

Volume and homology growth of aspherical manifolds

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(1) We provide upper bounds on the size of the homology of a closed aspherical Riemannian manifold that only depend on the systole and the volume of balls. (2) We show that linear growth of mod p Betti numbers or exponential growth of torsion homology imply that a closed aspherical manifold is “large”.

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1 Introduction and statement of results

1.1 Introduction

Let M be a manifold whose fundamental group $\Gamma = \pi_1(M)$ is *residually finite*. That is, Γ possesses a decreasing sequence — called a *residual chain* — of normal subgroups $\Gamma_i < \Gamma$ of finite index whose intersection is trivial. By covering theory there is an associated sequence of finite regular coverings $\cdots \rightarrow M_2 \rightarrow M_1 \rightarrow M$ of M such that $\pi_1(M_i) \cong \Gamma_i$ and $\deg(M_i \rightarrow M) = [\Gamma : \Gamma_i]$, which we call a *residual tower of finite covers*. A basic question is:

How does the size of the homology of M_i grow as $i \rightarrow \infty$?

The growth behavior of the first homology was connected to largeness of groups by Lackenby [14; 13]. It is also related to the cost; see Abért and Nikolov [1]. Number-theoretic connections of homology growth in the context of arithmetic locally symmetric spaces are discussed in Bergeron and Venkatesh [3].

What do we mean by size? If we measure size by Betti numbers

$$b_k(M_i) = \text{rk}_{\mathbb{Z}} H_k(M_i; \mathbb{Z}),$$

there is a general answer: the limit of $b_k(M_i)/[\Gamma : \Gamma_i]$ is the k^{th} ℓ^2 -Betti number of M , by a result of Lück [15]. If we measure size by mod p Betti numbers or in terms of the cardinality of the torsion subgroups $\text{tors } H_k(M_i; \mathbb{Z}) \subset H_k(M_i; \mathbb{Z})$, no general answer is available.

We shall consider throughout this paper the case that M is a closed *aspherical* manifold, that is, its universal cover \tilde{M} is contractible. The manifold M is aspherical if and only if M is a model for the classifying space of its fundamental group.

Our aim is to establish upper bounds for the homology and the homology growth of aspherical manifolds. Our gap theorem ([Theorem 1.5](#)) shows that the volume of a closed aspherical Riemannian manifold whose growth of torsion homology is exponential and whose Ricci curvature is bounded from below by -1 is greater than a positive universal constant. This establishes a link between the homology growth of aspherical manifolds and largeness of Riemannian manifolds in the sense of Gromov. Rather than being a precise notion, *largeness* stands here for a variety of phenomena in Riemannian geometry [9]. For instance, we would regard an aspherical manifold as large if it has non-zero minimal volume.

Important examples of aspherical manifolds are locally symmetric spaces of non-compact type. Bergeron and Venkatesh [3] develop a detailed, yet largely conjectural, picture for the (torsion) homology growth of arithmetic locally symmetric spaces of non-compact type. By relating the growth of the torsion homology of M_i to the analytic ℓ^2 -torsion, they are able to compute precisely the growth of torsion homology for special coefficient systems.

1.2 Statement of results

The *systole* of a Riemannian manifold is the minimal length of a non-contractible loop. The first result, which is based on the remarkable work of Guth [11], is for the homology of one manifold at a time but has an immediate consequence for the homology growth ([Corollary 1.3](#)).

In the sequel upper bounds for the \mathbb{F}_p -Betti numbers are formulated, where p stands for an arbitrary prime. By the universal coefficient theorem, the Betti number in degree k is bounded above by the \mathbb{F}_p -Betti number in degree k for any p . So each theorem below also yields a bound on the Betti numbers. The theorems of this section can be easily extended from constant to unitary coefficients but we refrain from doing so to keep the exposition short and easier to read.

Theorem 1.1 *For every $n \in \mathbb{N}$ and $V_0 > 0$ there exists a constant $\text{const}(n, V_0) > 0$ with the following property: Let M be an n -dimensional closed aspherical Riemannian manifold such that every 1-ball of M has volume at most V_0 and the systole of M is at least 1. Then for every $k \in \mathbb{N}$,*

$$\dim_{\mathbb{F}_p} H_k(M; \mathbb{F}_p) < C(n, V_0) \text{vol}(M), \quad \text{and}$$

$$\log |\text{tors } H_k(M; \mathbb{Z})| < C(n, V_0) \text{vol}(M).$$

Remark 1.2 Gromov [2] proved that all Betti numbers of a real-analytic, closed or finite-volume Riemannian manifold M whose sectional curvature is between -1 and 0 are bounded by $C(n) \text{vol}(M)$. In contrast, our assumptions are curvature-free but require a condition on the systole. For estimates of the torsion homology for non-compact arithmetic locally symmetric manifolds we refer to Gelander [7] and Emery [6].

Corollary 1.3 For every $n \in \mathbb{N}$ and $V_0 > 0$ there is $C(n, V_0) > 0$ with the following property: Let M be an n -dimensional closed connected aspherical Riemannian manifold such that every 1 -ball of the universal cover \tilde{M} has volume at most V_0 . Assume that the fundamental group is residually finite, and let (M_i) be a residual tower of finite covers. Then for every $k \in \mathbb{N}$,

$$\limsup_{i \rightarrow \infty} \frac{\dim_{\mathbb{F}_p} H_k(M_i; \mathbb{F}_p)}{\deg(M_i \rightarrow M)} < C(n, V_0) \text{vol}(M), \quad \text{and}$$

$$\limsup_{i \rightarrow \infty} \frac{\log |\text{tors } H_k(M_i; \mathbb{Z})|}{\deg(M_i \rightarrow M)} < C(n, V_0) \text{vol}(M).$$

Proof of corollary It is clear that the systole of M_i converges to ∞ as $i \rightarrow \infty$ in a residual tower of finite covers. Note also that the maximal volume of a 1 -ball of M coincides with the maximal volume of a 1 -ball in the universal cover \tilde{M} provided the systole is at least 1 . Now apply Theorem 1.1. □

Remark 1.4 There is a natural tension between the volume on M and on \tilde{M} in the above statements: if one scales the metric of M by a factor < 1 , then the volume of M decreases, but the curvature and so the volume of balls in the universal covering increase.

If the Ricci curvature is ≥ -1 (short for $\geq -g$ as quadratic forms), the volume of 1 -balls in M and \tilde{M} is bounded from above by a positive constant only depending on the dimension, according to the Bishop–Gromov inequality.

Our next result exhibits a gap phenomenon for the homology growth under a lower Ricci curvature bound.

Theorem 1.5 For every $n \in \mathbb{N}$ there is a constant $\epsilon(n) > 0$ with the following property: Let M be a closed connected aspherical n -dimensional Riemannian manifold M such that $\text{Ricci}(M) \geq -1$ and the volume of every 1 -ball in M is at most $\epsilon(n)$. Assume that the fundamental group is residually finite, and let (M_i) be a residual tower of finite covers. Then for every $k \in \mathbb{N}$,

$$\lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{F}_p} H_k(M_i; \mathbb{F}_p)}{\deg(M_i \rightarrow M)} = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \frac{\log |\text{tors } H_k(M_i; \mathbb{Z})|}{\deg(M_i \rightarrow M)} = 0.$$

Gromov showed [8, Section 3.4] that for every dimension n there is a constant $\epsilon(n) > 0$ with the following property: Every closed n -dimensional Riemannian manifold M such that $\text{Ricci}(M) \geq -1$ and the volume of every 1-ball in M is at most $\epsilon(n)$ can be covered by open, amenable sets with multiplicity $\leq n$. Here a subset U of a topological space X is called *amenable* if the image of the map $\pi_1(U; x) \rightarrow \pi_1(X; x)$ on fundamental groups is amenable for any base point $x \in U$. Hence [Theorem 1.5](#) is a direct consequence of the following topological result.

