The Prym eigenform locus $\Omega E_D(\kappa)$ has complex dimension 3. Observe that the stratum $\mathcal{H}(1,1)$ in genus 2 is a particular case of Prym locus.

Observe that the case $\kappa = (1,1)$ is part of the classification in genus two, which is obtained via decompositions of translation surfaces of genus two into connected sums of two tori (see McMullen [26]).

Remark 1.2 We will address the classification of connected components of $\Omega E_D(2,2)$ and $\Omega E_D(1,1,2)$ in a forthcoming paper [14] (see also [15] for related work). The statement is the following: for any discriminant $D \geq 8$ and $\kappa \in \{(2,2),(1,1,2)\}$, the locus $\Omega E_D(\kappa)$ is nonempty if and only if $D \equiv 0,1,4 \bmod 8$, and it is connected if $D \equiv 0,4 \bmod 8$, and has two connected components otherwise.

Even though Theorem 1.1 is a particular case of the recent results of Eskin and Mirzakhani [7] and Eskin, Mirzakhani and Mohammadi [8], our proof is independent from these works. It is based on the geometry of the kernel foliation on the space of Prym eigenforms. It also seems likely to us that the method introduced here can be generalized to yield Eskin, Mirzakhani and Mohammadi's result in invariant submanifolds which possess the *complete periodicity* property (see Section 2D).

We will also prove a finiteness result for Teichmüller curves in the locus $\Omega E_D(2,2)^{\mathrm{odd}}$; this is our second main result:

Theorem 1.3 *If D is not a square, there exist only finitely many closed* $\mathrm{GL}^+(2,\mathbb{R})$*–orbits in $\Omega E_D(2,2)^{\mathrm{odd}}$.*

We end with a few remarks on Theorem 1.3.

Remark 1.4

- To the authors' knowledge, such a finiteness result is not a direct consequence of the work by Eskin, Mirzakhani and Mohammadi.
- In $\mathrm{Prym}(1,1)$ a stronger statement holds: there exist only finitely many closed $\mathrm{GL}^+(2,\mathbb{R})$–orbits in $\bigsqcup_{D \text{ not a square}} \Omega E_D(1,1)$ (see McMullen [23; 25]). The same result holds for $\mathrm{Prym}(1,1,2)$: this is proved in a forthcoming paper by the first author and M Möller [13]. However, this is no longer true in $\mathrm{Prym}(2,2)^{\mathrm{odd}}$, as we will see in Theorem A.1.
- Other finiteness results on Teichmüller curves have been obtained in other situations by different methods; see for instance Möller [28], Bainbridge and Möller [1] and Matheus and Wright [19].

Outline of the paper We end this section with a sketch of the proofs of Theorem 1.1 and Theorem 1.3. Before going into the details, we single out the relevant properties of $\Omega E_D(\kappa)$ for our purpose. In what follows, (X,ω) will denote a surface in $\Omega E_D(\kappa)$ (sometimes we will simply use X when there is no confusion).

(1) Each locus is preserved by the *kernel foliation*, that is, $(X,\omega)+v$ is well defined for any sufficiently small vector $v \in \mathbb{R}^2$ (see Section 3). Up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, there exists $\varepsilon > 0$ such that a neighborhood of (X,ω) in $\Omega E_D(\kappa)$ can be identified with the set

$$\{(X,\omega)+v \mid |v| < \varepsilon\}.$$

(2) Every surface in $\Omega E_D(\kappa)$ is completely periodic in the sense of Calta: any direction of a simple closed geodesic is actually completely periodic, which means that the surface is decomposed into cylinders in this direction. The number of cylinders is bounded from above by $g+|\kappa|-1$, where $|\kappa|$ is the number of zeros of ω (see Section 2).

(3) Assume that (X,ω) decomposes into cylinders in the horizontal direction. Then the moduli of those cylinders are related by some equations with rational coefficients (see Proposition 5.1).

(4) The cylinder decomposition in a completely periodic direction is said to be *stable* if there is no saddle connection connecting two different zeros in this direction. The stable periodic directions are *generic* for the kernel foliation in the following sense: if the horizontal direction is stable for (X,ω), then there exists $\varepsilon > 0$ such that for any $v \in \mathbb{R}^2$ with $|v| < \varepsilon$, the horizontal direction is also periodic and stable on $X+v$. If the horizontal direction is *unstable* then there exists $\varepsilon > 0$ such that for any $v=(x,y)$ with $|v| < \varepsilon$ and $y \neq 0$ the horizontal direction is periodic and stable on $X+v$.

The properties (1)–(3) are explained in Lanneau and Nguyen [16] (see Section 3.1 and Corollary 3.2; Theorem 1.5; Theorem 7.2, respectively). We will give more details on property (4) in Section 4.

We now give a sketch of the proof of our results. The first part of the paper (Sections 3–7) is devoted to the proof of Theorem 1.1, while the second part (Sections 8–12) is concerned with Theorem 1.3.

Sketch of proof of Theorem 1.1 Let $(X,\omega) \in \Omega E_D(\kappa)$ be a Prym eigenform and let $\mathcal{O} := \mathrm{GL}^+(2,\mathbb{R}) \cdot (X,\omega)$ be the corresponding $\mathrm{GL}^+(2,\mathbb{R})$–orbit. We will show that if $\mathcal{O}$ is not a closed subset in $\Omega E_D(\kappa)$ then it is dense in a connected component of $\Omega E_D(\kappa)$.

We first prove a weaker version of Theorem 1.1 (see Section 6) under the additional condition that there exists a completely periodic direction θ on (X,ω) that is not parabolic. We start by applying the horocycle flow in that periodic direction, and use the classical Kronecker's theorem to show that the orbit closure contains the set

$(X,\omega)+x\vec{v}$, where $\vec{v}$ is the unit vector in direction θ, and $x\in(-\varepsilon,\varepsilon)$ with $\varepsilon>0$ small enough. Then we apply the same argument to the surfaces $(X,\omega)+x\vec{v}$ in another periodic direction that is transverse to θ. It follows that $\overline{\mathcal{O}}$ contains a neighborhood of (X,ω), and hence, for any $g\in \mathrm{GL}^+(2,\mathbb{R})$, $\overline{\mathcal{O}}$ contains a neighborhood of $g\cdot(X,\omega)$. Using this fact, we show that for any $(Y,\eta)\in\overline{\mathcal{O}}\setminus\mathcal{O}$, the closure $\overline{\mathcal{O}}$ also contains a neighborhood of (Y,η), from which we deduce that $\overline{\mathcal{O}}$ is an open subset of $\Omega E_D(\kappa)$. Hence $\overline{\mathcal{O}}$ must be a connected component of $\Omega E_D(\kappa)$.

In full generality (see Section 7), we show that if the orbit is not closed and all the periodic directions are parabolic, then it is also dense in a component of $\Omega E_D(\kappa)$. For this, we consider a surface $(Y,\eta)\in\overline{\mathcal{O}}\setminus\mathcal{O}$ for which the horizontal direction is periodic. From property (1), we see that there is a sequence $((X_n,\omega_n))_{n\in\mathbb{N}}$ of surfaces in $\mathcal{O}$ converging to (Y,η) such that we can write $(X_n,\omega_n)=(Y,\eta)+(x_n,y_n)$, where $(x_n,y_n)\to(0,0)$. Property (4) then implies that the horizontal direction is periodic for (X_n,ω_n). Moreover, we can assume that the corresponding cylinder decomposition in (X_n,ω_n) is stable (for n large enough).

For any $x\in(-\varepsilon,\varepsilon)$, where $\varepsilon>0$ is small enough, we show that (up to taking a subsequence) the orbit of the horocycle flow though (X_n,ω_n) contains a surface $(X_n,\omega_n)+(x_n,0)$ such that the sequence (x_n) converges to x. As a consequence, we see that $\overline{\mathcal{O}}$ contains $(Y,\eta)+(x,0)$ for every $x\in(-\varepsilon,\varepsilon)$. We can now conclude that $\overline{\mathcal{O}}$ is a component of $\Omega E_D(\kappa)$ by the weaker version of Theorem 1.1.

Sketch of proof of Theorem 1.3 We first show a finiteness result up to the (real) kernel foliation for surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$ (see Theorem 12.2): if D is not a square then there exists a finite family $\mathbb{P}_D\subset\Omega E_D(2,2)^{\mathrm{odd}}$ such that any $(X,\omega)\in\Omega E_D(2,2)^{\mathrm{odd}}$ having an unstable cylinder decomposition, up to rescaling by $\mathrm{GL}^+(2,\mathbb{R})$, satisfies

$$(X,\omega)=(X_k,\omega_k)+(x,0)\quad \text{for some } (X_k,\omega_k)\in\mathbb{P}_D.$$

Compare to McMullen [22] and Lanneau and Nguyen [15], where a similar result is established.

Now let us assume that there exists an infinite family, say

$$\mathcal{Y}=\bigcup_{i\in I}\mathrm{GL}^+(2,\mathbb{R})\cdot(X_i,\omega_i),$$

of closed $\mathrm{GL}^+(2,\mathbb{R})$–orbits, generated by Veech surfaces (X_i,ω_i), $i\in I$.

By the previous finiteness result, up to taking a subsequence, we assume that $(X_i,\omega_i)=(X,\omega)+(x_i,0)$ for some $(X,\omega)\in\mathbb{P}_D$, where x_i belongs to a finite open interval (a,b) which is independent of i (see Theorem 9.1). Up to taking a subsequence, one can assume that the sequence (x_i) converges to some $x \in [a,b]$. Hence the sequence $(X_i,\omega_i) = (X,\omega) + (x_i,0)$ converges to $(Y,\eta) := (X,\omega) + (x,0)$.

If $x \in (a,b)$ then (Y,η) belongs to $\Omega E_D(2,2)^{\mathrm{odd}}$; otherwise (that is, if $x \in \{a,b\}$), (Y,η) belongs to one of the loci $\Omega E_D(0,0,0)$, $\Omega E_D(4)$, or $\Omega E_{D'}(2)^*$, with $D' \in \{D, D/4\}$ (see Section 9). Then by using a by-product of the proof of Theorem 1.1, replacing $\mathcal{O}$ by $\mathcal{Y}$ (see Theorem 7.2 and Theorem 10.4) we obtain that $\mathcal{Y}$ is dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$. We conclude with Theorem 11.1, which asserts that the set of closed $\mathrm{GL}^+(2,\mathbb{R})$–orbits is not dense in any component of $\Omega E_D(2,2)^{\mathrm{odd}}$ when D is not a square.

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2 Background

For an introduction to translation surfaces, and a nice survey, see eg [17; 31]. In this section we recall necessary background and relevant properties of $\Omega E_D(\kappa)$ for our purpose. For a general reference on Prym eigenforms, see [24].

We will use the following notation throughout the paper:

- $|\cdot|$ is the Euclidean norm on $\mathbb{R}^2$.
- $\|\cdot\|$ is some norm on $\boldsymbol{M}(2,\mathbb{R})$.
- $\boldsymbol{B}(\varepsilon) = \{v \in \mathbb{R}^2 \mid |v| < \varepsilon\}$.
- $\boldsymbol{B}(M,\varepsilon) = \{A \in \mathrm{GL}^+(2,\mathbb{R}) \mid \|A - M\| < \varepsilon\}$.
- $\omega(\gamma) := \int_\gamma \omega$ for any $\gamma \in H_1(X,\mathbb{Z})$.

2A Prym loci and Prym eigenforms

Let X be a compact Riemann surface, and $\tau\colon X \to X$ be a holomorphic involution of X. We define the *Prym variety* of X:

$$\mathrm{Prym}(X,\tau) = (\Omega^-(X,\tau))^*/H_1(X,\mathbb{Z})^-,$$

where $\Omega^-(X,\tau)=\{\eta\in\Omega(X)\mid\tau^*\eta=-\eta\}$. It is an abelian subvariety of the Jacobian variety $\mathrm{Jac}(X):=\Omega(X)^*/H_1(X,\mathbb{Z})$.

For any integer vector $\kappa=(k_1,\dots,k_n)$ with nonnegative entries, we denote by $\mathrm{Prym}(\kappa)\subset\mathcal{H}(\kappa)$ the subset of pairs (X,ω) such that there exists an involution $\tau\colon X\to X$ satisfying $\tau^*\omega=-\omega$, and $\dim_{\mathbb{C}}\Omega^-(X,\tau)=2$. Following McMullen [24], we will call an element of $\mathrm{Prym}(\kappa)$ a *Prym form*. For instance, in genus two, one has $\mathrm{Prym}(2)\simeq\mathcal{H}(2)$ and $\mathrm{Prym}(1,1)\simeq\mathcal{H}(1,1)$ (the Prym involution being the hyperelliptic involution).

Let Y be the quotient of X by the Prym involution (here $g(Y)=g(X)-2$) and π the corresponding (possibly ramified) double covering from X to Y. By pushforward, there exists a meromorphic quadratic differential q on Y (with at most simple poles) so that $\pi^*q=\omega^2$. Let κ' be the integer vector that records the orders of the zeros and poles of q. Then there is a $\mathrm{GL}^+(2,\mathbb{R})$–equivariant bijection between $\mathcal{Q}(\kappa')$ and $\mathrm{Prym}(\kappa)$ [12, page 6].

All the strata of quadratic differentials of dimension 5 are recorded in Table 1. It turns out that the corresponding Prym varieties have complex dimension 2 (ie if (X,ω) is the orientation double covering of (Y,q), then $g(X)-g(Y)=2$).

We now give the definition of Prym eigenforms. Recall that a quadratic order is a ring isomorphic to $\mathcal{O}_D=\mathbb{Z}[X]/(X^2+bX+c)$, where $D=b^2-4c>0$ (quadratic orders being classified by their discriminant D).

Definition 2.1 (real multiplication) Let A be an abelian variety of dimension 2. We say that A admits a real multiplication by $\mathcal{O}_D$ if there exists an injective homomorphism $\mathfrak{i}\colon\mathcal{O}_D\to\mathrm{End}(A)$ such that $\mathfrak{i}(\mathcal{O}_D)$ is a self-adjoint, proper subring of $\mathrm{End}(A)$ (ie for any $f\in\mathrm{End}(A)$, if there exists $n\in\mathbb{Z}\setminus\{0\}$ such that $nf\in\mathfrak{i}(\mathcal{O}_D)$ then $f\in\mathfrak{i}(\mathcal{O}_D)$).

Definition 2.2 (Prym eigenform) For any quadratic discriminant $D>0$, we denote by $\Omega E_D(\kappa)$ the set of $(X,\omega)\in\mathrm{Prym}(\kappa)$ such that $\dim_{\mathbb{C}}\mathrm{Prym}(X,\tau)=2$, $\mathrm{Prym}(X,\tau)$ admits a multiplication by $\mathcal{O}_D$, and ω is an eigenvector of $\mathcal{O}_D$. Surfaces in $\Omega E_D(\kappa)$ are called *Prym eigenforms*.

Prym eigenforms do exist in each Prym locus described in Table 1, as real multiplications arise naturally from pseudo-Anosov homeomorphisms commuting with τ (see [24]).

We now collect several results concerning surfaces having a decomposition into periodic cylinders.

2B Periodic directions and cylinder decompositions

Let (X,ω) be a translation surface. A *cylinder* is a topological annulus embedded in X, isometric to a flat cylinder $\mathbb{R}/w\mathbb{Z}\times(0,h)$. In what follows all cylinders are supposed to be *maximal*, that is, they are not properly contained in a larger one. If $g\geq 2$, the boundary of a maximal cylinder is a finite union of saddle connections. If $\mathcal{C}$ is a cylinder, we will denote by $w(\mathcal{C})$, $h(\mathcal{C})$, $\mu(\mathcal{C})$ the width, height, and modulus of $\mathcal{C}$, respectively ($\mu(\mathcal{C})=h(\mathcal{C})/w(\mathcal{C})$).

Another important parameter of a cylinder is its twist $t(\mathcal{C})$. Note that we only define $t(\mathcal{C})$ when $\mathcal{C}$ is a horizontal cylinder. For that, we first mark a pair of oriented saddle connections on the bottom and the top boundaries of $\mathcal{C}$. This allows us to define a saddle connection contained in $\mathcal{C}$ joining the origins of the marked saddle connections. This gives us a twist vector; its vertical component equals $h(\mathcal{C})$ and its horizontal component is $t(\mathcal{C})$. We emphasize that $t(\mathcal{C})$ depends on the marking (see [10, Section 3]). However, the choice of the marking is irrelevant for our arguments throughout this paper. Therefore, we will refer to $t(\mathcal{C})$ as the twist associated to any marking.

A direction θ is *completely periodic* or simply *periodic* on X if all regular geodesics in this direction are closed. This means that X is the closure of a finite number of cylinders in direction θ, we will say that X admits a *cylinder decomposition* in this direction. Since the Prym involution τ preserves the set of cylinders, it naturally induces an equivalence relation on this set. We will often use the term "number of cylinders up to Prym involution" for the number of τ–equivalence classes of cylinders.

A *separatrix* is a geodesic ray emanating from a zero of ω. It is a well-known fact that a direction is periodic if and only if all the separatrices in this direction are saddle connections. We will often call a separatrix in direction $(1,0)$ a *positive horizontal separatrix*, and a separatrix in direction $(-1,0)$ a *negative horizontal separatrix*.

2C Combinatorial data of a cylinder decomposition

We can associate to any cylinder decomposition a separatrix diagram which encodes the way the cylinders are glued together; see [11]. Given such a diagram, one can reconstruct the surface (X,ω) (up to a rotation) from the widths, heights, and twists of the cylinders. More precisely, if (X,ω) is horizontally periodic, each saddle connection is contained in the upper (respectively, lower) boundary of a unique cylinder. We associate to this cylinder decomposition the following data:

- two partitions of the set of saddle connections into k subsets, where k is the number of cylinders, each subset in these partitions is equipped with a cyclic ordering, and
- a pairing of subsets in these two partitions.

We will call these data the *combinatorial data* or *topological model* of the cylinder decomposition.

2D Complete periodicity

A translation surface (X,ω) is said to be *completely periodic* if it satisfies the following property: for a direction $\theta \in \mathbb{RP}^1$, if the linear flow $\mathcal{F}_\theta$ in the direction θ has a regular closed orbit on X, then θ is a periodic direction. Flat tori and their ramified coverings are completely periodic, as well as Veech surfaces.

It turns out that, if the genus is at least two, the set of surfaces having this property has measure zero. Indeed, complete periodicity is locally expressed via proportionality of a nonempty set of relative periods, and thus is defined by some quadratic equations in the period coordinates. This property was introduced by Calta [2] (see also [3]); she proved that *any* surface in $\Omega E_D(2)$ and $\Omega E_D(1,1)$ is completely periodic. Note that this property can also be deduced from the characterization of eigenforms by McMullen (see [26, Section 6]). Later the authors extended this property to the Prym eigenforms given in Table 1. This property is also proved by A Wright [30] in a more general context by a different argument.

Theorem 2.3 [2; 16; 30] *Any Prym eigenform in the loci $\Omega E_D(\kappa) \subset \mathrm{Prym}(\kappa)$ of Table 1 is completely periodic.*

2E Stable and unstable cylinder decompositions

A cylinder decomposition of (X,ω) is said to be *stable* if every separatrix joins a zero of ω to itself. The decomposition is said to be *unstable* otherwise.

Lemma 2.4 *Let θ be a periodic direction for $(X,\omega) \in \mathcal{H}(\kappa)$ and g be the genus of X. If X has $g+|\kappa|-1$ cylinders in the direction θ, then the cylinder decomposition in this direction is stable ($|\kappa|$ is the number of zeros of ω).*

Proof Let $C_1,\dots,C_n$ be the cylinders in the direction θ of X. For $i=1,\dots,n$, let c_i be a core curve of C_i. Cutting X along c_i, we obtain r compact surfaces with boundary, denoted by $X_1,\dots,X_r$. Since each of X_i must contain at least a zero of ω, we have $r \le |\kappa|$. Let n_i be the number of boundary components of X_i. Remark that $\sum_{1\le i\le r} n_i = 2n$, and $\chi(X_i) \le 2-n_i$, where $\chi(\cdot)$ is the Euler characteristic. By construction,

$$2-2g = \chi(X) = \sum_{i=1}^{r} \chi(X_i) \le \sum_{i=1}^{r}(2-n_i) = 2r - \sum_{i=1}^{r} n_i = 2r-2n.$$

It follows immediately that

$$n \le g + r - 1 \le g + |\kappa| - 1.$$

From the previous inequalities, we see that the equality $n = g + |\kappa| - 1$ is realized if and only if $r = |\kappa|$ and each X_i has genus zero. In particular, if $n = g + |\kappa| - 1$, then each component X_i contains a unique zero of ω. If there is a saddle connection joining two distinct zeros of ω, then these two zeros must belong to the same X_i, and we draw a contradiction. Therefore, the cylinder decomposition must be stable. □

Remark 2.5 In $\mathcal{H}(1,1)$ the maximal number of cylinders in a cylinder decomposition is three, and a cylinder decomposition is stable if and only if this maximal number is attained. In higher genus, there are stable cylinder decompositions with less than $n + |\kappa| - 1$ cylinders.

Lemma 2.6 *Let $(X,\omega) \in \mathrm{Prym}(\kappa)$ be a surface in one of the strata given by Table 1. If the horizontal direction is periodic for (X,ω) then the number n of horizontal cylinders, counted up to the Prym involution, satisfies $n \le 3$. Moreover, if $\kappa \ne (1,1,2,2)$ and $n = 3$ then the cylinder decomposition in the horizontal direction is stable.*

Remark 2.7 Observe that Lemma 2.6 is false for the stratum $\mathrm{Prym}(1,1,2,2)$. However, using the identification $\mathrm{Prym}(1,1,2,2) \simeq \mathcal{H}(0,0,2)$, the statement becomes true with the convention that a cylinder decomposition of $(X,\omega) \in \mathrm{Prym}(1,1,2,2)$ is stable if and only if the decomposition of the corresponding surface in $\mathcal{H}(0,0,2)$ is.

Proof Let us assume that the horizontal direction is completely periodic. We first show that the number n of horizontal cylinders, counted up to the Prym involution, satisfies $n \le 3$. Let n_f be the number of fixed cylinders (by the Prym involution) and let $2 \cdot n_p$ be the number of noninvariant cylinders. Obviously $n = n_f + n_p$.

The next observation is that each fixed cylinder contains exactly two *regular* fixed points of the Prym involution, which project to simple poles of the corresponding quadratic differential. Hence if $\mathrm{Prym}(\kappa)$ is the covering of $\mathcal{Q}(-1^p, k_1, \dots, k_m)$ where $k_i \ge 0$ then $n_f \le \lfloor \frac{1}{2} p \rfloor$. Now since the number of cylinders is at most $g + |\kappa| - 1$, we get $n_p \le \lfloor \frac{1}{2}(g + |\kappa| - 1 - n_f) \rfloor$. Hence

$$n = n_f + n_p \le \left\lfloor \tfrac{1}{2}(g + |\kappa| - 1 + n_f) \right\rfloor.$$

The values of $g + |\kappa| - 1$ for the different cases of Table 1 are given in Table 2. In the first four entries of the table, the inequality $p \le 1$ holds for all cases, thus $n_f = 0$. Therefore $n \le \lfloor \frac{7}{2} \rfloor = 3$.

$\mathcal{Q}(\kappa')$	$\mathrm{Prym}(\kappa)$	$g+\lvert\kappa\rvert-1$
$\mathcal{Q}_0(-1^6,2)$	$\mathrm{Prym}(1,1)$	3
$\mathcal{Q}_1(-1^3,1,2)$	$\mathrm{Prym}(1,1,2)$	5
$\mathcal{Q}_1(-1^4,4)$	$\mathrm{Prym}(2,2)^{\mathrm{odd}}$	4
$\mathcal{Q}_2(-1^2,6)$	$\mathrm{Prym}(3,3)$	5
$\mathcal{Q}_2(1^2,2)$	$\mathrm{Prym}(1^2,2^2)$	7
$\mathcal{Q}_2(-1,2,3)$	$\mathrm{Prym}(1,1,4)$	6
$\mathcal{Q}_2(-1,1,4)$	$\mathrm{Prym}(2,2,2)^{\mathrm{even}}$	6
$\mathcal{Q}_3(8)$	$\mathrm{Prym}(4,4)^{\mathrm{even}}$	6

Table 2: Values of $g+|\kappa|-1$ for the different cases of Table 1.

For the last four entries of the table, one has, respectively:

(1) If $\kappa=(1,1)$ then $n_f\le 3$ and $n\le\left\lfloor\frac{1}{2}(3+n_f)\right\rfloor\le 3$.

(2) If $\kappa=(1,1,2)$ then $n_f\le 1$ and $n\le\left\lfloor\frac{1}{2}(5+n_f)\right\rfloor\le 3$.

(3) If $\kappa=(2,2)$ then $n_f\le 2$ and $n\le\left\lfloor\frac{1}{2}(4+n_f)\right\rfloor\le 3$.

(4) If $\kappa=(3,3)$ then $n_f\le 1$ and $n\le\left\lfloor\frac{1}{2}(5+n_f)\right\rfloor\le 3$.