Theorem 1.6 *Let M be a closed connected aspherical n -dimensional manifold. Assume that M is covered by open, amenable sets such that every point is contained in no more than n such subsets. Assume that the fundamental group is residually finite, and let (M_i) be a residual tower of finite covers. Then for every $k \in \mathbb{N}$,*

$$\lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{F}_p} H_k(M_i; \mathbb{F}_p)}{\deg(M_i \rightarrow M)} = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \frac{\log |\text{tors } H_k(M_i; \mathbb{Z})|}{\deg(M_i \rightarrow M)} = 0.$$

Remark 1.7 (Relation to Gromov's vanishing theorem) [Theorems 1.5](#) and [1.6](#) are reminiscent of the *isolation theorem* and the *vanishing theorem* in Gromov's seminal work [8, 0.5 and 3.1], which under the same assumptions conclude that the simplicial volume vanishes. The formal deduction of [Theorem 1.5](#) from [Theorem 1.6](#) corresponds to Gromov's deduction of the isolation theorem from the vanishing theorem. Gromov's proof of the vanishing theorem is based on bounded cohomology and I do not see how to deduce something for the integral homology with this method. We develop a different approach in [Section 5](#).

Remark 1.8 Vanishing results for the homology growth of aspherical spaces whose fundamental groups contain an infinite, normal, elementary amenable subgroup are proved in Lück [16]. The methods there are completely different from ours. If a normal amenable subgroup of $\pi_1(M)$ arises as the fundamental group of the fiber of a fiber bundle $F \rightarrow M \rightarrow N$ of closed aspherical manifolds, the assumptions in the previous theorem are satisfied for M according to [8, Corollaries (2), page 41].

Remark 1.9 (Consequences for ℓ^2 -Betti numbers) Lück's approximation theorem [15] yields as a corollary of [Theorems 1.5](#) and [1.6](#) that all ℓ^2 -Betti numbers of M vanish. This has been proved earlier by the author [18, Corollary of Theorem B]. Similarly, in [Corollary 1.3](#) all ℓ^2 -Betti numbers of M are bounded by $C(n, V_0) \text{vol}(M)$, which generalizes [18, Corollary of Theorem A].

1.3 On the proofs

As explained before, [Theorem 1.5](#) is a consequence of [Theorem 1.6](#). The proofs of [Theorems 1.1](#) and [1.6](#) start by a reduction to the orientable case. If there was no 2-torsion in the homology groups in question, the reduction would be just an easy transfer argument. Of course, we do not want to assume that, so the reduction argument requires more care. This is done in [Section 2](#). We may henceforth assume that all manifolds are oriented.

The broad theme of this paper is the relation between the homology growth on aspherical manifolds and volume. Since the volume of an oriented Riemannian manifold M is related to its homology through the volume form or its dual, the fundamental class, this suggests an important role of the fundamental class in our proofs. We first present an outline of the proof of [Theorem 1.6](#).

Let (M_i) be a residual tower of finite coverings of the closed aspherical manifold M in question which is associated to a residual chain (Γ_i) of the fundamental group. The proof consists of two major steps.

- (1) Bound the integral complexity of the fundamental class

$$[M_i] \in H_n(M_i; \mathbb{Z}) \cong H_n(\Gamma_i; \mathbb{Z}).$$

By *integral complexity* we mean the minimal number $l \in \mathbb{N}$ such that the fundamental class is represented as an integral linear combination of l singular simplices.

- (2) Bound the size of the homology (torsion and free part) of M_i , thus Γ_i , in arbitrary degrees in terms of the integral complexity of $[M_i]$.

The second step is dealt with in [Section 3](#). The first step, on which we elaborate now, is done in [Section 5](#). A central object of the proof is the profinite topological Γ -space

$$(1-1) \quad X := \varprojlim (\Gamma / \Gamma_0 \leftarrow \Gamma / \Gamma_1 \leftarrow \Gamma / \Gamma_2 \leftarrow \cdots).$$

The space X is also a compact topological group; we endow X with the normalized Haar measure μ . The idea to consider the space X is motivated by the work of Abért and Nikolov [\[1\]](#) who relate the cost of the Γ -action on X to the rank gradient of (Γ_i) . We take the cover of M by amenable subsets that figures in the assumption of [Theorem 1.6](#) and produce from it—by dynamical considerations—a different, Γ -equivariant measurable cover of a different object, namely the space $X \times \tilde{M}$ endowed with the diagonal Γ -action. Each set in this cover is a product of a measurable set in X and an open set in \tilde{M} . We modify the measurable cover a little ([Lemma 5.9](#)) so that the measurable set in X is cylindrical with respect to the profinite topology.

Next we convert this measurable cover to a cover of the diagonal Γ -space $\Gamma/\Gamma_i \times \tilde{M}$ for sufficiently large $i \in \mathbb{N}$ — see (5-9) for the definition of the associated cover of $\Gamma/\Gamma_i \times \tilde{M}$. The nerve of the cover of $\Gamma/\Gamma_i \times \tilde{M}$ is a Γ -space; we denote its orbit space, which is a Δ -complex, by $S(i)$. With an open cover and an associated partition of unity comes a map from the space to the nerve; in our situation the nerve map is equivariant and we will study the induced map f on orbit spaces:

$$\Gamma \backslash (\Gamma/\Gamma_i \times \tilde{M}) \cong \Gamma_i \backslash \tilde{M} = M_i \begin{array}{c} \xrightarrow{f} \\ \longleftarrow g \end{array} S(i).$$

From asphericity we conclude that there is a homotopy retract $g \circ f \simeq \text{id}$. The construction of the measurable cover above has been set up in such a way that the number of n -simplices in $S(i)$ is of magnitude $o([\Gamma : \Gamma_i])$. Finally, the homotopy retract implies that the integral complexity of $[M_i]$ is $o([\Gamma : \Gamma_i])$, concluding the argument for the first step.

The proof of Theorem 1.1 also uses nerves of covers (albeit very different ones). It is much shorter than the proof of Theorem 1.6 since the assertion is a rather direct consequence of Sections 2 and 3 and the remarkable work of Guth [11].

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2 Reduction to the orientable case

2.1 Estimates by subgroups of index 2

Lemma 2.1 *If $0 \rightarrow A \rightarrow B \rightarrow C$ is an exact sequence of finitely generated abelian groups, then $|\text{tors } B| \leq |\text{tors } A| \cdot |\text{tors } C|$.*

Proof The given exact sequence restricts to an exact sequence

$$0 \rightarrow \text{tors } A \rightarrow \text{tors } B \rightarrow \text{tors } C$$

of finite abelian groups, from which the assertion follows. □

Lemma 2.2 *If $A \rightarrow B \rightarrow C$ is an exact sequence of finitely generated abelian groups and A is torsion, then $|\text{tors } B| \leq |\text{tors } A| \cdot |\text{tors } C|$.*

Proof Let f be the map from B to C . Then $0 \rightarrow A/\ker(f) \rightarrow B \rightarrow C$ is exact and $|\text{tors } A| = |A| \geq |A/\ker(f)|$. Now apply the previous lemma. \square

The following two lemmas are only needed for trivial coefficients but their proofs do not become more complicated in the stated generality.

Lemma 2.3 *Let Γ be a group of type F_∞ and $\Lambda < \Gamma$ a subgroup of index 2. Let W be a $\mathbb{Z}[\Gamma]$ -module that is finitely generated free as a \mathbb{Z} -module. For $n \in \mathbb{N}$ we have*

$$\log |\text{tors } H_n(\Gamma, W)| \leq \sum_{k \leq n} \text{rk}_{\mathbb{Z}} H_k(\Lambda; W) + 2^n \sum_{k \leq n} \log |\text{tors } H_k(\Lambda; W)|.$$

Proof The finiteness condition F_∞ ensures that all homology groups to be considered are finitely generated. We consider the Lyndon–Hochschild–Serre spectral sequence for the group extension $0 \rightarrow \Lambda \rightarrow \Gamma \rightarrow C_2 \rightarrow 0$, where C_2 is the cyclic group of order 2:

$$E_{p,q}^2 = H_p(C_2; H_q(\Lambda; W)) \Rightarrow H_{p+q}(\Gamma; W).$$

For $p > 0$ the group $E_{p,q}^2$ is torsion, hence also $E_{p,q}^\infty$. The convergence of the spectral sequence means that there is an increasing filtration $F^i H_k(\Gamma; W)$ of $H_k(\Gamma; W)$ with $F^k H_k(\Gamma; W) = H_k(\Gamma; W)$ and $E_{0,k}^\infty \cong F^0 H_k(\Gamma; W)$ and short exact sequences

$$0 \rightarrow F^i H_k(\Gamma; W) \rightarrow F^{i+1} H_k(\Gamma; W) \rightarrow E_{i+1, k-i-1}^\infty \rightarrow 0$$

for $i = 0, \dots, k - 1$. Let $N_j := H_j(\Lambda; W)$. It follows inductively from these short exact sequences and [Lemma 2.1](#) that

$$(2-1) \quad \log |\text{tors } H_k(\Gamma; W)| = \sum_{i=0}^k \log |\text{tors } E_{i, k-i}^\infty| \leq \sum_{i=0}^k \log |\text{tors } H_i(C_2; N_{k-i})|.$$

The torsion submodule of N_j is a $\mathbb{Z}[C_2]$ -submodule of N_j . So there is an exact sequence of $\mathbb{Z}[C_2]$ -modules

$$(2-2) \quad 0 \rightarrow \text{tors } N_j \rightarrow N_j \rightarrow F_j \rightarrow 0,$$

where F_j is \mathbb{Z} -free. The $\mathbb{Z}[C_2]$ -module F_j decomposes as a direct sum of indecomposable $\mathbb{Z}[C_2]$ -modules:

$$F_j \cong F_j^1 \oplus \dots \oplus F_j^l.$$

By a (much more general) result of Diederichsen and Reiner [[5](#), Theorem 34.31, page 729] there are only three isomorphism types of indecomposable $\mathbb{Z}[C_2]$ -modules which are finitely generated free as \mathbb{Z} -modules: the trivial module \mathbb{Z} , the module \mathbb{Z}^{tw} with underlying \mathbb{Z} -module \mathbb{Z} on which the generator of C_2 acts by multiplication with -1 , and $\mathbb{Z}[C_2]$. Our use of the Diederichsen–Reiner result is motivated by [Remark 2](#)

of [6]. If $N \in \{\mathbb{Z}, \mathbb{Z}^{\text{tw}}, \mathbb{Z}[C_2]\}$, then by direct computation $|\text{tors } H_j(C_2; N)| \leq 2$. Since $l \leq \text{rk}_{\mathbb{Z}} N_j$, we obtain that

$$\log |\text{tors } H_k(C_2; F_j)| \leq \text{rk}_{\mathbb{Z}} N_j \cdot \log 2 \leq \text{rk}_{\mathbb{Z}} N_j.$$

The k^{th} chain group in the standard bar resolution computing $H_k(C_2; \text{tors } N_j)$ is isomorphic to $(\text{tors } N_j)^{2^k}$, which implies that

$$\log |\text{tors } H_k(C_2; \text{tors } N_j)| \leq 2^k \log |\text{tors } N_j|.$$

The short exact sequence (2-2) induces a long exact sequence in group homology:

$$\begin{aligned} \cdots \rightarrow H_{k+1}(C_2; F_j) \rightarrow H_k(C_2; \text{tors } N_j) \rightarrow H_k(C_2; N_j) \\ \rightarrow H_k(C_2; F_j) \rightarrow H_{k-1}(C_2; \text{tors } N_j) \rightarrow \cdots . \end{aligned}$$

Since $H_k(C_2; \text{tors } N_j)$ is torsion for $k \geq 0$, Lemma 2.2 implies that

$$\begin{aligned} \log |\text{tors } H_k(C_2; N_j)| &\leq \log |\text{tors } H_k(C_2; \text{tors } N_j)| + \log |\text{tors } H_k(C_2; F_j)| \\ &\leq 2^k \log |\text{tors } N_j| + \text{rk}_{\mathbb{Z}} N_j \end{aligned}$$

for $k \geq 0$. Combined with (2-1) this concludes the proof. □

Lemma 2.4 *Let Γ be a group of type F_{∞} and $\Lambda < \Gamma$ a subgroup of index 2. Let W be a $\mathbb{F}_p[\Gamma]$ -module that is finitely generated free as a \mathbb{F}_p -module. For $n \in \mathbb{N}$ we have*

$$\dim_{\mathbb{F}_p} H_n(\Gamma, W) \leq 2^n \sum_{k \leq n} \dim_{\mathbb{F}_p} H_k(\Lambda; W).$$

Proof The proof is easier than the one of the previous lemma. The Lyndon–Hochschild–Serre spectral sequence yields that

$$\dim_{\mathbb{F}_p} H_n(\Gamma; W) \leq \sum_{i=0}^n \dim_{\mathbb{F}_p} H_i(C_2; H_{n-i}(\Lambda; W)).$$

As the k^{th} chain group in the standard bar resolution computing $H_i(C_2; H_{n-i}(\Lambda; W))$ is isomorphic to a sum of 2^i copies of $H_{n-i}(\Lambda; W)$, we have

$$\dim_{\mathbb{F}_p} H_i(C_2; H_{n-i}(\Lambda; W)) \leq 2^i \dim_{\mathbb{F}_p} H_{n-i}(\Lambda; W). \quad \square$$

2.2 Reduction of Theorems 1.1 and 1.6 to the orientable case

Let us assume that Theorem 1.6 holds for the orientable case. Let M be a connected n -dimensional closed aspherical non-orientable manifold with fundamental group Γ . By asphericity the group homology of Γ and the homology of M are isomorphic.