The first statement of the lemma is proved. Now we notice that if $n=3$ then, in every case but $\kappa=(1,1,2,2)$, one has $n_f+2\cdot n_p=g+|\kappa|-1$. We conclude with Lemma 2.4. □

2F Action of the horizontal horocycle flow on cylinders

The (horizontal) horocycle flow is defined as the action of the one-parameter subgroup $U=\{u_s\mid s\in\mathbb{R}\}$ of $\mathrm{GL}^+(2,\mathbb{R})$, where $u_s=\left(\begin{smallmatrix}1&s\\0&1\end{smallmatrix}\right)$. If the horizontal direction on (X,ω) is completely periodic, then obviously the action of u_s on (X,ω) preserves the cylinder decomposition topologically. Moreover, each cylinder C_i with parameters $(w_i,h_i,t_i \mod w_i)$ is mapped to a cylinder $C_i(s):=u_s(C_i)$ of $u_s\cdot(X,\omega)$ with the same width and height, while the twist is given by

$$t(C_i(s))=t_i+sh_i \mod w_i. \tag{1}$$

3 Kernel foliation on Prym loci

We briefly recall the kernel foliation for Prym loci (see [6; 18; 2; 27; 31, Section 9.6] for related constructions). We refer to [16, Section 3.1] for details. This notion was introduced by Eskin, Masur and Zorich, and was certainly known to Kontsevich.

3A Kernel foliation

Let $(X,\omega)\in\mathcal{H}(\kappa)$ be a translation surface with *several distinct* zeros. Using the period mapping, we can identify a neighborhood of (X,ω) in $\mathcal{H}(\kappa)$ with an open subset $U\subset\mathbb{C}^d$, where $d=\dim\mathcal{H}(\kappa)$. There is a foliation of U by subsets consisting of surfaces having the same absolute periods. The set of surfaces in this neighborhood that have the same absolute coordinates as X corresponds to the intersection of U with an affine subspace of dimension $|\kappa|-1$. Therefore the leaves of this foliation have dimension $|\kappa|-1$. It is not difficult to see that this foliation is invariant under the coordinate changes of the period mappings; this globally defines a foliation in $\mathcal{H}(\kappa)$. We call it the *kernel foliation*.

It turns out that the kernel foliation also exists in $\mathrm{Prym}(\kappa)$ and $\Omega E_D(\kappa)$, for all κ in Table 1. In particular, in our situation, the leaves of the kernel foliation in $\Omega E_D(\kappa)$ have dimension 1. Hence there is a local action of $\mathbb{C}$ on $\Omega E_D(\kappa)$ as follows: for any Prym eigenform (X,ω) and $w\in\mathbb{C}$ with $|w|$ small enough, $(X',\omega'):=(X,\omega)+w$ is the unique surface in the neighborhood of (X,ω) (in $\Omega E_D(\kappa)$) such that ω' has the same absolute periods as ω, and for a chosen relative cycle $c\in H_1(X,\Sigma,\mathbb{Z})$, $\omega'(c)=\omega(c)+w$ (Σ is the set of zeros of ω). An explicit construction for $(X,\omega)+w$ will be given in Section 4.

Remark 3.1 There is no global action of $\mathbb{C}$ on each leaf of the kernel foliation, ie even if $(X,\omega)+w_1$ and $(X,\omega)+w_2$ are well defined, $(X,\omega)+w_1+w_2$ may not be well defined. Nevertheless, there exists a local action of $\mathbb{C}$: there exists a neighborhood U of $0\in\mathbb{C}$ such that for any $w_1,w_2\in U$ we have

$$(X,\omega)+(w_1+w_2)=((X,\omega)+w_1)+w_2=((X,\omega)+w_2)+w_1.$$

Convention In the sequel we only consider the intersection of kernel foliation leaves with a neighborhood of (X,ω) on which this local action of $\mathbb{C}$ is well defined. Hence by $(X,\omega)+w$ we will mean the surface obtained from (X,ω) by the construction described above.

The relative periods of $(X',\omega'):=(X,\omega)+w$ are characterized by the following lemma (see Figure 1 for an example in $\mathrm{Prym}(1,1,2)$).

Lemma 3.2 *If c is any path on X joining two zeros of ω, and c' is the corresponding path on X', then:*

1. *If the two endpoints of c are exchanged by τ then $\omega'(c')-\omega(c)=\pm w$.*
2. *If one endpoint of c is fixed by τ, but the other is not, then $\omega'(c')-\omega(c)=\pm\frac{1}{2}w$.*

The sign of the difference is determined by the orientation of c.

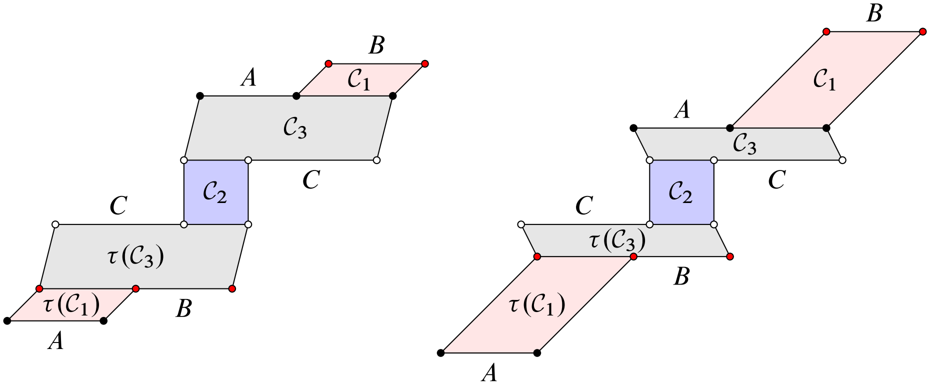

Figure 1: Decomposition of a surface $(X,\omega)\in\mathrm{Prym}(1,1,2)$. The cylinder $\mathcal{C}_2$ is fixed by the Prym involution τ, while the cylinders $\mathcal{C}_i$ and $\tau(\mathcal{C}_i)$ are exchanged for $i=1,3$. Along a kernel foliation leaf $(X,\omega)+(s,t)$ the twists and heights change as follows: $t_1(s)=t_1-s$, $t_2(s)=t_2$, $t_3(s)=t_3+\frac{1}{2}s$ and $h_1(t)=h_1-t$, $h_2(t)=h_2$, $h_3(t)=h_3+\frac{1}{2}t$. We emphasize that the formula for the twists does not depend on the choice of the marking.

We close this section with a description of a neighborhood of a Prym eigenform: up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, a neighborhood of a point (X,ω) in $\Omega E_D(\kappa)$ can be identified with the ball $\{(X,\omega)+w \mid |w|<\varepsilon\}$.

Proposition 3.3 [16] *For any $(X,\omega)\in\Omega E_D(\kappa)$, if (X',ω') is a Prym eigenform in $\Omega E_D(\kappa)$ close enough to (X,ω), then there exists a unique pair (g,w), where $g\in\mathrm{GL}^+(2,\mathbb{R})$ is close to Id, and $w\in\mathbb{R}^2$ is close to 0, such that $(X',\omega')=g\cdot((X,\omega)+w)$.*

Proof For completeness we include the proof here (see [16, Section 3.2]).

Let $(Y,\eta)=(X,\omega)+w$, with $|w|$ small, be a surface in the leaf of the kernel foliation through (X,ω). We denote by $[\omega]$ and $[\eta]$ the classes of ω and η in $H^1(X,\Sigma;\mathbb{C})^-$. Let $\rho\colon H^1(X,\Sigma;\mathbb{C})^-\to H^1(X,\mathbb{C})^-$ be the natural projection. We then have $[\eta]-[\omega]\in\ker\rho$. On the other hand, the action of $g\in\mathrm{GL}^+(2,\mathbb{R})$ on $H^1(X,\Sigma;\mathbb{C})^-$ satisfies $\rho(g\cdot[\omega])=g\cdot\rho([\omega])$. Therefore the leaves of the kernel foliation and the orbits of $\mathrm{GL}^+(2,\mathbb{R})$ are transversal. Since their dimensions are complementary, the proposition follows. $\square$

3B Horizontal and vertical kernel foliation

The horizontal and vertical kernel foliations were studied in some restricted settings in [5; 4; 27], where they were called the real and imaginary foliations, respectively.

Two nearby translation surfaces $(X,\omega),(X',\omega')\in\Omega E_D(\kappa)$ are in the same leaf of the horizontal (respectively, vertical) kernel foliation if the integrals of the flat structures along all closed curves are the same on X and X', and if the integrals along curves joining distinct singularities only differ in their horizontal (respectively, vertical) component. More precisely, a leaf of the horizontal kernel foliation is parametrized by a continuous map $I\to\Omega E_D(\kappa)$, $s\mapsto(X_s,\omega_s)$ on a maximal interval containing 0 such that

- $(X_0,\omega_0)=(X,\omega)$,
- $\omega_s(\gamma)=\omega(\gamma)$ for all $s\in I$ and any $\gamma\in H_1(X,\mathbb{Z})$,
- $\omega_{s_1}(c)-\omega_{s_2}(c)=(s_1-s_2,0)$ for a fixed relative class $c\in H_1(X,\Sigma,\mathbb{Z})$ and for all $s_1,s_2\in I$.

We will write $(X,\omega)+(s,0)=(X_s,\omega_s)$. If $I=\mathbb{R}$ then $(X,\omega)+(s,0)$ is defined for all s. The same description holds for the vertical kernel foliation.

Remark 3.4 If the horizontal direction on (X,ω) is stable then the horizontal kernel foliation is well defined for all times $s\in\mathbb{R}$.

3C Effect of the kernel foliation on cylinders

Assume that (X,ω) admits a stable cylinder decomposition in the horizontal directions, with cylinders denoted by $C_1,\dots,C_k$. Let $v=(s,0)$ be a vector such that $(X,\omega)+(s,0)$ is well defined and admits a stable cylinder decomposition (in the horizontal direction) with the same combinatorial data and the same widths of cylinders. This is the case if v is small enough (see Proposition 4.1 below). Let $C_i(s,0)$ denote the cylinder in $(X,\omega)+(s,0)$ corresponding to C_i. Analogously, let $C_i(0,t)$ denote the cylinder in $(X,\omega)+(0,t)$ that corresponds to C_i.

The widths of cylinders $C_i(s,0)$ and $C_i(0,t)$ are constant functions of s and t, respectively, since they correspond to absolute periods.

Similarly, the heights of the cylinder $C_i(s,0)$ are constant functions of s (there is no vertical deformations along the horizontal kernel foliation) and the twists of the cylinders $C_i(0,t)$ are also constant functions of t.

Lemma 3.5 *The twist (respectively, height) of the cylinder $C_i(s,0)$ (respectively, $C_i(0,t)$) is given by $t_i+\alpha_i s$ (respectively, $h_i+\alpha_i t$), where t_i is the twist of C_i, h_i is the height of C_i and*

$$\alpha_i=\begin{cases}0 & \text{if the zeros in the upper and lower boundaries of } C_i \text{ are the same,}\\ \pm 1 & \text{if the zeros are exchanged by the Prym involution,}\\ \pm\frac{1}{2} & \text{if one zero is fixed and the other is mapped to the third one}\\ & \text{by the Prym involution.}\end{cases}$$

We emphasize that, despite the fact that the twists depend on the marking, the formulas above do not depend on the marking. The proof of Lemma 3.5 is elementary and left to the reader.

4 Kernel foliation and cylinder decompositions

In this section we investigate the kernel foliation leaf near surfaces (X,ω) for which the horizontal direction is periodic. We separate the discussion into two cases: when the direction is stable or unstable.

Proposition 4.1 (stable case) *If the horizontal direction on (X,ω) is periodic and stable then there exists a small neighborhood U of $0 \in \mathbb{C}$ such that, for every $v = (s,t) \in U$, the horizontal direction on $(X,\omega)+v$ is periodic, stable and has the same combinatorial data and the same widths of cylinders.*

Proposition 4.2 (unstable case) *If the horizontal direction on (X,ω) is periodic and unstable then there exists a small neighborhood U of $0 \in \mathbb{C}$ such that, for every $v = (s,t) \in U$ with $t \neq 0$, the horizontal direction on $(X,\omega)+v$ is periodic and stable. Moreover, the combinatorial data and the widths of the cylinder decomposition on $(X,\omega)+v$ depend only on the sign of t.*

The difficulty lies in the fact that the horizontal decomposition on $(X,\omega)+v$ is not obviously given from the horizontal decomposition on (X,ω) if v is not horizontal. At the opposite, for horizontal vectors, the following lemma holds:

Lemma 4.3 *If $s \in \mathbb{R}$ with $|s| < \varepsilon$ then the cylinder decompositions of $(X,\omega)+(s,0)$ and (X,ω) have the same combinatorial data and the same width.*

Proof of Lemma 4.3 Obviously all the horizontal saddle connections in (X,ω) persist in $(X,\omega)+(s,0)$ (the lengths of some of them may be changed). Hence $(X,\omega)+(s,0)$ also has a cylinder decomposition in the horizontal direction with the same combinatorial data as the one of (X,ω). The widths of the corresponding cylinders must be the same since they are absolute periods of ω. $\square$

Since for small $s,t \in \mathbb{R}$ the relation $(X,\omega)+(s,t) = ((X,\omega)+(0,t))+(s,0)$ holds, in view of Lemma 4.3, it suffices to prove Propositions 4.2 and 4.1 only for vectors $v = (0,t)$.

The key to the two propositions is a careful analysis of the kernel foliation leaf near a surface (X,ω) in term of *flat geometry*.

Remark 4.4 The construction we present is general and can be used to describe the kernel foliation for an affine invariant subvariety obtained by covering. To simplify the exposition we present the construction when ω has two zeros permuted by the involution, denoted by P and Q. The general construction is obtained by considering as many Euclidean discs as the number of zeros of ω.

4A Flat geometry around a singularity

Let $\varepsilon > 0$ small enough so that the two discs

$$D(P,\varepsilon) := \{x \in X \mid \boldsymbol{d}(x,P) < \varepsilon\} \quad \text{and} \quad D(Q,\varepsilon) := \{x \in X \mid \boldsymbol{d}(x,Q) < \varepsilon\}$$

are embedded and disjoint in X. Taking ε smaller if necessary, we can assume that for any vector $v \in \boldsymbol{B}(\varepsilon)$ there is a unique surface (X',ω') (denoted by $(X,\omega)+v$) in a neighborhood of (X,ω) so that ω' and ω have the same absolute periods and $\omega'(c)-\omega(c) = v$, where c is a fixed relative cycle. Moreover, for any $v_1, v_2 \in \boldsymbol{B}(\varepsilon)$ such that $v_1+v_2 \in \boldsymbol{B}(\varepsilon)$, the equality $(X,\omega)+(v_1+v_2) = ((X,\omega)+v_1)+v_2$ holds.

Each of the discs $D(P,\varepsilon)$ and $D(Q,\varepsilon)$ is homeomorphic to a topological disc. However, metrically, each has the structure of a regular cone with a cone angle $2\pi m$, where $m-1 \geq 1$ is the multiplicity of the zero P (or Q, since the multiplicity is the same: the Prym involution permutes P and Q). Each cone can be glued from $2m$ disjoint copies of Euclidean half-discs whose boundaries are isometrically glued together in a circular fashion. Hence their centers are identified with the zero. More precisely, let

$$D_i^- = \{z \in \boldsymbol{B}(\varepsilon) \mid -\varepsilon \leq \mathrm{Re}(z) \leq 0\} \quad \text{and} \quad D_i^+ = \{z \in \boldsymbol{B}(\varepsilon) \mid 0 \leq \mathrm{Re}(z) \leq \varepsilon\}$$

be $2m$ disjoint Euclidean half-discs. We construct $D(P,\varepsilon)$ by gluing the half-discs $D_1^\pm, \dots, D_m^\pm$ as follows:

- D_i^+ is glued to D_i^- along the segment $\{\mathrm{Re}(z) = 0,\ 0 \leq \mathrm{Im}(z) < \varepsilon\}$, for $i = 1, \dots, m$.
- D_i^- is glued to D_{i+1}^+ along the segment $\{\mathrm{Re}(z) = 0,\ -\varepsilon < \mathrm{Im}(z) \leq 0\}$, for $i = 1, \dots, m-1$.
- D_m^- is glued to D_1^+ along the segment $\{\mathrm{Re}(z) = 0,\ -\varepsilon < \mathrm{Im}(z) \leq 0\}$.

Similarly, for $D(Q,\varepsilon)$, we glue $2m$ half-discs $D_{m+1}^\pm, \dots, D_{2m}^\pm$:

- D_i^- is glued to D_i^+ along the segment $\{\mathrm{Re}(z) = 0, -\varepsilon < \mathrm{Im}(z) \leq 0\}$, for $i = m+1, \dots, 2m$.
- D_i^+ is glued to D_{i+1}^- along the segment $\{\mathrm{Re}(z) = 0,\ 0 \leq \mathrm{Im}(z) < \varepsilon\}$, for $i = m+1, \dots, 2m-1$.
- D_{2m}^+ is glued to D_{m+1}^- along the segment $\{\mathrm{Re}(z) = 0,\ 0 \leq \mathrm{Im}(z) < \varepsilon\}$.

Observe that, by construction, $\tau(D_i^+) = D_{i+m}^-$ and $\tau(D_i^-) = D_{i+m}^+$ for $i = 1, \dots, m$.

4B Combinatorial data

Any horizontal separatrix from a zero of ω must end in another zero. Thus a positive horizontal saddle connection connects the "center" of some half-disc D_i^+ to the "center" of some $D_j^- = D_{\pi_c(i)}^-$. This defines a permutation π_c of $\{1, \dots, 2m\}$.

We can perform the same construction for the "top" and the "bottom" of the half-discs. More precisely, a positive horizontal ray emanating from the top of the half-disc D_i^+ will pass through the top of some half-disc D_j^-, which is identified to the top of $D_{j+1}^+ = D_{\pi_t(i)}^+$. This defines a permutation π_t of $\{1, \dots, 2m\}$. We have a similar permutation π_b of $\{1, \dots, 2m\}$.

By construction, the set of cylinders in the horizontal direction is in bijection with the set of cycles of π_t (or π_b). Moreover, the tuple (π_c, π_t, π_b) is independent of ε (as long as $D(P, \varepsilon)$ and $D(Q, \varepsilon)$ are embedded and disjoint) and it clearly determines the combinatorial data of the cylinder decomposition of (X, ω).

4C The "moving singularity" surgery

We assume that the horizontal direction on (X, ω) is periodic. Let h be the minimal height among the heights of the cylinders, and ℓ be the length of the shortest horizontal saddle connection. For any $0 < \varepsilon < \frac{1}{2}\min\{h, \ell\}$ we describe a local surgery of the flat structure of (X, ω), without changing the flat structure outside the union of the discs $D(P, \varepsilon)$ and $D(Q, \varepsilon)$, in order to recover $(X, \omega) + (0, t)$ for any $|t| < \varepsilon$ (see Figure 2).

Let us assume that $t > 0$ (the case $t < 0$ is completely similar). We change the way of gluing the half-discs as follows: as patterns we still use the Euclidean half-discs, but we move slightly the "centers": the center of $D_i^{\pm}$, for $i = 1, \dots, m$, will be moved by the vector $\left(0, -\frac{1}{2}t\right)$, while the center of $D_i^{\pm}$, for $i = m+1, \dots, 2m$, will be moved by the vector $\left(0, \frac{1}{2}t\right)$. We alternate half-discs with their centers moved up and down. All the lengths along identifications are matching:

- D_i^+ is glued to D_i^- along the segment $\left\{\mathrm{Re}(z) = 0,\ -\frac{1}{2}t \le \mathrm{Im}(z) < \varepsilon\right\}$, for $i = 1, \dots, m$.
- D_i^- is glued to D_{i+1}^+ along the segment $\left\{\mathrm{Re}(z) = 0,\ -\varepsilon < \mathrm{Im}(z) \le -\frac{1}{2}t\right\}$, for $i = 1, \dots, m-1$.
- D_m^- is glued to D_1^+ along the segment $\left\{\mathrm{Re}(z) = 0,\ -\varepsilon < \mathrm{Im}(z) \le -\frac{1}{2}t\right\}$.

Similarly, for $D(Q,\varepsilon)$, we glue the discs $D_{m+1},\dots,D_{2m}$:

- D_i^- is glued to D_i^+ along the segment $\{\mathrm{Re}(z)=0,-\varepsilon<\mathrm{Im}(z)\le\frac{1}{2}t\}$, for $i=m+1,\dots,2m$.
- D_i^+ is glued to D_{i+1}^- along the segment $\{\mathrm{Re}(z)=0,\ \frac{1}{2}t\le\mathrm{Im}(z)<\varepsilon\}$, for $i=m+1,\dots,2m-1$.
- D_{2m}^+ is glued to D_{m+1}^- along the segment $\{\mathrm{Re}(z)=0,\ \frac{1}{2}t\le\mathrm{Im}(z)<\varepsilon\}$.

Gluing the half-discs in this latter way, we obtain two topological discs, each of them having a flat metric with a cone-type singularity of angle $2m\pi$. Note that a small tubular neighborhood of the boundary of the initial cone is isometric to the corresponding tubular neighborhood of the boundary of the resulting object. Thus we can paste it back into the surface (with the same angle).

Since this surgery does not change the flat metric outside of $D(P,\varepsilon)$ and $D(Q,\varepsilon)$, it is not difficult to see that the resulting surface is $(X',\omega')=(X,\omega)+(0,t)$ (see [16]).

Example 4.5 In Figure 2 we provide an example of a deformation by the kernel foliation near an unstable decomposition. In this case $\kappa=(2,2)$, thus $m=3$. The permutation π_c is given by $(1,4,3,5,2,6)$. Similarly, the permutations π_b and π_t are $\pi_b=(1,4)(2,6)(3,5)$ and $\pi_t=(1,6)(2,5)(3,4)$. Hence the saddle connections emanating from P' (respectively, Q') correspond to cycles of π_b (respectively, π_t). The new cylinder on (X',ω') corresponds to the unique cycle of π_c.

Example 4.6 Similarly, in Figure 1 we can encode the (stable) cylinder decomposition with the help of the permutations π_c, π_t, π_b. In this situation, the two zeros permuted by the Prym involution have degree 1, thus $m=2$. The third (fixed) zero has degree 2. Hence we need $2+2+3$ discs. By using a suitable labeling, we find

$$\pi_c=(1)(2)(3)(4)(5,7)(6),\quad \pi_t=(1)(2)(3,4)(5,7)(6),\quad \pi_b=(1,2)(3)(4)(5)(6,7).$$

The lemma below summarizes how the flat metric changes in the discs. Recall that $t>0$.

Lemma 4.7 *For $i=1,\dots,m$, all the points at the coordinates $\left(0,-\frac{1}{2}t\right)$ in $D_i^\pm$ are identified to give a point P' in (X',ω') with cone angle $2\pi m$.*

For $i=m+1,\dots,2m$, all the points at the coordinates $\left(0,\frac{1}{2}t\right)$ in $D_i^\pm$ are identified to give a point Q' in (X',ω') with cone angle $2\pi m$.

All the other points of $D_i^\pm$ give regular points in (X',ω').

Moreover, for any $i=1,\dots,2m$, there is a positive horizontal ray in (X',ω') from the point at the coordinates $\left(0,-\frac{1}{2}t\right)$ in D_i^+ to the point at the coordinates $\left(0,-\frac{1}{2}t\right)$ in $D_{\pi_c(i)}^-$. The same conclusion holds for the point at the coordinates $\left(0,\frac{1}{2}t\right)$.

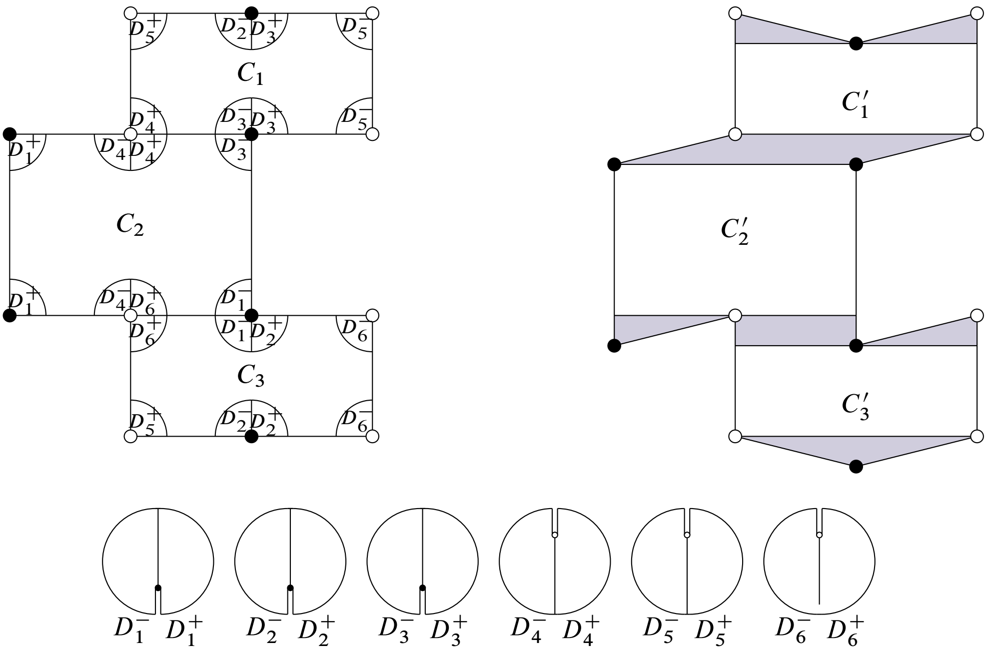

Figure 2: Kernel foliation and unstable direction.

4D Proof of Proposition 4.1 and Proposition 4.2

Proof of Proposition 4.1 (stable case) Let $0 < \varepsilon < \frac{1}{2}\min\{h, \ell\}$. Let $v = (0, t)$ with $0 < t < \varepsilon$.

Since the cylinder decomposition is stable, any positive horizontal saddle connection connects a zero to itself. Namely, the permutation π_c leaves invariant the subsets $\{1, \dots, m\}$ and $\{m+1, \dots, 2m\}$. By Lemma 4.7, for $i = 1, \dots, m$, there is a positive horizontal saddle connection from the singularity in D_i^+ to the point at the coordinate $\left(0, -\frac{1}{2}t\right)$ in $D^-_{\pi_c(i)}$, ie from P' to P'. The same is true for Q'. Thus the horizontal direction on $(X', \omega') = (X, \omega) + v$ is periodic and stable.

Clearly the permutations π_c, π_t, π_b and π'_c, π'_t, π'_b coincide (do not depend on ε nor t). Hence the cylinder decomposition of $(X, \omega) + v$ has the same combinatorial data as the one of (X, ω). This ends the proof for the case $t > 0$. If $v = (0, t)$ with $-\varepsilon < t < 0$, the proof is similar. $\square$

Proof of Proposition 4.2 (unstable case) Let $0 < \varepsilon < \frac{1}{2}\min\{h, \ell\}$. Let $0 < t < \varepsilon$ (if $-\varepsilon < t < 0$ the proof is similar).

We first claim that any positive horizontal ray emanating from P' ends in P'. The key remark is the following: for any $i = 1, \dots, m$ one can identify the singularity at the coordinates $\left(0,-\frac{1}{2}t\right)$ in $D_i^{\pm}$ (P' in (X',ω')) to the bottom of a disc with radius $\frac{1}{2}t$. Let γ be a positive horizontal separatrix from P' emanating from D_i^+ for some $i=1,\dots,m$. Let $c=(i,i_2,i_3,\dots,i_k,\dots,i_l)$ be the cycle of π_b containing i, where $i_2,i_3,\dots,i_k\in\{m+1,\dots,2m\}$ and $i_{k+1}\in\{1,\dots,m\}$. Hence γ will intersect the sequence of discs $D_{i_2},\dots,D_{i_k}$, and will pass through the point at the coordinates $\left(0,-\frac{1}{2}t\right)$ (which is regular). Then γ will intersect the disc $D_{i_{k+1}}$, and will pass through the point $\left(0,-\frac{1}{2}t\right)$ (which is identified to P' in (X',ω')).

The same argument shows that any horizontal ray emanating from Q' ends in Q' (and those saddle connections are encoded in π_t). In conclusion, $(X,\omega)+(0,t)$ admits a stable cylinder decomposition in the horizontal direction.

It remains to show that the combinatorial data does not depend on t (recall that $t>0$). There are two kinds of cylinders in $(X,\omega)+(0,t)$:

- A cylinder of the first kind corresponds to a cylinder in (X,ω); its central core curve does not intersect $D(P,\varepsilon)\sqcup D(Q,\varepsilon)$. These cylinders are encoded by the cycles of π_t and π_b.
- The other possibility is that a cylinder in $(X,\omega)+(0,t)$ contains some of the "centers" of the discs D_i^+. Hence its core curve is a concatenation of positive ray passing trough the centers of the discs D_i, and thus is encoded by a cycle of the permutation π_c (see Example 4.5 and Figure 2).

Thus the cylinder decomposition depends only on π_c,π_t,π_b; this finishes the proof. □

5 Cylinder decomposition: relation of moduli

The aim of this section is to establish the following result:

Proposition 5.1 *Let $(X,\omega)\in\Omega E_D(\kappa)$ be a Prym eigenform with κ in Table 1 such that the horizontal direction is periodic. Let n be the number of τ–equivalence classes of horizontal cylinders (recall that $n\le 3$), and $C_1,\dots,C_n$ be a family of cylinders representing the n equivalence classes.*

- *If $n=3$ then there exists $(r_1,r_2,r_3)\in\mathbb{Q}^3\setminus\{0\}$ such that*

$$r_1\mu_1+r\mu_2+r_3\mu_3=0. \tag{2}$$

  Moreover, let $\alpha_i\in\left\{0,\pm\frac{1}{2},\pm1\right\}$ be the coefficient given by Lemma 3.5 associated to C_i. Then (r_1,r_2,r_3) satisfies

$$r_1\frac{\alpha_1}{w_1}+r_2\frac{\alpha_2}{w_2}+r_3\frac{\alpha_3}{w_3}=0. \tag{3}$$

- *If the cylinder decomposition is unstable then the horizontal direction is parabolic.*

We first recall the following result dealing with the case when D is not a square.

Theorem 5.2 (McMullen [21]) *Let $K=\mathbb{Q}(\sqrt{D})\subset\mathbb{R}$ be a real quadratic field and let $(X,\omega)\in\Omega E_D(\kappa)$ be a Prym eigenform such that all the absolute periods of ω belong to $K(\imath)$. If the horizontal direction is periodic with k cylinders then*

$$\sum_{i=1}^{k} w_i' h_i = 0,$$

where w_i, h_i are respectively the width and the height of the i^{th} cylinder, and w_i' is the Galois conjugate of w_i in K.

Sketch of proof A remarkable property of Prym eigenform is that the *complex flux* vanishes. Namely (see [21, Theorem 9.7])

$$\int_X \omega\wedge\omega' = \int_X \omega\wedge\overline{\omega}' = 0.$$

Here $\overline{\omega}$ and ω' are respectively the complex conjugate and the Galois conjugate of ω. The argument is as follows: let T be a generator of the order $\mathcal{O}_D$. The vector space $H^1(X,\mathbb{R})^-$ splits into a pair of 2–dimensional eigenspaces $S\oplus S' = H^1(X,\mathbb{R})^-$ on which T acts by multiplication by a scalar. More precisely, S is spanned by $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$, and S' is spanned by $\mathrm{Re}(\omega')$ and $\mathrm{Im}(\omega')$. Since T is self-adjoint, S and S' are orthogonal with respect to the cup product. This shows the equalities above. Now

$$\int_{\mathcal{C}_i} \mathrm{Im}(\omega)\wedge\mathrm{Re}(\omega') = w_i' h_i,$$

where $\mathcal{C}_1,\dots,\mathcal{C}_k$ are the horizontal cylinders in X. Since the surface X is covered by those cylinders, it follows that

$$\begin{aligned}\sum_{i=1}^{k} w_i' h_i &= \sum_{i=1}^{k}\int_{\mathcal{C}_i} \mathrm{Im}(\omega)\wedge\mathrm{Re}(\omega')\\ &= \int_X \mathrm{Im}(\omega)\wedge\mathrm{Re}(\omega')\\ &= \frac{1}{4\imath}\int_X (\omega-\overline{\omega})\wedge(\omega'+\overline{\omega}') = 0.\end{aligned}$$

□

5A Proof of Proposition 5.1 when D is not a square

Proof Let $\beta_i\in\{1,2\}$ be the number of cylinders in the τ–equivalence class of C_i ($\beta_i=1$ if C_i is fixed by τ, $\beta_i=2$ if C_i is exchanged with another cylinder). Set $r_i=\beta_i w_i w_i'\in\mathbb{Q}$.

For the case $n=3$, the first equality follows directly from Theorem 5.2. Namely,

$$0=\sum_{i=1}^{k} w_i' h_i = \sum_{i=1}^{3} \beta_i (w_i w_i')\mu_i = \sum_{i=1}^{3} r_i \mu_i .$$

When $n=3$, Lemma 2.6 implies that the cylinder decomposition is stable. Thus we can associate to each cylinder C_i a coefficient $\alpha_i \in \left\{0, \pm\frac{1}{2}, \pm 1\right\}$ (by Lemma 3.5). Observe that moving in the leaves of the kernel foliation does not change the area of the surface, therefore

$$\mathrm{Area}(X,\omega) = \mathrm{Area}((X,\omega)+(0,s)),$$

and hence

$$\sum_{i=1}^{k} w_i h_i = \sum_{i=1}^{k} w_i (h_i + \alpha_i s),$$

which implies that

$$\sum_{i=1}^{k} \alpha_i w_i = \sum_{i=1}^{3} \alpha_i \beta_i w_i = 0. \tag{4}$$

Thus, one has

$$\sum_{i=1}^{3} r_i \frac{\alpha_i}{w_i} = \sum_{i=1}^{3} \beta_i \alpha_i w_i' = \left(\sum_{i=1}^{3} \alpha_i \beta_i w_i\right)' = 0,$$

and (3) is proved.

Consider now the case that the cylinder decomposition is unstable, which means that $n \le 2$. If $n=1$ then X has either a unique horizontal cylinder, or two horizontal cylinders which are exchanged by τ. In both cases, the horizontal direction is clearly parabolic. If $n=2$, then Theorem 5.2 implies that the ratio μ_1/μ_2 is rational, which means that the horizontal is also parabolic. Proposition 5.1 is then proved for the case that D is not a square. □

5B Proof of Proposition 5.1 when D is a square

We will need a technical lemma.

Lemma 5.3 *For every $i \in \{1,\dots,k\}$, either h_i is an absolute period, or there exists $j \neq i$ and some integers $x_i, x_j \in \{1,2\}$ such that $x_i h_i + x_j h_j$ is an absolute period. Moreover, if the cylinder decomposition is stable, and α_i, α_j are the coefficients associated to C_i and C_j (by Lemma 3.5), then $x_i\alpha_i + x_j\alpha_j = 0$.*

Proof If there is a zero of ω that is contained in both the top and bottom borders of C_i, then h_i is an absolute period. Let us assume that this does not occur. There are two cases.

First case: ω has two zeros P_1, P_2 Note that in this case P_1 and P_2 are exchanged by the Prym involution τ. We can assume that the bottom border of C_i contains P_1, and its top border contains P_2. By connectedness of X, there must exist a cylinder C_j whose bottom border contains P_2 and whose top border contains P_1. Note that $i \neq j$; otherwise P_1 is contained in both top and bottom borders of C_i. Let σ_i and σ_j be respectively some saddle connections in C_i and C_j which join P_1 to P_2. Then $c = \sigma_i \cup \sigma_j$ is a simple closed curve in X and we conclude that $h_1 + h_2 = \operatorname{Im} \omega(c)$.

Second case: ω has 3 zeros In this case two zeros are permuted by τ; we denote them by P_1, P_2. The third one is fixed by τ; let us denote this one by Q. We can always assume that P_1 is contained in the bottom border of C_i, but not in the top border of C_i.

Assume that the top border of C_i contains P_2, and let σ_i be a saddle connection in C_i which joins P_1 to P_2. If there exists another cylinder whose bottom border contains P_2 and top border contains P_1 then we are done. Otherwise, there must exist a cylinder C_j whose bottom border contains P_2 and top border contains Q. Let $C_{j'}$ be the cylinder which is permuted with C_j by τ. Then the top border of $C_{j'}$ contains P_1 and the bottom border of $C_{j'}$ contains Q. In particular, $C_{j'} \neq C_i$.

If $C_{j'} = C_j$, then the top border of C_j contains P_1, contradicting our hypothesis. Thus $C_{j'} \neq C_j$. Let σ_j be a saddle connection in C_j which joins P_2 to Q. Then $\tau(\sigma_j)$ is a saddle connection in $C_{j'}$ that joins Q to P_1. Consequently, $c := \tau(\sigma_j) \cup \sigma_j \cup \sigma_i$ is a simple closed curve in X, and $\operatorname{Im} \omega(c) = h_i + h_j + h_{j'} = h_i + 2h_j$.

We are left with the case where the top border of C_i contains Q. Let $C_{i'}$ be the cylinder which is permuted with C_i by τ. Then the top border of $C_{i'}$ contains P_2 and the bottom border contains Q. By assumption, $C_{i'} \neq C_i$. By connectedness of X, there exists a cylinder $C_j \neq C_i$ which contains P_1 in the top border, and P_2 or Q in the bottom border. If P_2 is contained in the bottom border of C_j then $h_j + h_i + h_{i'} = h_j + 2h_i$ is an absolute period. If Q is an contained in the bottom border of C_j then $h_i + h_j$ is an absolute period. Since $x_i h_i + x_j h_j$ is an absolute period, it is unchanged by the kernel foliation; Lemma 3.5 then implies that $x_i \alpha_i + x_j \alpha_j = 0$. □

Proof of Proposition 5.1 when D is a square We first consider the case $n = 3$. Since D is a square, one can normalize, using $\mathrm{GL}^+(2, \mathbb{R})$, so that all the absolute periods of ω belong to $\mathbb{Q}(\imath)$. By Lemma 5.3, one can find (x_1, x_2, x_3) and (y_1, y_2, y_3) with $x_i, y_i \in \{0, 1, 2\}$ such that $x_1 h_1 + x_2 h_2 + x_3 h_3$ and $y_1 h_1 + y_2 h_2 + y_3 h_3$ are absolute periods. The vectors (x_1, x_2, x_3) and (y_1, y_2, y_3) are chosen so that they are not collinear. Since all the absolute periods are in $\mathbb{Q}$, there exists $r \in \mathbb{Q}$, $r > 0$, such that

$$x_1h_1 + x_2h_2 + x_3h_3 = r(y_1h_1 + y_2h_2 + y_3h_3),$$

or equivalently

$$\sum_{i=1}^{3}(x_i - ry_i)h_i = 0.$$

Setting $r_i := (x_i - ry_i)w_i$, we get

$$\sum_{i=1}^{3} r_i\mu_i = 0.$$

Lemma 5.3 implies that $\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 = \alpha_1y_1 + \alpha_2y_2 + \alpha_3y_3 = 0$. Hence

$$\sum_{i=1}^{3}(x_i - ry_i)\alpha_i = \sum_{i=1}^{3} r_i\frac{\alpha_i}{w_i} = 0.$$

Now let us assume that the horizontal direction is unstable (hence $n \leq 2$). We will show that the horizontal direction is parabolic. Obviously, we only need to consider the case $n = 2$. Recall that we can normalize so that all the absolute periods of ω are in $\mathbb{Q}(\imath)$. In particular, $w_1, w_2 \in \mathbb{Q}$. We will show that both h_1, h_2 are also absolute periods.

First case: ω has two zeros P_1, P_2 Since the cylinder decomposition is unstable, there exists a horizontal saddle connection γ from P_2 to P_1. We can assume that P_1 is contained in the bottom border of C_1. If the top border of C_1 also contains P_1, then h_1 is an absolute period. Otherwise, let σ be a saddle connection joining P_1 to P_2 which is contained in C_1. Since $c := \gamma \cup \sigma$ is a closed curve and $h_1 = \operatorname{Im}\omega(c)$, we conclude that $h_1 \in \mathbb{Q}$. The same arguments show that $h_2 \in \mathbb{Q}$, hence the horizontal direction is parabolic.

Second case: ω has 3 zeros Let P_1, P_2 denote the zeros which are permuted and Q the zero fixed by τ. We first observe that there exists a path from P_1 and P_2 which is a union of horizontal saddle connection. Indeed, by assumption there exists a horizontal saddle connection γ which joins two different zeros. If γ joins P_1 to P_2 then we are done. Otherwise, γ joins Q to either P_1 or P_2. In both cases, the union of γ and $\tau(\gamma)$ is the desired path. Let us denote this path by η.

Let us assume that P_1 is contained in the bottom border of C_1 but not in the top border. If the top border of C_1 contains P_2, then the union of η and a saddle connection in C_1 joining P_1 to P_2 is a closed curve c such that $\operatorname{Im}\omega(c) = h_1$, which implies $h_1 \in \mathbb{Q}$.

If the top border of C_1 contains Q, then let C_3 be the cylinder which is permuted with C_1 by τ. Note that the bottom border of C_3 contains Q, and the top border contains P_2. Let σ_1 be a saddle connection in $\mathcal{C}_1$ joining P_1 to Q, and σ_3 be the image of σ_1 by τ in C_3. The union $c := \eta \cup \sigma_3 \cup \sigma_1$ is then a closed curve such that $\operatorname{Im}\omega(c) = 2h_1$, hence $h_1 \in \mathbb{Q}$. Similar arguments show that $h_2 \in \mathbb{Q}$. The horizontal direction is then parabolic. □

6 Proof of a weaker version of Theorem 1.1

In this section, we prove a weaker version of Theorem 1.1. We say that (X,ω) is not a Veech surface (or the orbit is not closed) for "*the most obvious reason*" if there exists a periodic direction on (X,ω) that is not parabolic (it is a theorem of Veech [29] that on a Veech surface any periodic direction is parabolic). We will prove Theorem 1.1 under this additional assumption.

Theorem 6.1 *Let $(X,\omega) \in \Omega E_D(\kappa)$ and let us denote by $\mathcal{O}$ its $\mathrm{GL}^+(2,\mathbb{R})$–orbit. If $\mathcal{O}$ is not closed for the most obvious reason then $\overline{\mathcal{O}}$ is a connected component of $\Omega E_D(\kappa)$.*

We begin with the following key lemma. The proof is classical, but is included here for completeness.

Lemma 6.2 *Let $(X,\omega) \in \Omega E_D(\kappa)$ be a Prym eigenform. We assume that the horizontal direction is completely periodic but not parabolic. Then for all $s \in \mathbb{R}$ the surface $(X,\omega) + (s,0)$ is well defined, and one has*

$$(X,\omega) + (s,0) \in \overline{U \cdot (X,\omega)}.$$

Before proving the lemma, let us state the following corollary:

Corollary 6.3 *Let $(X,\omega) \in \Omega E_D(\kappa)$ be a Prym eigenform. We assume that there exists*

$$(Y,\eta) \in \overline{\mathrm{GL}^+(2,\mathbb{R}) \cdot (X,\omega)}$$

and $\varepsilon > 0$ such that

$$(Y,\eta) + (s,0) \in \overline{\mathrm{GL}^+(2,\mathbb{R}) \cdot (X,\omega)}$$

for all $s \in \mathbb{R}$ with $|s| < \varepsilon$. Then there exists $\varepsilon' > 0$ such that

$$(Y,\eta) + v \in \overline{\mathrm{GL}^+(2,\mathbb{R}) \cdot (X,\omega)}$$

for any $v \in \mathbb{R}^2$ such that $|v| < \varepsilon'$.

Proof of Lemma 6.2 Let $C_1,\dots,C_k$ be the horizontal cylinders in X. Let n be the number of equivalence classes of cylinders that are permuted by the Prym involution τ. Recall that for all the cases in Table 1 the inequality $n\le 3$ holds. Assume that $\{C_1,\dots,C_n\}$ is a representative family for the τ–equivalence classes of cylinders.

Let us consider the case $n=3$. Lemma 2.6 implies in particular that the cylinder decomposition is stable. The surface is encoded by the topological gluings of the cylinders C_i, and the width, height and twist of C_i (which will be denoted by w_i, h_i, t_i, respectively).

The set of surfaces admitting a cylinder decomposition in the horizontal direction with the same topological gluings, and the same widths and heights of the cylinders as X, is parametrized by the 3–dimensional torus

$$\mathcal{X}=N(\mathbb{R})\times N(\mathbb{R})\times N(\mathbb{R})/N(w_1\mathbb{Z})\times N(w_2\mathbb{Z})\times N(w_3\mathbb{Z}),$$

where $N(A)=\{u_s\mid s\in A\}$.

The horocycle flow u_s preserves the topological decomposition as well as all the parameters but the twists t_i. The new twists $\tilde{t_i}$ are given by $\tilde{t_i}=t_i+sh_i \bmod w_i$. Hence surfaces in the U–orbit of (X,ω) are parametrized by the line

$$\{(t_1,t_2,t_3)+(h_1,h_2,h_3)s\mid s\in\mathbb{R}\}.$$

By Kronecker's theorem, the orbit closure $\overline{U\cdot(X,\omega)}$ is a subtorus of $\mathcal{X}$. Since the moduli are not commensurable (the horizontal direction is not parabolic) the dimension of this subtorus is at least two. More precisely, the orbit closure $\overline{U\cdot(X,\omega)}$ consists of the set of all twists $(\tilde{t_1},\tilde{t_2},\tilde{t_3})$ such that the normalized twists $(\tilde{t_i}-t_i)/w_i$ verify all nontrivial homogeneous linear relations with rational coefficients that are satisfied by the moduli $\mu_i=h_i/w_i$. Let $\mathbb{P}$ be the subspace of $\mathbb{R}^3$ which is defined by all such rational relations. By assumption, $\dim_{\mathbb{R}}\mathbb{P}\ge 2$. But we know from Proposition 5.1 that there exists $(r_1,r_2,r_3)\in\mathbb{Q}^3\setminus\{(0,0,0)\}$ such that $\sum_{i=1}^n r_i\mu_i=0$. Therefore $\dim_{\mathbb{R}}\mathbb{P}=2$ and

$$\mathbb{P}=\left\{(\tilde{t_1},\tilde{t_3},\tilde{t_3})\in\mathbb{R}^3\;\middle|\;\sum_{i=1}^{3}r_i\left(\frac{\tilde{t_i}-t_i}{w_i}\right)=0\right\}. \tag{5}$$

It follows that $\overline{U\cdot(X,\omega)}$ is the projection to $\mathcal{X}$ of the plane $\mathbb{P}\subset\mathbb{R}^3$ defined by Equation (5). Hence, all surfaces constructed from the cylinders with the same widths and heights as those of (X,ω) (by the same gluings), and with the twists $\tilde{t_i}$ satisfying Equation (5) above, belong to $\overline{U\cdot(X,\omega)}$.

Recall that in the horizontal kernel foliation leaf, a surface $(X,\omega)+(s,0)$ is still completely periodic (for the horizontal direction), and all the data (topological gluings of the cylinders, widths, heights) are preserved, except for the twists (see Lemma 3.5). To be more precise, if C_i^s is the horizontal cylinder in $(X,\omega)+(s,0)$ corresponding to $C_i = C_i^0$, then $t_i(s) = t_i + \alpha_i s$ (where the range of α_i is $\{-1,0,1\}$ or $\{-1,-\frac{1}{2},0,\frac{1}{2},1\}$ if ω has two or three zeros, respectively). It remains to show that

$$(t_1+\alpha_1 s, t_2+\alpha_2 s, t_3+\alpha_3 s) = (t_1,t_2,t_3)+(\alpha_1,\alpha_2,\alpha_3)s$$

belongs to $\mathbb{P}$. But

$$\sum_{i=1}^{3} r_i\left(\frac{(t_i+s\alpha_i)-t_i}{w_i}\right) = s\sum_{i=1}^{3} r_i\frac{\alpha_i}{w_i} = 0$$

by Equation (3). Thus the lemma is proved for the case $n=3$.

Let us now consider the case $n=2$. Note that if D is not a square then the horizontal direction is parabolic in this case (see Theorem 5.2). Therefore, D must be a square. By Proposition 5.1 we know that the cylinder decomposition is stable, which implies that $(X,\omega)+(s,0)$ is defined for all s. In this case, the closure of $U\cdot(X,\omega)$ can be identified with the torus

$$\mathcal{X}' = N(\mathbb{R})\times N(\mathbb{R})/N(w_1\mathbb{Z})\times N(w_2\mathbb{Z}).$$

Using this identification, the horizontal kernel foliation leaf through (X,ω) corresponds to the projection of the affine line $\{(t_1,t_2)+(\alpha_1,\alpha_2)s \mid s\in\mathbb{R}\}$. Hence

$$(X_s,\omega_s) = (X,\omega)+(s,0)\in\overline{U\cdot(X,\omega)},$$

which concludes the proof of Lemma 6.2. □

Proof of Corollary 6.3 We will apply Lemma 6.2 to a transverse direction to $(1:0)$. By Theorem 2.3, let θ be a completely periodic direction on Y which is transverse to the horizontal direction. Up to the action of U, we can assume that $\theta=(0:1)$.

By Proposition 4.1 and Proposition 4.2, there exists $\varepsilon'>0$ such that $(Y,\eta)+v$ is well defined, and the direction $(0:1)$ is completely periodic on $(Y,\eta)+v$ for all $v\in(-\varepsilon',\varepsilon')\times(-\varepsilon',\varepsilon')$. If $s\neq 0$ then the cylinder decomposition of $(Y,\eta)+(s,0)$ in the direction of $(0:1)$ is stable. Moreover, the combinatorial data of this decomposition is preserved when s varies in the intervals $(-\varepsilon',0)$ and $(0,\varepsilon')$. In conclusion, if the decomposition of (Y,η) in the vertical direction is stable, then the combinatorial data of $(Y,\eta)+(s,0)$ is the same for any $s\in(-\varepsilon',\varepsilon')$.