There is a unique connected 2-sheeted cover $\bar{M} \rightarrow M$ such that \bar{M} is orientable. The fundamental group $\Gamma' = \pi_1(\bar{M})$ embeds into Γ as a subgroup of index 2. If (Γ_i) is a residual chain of Γ , then $(\Gamma' \cap \Gamma_i)$ is a residual chain of Γ' . Moreover, $\Gamma' \cap \Gamma_i$ is either equal to Γ_i or a subgroup of Γ_i of index 2.

Let us assume that M is covered by open, amenable subsets with multiplicity $\leq n$. Then their preimages cover \bar{M} with multiplicity $\leq n$ and are amenable. By [Theorem 1.6](#) applied to \bar{M} and the universal coefficient theorem we have

$$(2-3) \quad \lim_{i \rightarrow \infty} \frac{1}{[\Gamma' : \Gamma' \cap \Gamma_i]} \sum_{k=0}^n \text{rk}_{\mathbb{Z}} H_k(\Gamma' \cap \Gamma_i; \mathbb{Z}) = 0,$$

$$\lim_{i \rightarrow \infty} \frac{1}{[\Gamma' : \Gamma' \cap \Gamma_i]} \sum_{k=0}^n \log |\text{tors } H_k(\Gamma' \cap \Gamma_i; \mathbb{Z})| = 0.$$

Thus [Lemma 2.3](#) yields

$$\lim_{i \rightarrow \infty} \frac{1}{[\Gamma : \Gamma_i]} \log |\text{tors } H_k(\Gamma_i; \mathbb{Z})| = 0$$

for every k . From (2-3) with \mathbb{Z} replaced by \mathbb{F}_p and [Lemma 2.4](#) we obtain that

$$\lim_{i \rightarrow \infty} \frac{1}{[\Gamma : \Gamma_i]} \dim_{\mathbb{F}_p} H_k(\Gamma_i; \mathbb{F}_p) = 0.$$

The reduction for [Theorem 1.1](#) via [Lemmas 2.3](#) and [2.4](#) is along similar lines.

3 Bounds by the fundamental class

The following lemma is due to Soulé [[19](#), Lemma 1] who gives credit to Gabber.

Lemma 3.1 *Let A and B be finitely generated free \mathbb{Z} -modules. Let a_1, \dots, a_n and b_1, \dots, b_m be \mathbb{Z} -bases of A and B , respectively. We endow $B_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} B$ with the Hilbert space structure for which b_1, \dots, b_m is a Hilbert basis. Let $f: A \rightarrow B$ be a homomorphism. Let $I \subset \{1, \dots, n\}$ be a subset such that $\{f(a_i) \mid i \in I\}$ is a basis of $\text{im}(f)_{\mathbb{C}}$. Then*

$$|\text{tors coker}(f)| \leq \prod_{i \in I} \|f(a_i)\|.$$

3.1 The torsion estimate

This subsection is devoted to the proof of the following result.

Theorem 3.2 *Let M be a closed n -dimensional oriented manifold. If the fundamental class of M is represented by an integral cycle with k singular n -simplices, then*

$$\log |\text{tors } H_j(M; \mathbb{Z})| \leq \log(n + 1) \cdot \binom{n+1}{j+1} \cdot k$$

for every $j \geq 0$.

Definition 3.3 Suppose $\tau = \sum_{i=1}^k a_i \cdot \sigma_i$ is an integral singular cycle representing the fundamental class $[M] \in H_n(M; \mathbb{Z})$. We define $C_*^\tau(M)$ as the subcomplex of the integral singular chain complex $C_*(M)$ that is generated, in degree j , by all j -dimensional faces of the n -simplices $\sigma_1, \dots, \sigma_k$.

Lemma 3.4 *Let τ be a representing cycle of the fundamental class of M . The restriction chain homomorphism*

$$\text{hom}_{\mathbb{Z}}(C_*(M), \mathbb{Z}) \rightarrow \text{hom}_{\mathbb{Z}}(C_*^\tau(M), \mathbb{Z})$$

induces injective maps in cohomology in all degrees.

Proof Let $\tau = \sum_{i=1}^k a_i \cdot \sigma_i$. The chain homomorphism

$$_-\cap \tau: \text{hom}_{\mathbb{Z}}(C_{n-*}(M), \mathbb{Z}) \rightarrow C_*(M),$$

$$\phi \mapsto \sum_{i=1}^k a_i \cdot \phi(\sigma_i|_{[n-l]} \otimes \sigma_i|_l \quad \text{for } \phi \in \text{hom}_{\mathbb{Z}}(C_{n-l}(M), \mathbb{Z})$$

given by the cap product with τ is a homology isomorphism by Poincaré duality. Let ψ be a singular cocycle such that $[\psi]$ is in the kernel of the cohomology homomorphism induced by the restriction homomorphism. Upon subtracting a coboundary, we may assume that ψ vanishes on $C_j^\tau(M)$. By the formula above, $[\psi \cap \tau] = 0$. Hence $[\psi] = 0$, so injectivity follows. \square

Proof of Theorem 3.2 Let $\tau = \sum_{i=1}^k a_i \cdot \sigma_i$ with $\sigma_i \neq \sigma_j$ for $i \neq j$ be an integral singular cycle representing the fundamental class of M . The set of j -dimensional faces of the singular simplices $\sigma_1, \dots, \sigma_k$ is a \mathbb{Z} -basis of $C_j^\tau(M)$. This turns $C_*^\tau(M)$ into a based \mathbb{Z} -chain complex.

For every based finitely generated free \mathbb{Z} -module we consider the Hilbert space structure on its complexification for which the \mathbb{Z} -basis becomes a Hilbert basis. The norm of the image of every basis element under the differential

$$\mathbb{C} \otimes \partial_j: \mathbb{C} \otimes_{\mathbb{Z}} C_j^\tau(M) \rightarrow \mathbb{C} \otimes_{\mathbb{Z}} C_{j-1}^\tau(M)$$

is at most $(j + 1)$. Since an n -simplex has $\binom{n+1}{j}$ -many $(j-1)$ -dimensional faces, we have

$$\text{rk}_{\mathbb{Z}} H_{j-1}(C_*^\tau(M)) \leq \text{rk}_{\mathbb{Z}} C_{j-1}^\tau(M) \leq \binom{n+1}{j} \cdot k.$$

By Lemma 3.1 we get that

$$\log |\text{tors } H_{j-1}(C_*^\tau(M))| \leq \log |\text{tors coker}(\partial_j)| \leq \log(j + 1) \binom{n+1}{j} \cdot k.$$

The universal coefficient theorem implies

$$\log |\text{tors } H^j(\text{hom}_{\mathbb{Z}}(C_*^\tau(M), \mathbb{Z}))| \leq \log(j + 1) \binom{n+1}{j} \cdot k.$$

By Lemma 3.4, $H^j(M; \mathbb{Z})$ embeds into $H^j(\text{hom}_{\mathbb{Z}}(C_*^\tau(M), \mathbb{Z}))$, and this yields an embedding of the corresponding torsion subgroups. By Poincaré duality we conclude that

$$\log |\text{tors } H_j(M; \mathbb{Z})| = \log |\text{tors } H^{n-j}(M; \mathbb{Z})| \leq \log(n + 1) \cdot \binom{n+1}{j+1} \cdot k. \quad \square$$

3.2 The rank estimate

The corresponding result for Betti numbers is stated in [10, pages 301 and 307] and [15, Example 14.28] with a different constant. For convenience we provide a quick proof.

Theorem 3.5 *Let M be a closed n -dimensional oriented manifold. If the fundamental class of M is represented by an integral cycle with k singular n -simplices, then*

$$\dim_{\mathbb{F}_p} H_j(M; \mathbb{F}_p) \leq \binom{n+1}{j} \cdot k$$

for every $j \geq 0$.

Proof We retain the notation from the previous subsection. The estimate

$$\dim_{\mathbb{F}_p} H^j(\text{hom}_{\mathbb{Z}}(C_*^\tau(M), \mathbb{Z})) \leq \binom{n+1}{j+1} \cdot k$$

holds since $\dim_{\mathbb{F}_p} \text{hom}_{\mathbb{F}_p}(C_*^\tau(M), \mathbb{Z})$ satisfies the same upper bound. The \mathbb{F}_p -analog of Lemma 3.4 holds true by the same proof. Hence the group $H^j(M; \mathbb{Z})$ injects into $H^j(\text{hom}_{\mathbb{Z}}(C_*^\tau(M), \mathbb{Z}))$ and so

$$\dim_{\mathbb{F}_p} H^j(M; \mathbb{Z}) \leq \binom{n+1}{j+1} \cdot k.$$

The statement now follows from Poincaré duality. □

4 Proof of Theorem 1.1

The proof is based on the following result, which is implicitly contained in Guth's paper [11]; a weaker version, where one replaces the bound on the volume of 1-balls by the stronger assumption $\text{Ricci}(M) \geq -1$, can be extracted from Gromov's paper [8].

Theorem 4.1 *For every $n \in \mathbb{N}$ and $V_0 > 0$ there is a constant $C(n, V_0) > 0$ with the following property:*

If M is an n -dimensional connected oriented closed aspherical Riemannian manifold such that every 1-ball of M has volume at most V_0 and the systole of M is at least 1, then there is an integral cycle that represents the fundamental class of M and has at most $C(n, V_0)$ singular n -simplices.

The following result about the simplicial volume, that is, about the complexity of real-valued cycles representing the fundamental class, is explicitly contained in Guth's work.

Theorem 4.2 (Guth) *For every $n \in \mathbb{N}$ and $V_0 > 0$ there is a constant $C(n, V_0) > 0$ with the following property:*

If M is an n -dimensional connected oriented closed aspherical Riemannian manifold such that every 1-ball of M has volume at most V_0 and the systole of M is at least 1, then the simplicial volume is at most $C(n, V_0) \text{vol}(M)$.

Proof Just combine Lemmas 7 and 9 in [11]. □

Guth's proof of this theorem also implies Theorem 4.1, as we explain now. Guth considers a good cover of M in the sense of Gromov [11, Section 1] and introduces a modification of the usual nerve construction of a cover, which is particularly adapted to covers by balls with varying radii, called the *rectangular nerve* [11, Section 3]. He shows that, for the map f from M to the rectangular nerve associated to a suitable partition of unity, the n -volume of $f(M)$ is bounded by $C(n) \text{vol}(M)$ [11, Lemma 5] for some universal constant $C(n) > 0$. He concludes that the image of the fundamental class is homologous to a cycle in the n -skeleton whose number of singular n -simplices is bounded by $C'(n) \text{vol}(M)$ for some universal constant $C'(n) > 0$ [11, Proof of Lemma 9]. By asphericity the map f has a left homotopy inverse [11, Lemma 7], and this implies Theorem 4.1.

Finally, we deduce Theorem 1.1 from Theorem 4.1.

Proof of Theorem 1.1 Let M satisfy the assumptions in Theorem 1.1. According to Section 2.2 we may assume that M is oriented. Theorem 1.1 is now a consequence of Theorem 4.1 and Theorem 3.2 (for the statement about torsion) or Theorem 3.5 (for the statement about \mathbb{F}_p -dimension), respectively. □

5 Proof of Theorem 1.6

5.1 Setup

Throughout Section 5 we shall consider a connected closed n -dimensional oriented aspherical manifold M and adhere to the following notation.

- (1) \mathcal{U} denotes a cover of M of multiplicity $\leq n$ by open, amenable subsets.
- (2) $(\Gamma_i)_{i \in \mathbb{N}}$ is a residual chain of the fundamental group $\Gamma := \pi_1(M)$.
- (3) (X, μ) denotes the profinite topological group in (1-1) with the normalized Haar measure μ , and $\pi_i: X \rightarrow \Gamma/\Gamma_i$ is the canonical projection.

The sets $\pi_i^{-1}(\gamma\Gamma_i)$ with $i \in \mathbb{N}$ and $\gamma \in \Gamma$ define a basis \mathcal{O} of the topology of X . A subset of X is called *cylindrical* if it is a finite union of elements in \mathcal{O} . For every cylindrical set A there is a smallest number $i \in \mathbb{N}$ such that A is the π_i -preimage of a subset of Γ/Γ_i . We call this number the *level* of A and denote it by $l(A)$. Note that, for every $i \geq l(A)$,

$$(5-1) \quad \pi_i^{-1}(\pi_i(A)) = A.$$

We will not use the topological group structure on X and only regard X as a Γ -space with a Γ -invariant probability measure μ .

5.2 Constructing measurable covers

The topological space $X \times \tilde{M}$ is endowed with the diagonal Γ -action. We say that a Γ -invariant collection \mathcal{V} of subsets of $X \times \tilde{M}$ is a Γ -equivariant measurable cover if the following conditions are satisfied.

- (1) An element of \mathcal{V} is a product of a measurable subset of X and an open subset of \tilde{M} .
- (2) The union of elements of \mathcal{V} is $X \times \tilde{M}$.
- (3) The Γ -set \mathcal{V} has only finitely many Γ -orbits.

We construct such covers by appealing to the generalized Rokhlin lemma by Ornstein and Weiss, which we recall first.

A *monotile* T in a group Λ is a finite subset for which there is a subset C such that $\{T \cdot c \mid c \in C\}$ is a partition of Λ . For finite subsets $F, T \subset \Lambda$ we say that T is (F, δ) -invariant if the F -boundary $\partial T = \{\lambda \in T \mid \exists \gamma \in F \text{ with } \gamma\lambda \notin T\}$ satisfies

$$|\partial T|/|T| < \delta.$$

Weiss showed that residually finite amenable groups possess (F, δ) -invariant monotiles for arbitrarily small $\delta > 0$ [21]. The relevance of monotiles stems from the following result of Ornstein and Weiss.