Let $\{w_i(s)\}_{i=1,\dots,k}$ and $\{h_i(s)\}_{i=1,\dots,k}$ be the widths and heights of the cylinders in the *vertical* direction of $(Y,\eta)+(s,0)$, $s\neq 0$. Note that the functions $w_i(s)$ are constant on each of intervals $(-\varepsilon,0)$ and $(0,\varepsilon)$. However, the set of heights $h_i(s)$

define nonconstant continuous functions of s. To be more precise, $h_i(s) = h_i + \alpha_i s$, where $\alpha_i \in \{-1, 0, 1\}$ or $\alpha_i \in \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$ depending on whether η has two or three zeros. Obviously, at least two of the α_i are different. Hence the set of moduli

$$\mu_i(s) = \frac{h_i + s\alpha_i}{w_i}$$

of cylinders (in the vertical direction) define also nonconstant continuous functions of s. In particular, for almost every s in $(-\varepsilon', 0)$ (resp. $(0, \varepsilon')$), the direction $(0:1)$ is completely periodic and not parabolic on $(Y, \eta) + (s, 0)$. Applying Lemma 6.2 to the vertical direction on $(Y, \eta) + (s, 0)$, we get that, for any $t \in (-\varepsilon', \varepsilon')$, one has

$$(Y, \eta) + (s, t) \in \overline{\mathrm{GL}^+(2,\mathbb{R}) \cdot ((Y, \eta) + (s, 0))}.$$

It follows immediately that $(Y, \eta) + v \in \overline{\mathrm{GL}^+(2,\mathbb{R}) \cdot (X, \omega)}$ for every $v = (s, t) \in (-\varepsilon', \varepsilon') \times (-\varepsilon', \varepsilon')$. This completes the proof of Corollary 6.3. □

One can now prove the main result of this section.

Proof of Theorem 6.1 We will show that any $(Y, \eta) \in \overline{\mathrm{GL}^+(2,\mathbb{R}) \cdot (X, \omega)} = \overline{\mathcal{O}}$ has an open neighborhood contained in $\overline{\mathcal{O}}$. Let $\boldsymbol{B}(\varepsilon) = \{v \in \mathbb{R}^2 \mid |v| < \varepsilon\}$.

First case: $(Y, \eta) \in \mathrm{GL}^+(2,\mathbb{R}) \cdot (X, \omega)$ By assumption, there exists a periodic direction for (X, ω) which is not parabolic. Lemma 6.2 and Corollary 6.3 then imply that there exists $\varepsilon > 0$ such that $(X, \omega) + v \in \overline{\mathcal{O}}$ for any $v \in \boldsymbol{B}(\varepsilon)$. It follows that $g \cdot ((X, \omega) + v) \in \overline{\mathcal{O}}$ for all $g \in \mathrm{GL}^+(2,\mathbb{R})$. In particular, there exists a neighborhood $\mathcal{U}$ of Id in $\mathrm{GL}^+(2,\mathbb{R})$ such that $g \cdot ((X, \omega) + v) \in \overline{\mathcal{O}}$, for any $(g, v) \in \mathcal{U} \times \boldsymbol{B}(\varepsilon)$. But by Proposition 3.3 the set $\{g \cdot ((X, \omega) + v) \mid (g, v) \in \mathcal{U} \times \boldsymbol{B}(\varepsilon)\}$ is a neighborhood of (X, ω) in $\Omega E_D(\kappa)$. Hence (X, ω) (and thus (Y, η)) has an open neighborhood contained in $\overline{\mathcal{O}}$.

Second case: $(Y, \eta) \notin \mathrm{GL}^+(2,\mathbb{R}) \cdot (X, \omega)$ Let $(X_n, \omega_n) = g_n \cdot (X, \omega)$ be a sequence converging to (Y, η) with $g_n \in \mathrm{GL}^+(2,\mathbb{R})$. By Proposition 3.3, there exist $\varepsilon > 0$ and a neighborhood $\mathcal{U}$ of Id in $\mathrm{GL}^+(2,\mathbb{R})$ such that $\mathcal{U} \times \boldsymbol{B}(\varepsilon)$ is identified with a neighborhood of (Y, η) via the mapping $(g, v) \mapsto g \cdot ((Y, \eta) + v)$. Thus for n large enough there is a pair (a_n, v_n), where $a_n \in \mathcal{U}$ and $v_n \in \boldsymbol{B}(\varepsilon) \subset \mathbb{R}^2$, such that $(X_n, \omega_n) = a_n \cdot ((Y, \eta) + v_n)$. Since (X_n, ω_n) converges to (Y, η), $(a_n)_n$ converges to Id, and $(v_n)_n$ converges to 0. Multiplying by a_n^{-1}, we get

$$\mathrm{GL}^+(2,\mathbb{R}) \cdot (X, \omega) \ni (X'_n, \omega'_n) = a_n^{-1} \cdot (X_n, \omega_n) = (Y, \eta) + v_n.$$

Without loss of generality, we also assume that the horizontal direction is completely periodic on Y. By Propositions 4.1 and 4.2, we can choose $r > 0$ such that for all

$v=(s,t)\in \boldsymbol{B}(r)$ the surface $(Y,\eta)+v$ also admits a cylinder decomposition in the horizontal direction. When $t\neq 0$ this decomposition is stable with combinatorial data depending only on the sign of t. We can assume that $v_n\in \boldsymbol{B}(r)$ (for n large enough).

Since $(X'_n,\omega'_n)\in \mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)$, the first case implies that $\overline{\mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)}$ contains a neighborhood of (X'_n,ω'_n). Hence for each n there exists $\varepsilon_n>0$ such that $(X'_n,\omega'_n)+v\in\overline{\mathcal{O}}$ for any $v\in\boldsymbol{B}(\varepsilon_n)$. Now for each n we choose $\delta_n\in(0,\varepsilon_n)$ small enough such that:

(a) $u_n=v_n+(0,\delta_n)\in\boldsymbol{B}(r)$.

(b) If $v_n=(s_n,t_n)$ with $t_n\neq 0$, then $\delta_n<|t_n|$.

In particular, since $u_n\in\boldsymbol{B}(r)$, (a) implies that $(Y,\eta)+u_n$ also admits a cylinder decomposition in the horizontal direction. Since the ratio of moduli is a continuous (nonconstant) function of δ_n, one can choose $\delta_n\in(0,\varepsilon_n)$ satisfying (a), (b) and the following conditions:

(c) The horizontal direction is stable and not parabolic for $(Y,\eta)+u_n$.

(d) $\lim_{n\to\infty}\delta_n=0$.

By construction, $\delta_n\in(0,\varepsilon_n)$, hence $(X''_n,\omega''_n):=(X'_n,\omega'_n)+(0,\delta_n)=(Y,\eta)+u_n\in\overline{\mathcal{O}}$. Since the horizontal direction is not parabolic on (X''_n,ω''_n), by Lemma 6.2, we derive that $(X''_n,\omega''_n)+(s,0)\in\overline{\mathcal{O}}$ for any $s\in\mathbb{R}$ (see Figure 3). Thus

$$(X''_n,\omega''_n)+(s,0)\in\overline{\mathcal{O}}\quad\text{for any } s\in\left(-\tfrac{1}{2}r,\tfrac{1}{2}r\right).$$

Since $(\delta_n)_n$ converges to 0 the sequence $(X''_n,\omega''_n)=(X'_n,\omega'_n)+(0,\delta_n)$ converges to (Y,η). It follows that

$$(Y,\eta)+(s,0)\in\overline{\mathcal{O}}\quad\text{for all } s\in\left(-\tfrac{1}{2}r,\tfrac{1}{2}r\right).$$

The theorem then follows from Corollary 6.3. $\square$

7 Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1 in full generality, namely without the assumption that the orbit $\mathcal{O}:=\mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)$ is not closed for the most obvious reason. However, our proof says nothing about the converse of this assumption, ie the following question remains open in our setting:

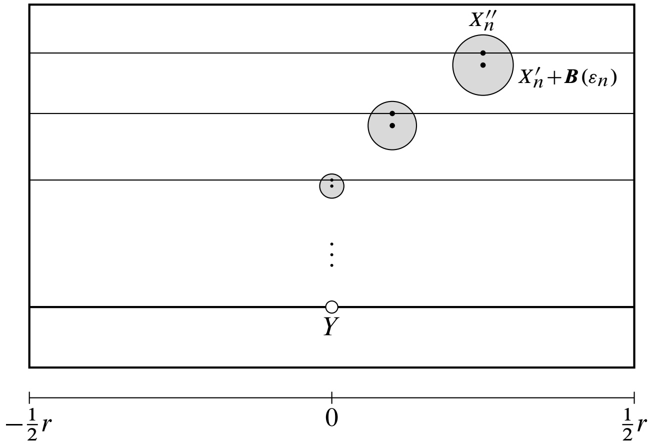

Figure 3: The convergence of (X_n',ω_n') and (X_n'',ω_n'') to (Y,η) in the kernel foliation leaf of (Y,η).

Question For an orbit $\mathcal{O} := \mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)$, is the property of being not closed equivalent to being not closed for the most obvious reason?

Proof of Theorem 1.1 We begin by fixing some notation and normalization. As usual, let $(X,\omega)\in\Omega E_D(\kappa)$ and let us assume that $\mathcal{O}:=\mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)$ is not closed. Let $(Y,\eta)\in\overline{\mathcal{O}}\setminus\mathcal{O}$ be some translation surface in the orbit closure, but not in the orbit itself.

Claim 1 *There exist a rotation R and a sequence $(X_n,\omega_n)_{n\in\mathbb{N}}$ converging to $R\cdot Y$ such that $(X_n,\omega_n)=R\cdot(Y,\eta)+v_n\in\mathcal{O}$ for every $n\in\mathbb{N}$, where $v_n=(x_n,y_n)$ with $y_n\neq 0$, and the horizontal direction on $R\cdot Y$ is completely periodic.*

Proof of the claim We choose a sequence $(X_n,\omega_n)\in\mathcal{O}$ converging to (Y,η). As in the proof of Theorem 6.1 we can assume that $(X_n,\omega_n)=(Y,\eta)+v_n$, where $v_n=(x_n,y_n)$ converges to $(0,0)\in\mathbb{R}^2$.

Again, up to replacing Y by $R_\theta\cdot Y$ for some suitable θ, without loss of generality we will also assume that the horizontal direction is completely periodic on Y. If $y_n\neq 0$ infinitely often then the claim follows by taking a subsequence. Otherwise we assume that $y_n=0$ for every $n>N$. We choose another (transverse) completely periodic direction on Y. We can assume that this direction is vertical by applying a matrix in U. Note that a matrix in U fixes the vectors $(x_n,0)$. Then, up to replacing (Y,η) and (X_n,ω_n) by $R_{\pi/2}\cdot(Y,\eta)$ and $R_{\pi/2}\cdot(X_n,\omega_n)$, respectively, the claim is proved (otherwise $x_n=0$ for n large enough, thus $(Y,\eta)=(X_n,\omega_n)\in\mathcal{O}$, which is a contradiction to our assumption). □

In the sequel, up to replacing Y by $R \cdot Y$, we assume that the conclusion of Claim 1 holds for Y. We choose some $\varepsilon > 0$ so that, for any $v = (x, y) \in \mathbb{R}^2$, if $v \in \boldsymbol{B}(\varepsilon)$ then the horizontal direction on $(Y, \eta) + v$ is periodic, and the cylinder decomposition is stable if $y \neq 0$. We can assume that $v_n \in \boldsymbol{B}(\varepsilon)$ and $y_n > 0$ for all n, which implies that the combinatorial data of the cylinder decomposition in the horizontal direction of (X_n, ω_n) are same for all n. Finally, we also assume that *all* the horizontal directions on X_n are parabolic (otherwise we are done by Theorem 6.1).

We sketch the idea of the proof. It makes use of the horocycle flow u_s acting on X_n. The key is to show that the actions of the kernel foliation and u_s coincide for a subsequence.

(1) Since *all* surfaces (X_n, ω_n) are horizontally parabolic, we will show that it is always possible to find a "good time" s_n so that $u_{s_n} \cdot X_n = X_n + (x_n, 0)$ for some vector $(x_n, 0) \in \boldsymbol{B}(\varepsilon)$.

(2) One can arrange that $(x_n, 0)$ converges to some arbitrary vector $(x, 0) \in \boldsymbol{B}(\varepsilon)$.

These two facts correspond, respectively, to Claim 3 and Claim 4 below. Once we achieve this, passing to the limit as $n \to \infty$, we get

$$u_{s_n} \cdot (X_n, \omega_n) = (X_n, \omega_n) + (x_n, 0) \to (Y, \eta) + (x, 0).$$

In other words, $(Y, \eta) + (x, 0) \in \overline{\mathcal{O}}$ for all $x \in (-\varepsilon, \varepsilon)$. Then Corollary 6.3 applies and this gives some $\varepsilon' > 0$ so that $(Y, \eta) + v \in \overline{\mathcal{O}}$ for any $v \in \boldsymbol{B}(\varepsilon')$, which proves the theorem.

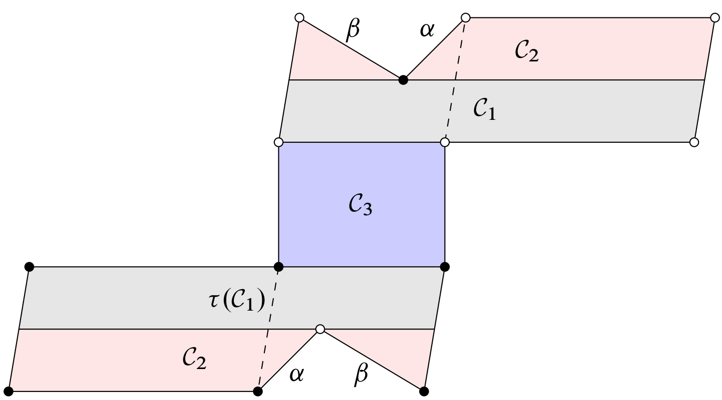

Figure 4: Decomposition into four cylinders of $(X_n, \omega_n) = (Y, \eta) + v_n$ near $(Y, \eta) \in \Omega E_D(2, 2)$, where $v_n = \int_\alpha \omega$. The cylinders $\mathcal{C}_2$ and $\mathcal{C}_3$ are fixed by the Prym involution τ, while the cylinders $\mathcal{C}_1$ and $\tau(\mathcal{C}_1)$ are exchanged. When $v_n \to 0$ the cylinder $\mathcal{C}_2$ is destroyed, while $\mathcal{C}_3$ remains in the limit (here we assume $h_3 > h_2$).

We now explain how to construct the sequence $(s_n)_{n\in\mathbb{N}}$. As usual, the cylinders on X_n are denoted by $\mathcal{C}_i^{(n)}$, $i=1,\dots,k$ (the numbering is such that for every $i\in\{1,2,3\}$, $\mathcal{C}_j^{(n)}=\tau(\mathcal{C}_i^{(n)})$ implies $j=i$ or $j>3$). The width, height, twist, and modulus of $\mathcal{C}_i^{(n)}$ are denoted by $w_i^{(n)},h_i^{(n)},t_i^{(n)},\mu_i^{(n)}$, respectively. Recall that, by Propositions 4.1 and 4.2, $w_i^{(n)}$ does not depend on n, therefore we can write $w_i^{(n)}=w_i$. Let us define

$$h_i^{\infty}=\lim_{n\to\infty} h_i^{(n)}.$$

Since the cylinder decomposition of X_n is stable, we can associate to each family of cylinders $(\mathcal{C}_i^{(n)})_n$ a coefficient $\alpha_i\in\left\{0,\pm\frac{1}{2},\pm 1\right\}$. Recall that the kernel foliation action of a vector $v=(x,y)$ changes the height $h_i^{(n)}$ of $\mathcal{C}_i^{(n)}$ to $h_i^{(n)}+\alpha_i y$, hence we can write

$$h_i^{(n)}=h_i^{\infty}+\alpha_i y_n.$$

Note that the horizontal direction on Y is not necessarily stable: some horizontal cylinders on X_n can be destroyed in the limit (as n tends to infinity). Therefore, some of the limits h_i^{∞} may be zero. However, there is at least one cylinder that remains in the limit, say it is $\mathcal{C}_3^{(n)}$ (see Figure 4 where the cylinder $\mathcal{C}_2^{(n)}$ is destroyed when performing the kernel foliation). Actually, since (X_n,ω_n) stays in a neighborhood of (Y,η), all the cylinders of (Y,η) persist in (X_n,ω_n). Thus, the number of horizontal cylinders of (X_n,ω_n) is always greater than (Y,η). We denote by $\mathcal{C}_3$ the cylinder on Y corresponding to $\mathcal{C}_3^{(n)}$ on X_n. Then the height of $\mathcal{C}_3$ is h_3^{∞}. In particular, $h_3^{\infty}>0$.

From Equation (4), we obtain

$$\sum_{i=1}^{3}\beta_i w_i\alpha_i=0.$$

Since all the α_i cannot vanish (otherwise for all $i\in\{1,\dots,k\}$ the upper and lower boundaries of $\mathcal{C}_i^{(n)}$ contain the same zero, which means that ω has only one zero), Equation (4) implies that there exist $i,j\in\{1,2,3\}$ such that α_i and α_j are nonzero and have opposite signs. In particular, there exists $i\in\{1,2,3\}$ such that $\alpha_i\neq 0$ and α_i has the opposite sign to α_3 if $\alpha_3\neq 0$. In what follows we suppose that α_1 satisfies this condition. By a slight abuse of language, we will say that α_1 and α_3 have opposite signs. Since $\alpha_1\neq 0$, $(t_1^{(n)},h_1^{(n)})$ is a relative coordinate. For the surface in Figure 1, ω has three zeros and $(\alpha_1,\alpha_3)=\left(-1,\frac{1}{2}\right)$, and for the one in Figure 4, ω has two zeros and $(\alpha_1,\alpha_3)=(-1,1)$.

Recall that, by Proposition 5.1, we know that there exists $(r_1,r_2,r_3)\in\mathbb{Q}^3\setminus\{(0,0,0)\}$ such that

$$r_1\mu_1^{(n)}+r_2\mu_2^{(n)}+r_3\mu_3^{(n)}=0 \quad\text{and}\quad r_1\frac{\alpha_1}{w_1}+r_2\frac{\alpha_2}{w_2}+r_3\frac{\alpha_3}{w_3}=0.$$

Obviously, we can assume that $(r_1, r_2, r_3) \in \mathbb{Z}^3$. Note that (r_1, r_2, r_3) does not depend on n. Set $\mu_i^\infty = h_i^\infty / w_i$. By continuity we have

$$r_1\mu_1^\infty + r_2\mu_2^\infty + r_3\mu_3^\infty = 0.$$

Claim 2 $r_2 \neq 0$.

Proof Suppose $r_2 = 0$. Then

$$\begin{cases} r_1\mu_1^{(n)} + r_3\mu_3^{(n)} = 0, \\ r_1\dfrac{\alpha_1}{w_1} + r_3\dfrac{\alpha_3}{w_3} = 0. \end{cases}$$

Since $\mu_i^{(n)} > 0$, $w_i > 0$ and $\alpha_1\alpha_3 \leq 0$, this system with unknowns (r_1, r_3) has a unique solution $r_1 = r_3 = 0$. This is a contradiction. □

From now on, we fix an integral vector $(r_1, r_2, r_3) \in \mathbb{Z}^3$ satisfying equations (2)–(3), with $r_2 \neq 0$.

Claim 3 *Let $(\widetilde{X}, \widetilde{\omega}) \in \Omega E_D(\kappa)$ be a surface which admits the same cylinder decomposition as X_n in the horizontal direction. We denote by $\mathcal{C}_i$ the cylinder in $\widetilde{X}$ which corresponds to the cylinder $\mathcal{C}_i^{(n)}$ of X_n. Let $w_i, h_i, t_i, \mu_i, \alpha_i$ be the parameters of $\mathcal{C}_i$. Given two integers k_1, k_3, if the real numbers s and $x(s)$ satisfy*

$$x(s) := \frac{1}{\alpha_3}(sh_3 - r_2k_3w_3) = \frac{1}{\alpha_1}(sh_1 - r_2k_1w_1), \tag{6}$$

then $u_s \cdot (\widetilde{X}, \widetilde{\omega}) = (\widetilde{X}, \widetilde{\omega}) + (x(s), 0)$.

Remark 7.1 If $\alpha_3 = 0$, we replace Equation (6) by the following system:

$$\begin{cases} sh_3 = r_2k_3w_3, \\ x(s) = \dfrac{sh_1 - r_2k_1w_1}{\alpha_1}. \end{cases}$$

Proof of the claim On one hand, the kernel foliation $\widetilde{X} + (x, 0)$, for small values of x, maps the twist of the cylinder $\mathcal{C}_i$ to $t_i(x) = t_i + \alpha_i x$. On the other hand, the action of u_s on the cylinder $\mathcal{C}_i$ maps the twist t_i to the twist $\widetilde{t}_i = t_i + sh_i \mod w_i$. Equation (6) implies

$$sh_1 = \alpha_1 x(s) + r_2k_1w_1 \quad \text{and} \quad sh_3 = \alpha_3 x(s) + r_2k_3w_3,$$

which is equivalent to

$$\begin{cases} s\mu_1 = \dfrac{\alpha_1}{w_1}x(s) + r_2k_1, \\ s\mu_3 = \dfrac{\alpha_3}{w_3}x(s) + r_2k_3. \end{cases} \tag{7}$$

We see that the twist of the cylinder $\mathcal{C}_i$ of $u_s \cdot \widetilde{X}$ is $\widetilde{t_i} = t_i + \alpha_i x(s) \bmod w_i$, for $i \in \{1,3\}$. It remains to show that $sh_2 = \alpha_2 x(s) \bmod w_2$. Using equations (2)–(3), (7) implies

$$-r_2 s\mu_2 = -r_2\frac{\alpha_2}{w_2}x(s) + r_2(r_1k_1 + r_3k_3).$$

It follows that

$$sh_2 = \alpha_2 x(s) - (r_1k_1 + r_3k_3)w_2.$$

Thus we can conclude that $u_s \cdot (\widetilde{X}, \widetilde{\omega}) = (\widetilde{X}, \widetilde{\omega}) + (x(s), 0)$. □

Equation (6) above reads

$$s = r_2\frac{w_1k_1\alpha_3 - w_3k_3\alpha_1}{h_1\alpha_3 - h_3\alpha_1}. \tag{8}$$

Note that since α_1 and α_3 have opposite signs, s is always defined. Substituting this last equation into (6), we derive the relation

$$x(s) = \frac{r_2}{\alpha_3}\left(\frac{w_1k_1\alpha_3 - w_3k_3\alpha_1}{h_1\alpha_3 - h_3\alpha_1}h_3 - k_3w_3\right) = \cdots = \frac{r_2h_3w_1}{h_1\alpha_3 - h_3\alpha_1}\left(k_1 - \frac{\mu_1}{\mu_3}\,k_3\right).$$

We now make the additional assumption that the horizontal direction is parabolic, ie the moduli μ_i are all commensurable. We thus write the last expression as

$$x(s) = \frac{r_2h_3w_1}{h_1\alpha_3 - h_3\alpha_1}\left(k_1 - \frac{p}{q}\,k_3\right), \quad \text{where } \frac{p}{q} = \frac{\mu_1}{\mu_3} \in \mathbb{Q}.$$

We perform this calculation for each surface X_n, so that given a sequence $(k_1^{(n)}, k_3^{(n)})_n$ we get a sequence

$$x_n = \frac{r_2h_3^{(n)}w_1^{(n)}}{h_1^{(n)}\alpha_3 - h_3^{(n)}\alpha_1}\left(k_1^{(n)} - \frac{p^{(n)}}{q^{(n)}}\,k_3^{(n)}\right), \tag{9}$$

where $(p^{(n)}, q^{(n)}) \in \mathbb{Z}^2$ and $\gcd(p^{(n)}, q^{(n)}) = 1$. We want to choose a suitable pair of integers $(k_1^{(n)}, k_3^{(n)}) \in \mathbb{Z}^2$ in order to make the sequence $(x_n)_n$ converge to some arbitrary x.

Claim 4 *There exists a constant C independent of n such that, for any $x \in (-\varepsilon, \varepsilon)$, there exists $(k_1^{(n)}, k_3^{(n)}) \in \mathbb{Z}^2$ satisfying the following: if x_n is defined by (9), then*

$$|x_n - x| < \frac{C}{q^{(n)}}.$$

Proof of the claim For each $n \in \mathbb{N}$, since $p^{(n)}$ and $q^{(n)}$ are coprime, we can choose $(k_1^{(n)}, k_3^{(n)}) \in \mathbb{Z}^2$ such that

$$\left| k_1^{(n)} - \frac{p^{(n)}}{q^{(n)}} k_3^{(n)} - \frac{h_1^{(n)}\alpha_3 - h_3^{(n)}\alpha_1}{r_2 h_3^{(n)} w_1^{(n)}} x \right| < \frac{1}{q^{(n)}}. \tag{10}$$

As n tends to infinity, the sequence $(h_3^{(n)})_n$ converges to h_3^∞. Since $w_1^{(n)}$ is constant, and $h_1^{(n)}\alpha_3 - h_3^{(n)}\alpha_1$ converges to a nonzero constant (recall that α_1 and α_3 have opposite signs), there exists some constant $C > 0$ such that

$$\frac{r_2 h_3^{(n)} w_1^{(n)}}{h_1^{(n)}\alpha_3 - h_3^{(n)}\alpha_1} < C. \tag{11}$$

From (10) and (11) we obtain

$$|x_n - x| < \frac{C}{q^{(n)}},$$

which is the desired inequality. The claim is proved. □

In order to conclude the proof of Theorem 1.1, one needs to show that $q^{(n)} \to \infty$. Indeed, we then have that $x_n \to x$ and since x was arbitrary, by Claim 3 this shows

$$(Y, \eta) + (x, 0) \in \overline{\mathcal{O}} \quad \text{for any } x \in (-\varepsilon, \varepsilon).$$

Then Corollary 6.3 applies and Y has an open neighborhood in $\overline{\mathcal{O}}$, which proves the theorem.