Theorem 5.1 [17] *Let T be a monotile in an amenable group Λ . Let Λ act freely and ν -preservingly on a standard probability space (Y, ν) . For every $\epsilon > 0$ there is a measurable subset $A \subset Y$ satisfying:*

- (1) *For $\lambda, \lambda' \in T$ with $\lambda \neq \lambda'$ the sets λA and $\lambda' A$ are disjoint.*
- (2) *$\nu(\bigcup_{\lambda \in T} \lambda A) > 1 - \epsilon$.*

We may assume that the elements $U_i \subset M$ of $\mathcal{U} = \{U_1, \dots, U_m\}$ are connected. By taking the connected components we possibly increase m but we do not increase the multiplicity. Define

$$\Lambda_i := \text{im}(\pi_1(U_i) \rightarrow \pi_1(M)) \subset \Gamma.$$

This involves the choice of a base point in each U_i and paths from them to the base point of M , which is not relevant for our discussion. By assumption, each Λ_i is amenable. Since Γ is residually finite, each Λ_i is residually finite. Let \bar{U}_i be the regular covering of U_i associated to the kernel of the homomorphism $\pi_1(U_i) \rightarrow \pi_1(M)$. Let $\text{pr}: \tilde{M} \rightarrow M$ denote the universal covering projection. The group Λ_i acts on \bar{U}_i by deck transformations. By covering theory we may and will choose a lift of the map $\bar{U}_i \rightarrow U_i \rightarrow M$ to \tilde{M} . This lift yields a homeomorphism

$$\Gamma \times_{\Lambda_i} \bar{U}_i \cong \text{pr}^{-1}(U_i)$$

of coverings of U_i which allows us to regard \bar{U}_i as a subset of \tilde{M} . We choose an open subset $K_i \subset \bar{U}_i$ which is relatively compact in \tilde{M} such that $\bar{U}_i = \Lambda_i \cdot K_i$ for every $i \in \{1, \dots, m\}$. Properness of the Γ -action allows us to fix a finite subset $F \subset \Gamma$ with the property that

$$(5-2) \quad \gamma K_r \cap K_s \neq \emptyset \implies \gamma \in F$$

for all $r, s \in \{1, \dots, m\}$. Our following construction of Γ -equivariant measurable covers depends on a parameter $\delta > 0$ (and on many other choices but they are less relevant). Later in the proof we consider a sequence of Γ -equivariant measurable covers for $\delta \rightarrow 0$.

Let us fix a parameter $\delta > 0$. For every $i \in \{1, \dots, m\}$ we choose a monotile $T_i \subset \Lambda_i$ of the residually finite amenable group Λ_i that is $(F \cap \Lambda_i, \delta)$ -invariant. Since each $T_i^{-1} K_i \cup K_i$ is relatively compact, there is a finite subset $E \subset \Gamma$ with the property that

$$(5-3) \quad (T_r^{-1} K_r \cup K_r) \cap \gamma (T_s^{-1} K_s \cup K_s) \neq \emptyset \implies \gamma \in E$$

for all $r, s \in \{1, \dots, m\}$. Set

$$(5-4) \quad \epsilon := 2^{-(n+1)}|E|^{-(n+1)}m^{-1}\delta.$$

By [Theorem 5.1](#) there are measurable subsets $A_i \subset X$ such that, for fixed i , the sets λA_i , $\lambda \in T_i$, are disjoint and $R_i = X \setminus T_i A_i$ has μ -measure at most ϵ .

Definition 5.2

$$\begin{aligned} \mathcal{W}_\delta &:= \{\gamma A_i \times \gamma T_i^{-1} K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma\}, \\ \mathcal{V}_\delta &:= \{\gamma A_i \times \gamma T_i^{-1} K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma\} \cup \{\gamma R_i \times \gamma K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma\}, \\ \mathcal{W}_\delta^0 &:= \{A_i \times T_i^{-1} K_i \mid i \in \{1, \dots, m\}\}, \\ \mathcal{V}_\delta^0 &:= \{A_i \times T_i^{-1} K_i \mid i \in \{1, \dots, m\}\} \cup \{R_i \times K_i \mid i \in \{1, \dots, m\}\}. \end{aligned}$$

The subsets \mathcal{V}_δ^0 and \mathcal{W}_δ^0 are Γ -transversals of \mathcal{V}_δ and \mathcal{W}_δ , respectively. Keeping in mind that $\Gamma K_i = \tilde{M}$ one immediately verifies:

Lemma 5.3 \mathcal{V}_δ is a Γ -equivariant measurable cover.

Remark 5.4 In [\[18\]](#) we proved that the ℓ^2 -Betti numbers of M vanish under the assumptions of [Theorem 1.6](#); this turns out to be a corollary to [Theorem 1.6](#) provided the fundamental group is residually finite (cf [Remark 1.9](#)). In [\[18\]](#) we use a similar construction of measurable covers but we do not use the topological structure on X and the methods therein are not suited for the consideration of mod p or torsion homology.

5.3 Managing expectations

A j -tuple with pairwise distinct entries from a set Y is called a j -configuration in Y . Let $(s_1, \dots, s_j) \in \{1, \dots, m\}^j$. A j -configuration in \mathcal{V}^δ has type (s_1, \dots, s_j) if its l^{th} entry is $\gamma A_{s_l} \times \gamma T_{s_l}^{-1} K_{s_l}$ or $\gamma R_{s_l} \times \gamma K_{s_l}$ for some $\gamma \in \Gamma$. A j -configuration in \mathcal{W}^δ has type (s_1, \dots, s_j) if its l^{th} entry is $\gamma A_{s_l} \times \gamma T_{s_l}^{-1} K_{s_l}$ for some $\gamma \in \Gamma$.

Definition 5.5 Let $\text{mult}^{\mathcal{V}_\delta}(j): X \rightarrow \mathbb{N}$ be the random variable whose value at $x \in X$ is the number of j -configurations in \mathcal{V}_δ with the property that the common intersection of all sets in the configuration with the set $\{x\} \times \tilde{M}$ is non-empty and the first set of the configuration is from \mathcal{V}_δ^0 . Correspondingly, $\text{mult}^{\mathcal{W}_\delta}(j): X \rightarrow \mathbb{N}$ is defined. For $(s_1, \dots, s_j) \in \{1, \dots, m\}^j$ the random variables $\text{mult}^{\mathcal{V}_\delta}(j; s_1, \dots, s_j)$ and $\text{mult}^{\mathcal{W}_\delta}(j; s_1, \dots, s_j)$ are similarly defined but only j -configurations of type (s_1, \dots, s_j) are counted.

Lemma 5.6 For every type (s_1, \dots, s_j) , $2 \leq j \leq n + 1$, the expected values satisfy

$$\mathbb{E} \text{mult}^{\mathcal{V}\delta}(j; s_1, \dots, s_j) \leq \mathbb{E} \text{mult}^{\mathcal{W}\delta}(j; s_1, \dots, s_j) + \delta.$$

Proof A crude estimate yields that $\text{mult}^{\mathcal{V}\delta}(j; s_1, \dots, s_j) \leq (2|E|)^j$, where $E \subset \Gamma$ is the subset from (5-3). The support of $\text{mult}^{\mathcal{V}\delta}(j; s_1, \dots, s_j) - \text{mult}^{\mathcal{W}\delta}(j; s_1, \dots, s_j)$ is a subset of $E \cdot (R_1 \cup \dots \cup R_m)$. Hence the difference of the corresponding expected values is at most

$$\mu(E \cdot (R_1 \cup \dots \cup R_m))(2|E|)^j \leq 2^j |E|^{j+1} m\epsilon \stackrel{(5-4)}{\leq} \delta. \quad \square$$

Lemma 5.7 Let $2 \leq j \leq n + 1$. If j is greater than the multiplicity of \mathcal{U} , then

$$\lim_{\delta \rightarrow 0} \mathbb{E} \text{mult}^{\mathcal{V}\delta}(j; s_1, \dots, s_j) = \lim_{\delta \rightarrow 0} \mathbb{E} \text{mult}^{\mathcal{W}\delta}(j; s_1, \dots, s_j) = 0$$

for every type (s_1, \dots, s_j) .