We now prove that $q^{(n)} \to \infty$. Recall that

$$\frac{p^{(n)}}{q^{(n)}} = \frac{\mu_1^{(n)}}{\mu_3^{(n)}} = \frac{w_3^{(n)}}{w_1^{(n)}} \cdot \frac{h_1^{(n)}}{h_3^{(n)}} = \frac{w_3}{w_1} \cdot \frac{h_1^\infty + \alpha_1 y_n}{h_3^\infty + \alpha_3 y_n}$$

and $\gcd(p^{(n)}, q^{(n)}) = 1$. Note that since α_1 and α_3 have opposite signs, $p^{(n)}/q^{(n)}$ cannot be a stationary sequence as y_n tends to 0. As n tends to infinity, $p^{(n)}/q^{(n)}$ converges to $p^\infty/q^\infty = w_3 h_1^\infty / w_1 h_3^\infty$. But as we have seen, $p^{(n)}/q^{(n)}$ cannot be stationary, therefore there are infinitely many n such that $p^{(n)}/q^{(n)} \neq p^\infty/q^\infty$, which implies that $q^{(n)} \to \infty$. □

In the remainder of this paper, we will apply Theorem 1.1 (more precisely, the techniques used in the proof) to show that, for any D which is not a square, there are at most finitely many closed $\mathrm{GL}^+(2,\mathbb{R})$–orbits in $\Omega E_D(2,2)^{\mathrm{odd}}$. Even though we only prove the result for this case, it seems very likely that one can also obtain similar results for all strata listed in Table 1. In higher "complexity" (genus and number singularities) the difficulty comes from the increasing number of degenerated surfaces. Along the way, we will give description of surfaces in a partial compactification of $\Omega E_D(2,2)^{\mathrm{odd}}$.

We end this section with a by-product theorem which follows from the same arguments as the proof of Theorem 1.1.

Theorem 7.2 *Let $(Y,\eta)\in\Omega E_D(\kappa)$ be a Prym eigenform (where $\Omega E_D(\kappa)$ has complex dimension 3) satisfying the following properties:*

(1) *The horizontal direction is completely periodic on (Y,η).*
(2) *There exists a sequence $(X_n,\omega_n)=(Y,\eta)+(x_n,y_n)$ converging to (Y,η), where $y_n\neq 0$ for any $n\in\mathbb{N}$.*
(3) *For every n, X_n is horizontally parabolic.*

Then there exists $\varepsilon>0$ such that $(Y,\eta)+(x,0)\in\overline{\mathcal{O}}$ for all $x\in(-\varepsilon,\varepsilon)$, where $\mathcal{O}=\bigcup_n \mathrm{GL}^+(2,\mathbb{R})\cdot(X_n,\omega_n)$.

8 Preparation of a surgery toolkit for the proof of Theorem 1.3

In this section we will describe several useful surgeries for Prym eigenforms. More precisely, let us fix a surface (X_0,ω_0) in the following list of strata of $\Omega E_D(\kappa)$:

- $\Omega E_D(0,0,0)$ (space of triple tori; see Section 8A),
- $\Omega E_D(4)$ (see Section 8B),
- $\Omega E_D(2)^*$ (set of $(M,\omega)\in\Omega E_D(2)$ with a marked Weierstrass point; see Section 8C).

For each case, we will construct a continuous locally injective map $\Psi\colon D^\circ(\varepsilon)\to\Omega E_D(2,2)^{\mathrm{odd}}$, where $D^\circ(\varepsilon)=\{z\in\mathbb{C}\mid 0<|z|<\varepsilon\}$, which induces an embedding of $D^\circ(\varepsilon)/(z\sim -z)$ into $\Omega E_D(2,2)^{\mathrm{odd}}$. Up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, the set $\Psi(D^\circ(\varepsilon))$ will be identified with a neighborhood of (X_0,ω_0) in $\Omega E_D(2,2)^{\mathrm{odd}}$.

We now describe these surgeries in detail (observe that the second one already appears in [11] as "breaking up a zero").

8A Space of triples of tori

We say that $(X,\omega)\in \mathrm{Prym}(2,2)^{\mathrm{odd}}$ admits a *three-torus decomposition* if there exists a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ on X, each of which connects the two zeros of ω, such that (X,ω) can be viewed as a connected sum of three tori which are glued together along the slits corresponding to σ_j. One can reduce the length of saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ to zero by moving in the kernel foliation leaf through (X,ω); the limit surface is then the union of three flat tori (X_j,ω_j), $j=0,1,2$, which are joined at a unique common point P.

Recall that $\mathcal{H}(0)$ is the space of triples (X,ω,P) where X is an elliptic curve, ω a nonzero abelian differential on X, and P is a marked point of X. We denote by $\mathrm{Prym}(0,0,0)$ the space of triples $\{(X_j,\omega_j,P_j)\mid j=0,1,2\}$, where $(X_j,\omega_j,P_j)\in\mathcal{H}(0)$, such that (X_1,ω_1,P_1) and (X_2,ω_2,P_2) are isometric. The geometric object corresponding to such a triple is the union of the three tori, where we identify P_0,P_1,P_2 to get a unique common point. By construction, there exists an involution τ on the "surface" $X:=\{(X_j,\omega_j,P_j)\mid j=0,1,2\}$ which preserves X_0 and exchanges X_1 and X_2. We will call τ the Prym involution.

We define $\Omega E_D(0,0,0)\subset\mathrm{Prym}(0,0,0)$ to be the space of all triples $\{(X_j,\omega_j,P_j)\mid j=0,1,2\}$ which can be obtained by collapsing triples of homologous saddle connections associated to three-torus decompositions of surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$. The aim of this section is to show:

Proposition 8.1 *For any triple of tori $\{(X_j,\omega_j,P_j)\mid j=0,1,2\}$ in $\Omega E_D(0,0,0)$, there exist $\varepsilon>0$ and a continuous locally injective map $\Psi\colon D^\circ(\varepsilon)\to\Omega E_D(2,2)^{\mathrm{odd}}$ satisfying:*

1. *For all $z\in D^\circ(\varepsilon)$, the surface $(X,\omega)=\Psi(z)$ has a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ decomposing X into three tori such that $\omega(\sigma_j)=z$.*
2. *The map Ψ is two-to-one and it induces an embedding of $D^\circ(\varepsilon)/(z\sim -z)$ into $\Omega E_D(2,2)^{\mathrm{odd}}$.*
3. *Up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, the set $\Psi(D^\circ(\varepsilon))$ can be viewed as a neighborhood of $\{(X_j,\omega_j,P_j)\mid j=0,1,2\}$ in $\Omega E_D(2,2)^{\mathrm{odd}}$.*

We postpone the proof of Proposition 8.1 and first provide a description of the space $\Omega E_D(0,0,0)$ (compare with [26, Theorem 8.3]).

Proposition 8.2 *Let $\{(X_j,\omega_j,P_j)\mid j=0,1,2\}$ be a triple of tori in $\Omega E_D(0,0,0)$ (where X_1,X_2 are exchanged by the Prym involution τ). Then there exist $(e,d)\in\mathbb{Z}^2$, with $d>0$, and a covering $p\colon X_1\to X_0$ of degree d such that*

- $D = e^2 + 8d$,
- $\gcd(e, p_{11}, p_{12}, p_{21}, p_{22}) = 1$, *where* (p_{ij}) *is the matrix of* p *in some symplectic bases of* $H_1(X_0,\mathbb{Z})$ *and* $H_1(X_1,\mathbb{Z})$,
- $p^*\omega_0 = \frac{1}{2}\lambda\omega_1$, *where* λ *satisfies* $\lambda^2 = e\lambda + 2d$.

Proof Let (a_j, b_j) be a symplectic basis of $H_1(X_j,\mathbb{Z})$, where

$$a_2 = -\tau(a_1), \quad b_2 = -\tau(b_1),$$

and set

$$\widehat{a} = a_1 + a_2, \quad \widehat{b} = b_1 + b_2.$$

Then $(a_0, b_0, \widehat{a}, \widehat{b})$ is a symplectic basis of $H_1(X,\mathbb{Z})^-$ (here X is the surface obtained by identifying $P_0 \sim P_1 \sim P_2$). There exists a unique generator T of $\mathcal{O}_D$ such that the matrix of T in the basis $(a_0, b_0, \widehat{a}, \widehat{b})$ is of the form

$$T = \begin{pmatrix} e\,\mathrm{Id}_2 & 2B \\ B^* & 0 \end{pmatrix},$$

where $e \in \mathbb{Z}$, $B \in \boldsymbol{M}_2(\mathbb{Z})$,

$$B^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot B \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $T^*\omega = \lambda\omega$, with $\lambda > 0$.

Observe that B can be regarded as a map from $H_1(X_1,\mathbb{Z})$ to $H_1(X_0,\mathbb{Z})$. Set

$$L_0 = \mathbb{Z}\omega_0(a_0) + \mathbb{Z}\omega_0(b_0), \quad L_1 = \mathbb{Z}\omega_1(a_1) + \mathbb{Z}\omega_1(b_1).$$

We can identify (X_0, ω_0) and (X_1, ω_1) with $(\mathbb{C}/L_0, dz)$ and $(\mathbb{C}/L_1, dz)$, respectively. The condition $T^*\omega = \lambda\omega$ reads

$$\omega_0(2B(a_1)) = \lambda \cdot \omega_1(a_1) \quad \text{and} \quad \omega_0(2B(b_1)) = \lambda \cdot \omega_1(b_1).$$

Hence $\frac{1}{2}\lambda L_1$ is a sublattice of L_0. It follows that there exists a covering map $p\colon \mathbb{C}/L_1 \to \mathbb{C}/L_0$ such that $p^*dz = \frac{1}{2}\lambda\, dz$. The degree of p is given by $d = \det(B) > 0$. Note that T satisfies

$$T^2 = eT + 2\det(B).$$

Since T is a generator of $\mathcal{O}_D$, we have $D = e^2 + 8\det(B)$, and λ satisfies the same equation since λ is an eigenvalue of T. □

Proof of Proposition 8.1 Let $\varepsilon > 0$ be small enough so that the set

$$D(P_j,\varepsilon) = \{x \in X_j \mid \boldsymbol{d}(x, P_j) < \varepsilon\}$$

is an embedded disc in X_j, $j = 0,1,2$. The map Ψ is defined as follows: for any $z \in D^\circ(\varepsilon)$, let σ_j be the geodesic segment in X_j whose midpoint is P_j such that $\omega(\sigma_j) = z$ (since $|z| < \varepsilon$, σ_j is an embedded segment). By slitting X_j along σ_j, and gluing X_0, X_1, X_2 along the slits in cyclic order, we get a surface (X,ω) in $\mathcal{H}(2,2)$. It is easy to check that $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$. We define $(X,\omega) = \Psi(z)$. Since we cannot distinguish the two zeros of ω, one has $\Psi(z) = \Psi(-z)$.

Clearly, any surface in $\Omega E_D(2,2)^{\mathrm{odd}}$ admitting a three-torus decomposition $\{(X'_j,\omega'_j) \mid j = 1,2,3\}$ such that $(X'_j,\omega'_j) = (X_j,\omega_j)$ and where the length of the slit is smaller than ε belongs to the image of Ψ. The proposition follows immediately from this observation. □

8B Collapsing surfaces to $\Omega E_D(4)$

This surgery already appears in [11] ("breaking up a zero"). As in the previous section, our aim is to show:

Proposition 8.3 *For any $(X_0,\omega_0) \in \Omega E_D(4)$, there exist $\varepsilon > 0$ and a continuous locally injective map $\Psi\colon D^\circ(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ satisfying:*

(1) *For all $z \in D^\circ(\varepsilon)$, the surface $(X,\omega) = \Psi(z)$ has the same absolute periods as (X_0,ω_0).*

(2) *There exists a saddle connection σ in X, joining the zeros of ω and invariant under the Prym involution, such that $\omega(\sigma) = z^5$.*

(3) $\Psi(z) = \Psi(-z)$.

(4) *Up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, a neighborhood of $(X_0,\omega_0) \in \Omega E_D(4)$ in $\Omega E_D(2,2)^{\mathrm{odd}}$ is identified with $\Psi(D^\circ(\varepsilon))$.*

The constructive proof we will give is on the level of abelian differentials, ie in $\mathrm{Prym}(2,2)$ and $\mathrm{Prym}(4)$. One can interpret this construction on the level of quadratic differentials, ie $\mathcal{Q}(-1^4,4)$ and $\mathcal{Q}(-1^3,3)$, respectively. This last approach is related to the surgery "breaking up a singularity" in [11] (breaking up the zero of degree 3 of the quadratic differential into a pole and a zero of degree 4).

Proof of Proposition 8.3 Let $(X_0,\omega_0) \in \Omega E_D(4)$ and let P_0 be the unique zero of ω_0. We consider $0 < \varepsilon < 1$ small enough so that the disc $D(P_0,\varepsilon) = \{x \in X_0 \mid \boldsymbol{d}(x, P_0) < \varepsilon\}$ is embedded into X_0. To define the map Ψ, we will deform the metric structure inside

$D(P_0,\varepsilon)$ in a similar manner as was done in Section 4. Let $v \in D(\varepsilon)$ be a vector with $v \neq 0$. We cut the Euclidean disc $D(\varepsilon) := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ by a line in the direction of v through its center. Let $D^+(\varepsilon)$ and $D^-(\varepsilon)$ be respectively the upper and lower half-discs. The disc $D(P_0,\varepsilon)$ can be constructed from five copies of $D^+(\varepsilon)$ and five copies of $D^-(\varepsilon)$.

We change the method of gluing the half-discs, as follows: we still use the same Euclidean half-discs, but we move slightly the centers on their diameters (in the direction of v). We will use two special half-discs as indicated in Figure 5. They have two marked points on the diameter at the distance $\frac{1}{2}|v|$ from the center. Each of the remaining half-discs has a single marked point at the distance $\frac{1}{2}|v|$ from the center. We alternate the half-discs with the marked point moved to the right and to the left from the center (in direction v). All the lengths along identifications match. We obtain a new topological disc, but now the flat metric has two cone-type singularities with the cone angle 6π. Note that a small tubular neighborhood of the boundary of the initial cone is isometric to the corresponding tubular neighborhood of the boundary of the resulting object. Thus we can paste it back into the surface.

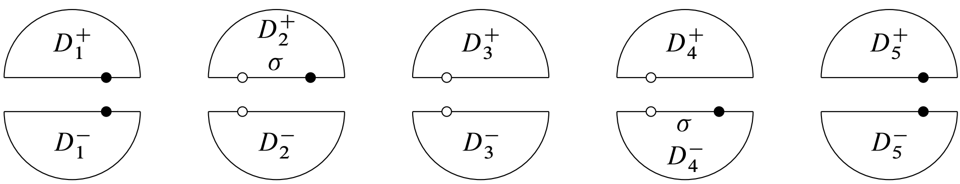

Figure 5: Splitting a zero of order 4 into two zeros of order 2.

Observe that, by construction, there exists an involution in $D(P_0,\varepsilon)$ that maps D_i^+ to $D^-_{(i+2)}$. Thus the resulting surface (X,ω) belongs to $\mathrm{Prym}(2,2)^{\mathrm{odd}}$. By construction, there is a saddle connection σ, invariant under the involution with $\omega(\sigma) = v$. Since we have five choices for the pair of half-discs which contain σ in their boundary, there are five surfaces $(X,\omega) \in \mathrm{Prym}(2,2)$ close to (X_0,ω_0) satisfying the following conditions:

- The absolute periods of ω and ω_0 coincide.
- There exists a saddle connection σ in X, invariant under the Prym involution and joining the two zeros of ω, such that $\omega(\sigma) = v$.

Since the absolute periods of ω and ω_0 coincide, the new surfaces actually belong to the same real multiplication locus as (X_0,ω_0), that is, $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$.

Let z be a complex number such that $z^5 = v$. We define the map Ψ by assigning $\Psi(z)$ to be one of the surfaces constructed above. By analytic continuation, this defines the desired map $\Psi\colon D^\circ(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$. Since we cannot distinguish the zeros of ω, the surfaces corresponding to $\pm z$ are the same (with different choices for the orientation of σ). The properties asserted in the statement of the proposition follow immediately from the definition of Ψ. □

Remark 8.4 The "breaking up a zero" surgery is clearly invertible: we can collapse the two zeros of (X,ω) along σ to get the surface $(X_0,\omega_0)\in\Omega E_D(4)$. More generally, let P, Q denote the zeros of ω, where $(X,\omega)\in\Omega E_D(2,2)^{\mathrm{odd}}$, and let σ be a saddle connection, which we assume to be horizontal, which joins P to Q and which is invariant under the involution τ (such a saddle connection always exists, for instance the union of a path of minimal length joining a fixed point of τ to P or Q, and its image under τ). If any other horizontal saddle connection σ' satisfies $|\sigma'|>2|\sigma|$ then one can collapse the zeros of ω along σ by using the kernel foliation (see Section 9). The resulting surface (X_0,ω_0) belongs to $\Omega E_D(4)$. However, if σ has twins, that is another saddle connection σ' such that $\omega(\sigma')=\omega(\sigma)$, then the limit surface is no longer in $\Omega E_D(4)$, as we will see in the next section.

8C Collapsing surfaces to $\Omega E_D(2)^*$

In this section, we investigate degenerations by shrinking a pair of saddle connections that are exchanged by the Prym involution. Let $\Omega E_{D'}(2)^*$ be the space of triples (X,ω,W), where $(X,\omega)\in\Omega E_{D'}(2)$ and W is a Weierstrass point of X which is not the zero of ω. We will prove:

Proposition 8.5 *For any $(X_0,\omega_0,W_0)\in\Omega E_{D'}(2)^*$ there exist $\varepsilon>0$, $D\in\{D',4D'\}$, and a continuous locally injective map $\Psi\colon D^\circ(\varepsilon)\to\Omega E_D(2,2)^{\mathrm{odd}}$ with the following properties:*

(1) *For all $z\in D^\circ(\varepsilon)$ the surface $(X,\omega)=\Psi(z)$ has the same absolute periods as (X_0,ω_0,W_0).*
(2) *There exists a pair of saddle connections (σ_1,σ_2) on X that are exchanged by the Prym involution and satisfy $\omega(\sigma_1)=\omega(\sigma_2)=z^3$.*
(3) $\Psi(z)=\Psi(-z)$.
(4) *Up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, $\Psi(D^\circ(\varepsilon))$ is a neighborhood of (X_0,ω_0,W_0) in $\Omega E_D(2,2)^{\mathrm{odd}}$.*

As for the surgeries described previously, we will describe how one can degenerate some $(X,\omega)\in\Omega E_D(2,2)^{\mathrm{odd}}$ to the boundary of the stratum, ie to $(X_0,\omega_0,W_0)\in\Omega E_{D'}(2)^*$, by using the kernel foliation. The inverse procedure will give the map Ψ of Proposition 8.5. Hence let us show:

Theorem 8.6 *Let (σ_1,σ_2) be a pair of nonhomologous saddle connections in X that are exchanged by the Prym involution τ. Assume that any other saddle connection σ' joining P to Q in the same direction as σ_1 satisfies $|\sigma'|>|\sigma_1|$. Then as the length of σ_1 tends to zero (in the leaf of the kernel foliation), (X,ω) tends to a point in the boundary of $\Omega E_D(2,2)^{\mathrm{odd}}$ which is represented by a triple $(X_0,\omega_0,W_0)\in\Omega E_{D'}(2)^*$ for some $D'\in\{D,D/4\}$.*

Observe that we consider θ and $-\theta$ ($\theta\in\mathbb{S}^1$) as two distinct directions. As usual, we choose the orientation for any saddle connection joining P and Q to be *from P to Q*. For the remainder of this section, we fix a pair of saddle connections (σ_1,σ_2) satisfying the assumptions of Theorem 8.6. We will need the following:

Lemma 8.7 *Construct the translation surface (X',ω') by first cutting (X,ω) along $c=\sigma_1*(-\sigma_2)$ and then gluing the resulting pair of geodesic segments in each boundary component. Then*

$$(X',\omega')\in\Omega E_{D'}(1,1)\quad\text{for some}\quad D'\in\{D,D/4\}$$

(the involution τ of X descends to the hyperelliptic involution of X').

Proof of Lemma 8.7 We first show that $(X',\omega')\in\mathcal{H}(1,1)$. For that, we remark that the pair of angles specified by these two rays at the zeros P and Q are $(2\pi,4\pi)$. Since τ sends σ_1 to $-\sigma_2$ and preserves the orientation of X, necessarily the angle 2π at P and the angle 2π at Q belong to the same side of c, which proves the first fact.

The surface (X',ω') has two marked segments c_1,c_2, where c_1 is a saddle connection, and c_2 is simply a geodesic segment which has the same length and the same direction as c_1. We denote the endpoints of c_1 (respectively, c_2) by P_1,Q_1 (respectively, P_2,Q_2), where P_1,P_2 correspond to P and Q_1,Q_2 correspond to Q. Note that P_1,Q_1 are the zeros of ω'. We choose the orientation of c_1 (respectively, c_2) to be from P_1 to Q_1 (respectively, from P_2 to Q_2).

With this notation, τ induces an involution τ' on X' such that $\tau'(c_1)=-c_1$ and $\tau'(c_2)=-c_2$. It turns out that τ' has six fixed points on X': these are the four fixed points of τ (none of them are contained in c) and two additional fixed points in c_1 and c_2. By uniqueness, τ' is therefore the hyperelliptic involution of X'.

To conclude the proof, we need to show that (X',ω') is an eigenform. For that we first need to choose a symplectic basis of $H_1(X',\mathbb{Z})$. We proceed as follows (see Figure 6). Let $\alpha_{1,1},\alpha_{1,2},\alpha_2,\beta_2$ be the simple closed curves, and $\beta_{1,1}$ and $\beta_{1,2}$ be simple arcs in X' as shown in Figure 6, where $\alpha_{1,2}=-\tau'(\alpha_{1,1})$ and $\beta_{1,2}=-\tau'(\beta_{1,1})$. Let β_1' denote the simple closed curve which is the concatenation $c_1\cup\beta_{1,1}\cup c_2\cup\beta_{1,2}$. Set

$\alpha'_1 = \alpha_{1,1}$ (the orientations are chosen so that $(\alpha'_1, \beta'_1, \alpha_2, \beta_2)$ is a symplectic basis of $H_1(X', \mathbb{Z})$).

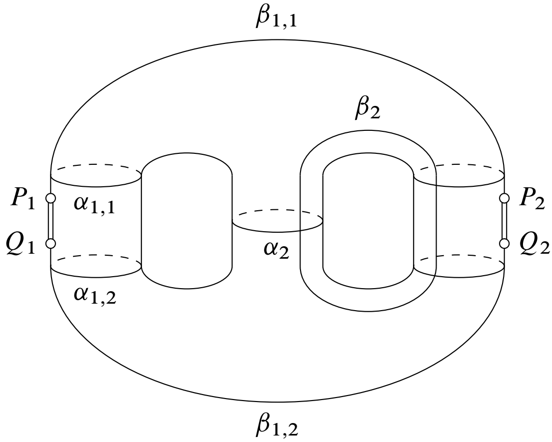

Figure 6: Surface in $\mathcal{H}(1,1)$ obtained by cutting and gluing along a pair of saddle connections exchanged by the Prym involution. The hyperelliptic involution τ' exchanges the upper and the lower halves of X'.

Observe that $\beta_{1,1}$ and $\beta_{1,2}$ correspond to two simple closed curves in X, and that $\alpha_{1,1}$ and $\alpha_{1,2}$ are not homologous in $H_1(X, \mathbb{Z})$. Set $\alpha_1 = \alpha_{1,1} + \alpha_{1,2}, \beta_1 = \beta_{1,1} + \beta_{1,2}$. Then $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ is a symplectic basis of $H_1(X, \mathbb{Z})^-$. In this basis, the intersection form is given by the matrix $\left(\begin{smallmatrix} 2J & 0 \\ 0 & J \end{smallmatrix}\right)$.

Since $(X, \omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$, by definition there exists a unique generator T of $\mathcal{O}_D$ that can be expressed (in the basis $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ of $H_1(X, \mathbb{Z})^-$) by the matrix

$$T = \begin{pmatrix} e & 0 & a & b \\ 0 & e & c & d \\ 2d & -2b & 0 & 0 \\ -2c & 2a & 0 & 0 \end{pmatrix},$$

where $D = e^2 + 8(ad - bc)$, $\gcd(a,b,c,d,e) = 1$ and $T^*\omega = \lambda \cdot \omega$, with $\lambda > 0$. In the symplectic basis $(\alpha'_1, \beta'_1, \alpha_2, \beta_2)$ of $H_1(X', \mathbb{Z})$ we define the endomorphism

$$T' = \begin{pmatrix} e & 0 & 2a & 2b \\ 0 & e & c & d \\ d & -2b & 0 & 0 \\ -c & 2a & 0 & 0 \end{pmatrix}.$$

It is easy to check that T' is self-adjoint with respect to the symplectic form $\left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$ and $T'^2 = eT' + 2(ad - bc)\,\mathrm{Id}$.

We now claim that ω' is an eigenform for T', namely $(T')^*\omega' = \lambda\cdot\omega'$, with $\lambda > 0$. Let (x, y, z, t) be the periods of $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ by ω. The condition $T^*\omega = \lambda\omega$ reads

$$(x, y, z, t)\cdot T = \lambda(x, y, z, t). \tag{12}$$

Elementary computation gives

$$\begin{aligned}
\omega'(\alpha_1') &= \omega(\alpha_{1,1}) = \tfrac{1}{2}\omega(\alpha_1) = \tfrac{1}{2}x,\\
\omega'(\beta_1') &= -\omega'(c_1) + \omega'(\beta_{1,1}) + \omega'(c_2) + \omega'(\beta_{1,2})\\
&= \omega(\beta_{1,1}) + \omega(\beta_{1,2}) = \omega(\beta_1) = y,\\
\omega'(\alpha_2) &= \omega(\alpha_2) = z,\\
\omega'(\beta_2) &= \omega(\beta_2) = t.
\end{aligned}$$

By simple computations, we see that (12) implies

$$\left(\tfrac{1}{2}x, y, z, t\right)\cdot T' = \lambda\left(\tfrac{1}{2}x, y, z, t\right), \tag{13}$$

which means that ω' is an eigenvector for T'. Actually (12) and (13) are equivalent.

Observe that T' generates a self-adjoint subring isomorphic to $\mathcal{O}_D$ in $\mathrm{End}(\mathrm{Jac}(X'))$ for which ω' is an eigenform. In other words, $(X', \omega') \in \Omega E_{D'}(1,1)$ for some D' dividing D. The proper subring isomorphic to $\mathcal{O}_{D'}$ is generated by the matrix $T'/k \in \mathrm{End}(\mathrm{Jac}(X'))$, where $k = \gcd(2a, 2b, c, d, e)$. By assumption $\gcd(a, b, c, d, e) = 1$, therefore $k \in \{1, 2\}$. Since $D = k^2 D'$, the lemma follows. □

We can now proceed to the proofs of our results.

Proof of Theorem 8.6 We keep the notation of Lemma 8.7. By construction, there is no obstruction to collapsing c_1 along the kernel foliation leaf through (X', ω'), and the resulting surface belongs to $\Omega E_{D'}(2)$. Note that when c_1 is shrunken to a point, so is c_2. Since c_2 is invariant under the hyperelliptic involution of X', in the limit c_2 becomes a marked Weierstrass point. □

Proof of Proposition 8.5 The surgery "collapse a pair of saddle connections exchanged by τ", as described above, is invertible: this is the map Ψ of the proposition. Let us give a more precise definition of this map.

We fix a point $(X_0, \omega_0, W_0) \in \Omega E_{D'}(2)^*$, and choose $\varepsilon > 0$ small enough so that the sets $D(P_0, \varepsilon) = \{x \in X_0 \mid \boldsymbol{d}(x, P_0) < \varepsilon\}$, where P_0 is the unique zero of ω_0, and $D(W_0, \varepsilon) = \{x \in X_0 \mid \boldsymbol{d}(x, W_0) < \varepsilon\}$, are two disjoint embedded discs.

Given any vector $v \in \mathbb{R}$ with $|v| < \varepsilon$, we construct a Prym form in $\mathrm{Prym}(2,2)$ as follows. We break up the zero P_0 into two zeros to get a surface $(X', \omega') \in \mathcal{H}(1,1)$

having the same absolute periods as ω, with a marked saddle connection, say σ_1, that is invariant under the hyperelliptic involution and such that $\omega'(\sigma_1) = v$. Note that, by assumption, σ_1 is disjoint from $D(W_0, \varepsilon)$. Let σ_2 be a geodesic segment in $D(W_0, \varepsilon)$ such that $\omega'(\sigma_2) = v$ and W_0 is the midpoint of σ_2. Cutting X' along σ_1 and σ_2, then regluing the resulting boundary components, we get a new surface $(X, \omega) \in \mathcal{H}(2,2)$ together with an involution $\tau\colon X \to X$ induced by the hyperelliptic involution of X'. Since $\tau^*\omega = -\omega$ by construction, one has $(X, \omega) \in \mathrm{Prym}(2,2)$.

The arguments of the proof of Lemma 8.7 actually show that $(X, \omega) \in \Omega E_D(2,2)$ for some $D \in \{D', 4D'\}$. We then define $\Psi(z) = (X, \omega)$, where z is a complex number such that $v = z^3$ (this condition is due to the fact that we have three choices for the segment σ_1), then extend Ψ to $D^\circ(\varepsilon)$ by analytic continuation. It is now straightforward to check that the map Ψ has the desired properties. □

9 Degenerating surfaces of $\Omega E_D(2,2)^{\mathrm{odd}}$

In this section, we show that the surgeries described in Section 8 are sufficient to describe all the degenerations (along the kernel foliation) of Prym eigenforms in $\Omega E_D(2,2)^{\mathrm{odd}}$ having an unstable cylinder decomposition when D is not a square (compare with [14]).

Theorem 9.1 *Assume that D is not a square, and $(X, \omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$ admits an unstable cylinder decomposition in the horizontal direction. Then there exists a finite interval $[s_{\min}, s_{\max}]$ such that for any $x \in]s_{\min}, s_{\max}[$, the surface $(X, \omega) + (x, 0)$ is well defined and belongs to $\Omega E_D(2,2)^{\mathrm{odd}}$. Moreover, when x tends to $\partial[s_{\min}, s_{\max}]$, $(X, \omega) + (x, \omega)$ converges to a surface (Y, η) which belongs to*

$$\Omega E_D(0,0,0), \quad \Omega E_D(4) \quad \text{or} \quad \Omega E_{D'}(2)^*, \quad \text{with } D' \in \left\{D, \tfrac{1}{4}D\right\}.$$

We will use the following elementary lemma.

Lemma 9.2 *Let $(X, \omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$. Assume that one of the following occurs:*

(1) *There exists a nontrivial homology class $c \in H_1(X, \mathbb{Z})^-$ such that $\omega(c) = 0$.*

(2) *There exist two twin saddle connections in X joining the two zeros of ω, both of which are invariant under the Prym involution.*

(3) *There exists a triple of twin saddle connections $(\sigma_0, \sigma_1, \sigma_2)$ (that is, $\omega(\sigma_0) = \omega(\sigma_1) = \omega(\sigma_2)$), where σ_0 is invariant and σ_1, σ_2 are exchanged by the Prym involution, such that $c_0 = \sigma_1 * (-\sigma_2)$ is nonseparating.*

Then D is a square.

Proof of Lemma 9.2 For the first condition, we set $K = \mathbb{Q}(\sqrt{D})$. If D is not a square then K is a real quadratic field over $\mathbb{Q}$ and, up to a rescaling by $\mathrm{GL}^+(2,\mathbb{R})$, the map $H_1(X,\mathbb{Q})^- \to K(i)$ given by $c \mapsto \omega(c)$ is an isomorphism of $\mathbb{Q}$–vector spaces. Thus $\omega(c) = 0$ implies $c = 0$ in $H_1(X,\mathbb{Z})^-$.

For the second condition, let σ_1, σ_2 be a pair of twin saddle connections which are both invariant under the Prym involution τ. If $c = \sigma_1 * (-\sigma_2) \in H_1(X,\mathbb{Z})^-$ is separating, then by cutting X along σ_1, σ_2 and regluing the segments of the boundary of the two components, we get a pair of translation surfaces each of which has a unique singularity with cone angle 4π. They thus belong to the stratum $\mathcal{H}(1)$. Since this stratum is empty, we get a contradiction. Therefore, c must be nonseparating, ie $c \neq 0 \in H_1(X,\mathbb{Z})^-$. One has $\omega(c) = \omega(\sigma_1) - \omega(\sigma_2) = 0$, hence the first condition applies and D is a square.

For the last condition, we set $c_j = \sigma_0 * (-\sigma_j)$, $j = 1, 2$. Remark that $\tau(c_1) = -c_2$ and $c_0 = c_2 - c_1$ in $H_1(X,\mathbb{Z})$. Since c_0 is nonseparating by assumption, it is a primitive element of $H_1(X,\mathbb{Z})$. Observe that if one of the curves c_1 or c_2 is separating then the other is also separating (as $\tau(c_1) = -c_2$) and in this case $c_0 = c_1 - c_2 = 0 \in H_1(X,\mathbb{Z})$, contradicting the assumption. Hence both c_1, c_2 are nonseparating. Let $c = c_1 + c_2$. Then $\tau(c) = -c$, or $c \in H_1(X,\mathbb{Z})^-$. If $c = 0 \in H_1(X,\mathbb{Z})$ then $c_2 = -c_1$, ie $c_0 = c_1 - c_2 = 2c_1$, contradicting the primitivity of $c_0 \in H_1(X,\mathbb{Z})$. Thus $c \neq 0 \in H_1(X,\mathbb{Z})^-$. Since $\sigma_0, \sigma_1, \sigma_2$ are twin saddle connections, we conclude

$$\omega(c) = \omega(c_1) + \omega(c_2) = 2\omega(\sigma_0) - \omega(\sigma_1) - \omega(\sigma_2) = 0.$$

Again the first condition applies and D is a square. □

Proof of Theorem 9.1 Let P, Q be the zeros of ω. We denote by $\{\sigma_i \mid i \in I\}$ the set of horizontal saddle connections on X connecting P to Q. Recall that we always define the orientation of such a saddle connection to be from P to Q; it is said to be *positively oriented* if the orientation is from the left to the right, otherwise it is said to be *negatively oriented*. The corresponding holonomy vectors are $\{(s_i, 0) = \omega(\sigma_i) \in \mathbb{R}^2 \mid i \in I\}$. For every $i \in I$, σ_i is contained on the lower boundary of a unique cylinder. If σ_i is positively oriented (namely $s_i > 0$) then there exists σ_j in the same lower boundary component as σ_i which is negatively oriented. In particular, all the numbers $\{s_i \mid i \in I\}$ cannot have the same sign.

Let us define

$$s_{\min} = -\min\{s_i \mid s_i > 0\} \quad \text{and} \quad s_{\max} = -\max\{s_i \mid s_i < 0\}.$$

If $(Y,\eta) = (X,\omega) + (x,0)$ then by construction $\eta(\sigma_i) = (s_i + x, 0)$ and the surface (Y,η) can be constructed from the same cylinders as (X,ω). For all $x \in]s_{\min}, s_{\max}[$,

$(X,\omega)+(x,0)$ is a well-defined surface in $\Omega E_D(2,2)^{\mathrm{odd}}$ since $s_i+x\neq 0$, proving the first statement.

We now prove the second assertion. Let us analyze the case when x tends to $s_{\min}$ (the case when x tends to $s_{\max}$ being similar). Let

$$\mathcal{C}_{\min}=\{\sigma_i \mid s_i=-s_{\min}\} \quad \text{and} \quad \mathcal{C}_{\max}=\{\sigma_i \mid s_i=-s_{\max}\}$$

(necessarily $|\mathcal{C}_{\min}|\leq 3$, and $|\mathcal{C}_{\max}|\leq 3$). When $x\to s_{\min}$, only the saddle connections of $\mathcal{C}_{\min}$ can collapse to a point. We thus have three cases, parametrized by the number of elements of $\mathcal{C}_{\min}$:

(1) $\mathcal{C}_{\min}=\{\sigma_{i_0}\}$: the unique saddle connection σ_{i_0} is invariant under τ and $(X,\omega)+(x,0)$ converges to a surface in $\Omega E_D(4)$.

(2) $\mathcal{C}_{\min}=\{\sigma_{i_1},\sigma_{i_2}\}$: σ_{i_1} and σ_{i_2} are exchanged by τ (otherwise the closed curve $c=\sigma_{i_1}*(-\sigma_{i_2})\in H_1(X,\mathbb{Z})^-$ represents a nonzero element, and since $\omega(c)=0$, Lemma 9.2 implies that D is a square). By Theorem 8.6, $(X,\omega)+(x,0)$ converges to a surface in $\Omega E_{D'}(2)^*$, for some $D'\in\{D,D/4\}$.

(3) $\mathcal{C}_{\min}=\{i_0,i_1,i_2\}$: if there are two saddle connections in $\{\sigma_{i_0},\sigma_{i_1},\sigma_{i_2}\}$ that are invariant under τ then D must be square (see Lemma 9.2). Hence one can assume that τ preserves σ_{i_0} while it exchanges σ_{i_1} and σ_{i_2}. If the closed curve $c_0=\sigma_{i_1}*(-\sigma_{i_2})$ is nonseparating then D must be a square (again by Lemma 9.2). Thus c_0 is separating and $\{\sigma_{i_0},\sigma_{i_1},\sigma_{i_2}\}$ are homologous saddle connections. We only need to show that X decomposes into three tori. Indeed, as x tends to $s_{\min}$ the length of these saddle connections tends to zero, and the limit surface is an element of $\Omega E_D(0,0,0)$.

Hence, in view of the above discussion, in order to finish the proof of the theorem, we need to show that, in case (3), the complement of $\sigma_{i_0}\cup\sigma_{i_1}\cup\sigma_{i_2}$ has three connected components, each of which is a one-holed torus.

We begin by observing that $\sigma_{i_1},\sigma_{i_2}$ determine a pair of angles $(2\pi,4\pi)$ at P and Q. Since τ exchanges P and Q and preserves the orientation of X, the angle 2π at P and the angle 2π at Q belong to the same side of c_0. Cut X along c_0, then glue the two segments in each boundary component together. We then obtain two closed translation surfaces. From the observation above, one of the new surfaces has no singularities, hence it must be a flat torus that will be denoted by (X',ω'). The remaining surface is then a surface (X'',ω'') in $\mathcal{H}(1,1)$.

There is a marked geodesic segment σ' in X' which is the identification of σ_1 and σ_2. We denote the endpoints of this segment by P' and Q', which correspond to P and Q, respectively. For (X'',ω''), we denote the zeros of ω'' by P'' and Q'', which correspond to P and Q, respectively. There is a pair of twin saddle connections σ_0 and σ'' in X'', where σ'' is the identification of σ_1 and σ_2.

The involution τ induces an involution τ' on X' and an involution τ'' on X''. We can consider τ' and τ'' as the restrictions of τ in X' and X'', respectively. Note that τ' exchanges P' and Q' and satisfies $\tau'(\omega') = -\omega'$. Since X' is an elliptic curve, there exists only one such involution. We deduce in particular that τ' has four fixed points in X', one of which is the midpoint of σ'; the other three are the fixed points of τ.

Recall that τ has four fixed points in X. Therefore, τ'' has exactly two fixed points, one of which is the midpoint of σ_0 by assumption (recall that σ_0 is invariant under τ), and the other one of which is the midpoint of σ''. Let ι denote the hyperelliptic involution of X''. Remark that ι has six fixed points. From the observations above, we can conclude that $\tau'' \neq \iota$.

We now claim that $\iota(\sigma_0) = -\sigma''$. Indeed, since ι belongs to the center of the group $\mathrm{Aut}(X'')$, we have $\iota \circ \tau'' = \tau'' \circ \iota$. Therefore ι preserves the set of fixed points of τ''. If ι fixes the midpoint of σ_0, then it follows that $\iota \circ \tau'' = \mathrm{Id}$, since both ι and τ'' are involutions, and hence $\tau'' = \iota$. This is a contradiction. Therefore, ι must send the midpoint of σ_0 to the midpoint of σ''. Remark that $\iota^*\omega'' = -\omega''$, which means that ι is an isometry of (X'', ω''). Thus ι maps σ_0 to another saddle connection such that $\omega''(\iota(\sigma_0)) = -\omega''(\sigma_0)$. Since ι exchanges the zeros of ω'', we conclude that $\iota(\sigma_0) = -\sigma''$.

Now, the element in $H_1(X'', \mathbb{Z})$ represented by the closed curve $\sigma_0 \cup \sigma''$ is preserved by ι, which implies that this curve is separating. Cut X'' along $\sigma_0 \cup \sigma''$, then glue the segments in the boundary of each component together. We then get two flat tori (X''_1, ω''_1) and (X''_2, ω''_2) which are exchanged by τ''. This finishes the proof of Theorem 9.1. □

10 Cylinder decomposition of surfaces near $\Omega E_D(4)$ and $\Omega E_D(2)^*$

Let (X_0, ω_0) be a surface in $\Omega E_D(4)$, and $\Psi\colon D^\circ(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ be the map in Proposition 8.3.

Proposition 10.1 *Assume that the horizontal direction is completely periodic for (X_0, ω_0). Then there exists $0 < \varepsilon_1 < \varepsilon$ such that for every $(X, \omega) \in \Psi(D^\circ(\varepsilon_1))$, the horizontal direction is also completely periodic. Set*

$$R_{(k,5)}(\varepsilon_1) = \{\varrho e^{k\imath\pi/5} \mid 0 < \varrho < \varepsilon_1\} \quad \text{for } k = 0, \dots, 9$$

and

$$D^{\circ}_{(k,5)}(\varepsilon_1) = \{\varrho e^{\imath\theta} \mid 0 < \varrho < \varepsilon_1,\ (k-1)\pi/5 < \theta < k\pi/5\} \quad \text{for } k = 1, \dots, 10.$$

Then:

(1) *The cylinder decompositions in the horizontal direction of all surfaces in* $\Psi(R_{(k,5)}(\varepsilon_1))$ *are unstable and have the same combinatorial data.*

(2) *The cylinder decompositions in the horizontal direction of all surfaces in* $\Psi(D^{\circ}_{(k,5)}(\varepsilon_1))$ *are stable and have the same combinatorial data.*

Proof This proposition follows from similar arguments as Proposition 4.2. Let $\mathcal{C}_i$, $i = 1, \dots, n$, denote the horizontal cylinders of X_0, and γ_i denote the simple closed geodesic in $\mathcal{C}_i$ whose distances to the two boundary components of $\mathcal{C}_i$ are equal. Choose ε_1 satisfying $0 < \varepsilon_1 < \min\{\varepsilon, 1\}$ small enough so that $D(P_0, \varepsilon_1) = \{x \in X_0 \mid \boldsymbol{d}(x, P_0) < \varepsilon_1\}$ is an embedded disc disjoint from the curves γ_i, where P_0 is the unique zero of ω_0. By construction, $\varepsilon_1^5 < \varepsilon_1 < \varepsilon$.

By definition, the surface $\Psi(\varrho e^{\imath\theta})$ has a small saddle connection (of length ϱ^5) in the direction 5θ. It follows immediately that the horizontal direction is periodic for the surfaces in $\Psi(R_{(k,5)}(\varepsilon_1))$. Since there is a horizontal saddle connection with distinct endpoints, the corresponding cylinder decomposition is unstable. Clearly, the combinatorial data of the decomposition of $\Psi(z)$ does not change as z varies in $R_{(k,5)}(\varepsilon_1)$ (see Lemma 4.3).

Let us now consider a surface $(X, \omega) = \Psi(z)$, where $z \in D^{\circ}_{(k,5)}(\varepsilon_1)$. We will assume in addition that $z^5 = 2\imath h$ with $0 < h < \frac{1}{2}\varepsilon_1$; the general case then follows from Lemma 4.3. Recall that $D(P_0, \varepsilon_1)$ is the union of ten half-discs $D_i^{\pm}$, with $i = 1, \dots, 5$, where D_i^+ is a copy of $\{z \in \mathbb{C} \mid |z| \leq \varepsilon_1,\ \mathrm{Re}(z) \geq 0\}$ and D_i^- is a copy of $\{z \in \mathbb{C} \mid |z| \leq \varepsilon_1,\ \mathrm{Re}(z) \leq 0\}$. Let $t_i^{\pm}, b_i^{\pm}, c_i^{\pm}$ denote the points in the border of $D_i^{\pm}$ that correspond to $\imath\varepsilon_1, -\imath\varepsilon_1, 0$, respectively.

Since the horizontal direction is periodic for (X_0, ω_0), each horizontal separatrix emanating from the "center" of a half-disc D_i^+ ends at the "center" of a half-disc $D_j^- := D_{\pi(i)}^-$. Thus we have a permutation π of the index set $\{1, \dots, 5\}$.

We have the same situation for horizontal rays emanating from the "top" (and similarly the"bottom") of D_i^+. The gluing rules then give rise to two permutations π_t (corresponding to the top of D_i^+) and π_b (corresponding to the bottom of D_i^+) of the set $\{1, \dots, 5\}$ (see Section 4).

Now, the surface $(X, \omega) = \Psi(z)$ is obtained from (X_0, ω_0) by replacing the disc $D(P_0, \varepsilon_1)$ by another disc constructed from the half-discs $D_i^{\pm}$ as follows: pick a

$j \in \{1,\dots,5\}$ and apply the following gluing rules (see Figure 7 for the case $j=2$; here we use the convention $i \sim (i-5)$ if $i > 5$):

- D_i^+ is glued to D_i^- along the segment $\{\mathrm{Re}(z)=0 \mid h \le \mathrm{Im}(z) < \varepsilon_1\}$ for $i \in \{j, j+1, j+2\}$.
- D_i^+ is glued to D_i^- along the segment $\{\mathrm{Re}(z)=0 \mid -h \le \mathrm{Im}(z) < \varepsilon_1\}$ for $i \notin \{j, j+1, j+2\}$.
- D_i^- is glued to $D_{(i+1)}^+$ along the segment $\{\mathrm{Re}(z)=0 \mid -\varepsilon_1 < \mathrm{Im}(z) \le h\}$ for $i \in \{j, j+1\}$.
- D_i^- is glued to $D_{(i+1)}^+$ along the segment $\{\mathrm{Re}(z)=0 \mid -\varepsilon_1 < \mathrm{Im}(z) \le -h\}$ for $i \notin \{j, j+1\}$.
- D_j^+ is glued to $D_{(j+2)}^-$ along the segment $\{\mathrm{Re}(z)=0, -h \le \mathrm{Im}(z) \le h\}$.

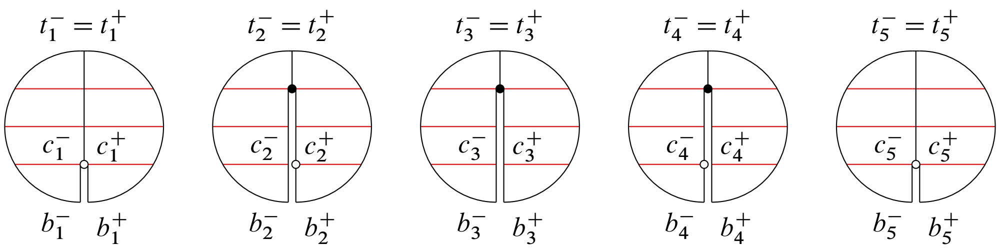

Figure 7: Splitting a zero of order 4 into two zeros of order 2 ($j=2$).

Let P and Q denote the zeros of ω which correspond to the points $-\imath h \in D_j^+$ and $\imath h \in D_j^+$, respectively. From the gluing rules, horizontal geodesic rays emanating from P end up at P, and similarly for Q. Moreover, those horizontal saddle connections are encoded in the permutations π_b and π_t. It follows that (X,ω) admits a stable cylinder decomposition in the horizontal direction.

By the choice of ε_1, (X,ω) has n cylinders associated to the geodesics γ_i, $i=1,\dots,n$, and some additional cylinders which contain some of the points $c_i^\pm$. The cylinders associated to γ_i are in bijection with the cycles of π_t and π_b. For the additional ones, we remark that the gluing rules imply the following identifications:

- c_i^- is identified with c_i^+ if $i \notin \{j, j+1, j+2\}$,
- c_i^- is identified with $c_{(i+1)}^+$ if $i \in \{j, j+1\}$,
- $c_{(j+2)}^-$ is identified with c_j^+.

Composing these identifications with π, we get a permutation π_c of the set $\{1,\dots,5\}$. The horizontal cylinders containing some of the points $c_i^\pm$ are in bijection with the cycles of π_c. It follows that the permutations π_t, π_b, π_c completely determine the combinatorial data of the cylinder decomposition of (X,ω). Hence these combinatorial data depend only on the sector $D^\circ_{k,5}(\varepsilon_1)$. The proposition is then proved. □

Remark 10.2 In general, the topological model of the decomposition of (X,ω) changes if we change the sector $D^\circ_{(k,5)}(\varepsilon_1)$.

By a saddle connection on $(X_0,\omega_0,W_0) \in \Omega E_{D'}(2)^*$, we refer to a geodesic segment whose endpoints are in the set $\{P_0, W_0\}$. We consider, by convention, a cylinder in (X_0,ω_0,W_0) as the union of all simple closed geodesics in the same free homotopy class in $X_0 \backslash \{P_0, W_0\}$. Obviously, a direction θ is periodic for (X_0,ω_0,W_0) if and only if it is periodic for (X_0,ω_0), but the associated cylinder decomposition of (X_0,ω_0,W_0) may have one more cylinder than the one of (X_0,ω_0), since a simple closed geodesic passing through W_0 will cut the corresponding cylinder in (X_0,ω_0) into two cylinders in (X_0,ω_0,W_0). The following proposition follows from completely similar arguments as Proposition 10.1.