Proof By Lemma 5.6 it suffices to show the second equality. The multiplicity of $\mathcal{U} = \{U_1, \dots, U_m\}$ is the same as the one of $\{\text{pr}^{-1}(U_1), \dots, \text{pr}^{-1}(U_m)\}$. Since $\Gamma K_i = \text{pr}^{-1}(U_i)$, $\text{mult}^{\mathcal{W}\delta}(j; s_1, \dots, s_j)$ is identically zero or the type (s_1, \dots, s_j) has two identical components. For the remaining proof we thus may assume that $s_1 = s_2$.

Fix $\delta > 0$. Let $C_l(x)$ be the l -configurations that contribute to $\text{mult}^{\mathcal{W}\delta}(l; s_1, \dots, s_l)(x)$ for $l \leq j$ and $x \in X$, so that $\text{mult}^{\mathcal{W}\delta}(l; s_1, \dots, s_l)(x) = |C_l(x)|$. Note that the first set of any l -configuration in $C_l(x)$ is $A_{s_1} \times T_{s_1}^{-1} K_{s_1}$. For $\lambda \in T_{s_1}^{-1}$ we define $C_l^\lambda(x)$ as the subset of $C_l(x)$ that consists of l -configurations with the additional property that the common intersection of all sets of the configuration with $A_{s_1} \times \lambda K_{s_1}$ is non-empty. Obviously, the union of all $C_l^\lambda(x)$ is $C_l(x)$ but it is not necessarily a disjoint union. At least we get that

$$\text{mult}^{\mathcal{W}\delta}(l; s_1, \dots, s_l)(x) = |C_l(x)| \leq \sum_{\lambda \in T_{s_1}^{-1}} |C_l^\lambda(x)|.$$

Let $3 \leq l \leq j$ and $\lambda \in T_{s_1}^{-1}$. Let us consider a fiber of the projection $C_l^\lambda(x) \rightarrow C_{l-1}^\lambda(x)$ that drops the last set of a configuration. The cardinality of any fiber is bounded by the number of elements $\gamma \in \Gamma$ such that $x \in \gamma A_{s_l}$ and $\lambda K_{s_1} \cap \gamma T_{s_l}^{-1} K_{s_l} \neq \emptyset$. By (5-2) the latter implies that $\gamma \in \lambda F T_{s_l}$. For every $\xi \in F$ the sets $\lambda \xi \theta A_{s_l}$, where θ runs through T_{s_l} , are pairwise disjoint, so the given x can only be in one of them. Thus any fiber has at most $|F|$ elements. We obtain inductively that

$$\sum_{\lambda \in T_{s_1}^{-1}} |C_j^\lambda(x)| \leq |F|^{j-2} \sum_{\lambda \in T_{s_1}^{-1}} |C_2^\lambda(x)|.$$

Let $s := s_1 = s_2$. Every 2-configuration in one of the sets $C_2^\lambda(x)$ has type (s, s) . Define

$$f(x) := \left| \{ (\lambda, \theta, \gamma) \in T_s^{-1} \times T_s^{-1} \times \Gamma \setminus \{e\} \mid x \in A_s \cap \gamma A_s \text{ and } \lambda K_s \cap \gamma \theta K_s \neq \emptyset \} \right|.$$

For every $x \in X$ we have

$$\sum_{\lambda \in T_s^{-1}} |C_2^\lambda(x)| \leq f(x).$$

Hence $\text{mult}^{\mathcal{V}_\delta}(j; s_1, \dots, s_j)$ is dominated by $|F|^{j-2} f$ and so

$$(5-5) \quad \mathbb{E} \text{mult}^{\mathcal{V}_\delta}(j; s_1, \dots, s_j) \leq |F|^{j-2} \mathbb{E} f.$$

We shall now use for the first time that the monotiles T_i are $(F \cap \Lambda_i, \delta)$ -invariant. Let $(\lambda, \theta, \gamma) \in T_s^{-1} \times T_s^{-1} \times \Gamma \setminus \{e\}$ be a triple such that $\lambda K_s \cap \gamma \theta K_s \neq \emptyset$. By (5-2) there is $\rho \in F$ with $\gamma = \lambda \rho \theta^{-1}$. Suppose that $\theta^{-1} \notin \partial T_s$. Then $\rho \theta^{-1} \in T_s$. Since $\gamma \neq e$ one has $\rho \theta^{-1} \neq \lambda^{-1}$, which yields $\lambda^{-1} A_s \cap \rho \theta^{-1} A_s = \emptyset$, thus $A_s \cap \gamma A_s = \emptyset$. Therefore,

$$\mathbb{E} f \leq \sum_{(\lambda, \theta, \rho)} \mu(A_s \cap \lambda \rho \theta^{-1} A_s) = \sum_{(\lambda, \theta, \rho)} \mu(\lambda^{-1} A_s \cap \rho \theta^{-1} A_s),$$

where the summation runs over all triples in $T_s^{-1} \times (\partial T_s)^{-1} \times F$. Since the sets $\lambda^{-1} A_s$, where λ runs through T_s^{-1} , are disjoint, we have

$$\sum_{(\lambda, \theta, \rho)} \mu(\lambda^{-1} A_s \cap \rho \theta^{-1} A_s) \leq \sum_{(\theta, \rho)} \mu(\rho \theta^{-1} A_s) = \sum_{(\theta, \rho)} \mu(A_s),$$

where (θ, ρ) runs over $(\partial T_s)^{-1} \times F$. Since $\mu(A_s) \leq 1/|T_s|$ we obtain that

$$\mathbb{E} f \leq |F| |\partial T_s| \mu(A_s) \leq |F| |\partial T_s| / |T_s| \leq |F| \delta.$$

With (5-5) the proof is completed. □

Theorem 5.8 *Let $2 \leq j \leq n + 1$. If j is greater than the multiplicity of \mathcal{U} , then $\lim_{\delta \rightarrow 0} \mathbb{E} \text{mult}^{\mathcal{V}_\delta}(j) = 0$.*

Proof The random variable $\text{mult}^{\mathcal{V}_\delta}(j)$ is the sum over all m^j possible types (s_1, \dots, s_j) of the random variables $\text{mult}^{\mathcal{V}_\delta}(j; s_1, \dots, s_j)$. Thus the statement follows from Lemma 5.7. □

5.4 From measurable to open covers on $X \times \tilde{M}$

Our next goal is to show that we can replace the measurable sets A_i and R_i in the definition of \mathcal{V}_δ by cylindrical (in particular, open) sets without losing the property stated in [Theorem 5.8](#).

Lemma 5.9 *Let $\delta > 0$, and let \mathcal{V}_δ be the measurable cover in [Definition 5.2](#). For every $\epsilon > 0$ there are cylindrical subsets A_i^c and R_i^c of X for every $i \in \{1, \dots, m\}$ such that:*

(1) *The collection*

$$\mathcal{V}_\delta^c := \{\gamma A_i^c \times \gamma T_i^{-1} K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma\} \cup \{\gamma R_i^c \times \gamma K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma\}$$

is a Γ -equivariant measurable cover.

(2) $\mathbb{E} \text{mult}^{\mathcal{V}_\delta^c}(k) < \mathbb{E} \text{mult}^{\mathcal{V}_\delta}(k) + \epsilon$ for $k \in \{1, \dots, n + 1\}$.