Proposition 10.3 *Let (X_0,ω_0,W_0) be a surface in $\Omega E_{D'}(2)^*$. Assume that the horizontal direction is periodic for (X_0,ω_0,W_0). Let $\Psi\colon D^\circ(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ be the map defined in Proposition 8.5. Then there exists $0 < \varepsilon_1 < \varepsilon$ such that for all $(X,\omega) \in \Psi(D^\circ(\varepsilon_1))$, the horizontal direction is also periodic. Set*

$$R_{(k,3)}(\varepsilon_1) = \{\varrho e^{k\imath\pi/3} \mid 0 < \varrho < \varepsilon_1\} \quad \text{for } k = 0,\dots,5$$

and

$$D^\circ_{(k,3)}(\varepsilon_1) = \{\varrho e^{\imath\theta} \mid 0 < \varrho < \varepsilon_1,\ (k-1)\pi/3 < \theta < k\pi/3\} \quad \text{for } k = 1,\dots,6.$$

Then the associated cylinder decompositions of surfaces in $\Psi(R_{(k,3)}(\varepsilon_1))$ or in $\Psi(D^\circ_{(k,3)}(\varepsilon_1))$ are unstable and have the same combinatorial data.

Having proved Propositions 10.1 and 10.3, using the arguments in Section 7, we get:

Theorem 10.4 *Let (X_0,ω_0) be a surface in $\Omega E_D(4)$ which is horizontally periodic, and $\Psi\colon D^\circ(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ be the map defined in Proposition 8.3. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers in a fixed sector $D^\circ_{(k,n)}(\varepsilon_1)$, where ε_1 is the constant in Proposition 10.1, such that $z_n \to 0$ as $n \to \infty$. Assume that for all $n \in \mathbb{N}$, the horizontal direction is parabolic for the surface $(X_n,\omega_n) = \Psi(z_n)$. Then the set*

$$\mathcal{O} := \bigcup_{n\in\mathbb{N}} \mathrm{GL}^+(2,\mathbb{R}) \cdot (X_n,\omega_n)$$

is dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$.

The same statement also holds with (X_0,ω_0)*,* $\Omega E_D(4)$*, and Propositions 8.3 and 10.1 replaced by* (X,ω_0,W_0)*,* $\Omega E_D(2)^*$*, and Propositions 8.5 and 10.3, respectively.*

Proof Since the arguments for the two cases are the same, we will only consider the case $(X_0,\omega_0)\in\Omega E_D(4)$. Recall that, by definition, all the surfaces in $\Psi(D^\circ(\varepsilon))$ belong to the same leaf of the kernel foliation. Set

$$\overline{D}^\circ_{(k,n)}(\varepsilon_1)=\{z=\varrho e^{\imath\theta}\in\mathbb{C}\mid 0<\varrho<\varepsilon_1,\ (k-1)\pi/5\le\theta\le k\pi/5\}.$$

By a slight abuse of notation, if $(X,\omega)=\Psi(z)$, with $z\in\overline{D}^\circ_{(k,n)}(\varepsilon_1)$, then we will write $(X,\omega)=(X_0,\omega_0)+z^5$. Using this convention, given z_1,z_2 in $\overline{D}^\circ_{(k,n)}(\varepsilon_1)$, we obtain

$$(X_0,\omega_0)+z_2^5=((X_0,\omega_0)+z_1^5)+(z_2^5-z_1^5),$$

where the expression in the right-hand side corresponds to a move in a leaf of the kernel foliation in $\Omega E_D(2,2)^{\mathrm{odd}}$.

By assumption, we can write $(X_n,\omega_n)=(X_0,\omega_0)+(s_n,t_n)$, with $(s_n,t_n)\to(0,0)$ as $n\to\infty$, $t_n\neq 0$, and (X_n,ω_n) admits a parabolic cylinder decomposition in the horizontal direction. By Proposition 10.1, we know that the topological data and the widths of the cylinders in this decomposition are the same for all n. Thus, the arguments in Section 7 allows us to conclude that $(X_0,\omega_0)+(x,0)\in\overline{\mathcal{O}}$, for all $x\in(-\varepsilon_1^5,\varepsilon_1^5)$.

Pick a point $x\in(-\varepsilon_1^5,\varepsilon_1^5)\setminus\{0\}$, and set $(X,\omega)=(X_0,\omega_0)+(x,0)$. We see that there exists $\varepsilon_0>0$ such that $(X,\omega)+(s,0)\in\overline{\mathcal{O}}$ for all $s\in(-\varepsilon_0,\varepsilon_0)$. Corollary 6.3 then allows us to conclude that $(X,\omega)+v\in\overline{\mathcal{O}}$ for any $v\in\mathbb{R}^2$, with v small enough. We can then choose v such that $(X,\omega)+v\in\Psi(D^\circ_{(k,n)}(\varepsilon_1))$ and the horizontal direction is not parabolic for $(X,\omega)+v$. We conclude with Theorem 6.1. □

11 The set of Veech surfaces is not dense

In this section we will prove the following theorem:

Theorem 11.1 *If* D *is not a square, then for any connected component* $\mathscr{C}$ *of* $\Omega E_D(2,2)^{\mathrm{odd}}$ *there exists an open subset* $\mathcal{U}\subset\mathscr{C}$ *which contains no Veech surfaces.*

11A Cylinder decomposition and prototypes

We first prove the following lemma. Informally, if (X,ω) has a three-torus decomposition such that the direction of the slits is periodic, then up to the action of $\mathrm{GL}^+(2,\mathbb{R})$ the surface belongs to the real kernel foliation leaf of some “prototypical surface” in a finite family.

Lemma 11.2 *Let $(X,\omega)\in\Omega E_D(2,2)^{\mathrm{odd}}$ be an eigenform with a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ so that (X,ω) admits a three-torus decomposition into tori (X_j,ω_j), $j=0,1,2$. Assume that (X,ω) is periodic in the direction of σ_0. Let $(\widetilde{a}_j,\widetilde{b}_j)$ be a basis of $H_1(X_j,\mathbb{Z})$ with $\widetilde{a}_j$ parallel to σ_j, and*

$$\tau(\widetilde{a}_1)=-\widetilde{a}_2,\quad \tau(\widetilde{b}_1)=-\widetilde{b}_2,$$

where τ is the Prym involution. Then there exists a 4–tuple $(w,h,t,e)\in\mathbb{Z}^4$ satisfying

$$(\mathcal{P}_D(0,0,0))\qquad \begin{cases} w>0,\ h>0,\ 0\le t<\gcd(w,h),\ \gcd(w,h,t,e)=1,\\ D=e^2+8wh\end{cases}$$

such that up to the action of $\mathrm{GL}^+(2,\mathbb{R})$ and Dehn twists, we have

$$\begin{aligned}\omega(\mathbb{Z}\widetilde{a}_0\oplus\mathbb{Z}\widetilde{b}_0)&=\lambda\cdot\mathbb{Z}^2,\\ \omega(\mathbb{Z}\widetilde{a}_j\oplus\mathbb{Z}\widetilde{b}_j)&=\mathbb{Z}(w,0)\oplus\mathbb{Z}(t,h)\quad \textit{for } j=1,2,\end{aligned}$$

where $\lambda\in\mathbb{Q}(\sqrt{D})$ is the unique positive root of the equation $\lambda^2-e\lambda-2wh=0$.

Proof Set $\widetilde{a}=\widetilde{a}_1+\widetilde{a}_2$ and $\widetilde{b}=\widetilde{b}_1+\widetilde{b}_2$. Then $(\widetilde{a}_0,\widetilde{b}_0,\widetilde{a},\widetilde{b})$ is a symplectic basis of $H_1(X,\mathbb{Z})^-$. The restriction of the intersection form is given by the matrix $\left(\begin{smallmatrix}J&0\\0&2J\end{smallmatrix}\right)$. Since $(X,\omega)\in\Omega E_D(2,2)^{\mathrm{odd}}$, let us denote by T a generator of the order $\mathcal{O}_D$. In the above coordinates, since T is self-adjoint, T has the following form (up to replacing T by $T-f\cdot\mathrm{Id}$):

$$T=\begin{pmatrix} e&0&2w&2t\\ 0&e&2c&2h\\ h&-t&0&0\\ -c&w&0&0\end{pmatrix},$$

for some $(w,h,t,e,c)\in\mathbb{Z}^5$. Since ω is an eigenform, $T^*\omega=\lambda\cdot\omega$ for some λ (that can be chosen to be positive by changing T to $-T$). Now, up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, one can always assume that $\omega(\mathbb{Z}\widetilde{a}_0\oplus\mathbb{Z}\widetilde{b}_0)=\lambda\cdot\mathbb{Z}^2$. In our coordinates, $\mathrm{Re}(\omega)=(\lambda,0,x,y)$ and $\mathrm{Im}(\omega)=(0,\lambda,0,z)$ for some $x,y,z>0$. Substituting into the equation $T^*\omega=\lambda\cdot\omega$, we obtain $x=2w$, $y=2t$, $z=2h$ and $c=0$. Since T satisfies the quadratic equation $T^2-eT-2wh\,\mathrm{Id}=0$, we get $D=e^2+8wh$. We can renormalize further using Dehn twists so that $0\le t<\gcd(w,h)$. Finally, properness of $\mathcal{O}_D$ implies $\gcd(w,h,t,e)=1$. All the conditions of $(\mathcal{P}_D(0,0,0))$ are now fulfilled and the lemma is proved (compare with [15, Proposition 4.2]). □

Definition 11.3 For each D, let $\mathcal{P}_D(0,0,0)$ denote the set

$$\{(w,h,t,e)\in\mathbb{Z}^4 \mid (w,h,t,e) \text{ satisfies } (\mathcal{P}_D(0,0,0))\}.$$

We call an element of $\mathcal{P}_D(0,0,0)$ a *prototype*. The set of prototypes is clearly finite.

11B Switching decompositions

Let (X,ω) be a surface in $\Omega E_D(2,2)^{\mathrm{odd}}$ which admits a three-torus decomposition by a triple of saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$. We also assume that the direction of σ_j is periodic. Let (X_j,ω_j) and $(\widetilde{a}_j,\widetilde{b}_j)$ be as in Lemma 11.2. We wish now to investigate the situation where X admits other three-torus decompositions.

By Proposition 8.2, for any primitive element $b_0\in H_0(X_0,\mathbb{Z})$, there exists a unique primitive element $b_j\in H_1(X_j,\mathbb{Z})$, $j=1,2$ such that

$$\omega(b_j)=\frac{2\beta_j}{\lambda}\omega(b_0),$$

with $\beta_j\in\mathbb{N}$. This is because $L(X_j,\omega_j)$ is a sublattice of $(2/\lambda)L(X_0,\omega_0)$ (here $L(X_j,\omega_j)$ is the lattice associated to (X_j,ω_j); see Proposition 8.2), hence it contains a vector parallel to $(2/\lambda)\omega_0(b_0)$. We call b_j the *shadow* of b_0 in X_j.

The following lemma provides us with a sufficient condition for the existence of many other three-torus decompositions. Its proof is inspired by [23, Theorem 5.3].

Lemma 11.4 *Let b_0 be a primitive element of $H_1(X_0,\mathbb{Z})\setminus\{\pm\widetilde{a}_0\}$ and let b_j be the shadow of b_0 in X_j, $j=1,2$. Set $c=b_0+b_1+b_2$. Then there exists $s_0>0$ such that if the ratio $s=|\sigma_0|/|\widetilde{a}_0|$ is smaller than s_0, then the surface (X,ω) admits a three-torus decomposition by a triple of saddle connections $\{\delta_0,\delta_1,\delta_2\}$ such that $\delta_j*(-\sigma_j)=c$.*

Proof For $v_j=(x_j,y_j)\in\mathbb{R}^2$, $j=1,2$, let us define $v_1\wedge v_2=\det\left(\begin{smallmatrix}x_1 & x_2\\ y_1 & y_2\end{smallmatrix}\right)$. By assumption $b_0\notin\mathbb{Z}\widetilde{a}_0$, hence $|\omega(b_0)\wedge\omega(\widetilde{a}_0)|>0$. Since $\omega(b_j)$ is parallel to $\omega(b_0)$, and $\omega(\widetilde{a}_j)$ is parallel to $\omega(\widetilde{a}_0)$, we also have $|\omega(b_j)\wedge\omega(\widetilde{a}_j)|>0$.

Choose s_0 small enough so that if $0<s<s_0$, then $0<s|\omega(b_j)\wedge\omega(\widetilde{a}_j)|<\mathrm{Area}(X_j)$. Assume that $|\sigma_j|<s_0|\widetilde{a}_j|$ for $j=0,1,2$. Note that $|\sigma_0|=|\sigma_1|=|\sigma_2|$, and $|\widetilde{a}_1|=|\widetilde{a}_2|=w/\lambda|\widetilde{a}_0|$.

Let $\widehat{\sigma}_j$ be the marked geodesic segment corresponding to $\{\sigma_0,\sigma_1,\sigma_2\}$ in the torus X_j, and let γ_j be a simple closed geodesic representing the homology class $b_j\in H_1(X_j,\mathbb{Z})$. By assumption $0<|\omega(\gamma_j)\wedge\omega(\widehat{\sigma}_j)|<\mathrm{Area}(X_j)$, hence γ_j intersects $\widehat{\sigma}_j$ at at most one point. Thus the union of all the geodesics representing b_j which intersect $\widehat{\sigma}_j$ is an embedded cylinder $\widehat{\mathcal{C}}_j$ in X_j.

Recall that (X,ω) is obtained from X_0,X_1,X_2 by slitting and regluing along $\widehat{\sigma}_j$. As a consequence, we see that the union of the cylinders $\widehat{\mathcal{C}}_j$, $j=0,1,2$, is an embedded cylinder $\mathcal{C}$ whose core curves represent the homology class $c=b_0+b_1+b_2$. Let δ_j be the image of σ_j under a Dehn twist in $\mathcal{C}$. Then $\{\delta_j\mid j=0,1,2\}$ is also a triple of homologous saddle connections which decompose X into three tori (see Figure 8). The lemma follows from $\delta_j*(-\sigma_j)=c$. □

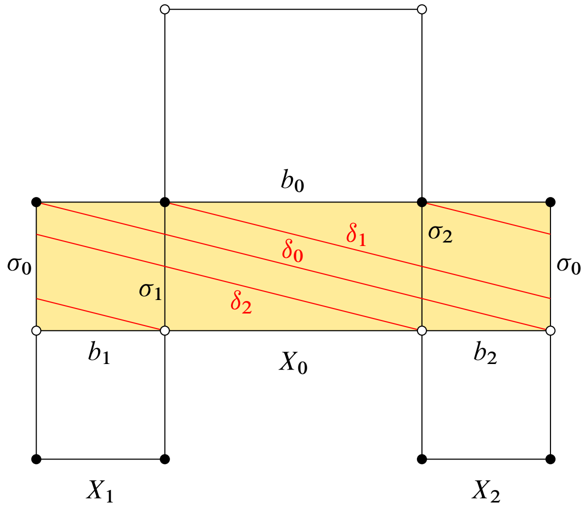

Figure 8: Switching three-torus decompositions.

Using the same notation as in Lemma 11.4, let (X'_j, ω'_j), $j = 0, 1, 2$, denote the tori in the decomposition specified by $\{\delta_0, \delta_1, \delta_2\}$ (X'_0 is the torus which is fixed by τ). We regard X_j and X'_j as subsurfaces of X. The following elementary lemma provides us with an explicit basis of $H_1(X'_0, \mathbb{Z})$. Its proof is left to the reader.

Lemma 11.5 *Let a_0 be a primitive element of $H_1(X_0, \mathbb{Z})$ such that (a_0, b_0) is a basis of $H_1(X_0, \mathbb{Z})$. Then $H_1(X'_0, \mathbb{Z}) = \mathbb{Z} \cdot (a_0 + c) + \mathbb{Z} \cdot b_0$.*

Lemma 11.6 *Let (X, ω) be a surface in $\Omega E_D(2,2)^{\mathrm{odd}}$ satisfying the hypothesis of Lemma 11.4. Let a_0 be a primitive element of $H_1(X_0, \mathbb{Z})$ such that (a_0, b_0) is a basis of $H_1(X_0, \mathbb{Z})$. There exists $(p, q) \in \mathbb{Z}^2$ such that $\widetilde{a}_0 = pa_0 + qb_0$. Set $\beta = 2\beta_1 + 2\beta_2 = 4\beta_1 \in \mathbb{Z}$, where $\omega(b_j) = (2\beta_j/\lambda)\omega(b_0)$. If the direction of δ_0 is completely periodic, then*

$$s = \frac{\lambda + \beta}{(rp + p - q)\lambda + p\beta} \quad \text{with } r \in \mathbb{Q}. \tag{14}$$

Proof We know that the saddle connections $\{\delta_0, \delta_1, \delta_2\}$ decompose X into three tori X'_0, X'_1, X'_2, where X'_0 is preserved by τ. By Lemma 11.5 we have

$$H_1(X'_0, \mathbb{Z}) = \mathbb{Z} \cdot (a_0 + b_0 + b_1 + b_2) + \mathbb{Z} \cdot b_0.$$

Set $A = \omega(a_0 + b_0 + b_1 + b_2)$ and $B = \omega(b_0)$. Then $L(X'_0) = \mathbb{Z}A + \mathbb{Z}B$, where $L(X'_0)$ is the lattice associated to X'_0. Setting $v = \omega(\sigma_0)$, $w = \omega(\delta_0)$, simple computation shows that

$$A = \omega(a_0) + \omega(b_0) + (\beta/\lambda)\omega(b_0) = \omega(a_0) + (1+\beta/\lambda)B.$$

Thus

$$\omega(a_0) = A - (1+\beta/\lambda)B.$$

Using $\widetilde{a}_0 = pa_0 + qb_0$, we obtain

$$\begin{aligned} v = s\omega(\widetilde{a}_0) &= s(p\omega(a_0) + q\omega(b_0)) \\ &= s\big(p(A-(1+\beta/\lambda)B) + qB\big) \\ &= s\big(pA + (q - p(1+\beta/\lambda))B\big). \end{aligned}$$

Now

$$\begin{aligned} w &= v + \omega(b_0 + b_1 + b_2) \\ &= spA + s(q - p(1+\beta/\lambda))B + (1+\beta/\lambda)B \\ &= spA + \big(sq + (1-sp)(1+\beta/\lambda)\big)B. \end{aligned}$$

The direction of δ_0 is periodic if and only if w is parallel to a vector in the lattice $\mathbb{Z}A + \mathbb{Z}B$, which is equivalent to

$$r = \frac{sq + (1-sp)(1+\beta/\lambda)}{sp} = \frac{sq\lambda + (1-sp)(\lambda+\beta)}{sp\lambda} \in \mathbb{Q}.$$

It follows that

$$srp\lambda = sq\lambda + (\lambda+\beta) - sp(\lambda+\beta),$$

or equivalently

$$s = \frac{\lambda+\beta}{rp\lambda - q\lambda + p(\lambda+\beta)} = \frac{\lambda+\beta}{(rp+p-q)\lambda + p\beta}. \qquad \square$$

We can now prove:

Proposition 11.7 *Let (X,ω) be a surface in $\Omega E_D(2,2)^{\mathrm{odd}}$, where D is not a square. Assume that there exists a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ which decompose (X,ω) into three tori, and the direction of σ_j is periodic. Set $s = |\sigma_0|/|\widetilde{a}_0|$, where $\widetilde{a}_0$ is a simple closed geodesic parallel to σ_0 in the torus which is preserved by the involution. Then there exists a constant $s_0 > 0$ depending only on D such that if $s < s_0$ then (X,ω) is not a Veech surface.*

Proof Let $(\widetilde{a}_j, \widetilde{b}_j)$, $j = 0, 1, 2$, be as in Lemma 11.2. Let (e, w, h, t) be the prototype in $\mathcal{P}_D(0,0,0)$ which is associated to the cylinder decomposition in the direction of σ_0. Set $(a_0, b_0) = (\widetilde{a}_0, \widetilde{b}_0)$, and $(a'_0, b'_0) = (\widetilde{a}_0 + \widetilde{b}_0, \widetilde{a}_0 + 2\widetilde{b}_0)$. Let b_j and b'_j be the shadows of b_0 and b'_0 in X_j, respectively, for $j = 1, 2$. Then

$$\omega(b_1 + b_2) = (\beta/\lambda)\omega(b_0), \quad \omega(b'_1 + b'_2) = (\beta'/\lambda)\omega(b'_0),$$

where $\beta, \beta' \in \mathbb{N}$ are determined by the prototype (e, w, h, t). From Lemma 11.4, there exists $s_1 > 0$ such that if $s < s_1$, then (X, ω) admits three-torus decompositions by the triples of saddle connections $\{\delta_j \mid j = 0, 1, 2\}$ and $\{\delta'_j \mid j = 0, 1, 2\}$, where δ_0 and δ'_0 satisfy

$$\delta_0 * (-\sigma_0) = b_0 + b_1 + b_2 \in H_1(X, \mathbb{Z}),$$
$$\delta'_0 * (-\sigma_0) = b'_0 + b'_1 + b'_2 \in H_1(X, \mathbb{Z}).$$

By definition, $\widetilde{a}_0 = a_0 = 2a'_0 - b'_0$. Assume that (X, ω) is a Veech surface. Then the directions of δ and δ' must be periodic. Lemma 11.6 then implies

$$s = \frac{\lambda + \beta}{(r+1)\lambda + \beta} = \frac{\lambda + \beta'}{(2r'+3)\lambda + 2\beta'} \tag{15}$$

with $r, r' \in \mathbb{Q}$. Set $R = r + 1$, $R' = 2r' + 3$. We see that (15) is equivalent to

$$R'\lambda^2 + (R'\beta + 2\beta')\lambda + 2\beta\beta' = R\lambda^2 + (R\beta' + \beta)\lambda + \beta\beta'.$$

Using $\lambda^2 = e\lambda + 2wh$, we get

$$R'(e\lambda + 2wh) + (R'\beta + 2\beta')\lambda + 2\beta\beta' = R(e\lambda + 2wh) + (\beta + R\beta')\lambda + \beta\beta',$$

which holds if and only if

$$(R'e + R'\beta + 2\beta')\lambda + (2whR' + 2\beta\beta') = (Re + \beta + R\beta')\lambda + (2whR + \beta\beta').$$

It follows that

$$\begin{cases} R'(e+\beta) + 2\beta' = R(e+\beta') + \beta, \\ 2whR' + 2\beta\beta' = 2whR + \beta\beta', \end{cases}$$

or

$$\begin{cases} R(e+\beta') - R'(e+\beta) = 2\beta' - \beta, \\ R - R' = \dfrac{\beta\beta'}{2wh}. \end{cases} \tag{16}$$

We first remark that $\beta \neq \beta'$, otherwise (15) would imply that $(R - R')\lambda = \beta$, and hence $R - R' \notin \mathbb{Q}$ since $\beta \neq 0$. It follows that the linear system (16) has a unique solution. Let s_2 be the value of s corresponding to this solution given by Equation (15). It follows that if $s < \min\{s_1, s_2\}$ then the directions of δ_0 and δ'_0 cannot be both periodic, hence (X,ω) cannot be a Veech surface. Since the set $\mathcal{P}_D(0,0,0)$ is finite, the proposition follows. □

The next proposition is a direct consequence of Proposition 11.7.

Proposition 11.8 *Let* $\{(X_j,\omega_j,P_j) \mid j=0,1,2\}$ *be an element of* $\Omega E_D(0,0,0)$, *and* $\Psi\colon D^\circ(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ *be the map in Proposition 8.1. Then there exists* $0<\delta<\varepsilon$ *such that if* $(X,\omega)\in\Psi(D^\circ(\delta))$, *then* (X,ω) *is not a Veech surface.*

Proof Let ℓ_0 be the length of the shortest simple closed geodesic in the torus (X_0,ω_0). Let s_0 be the constant in Proposition 11.7. Pick $\delta<\min\{\varepsilon, s_0\ell_0\}$. By definition, if $(X,\omega)=\Psi(z)$, then there is a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ which decompose X into three tori such that $\omega(\sigma_j)=z$. Assume that $z\in D^\circ(\delta)$. We claim that (X,ω) is not a Veech surface. There are two cases:

- z is not parallel to any vector in the lattice L_0 associated to (X_0,ω_0). In this case, the direction of σ_j is not periodic, hence (X,ω) is not a Veech surface.
- z is parallel to some vector in L_0. In this case, (X,ω) admits a decomposition into three cylinders, which correspond to the tori X_0, X_1, X_2, in the direction of z. Let v be the primitive vector in L_0 in the same direction as z. Then the width of the cylinder corresponding to X_0 is $|v|$. By assumption,
$$\frac{|\sigma_0|}{|v|} \le \frac{|\sigma_0|}{\ell_0} < s_0.$$
Therefore, (X,ω) cannot be a Veech surface by Proposition 11.7. □

Using Proposition 11.8, we can now prove the theorem announced at the beginning of the section.

Proof of Theorem 11.1 Fix a connected component $\mathscr{C}$ of $\Omega E_D(2,2)^{\mathrm{odd}}$. By the main result of [14], we know that there exists a surface $(X,\omega)\in\mathscr{C}$ which admits a three-torus decomposition by a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$.

By moving in the kernel foliation leaves, we can assume that the direction of σ_j is periodic on (X,ω). By Lemma 11.2, we get a corresponding prototype (w,h,t,e) in $\mathcal{P}_D(0,0,0)$. Set
$$L_0=\mathbb{Z}(\lambda,0)+\mathbb{Z}(0,\lambda),\quad L_1=L_2=\mathbb{Z}(w,0)+\mathbb{Z}(t,h),\quad (X_j,\omega_j)=(\mathbb{C}/L_j,dz),$$
for $j=0,1,2$. Let P_j be the projection of $0\in\mathbb{C}$ in X_j. Then the triple $\{(X_j,\omega_j,P_j) \mid j=0,1,2\}$ belongs to $\Omega E_D(0,0,0)$. Note that we obtain this triple of tori as the limit surface when $\sigma_0,\sigma_1,\sigma_2$ are collapsed.

Let $\Psi\colon D^\circ(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ be the map in Proposition 8.1. It is easy to see that $\Psi(D^\circ(\varepsilon)) \subset \mathscr{C}$. From Proposition 11.8, we know that there exists $0 < \delta < \varepsilon$ such that the set $\mathcal{V} = \Psi(D^\circ(\delta))$ does not contain any Veech surface. As a consequence, the set $\mathcal{U} = \mathrm{GL}^+(2,\mathbb{R}) \cdot \mathcal{V}$ does not contain any Veech surface either. It is easy to see that $\mathcal{U}$ is an open subset of $\mathscr{C}$. The theorem is then proved. $\square$

12 Finiteness of closed orbits

In this section we will prove our second main result, namely:

Theorem 12.1 *If D is not a square then the number of closed $\mathrm{GL}^+(2,\mathbb{R})$–orbits in $\Omega E_D(2,2)^{\mathrm{odd}}$ is finite.*

We first show a useful finiteness result up to the kernel foliation for surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$. Recall that (X,ω) admits an unstable cylinder decomposition in the horizontal direction if and only if this direction is periodic, and there exists at least one horizontal saddle connection whose endpoints are distinct zeros of ω.

Theorem 12.2 *If D is not a square then there exists a finite family $\mathbb{P}_D$ of surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$ such that if $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$ admits an unstable cylinder decomposition, then up to rescaling by $\mathrm{GL}^+(2,\mathbb{R})$, one has*

$$(X,\omega) = (X_i,\omega_i) + (x,0) \quad \textit{for some } (X_i,\omega_i) \in \mathcal{P}_D.$$

If we label the zeros of ω by P and Q, we always choose the orientation for any saddle connection joining P and Q to be *from P to Q*; this defines in a unique way the surface $(X,\omega) + (x,0)$.

Proof of Theorem 12.2 By [22], for any $D' \equiv 0, 1 \bmod 4$, $D' > 0$, the set $\Omega E_{D'}(2)^*$ is a finite union of $\mathrm{GL}^+(2,\mathbb{R})$–closed orbits. More precisely, there exists a finite family $\mathbb{P}_{D'}(2)$ of surfaces (*prototypical splittings*) such that any $(X,\omega) \in \Omega E_{D'}(2)^*$ which is horizontally periodic belongs to the P–orbit $\left(\text{here } P = \left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \subset \mathrm{GL}^+(2,\mathbb{R})\right\}\right)$ of some surface in $\mathbb{P}_{D'}(2)$.

In [15], we proved the same result for the stratum $\Omega E_D(4)$: there exists a finite family $\mathbb{P}_D(4)$ of surfaces such that any horizontally periodic surface $(X,\omega) \in \Omega E_D(4)$ belongs to the P–orbit of a surface in $\mathbb{P}_D(4)$. The corresponding statement for the stratum $\Omega E_D(0,0,0)$ is Lemma 11.2. Let $\mathbb{P}_D(0,0,0)$ be the set of corresponding surfaces in $\Omega E_D(0,0,0)$. We will call the surfaces in the families $\mathbb{P}_{D'}(2)$, $\mathbb{P}_D(4)$, $\mathbb{P}_D(0,0,0)$ *prototypical surfaces*.

Given a discriminant $D>0$, for each prototypical surface X_∞ in these finite families $\mathbb{P}_D(0,0,0)$, $\mathbb{P}_D(4)$ and $\mathbb{P}_{D'}(2)$, where $D'\in\{D,D/4\}$, we apply, respectively, Propositions 8.1, 8.3 and 8.5. This furnishes a map $\Psi\colon D^\circ(\varepsilon)\to\Omega E_D(2,2)^{\mathrm{odd}}$, where $\varepsilon>0$.

By construction, the surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$ whose horizontal kernel foliation leaves contain X_∞ (ie X_∞ is a limit of the real kernel foliation leaf through such surfaces) and which are close enough to X_∞ are contained in the set $\Psi(R_{(k,n)}(\varepsilon))$, where $n\in\{1,3,5\}$, $k\in\{0,\dots,2n-1\}$, depending on the space to which X_∞ belongs. For each prototypical surface, and each admissible pair (k,n), we pick a surface in $\Psi(R_{(k,n)}(\varepsilon))$. Let $\mathbb{P}_D$ denote this (finite) family. Note that for all the surfaces in this family, the cylinder decomposition in the horizontal direction is unstable.

Now, thanks to Theorem 9.1, if $(X,\omega)\in\Omega E_D(2,2)^{\mathrm{odd}}$ admits an unstable cylinder decomposition, then up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, the horizontal kernel foliation leaf through (X,ω) contains some prototypical surface. Therefore (X,ω) belongs to the same horizontal kernel foliation leaf of a surface in the family $\mathbb{P}_D$, and the theorem follows. □

We now have all necessary tools to prove our main result.

Proof of Theorem 12.1 Let $\{(X_i,\omega_i)\mid i\in I\}$ be a family of Veech surfaces that generates an infinite family of closed $\mathrm{GL}^+(2,\mathbb{R})$–orbits in $\Omega E_D(2,2)^{\mathrm{odd}}$. We will show that the set

$$\mathcal{O}=\bigcup_{i\in I}\mathrm{GL}^+(2,\mathbb{R})\cdot(X_i,\omega_i)$$

is dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$, contradicting Theorem 11.1.

Since the direction of any saddle connection on a Veech surface is periodic, each surface in the family $\{(X_i,\omega_i)\mid i\in I\}$ admits infinitely many unstable cylinder decompositions. Therefore, we can assume that each of the surfaces (X_i,ω_i) belongs to the horizontal kernel foliation leaf of one surface in the family $\mathbb{P}_D$ of Theorem 12.2. Since the set $\mathbb{P}_D$ is finite, there exists a surface $(X,\omega)\in\mathbb{P}_D$ and an infinite subfamily $I_0\subset I$ such that $(X_i,\omega_i)=(X,\omega)+(x_i,0)$ for any $i\in I_0$. By Theorem 9.1, $x_i\in\,]a,b[$, where a,b do not depend on i.

The compactness of the interval $[a,b]$ implies that there is a subsequence $\{i_k\}_{k\in\mathbb{N}}\subset I_0$ such that $\{x_{i_k}\}$ converges to some $x\in[a,b]$. The sequence $(X_{i_k},\omega_{i_k})=(X,\omega)+(x_{i_k},0)$ thus converges to $(Y,\eta):=(X,\omega)+(x,0)$. If $x\in\,]a,b[$, then (Y,η) belongs to $\Omega E_D(2,2)^{\mathrm{odd}}$. However, if $x\in\{a,b\}$, then by Theorem 9.1 (Y,η) belongs to a boundary component of $\Omega E_D(2,2)^{\mathrm{odd}}$, namely $\Omega E_D(4)$, $\Omega E_{D'}(2)^*$ with $D'\in\{D,D/4\}$, or $\Omega E_D(0,0,0)$. We distinguish separately the four cases below.

Case $(Y,\eta)\in\Omega E_D(2,2)^{\mathrm{odd}}$ Let θ be a periodic direction on (Y,η) that is different from $(\pm 1,0)$. Set

$$(Y^\theta,\eta^\theta) := R_{-\theta}\cdot(Y,\eta) \quad \text{and} \quad (X^\theta_{i_k},\omega^\theta_{i_k}) = R_{-\theta}\cdot(X_{i_k},\omega_{i_k}),$$

where $R_{-\theta}=\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$. Observe that (Y^θ,η^θ) is horizontally periodic and

$$(X^\theta_{i_k},\omega^\theta_{i_k}) = (Y^\theta,\eta^\theta)+v_k,$$

where $v_k = R_{-\theta}\cdot(x-x_{i_k},0)$. Thus $v_k\to(0,0)$ as $k\to\infty$. Note that, since $\theta\neq(\pm1,0)$, v_k does not belong to $\mathbb{R}\times\{0\}$.

By Propositions 4.1 and 4.2, for k large enough, $(X^\theta_{i_k},\omega^\theta_{i_k})$ admits a stable cylinder decomposition in the horizontal direction. Moreover, we can assume that the cylinder decompositions of $(X^\theta_{i_k},\omega^\theta_{i_k})$ in the horizontal direction share the same combinatorial data, and the same widths of cylinders. Finally, since $(X^\theta_{i_k},\omega^\theta_{i_k})$ are Veech surfaces, the horizontal direction is parabolic. The assumptions of Theorem 7.2 are therefore fulfilled, and we derive that there exists $\varepsilon_1>0$ such that $(Y^\theta,\eta^\theta)+(s,0)\in\overline{\mathcal{O}}$ for all $s\in(-\varepsilon_1,\varepsilon_1)$. It follows from Corollary 6.3 that there exists $\varepsilon'_1>0$ such that $(Y^\theta,\eta^\theta)+v\in\overline{\mathcal{O}}$ for any $v\in\mathbb{R}^2$ such that $|v|<\varepsilon'_1$. One can find a vector v with $|v|<\varepsilon'$ such that the surface $(Y^\theta,\eta^\theta)+v$ is horizontally periodic but not parabolic. By Theorem 6.1, the $\mathrm{GL}^+(2,\mathbb{R})$–orbit of $(Y^\theta,\eta^\theta)+v$ is dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$. Since this $\mathrm{GL}^+(2,\mathbb{R})$–orbit is contained in $\overline{\mathcal{O}}$, we conclude that $\overline{\mathcal{O}}$ contains a component of $\Omega E_D(2,2)^{\mathrm{odd}}$.

Case $(Y,\eta)\in\Omega E_D(4)$ In this case (Y,η) is a Veech surface. Choose a periodic direction θ for (Y,η) that is different from $(\pm1,0)$. We define (Y^θ,η^θ) and $(X^\theta_{i_k},\omega^\theta_{i_k})$ as in the previous case.

Let $\Psi\colon D^\circ(\varepsilon)\to\Omega E_D(2,2)^{\mathrm{odd}}$ be the map in Proposition 8.3 associated to (Y^θ,η^θ). Recall that, by construction, the set $\Psi(R_{(k,5)}(\varepsilon))$ consists of surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$ close to (Y^θ,η^θ) which have a small *horizontal* saddle connection invariant under the Prym involution.

By the choice of θ, $(X^\theta_{i_k},\omega^\theta_{i_k})$ is not contained in $\Psi(R_{(k,5)}(\varepsilon))$ for any $k\in\{0,\dots,9\}$. Thus, there must exist $k\in\{1,\dots,10\}$ such that the sector $\Psi(D^\circ_{(k,5)}(\varepsilon))$ contains infinitely many elements of the family $\{(X^\theta_{i_k},\omega^\theta_{i_k})\}$. Note that every surface in $\Psi(D^\circ_{(k,5)}(\varepsilon))$ admits a stable cylinder decomposition in the horizontal direction with the same combinatorial data and the same widths of cylinders (see Proposition 10.1). By assumption, the horizontal direction is parabolic for all $(X^\theta_{i_k},\omega^\theta_{i_k})$. Thus Theorem 10.4 allows us to conclude that $\mathcal{O}$ is dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$.

Case $(Y,\eta) \in \Omega E_{D'}(2)^*$ In particular, (Y,η) is a Veech surface (viewed as a surface of $\Omega E_{D'}(2)$). The same arguments as the case $(Y,\eta) \in \Omega E_D(4)$ show that $\overline{\mathcal{O}}$ contains a component of $\Omega E_D(2,2)^{\mathrm{odd}}$.

Case $(Y,\eta) \in \Omega E_D(0,0,0)$ In this case (X,ω) has a triple of horizontal saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ that decompose the surface into a connected sum of three tori, and (Y,η) can be viewed as the limit when the length of σ_j goes to zero. By Proposition 11.8, there is no Veech surface in the neighborhood of (Y,η). Thus this case does not occur.

From above discussion, we conclude that $\mathcal{O}$ is always dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$, but this is a contradiction with Theorem 11.1. The proof of Theorem 12.1 is now complete. □

Appendix: Existence of Veech surfaces in infinitely many Prym eigenform loci

It follows from the work of McMullen [25] that there exist only finitely many closed $\mathrm{GL}^+(2,\mathbb{R})$–orbits in the union $\bigcup_{D \text{ not a square}} \Omega E_D(1,1)$ (see [13] for a similar result in $\Omega E_D(1,1,2)$). However, the situation is different in $\Omega E_D(2,2)^{\mathrm{odd}}$. We will show that, for infinitely many discriminants D that are not squares, the locus $\Omega E_D(2,2)^{\mathrm{odd}}$ contains at least one closed $\mathrm{GL}^+(2,\mathbb{R})$–orbit (we will prove in [14] that $\Omega E_{D_1}(2,2)^{\mathrm{odd}}$ and $\Omega E_{D_2}(2,2)^{\mathrm{odd}}$ are disjoint if $D_1 \neq D_2$). Remark that the corresponding Veech surfaces we found are not primitive; they are double coverings of surfaces in $\Omega E_D(2)$. It is unknown to the authors if there exists any primitive Veech surface in $\bigcup_{D \text{ not a square}} \Omega E_D(2,2)^{\mathrm{odd}}$.

Following [22] we say that a quadruple of integers (w,h,t,e) is a *splitting prototype* of discriminant D if the conditions below are fulfilled:

$$\begin{cases} w>0,\ h>0,\ 0\leq t<\gcd(w,h),\\ \gcd(w,h,t,e)=1,\\ D=e^2+4wh,\\ 0<\lambda:=\frac{1}{2}(e+\sqrt{D})<w. \end{cases}$$

To each splitting prototype one can associate a Veech surface $(X,\omega)\in\Omega E_D(2)$ as follows (see Figure 9).

Define a pair of lattices in $\mathbb{C}$ by $\Lambda_1=\mathbb{Z}(\lambda,0)\oplus\mathbb{Z}(0,\lambda)$ and $\Lambda_2=\mathbb{Z}(w,0)\oplus\mathbb{Z}(t,h)$ (recall that $\lambda:=\frac{1}{2}(e+\sqrt{D})>0$). We construct the corresponding tori $(E_i,\omega_i)=(\mathbb{C}/\Lambda_i,dz)$ and the genus-two surface (X,ω), where $X=E_1\#E_2$ and $\omega=\omega_1+\omega_2$.

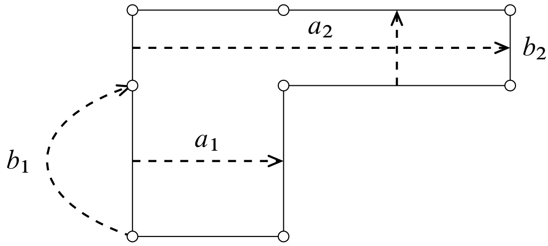

Figure 9: Prototypical splitting of type $(w,h,0,e)$, where $\omega(a_1)=(\lambda,0)$, $\omega(b_1)=(0,\lambda)$, $\omega(a_2)=(w,0)$ and $\omega(b_2)=(0,h)$. Parallel edges are identified to obtain a surface $(X,\omega)\in\Omega E_D(2)$.

Geometrically, the surface (X,ω) is made of two horizontal cylinders whose core curves are denoted by a_1 and a_2 (see [22] and Figure 9 for details).

Let $\{a_1,b_1,a_2,b_2\}$ be the symplectic basis of $H_1(X,\mathbb{Z})$ such that $\omega(a_1)=(\lambda,0)$, $\omega(b_1)=(0,\lambda)$, $\omega(a_2)=(w,0)$ and $\omega(b_2)=(t,h)$. A generator of the order $\mathcal{O}_D$ is given (in the above basis) by the matrix

$$T=\begin{pmatrix} e & 0 & w & t\\ 0 & e & 0 & h\\ h & -t & 0 & 0\\ 0 & w & 0 & 0\end{pmatrix}.$$

It is straightforward to check that T is self-adjoint with respect to the intersection form of $H_1(X,\mathbb{Z})$, that $T^2=eT+wh\,\mathrm{Id}$, and that T satisfies $T^*\omega=\lambda\omega$. It follows that T generates a proper subring in $\mathrm{End}(\mathrm{Jac}(X))$ for which ω is an eigenvector. Thus $(X,\omega)\in\Omega E_D(2)$, and therefore (X,ω) is a Veech surface (see [24] for more details).

Theorem A.1 *Let (w,h,t,e) be a splitting prototype for a discriminant D, and (X,ω) be the associated Veech surface in $\Omega E_D(2)$. Let (Y_1,η_1) and (Y_2,η_2) be two surfaces in $\mathcal{H}(2,2)$ constructed from (w,h,t,e) as shown in Figure 10. Then both (Y_1,η_1) and (Y_2,η_2) are Veech surfaces in some Prym eigenform loci in $\mathcal{H}(2,2)^{\mathrm{odd}}$. More specifically:*

(i) $(Y_1,\omega_1)\in\Omega E_{4D}(2,2)^{\mathrm{odd}}$ *if h is odd, otherwise* $(Y_1,\eta_1)\in\Omega E_D(2,2)^{\mathrm{odd}}$.

(ii) $(Y_2,\omega_2)\in\Omega E_{4D}(2,2)^{\mathrm{odd}}$ *if w is odd, otherwise* $(Y_2,\eta_2)\in\Omega E_D(2,2)^{\mathrm{odd}}$.

Remark A.2
- In general, the Teichmüller discs generated by (Y_1,ω_1) and by (Y_2,ω_2) are different, for instance when h is odd, and w is even.
- If $D\equiv 5 \bmod 8$, then it is easy to see that e,w,h are all odd. Therefore, in both constructions (Y_i,η_i) belongs to $\Omega E_{4D}(2,2)^{\mathrm{odd}}$.

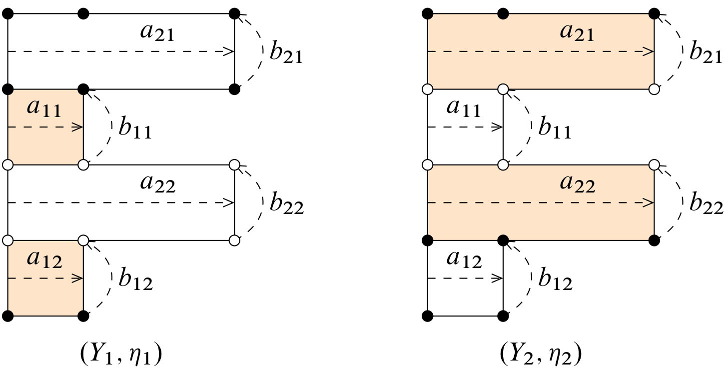

Figure 10: Double coverings of a surface in $\Omega E_D(2)$: $\eta_i(a_{11})=\eta_i(a_{12})=\lambda$, $\eta_i(b_{11})=\eta_i(b_{12})=\imath\lambda$, $\eta_i(a_{21})=\eta_i(a_{22})=w$, $\eta_i(b_{21})=\eta_i(b_{22})=t+\imath h$, $i=1,2$. The cylinders fixed by the Prym involution are colored.

Proof It is easy to see that both (Y_1,η_1) and (Y_2,η_2) are double coverings of (X,ω), and the deck transformation sends a_{ij} to $a_{i\,j+1}$ and b_{ij} to $b_{i\,j+1}$ (here we use the convention $(i3)\sim(i1)$). Since (X,ω) is a Veech surface, both (Y_1,ω_1) and (Y_2,ω_2) are Veech surfaces (see [9; 17]).

Remark that Y_i has an involution τ_i that exchanges the zeros of η_i such that $\tau_i^*\eta_i=-\eta_i$; in Figure 10 the cylinders fixed by τ_i are colored. It follows that (Y_i,η_i) belongs to the Prym locus $\mathrm{Prym}(2,2)\subset\mathcal{H}(2,2)^{\mathrm{odd}}$ ($\mathrm{Prym}(2,2)$ consists of double coverings of quadratic differentials in $\mathcal{Q}(-1^4,4)$). By some standard arguments (see [15; 24]), we can conclude that (Y_i,η_i) is a Prym eigenform, thus (Y_i,η_i) is contained in some locus $\Omega E_{\widetilde{D}}(2,2)^{\mathrm{odd}}$. It remains to determine the discriminant $\widetilde{D}$.

Set $H_1(Y_i,\mathbb{Z})^-=\{\alpha\in H_1(Y_i,\mathbb{Z})\mid\tau_i(\alpha)=-\alpha\}$. Since $(Y_i,\eta_i)\in\mathrm{Prym}(2,2)$, we have $H_1(Y_i,\mathbb{Z})^-\simeq\mathbb{Z}^4$. We choose a basis of $H_1(Y_i,\mathbb{Z})^-$ as follows:

- For (Y_1,η_1), set $\alpha_1=a_{11}=a_{12}$ and $\alpha_2=a_{21}+a_{22}$. We choose $\beta_1=b_{11}+b_{12}$ and $\beta_2=b_{21}+b_{22}$. In particular, the restriction of the symplectic form has the matrix $\left(\begin{smallmatrix}J&0\\0&2J\end{smallmatrix}\right)$.
- For (Y_2,η_2), set $\alpha_1=a_{11}+a_{12}$, $\alpha_2=a_{21}=a_{22}$, $\beta_1=b_{11}+b_{12}$, $\beta_2=b_{21}+b_{22}$. In this basis, the restriction of the intersection form to $H_1(Y_2,\mathbb{Z})^-$ is given by $\left(\begin{smallmatrix}2J&0\\0&J\end{smallmatrix}\right)$.

In the above bases, the coordinates of η_i are the following:

$$\mathrm{Re}(\eta_1)=(\lambda,0,2w,2t)\quad\text{and}\quad\mathrm{Im}(\eta_1)=(0,2\lambda,0,2h),$$
$$\mathrm{Re}(\eta_2)=(2\lambda,0,w,2t)\quad\text{and}\quad\mathrm{Im}(\eta_2)=(0,2\lambda,0,2h).$$

Let $\widetilde{T}_1$ be the following self-adjoint endomorphism of $H_1(Y_1,\mathbb{Z})^-$ (given in the basis $\{\alpha_1,\beta_1,\alpha_2,\beta_2\}$):

$$\widetilde{T}_1=\begin{pmatrix}2e & 0 & 4w & 4t\\ 0 & 2e & 0 & 2h\\ h & -2t & 0 & 0\\ 0 & 2w & 0 & 0\end{pmatrix}.$$

Similarly, let $\widetilde{T}_2$ be the self-adjoint endomorphism of $H_1(Y_2,\mathbb{Z})^-$ (given in the basis $\{\alpha_1,\beta_1,\alpha_2,\beta_2\}$) by the following matrix:

$$\widetilde{T}_2:=\begin{pmatrix}2e & 0 & w & 2t\\ 0 & 2e & 0 & 2h\\ 4h & -4t & 0 & 0\\ 0 & 2w & 0 & 0\end{pmatrix}.$$

It is straightforward to check that $\widetilde{T}_i^*\eta_i=(2\lambda)\cdot\eta_i$, thus η_i is an eigenform of $\widetilde{T}_i$. Both $\widetilde{T}_i$ satisfy $\widetilde{T}_i^2-2e\widetilde{T}_i-4wh\,\mathrm{Id}=0$, which implies that $\widetilde{T}_i$ generates a self-adjoint subring of $\mathrm{End}(\mathrm{Prym}(Y_i))$ isomorphic to $\mathcal{O}_{D'}$, where

$$D'=(2e)^2+16wh=4(e^2+4wh)=4D.$$

There exists a unique proper subring of $\mathrm{End}(\mathrm{Prym}(Y_i))$ for which η_i is an eigenform; this proper subring is isomorphic to a quadratic order $\mathcal{O}_{\widetilde{D}_i}$. Clearly, this subring must contain $\widetilde{T}_i$, hence it is generated by $\widetilde{T}_i/k_i$, where

$$k_1=\gcd(2e,4w,2h,2w,h,4t,2t)=\gcd(2e,2w,h,2t),\quad k_2=\gcd(2e,w,2h,2t).$$

The relation $\gcd(w,h,t,e)=1$ implies $k_i\in\{1,2\}$. Note that $4D=k_i^2\widetilde{D}_i$, therefore $\widetilde{D}_i=4D$ if $k_i=1$, and $\widetilde{D}_i=D$ if $k_i=2$. We can now conclude by noticing that $k_1=1$ if and only if h is odd, and $k_2=1$ if and only if w is odd. □

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