Proof Let $\epsilon_0 > 0$. Being the Haar measure on the compact topological group (1-1), μ is regular. Hence there are open subsets $A_i^o \supset A_i$ and $R_i^o \supset R_i$ in X for every $i \in \{1, \dots, m\}$ such that $\mu(A_i^o \setminus A_i) < \epsilon_0$ and $\mu(R_i^o \setminus R_i) < \epsilon_0$. For any $i \in \{1, \dots, m\}$ we can write A_i^o and R_i^o as increasing countable unions of cylindrical subsets:

$$A_i^o = \bigcup_{q \in \mathbb{N}} A_i^{(q)}, \quad R_i^o = \bigcup_{q \in \mathbb{N}} R_i^{(q)},$$

since \mathcal{O} is a countable subbasis of the topology of X . Let $\text{proj}: X \times \tilde{M} \rightarrow X \times_\Gamma \tilde{M}$ be the quotient map for the diagonal action. Let

$$S_q := \bigcup_{i \in \{1, \dots, m\}} \bigcup_{\gamma \in \Gamma} (\gamma A_i^{(q)} \times \gamma T_i^{-1} K_i \cup \gamma R_i^{(q)} \times \gamma K_i).$$

Since the sets in \mathcal{V}_δ cover $X \times \tilde{M}$, we have the following increasing unions:

$$X \times \tilde{M} = \bigcup_{q \in \mathbb{N}} S_q, \quad X \times_\Gamma \tilde{M} = \bigcup_{q \in \mathbb{N}} \text{proj}(S_q).$$

The orbit space $X \times_\Gamma \tilde{M}$ is compact since M and X are compact. Further, proj is an open map. By compactness there is $q_0 \in \mathbb{N}$ such that $X \times_\Gamma \tilde{M} = \text{proj}(S_q)$ for every $q \geq q_0$. Hence $X \times \tilde{M} = S_q$ for $q \geq q_0$ and

$$\mathcal{W}_q := \{\gamma A_i^{(q)} \times \gamma T_i^{-1} K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma\} \cup \{\gamma R_i^{(q)} \times \gamma K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma\}$$

is a Γ -equivariant measurable cover for $q \geq q_0$. We claim that

$$(5-6) \quad \mathbb{E} \text{mult}^{\mathcal{W}_q}(k) < \mathbb{E} \text{mult}^{\mathcal{V}_\delta}(k) + \epsilon$$

for all $k \in \{1, \dots, n + 1\}$ provided $\epsilon_0 > 0$ is sufficiently small and $q \geq q_0$ is sufficiently large. For such ϵ_0 and $q \geq q_0$ we then set $A_i^c := A_i^{(q)}$ and $R_i^c := R_i^{(q)}$, which finishes the proof.

To show (5-6) it suffices to verify that for a fixed k -configuration (s_1, \dots, s_k) we have

$$(5-7) \quad \mathbb{E} \text{mult}^{\mathcal{V}_q}(k; s_1, \dots, s_k) < \mathbb{E} \text{mult}^{\mathcal{V}_\delta}(k; s_1, \dots, s_k) + \epsilon$$

provided $\epsilon_0 > 0$ is sufficiently small and $q \geq q_0$ is sufficiently large. To this end, we rewrite the expected values using Fubini’s theorem. To formulate the result, we set

$$B_i(b) := \begin{cases} A_i & \text{if } b = 0, \\ R_i & \text{if } b = 1, \end{cases} \quad \text{and} \quad C_i(b) := \begin{cases} T_i^{-1} K_i & \text{if } b = 0, \\ K_i & \text{if } b = 1. \end{cases}$$

Similarly, we define $B_i^{(q)}(b)$ with A_i and R_i being replaced by $A_i^{(q)}$ and $R_i^{(q)}$, respectively. By Fubini’s theorem we have

$$\mathbb{E} \text{mult}^{\mathcal{V}_\delta}(k; s_1, \dots, s_k) = \sum_{\substack{\gamma_2, \dots, \gamma_k \\ b_1, \dots, b_k}} \mu(B_{s_1}(b_1) \cap \gamma_2 B_{s_2}(b_2) \cap \dots \cap \gamma_k B_{s_k}(b_k)),$$

where the sum runs through $\gamma_2, \dots, \gamma_k \in \Gamma$ and $b_1, \dots, b_k \in \{0, 1\}$ with the property that

$$C_{s_1}(b_1) \cap \gamma_2 C_{s_2}(b_2) \cap \dots \cap \gamma_k C_{s_k}(b_k) \neq \emptyset.$$

For $\mathbb{E} \text{mult}^{\mathcal{V}_q}(k; s_1, \dots, s_k)$ we have a similar expression. Letting $\epsilon_0 \rightarrow 0$ and $q \rightarrow \infty$, each summand

$$\mu(B_1^{(q)}(b_1) \cap \gamma_2 B_2^{(q)}(b_2) \cap \dots \cap \gamma_k B_k^{(q)}(b_k))$$

tends to

$$\mu(B_1(b_1) \cap \gamma_2 B_2(b_2) \cap \dots \cap \gamma_k B_k(b_k)).$$

This yields (5-7) and thus (5-6). □

5.5 The passage to open covers on $\Gamma/\Gamma_q \times \tilde{M}$

Next we describe how we produce from a Γ -equivariant measurable cover on $X \times \tilde{M}$ a sequence of Γ -equivariant open covers on $\Gamma/\Gamma_q \times \tilde{M}$, indexed by $q \in \mathbb{N}$, which are compatible with respect to the natural projections $\Gamma/\Gamma_q \times \tilde{M} \rightarrow \Gamma/\Gamma_{q-1} \times \tilde{M}$.

Let us fix $\delta > 0$. By Lemma 5.9 one can replace the sets A_i and R_i in \mathcal{V}_δ by cylindrical sets A_i^c and R_i^c such that

$$\mathcal{V}_\delta^c = \{\gamma A_i^c \times \gamma T_i^{-1} K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma\} \cup \{\gamma R_i^c \times \gamma K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma\}$$

is a Γ -equivariant measurable cover with

$$(5-8) \quad \mathbb{E} \text{mult}^{\mathcal{V}_\delta^c}(n+1) < 2 \cdot \mathbb{E} \text{mult}^{\mathcal{V}_\delta}(n+1).$$

From \mathcal{V}_δ^c we obtain the Γ -equivariant open cover of $\Gamma/\Gamma_q \times \tilde{M}$:

$$(5-9) \quad \mathcal{V}_\delta(q) := \{ \{ \pi_q(\gamma x) \} \times \gamma T_i^{-1} K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma, x \in A_i^c \} \\ \cup \{ \{ \pi_q(\gamma x) \} \times \gamma K_i \mid i \in \{1, \dots, m\}, \gamma \in \Gamma, x \in R_i^c \}.$$

We define the function $\text{mult}^{\mathcal{V}_\delta(q)}(j): \Gamma/\Gamma_q \rightarrow \mathbb{N}$ similarly to $\text{mult}^{\mathcal{V}_\delta^c}(j)$. The value $\text{mult}^{\mathcal{V}_\delta(q)}(j)(y)$ at $y \in \Gamma/\Gamma_q$ is the number of j -configurations in $\mathcal{V}_\delta(q)$ with the property that the common intersection of all sets in the configuration with $\{y\} \times \tilde{M}$ is non-empty and the first set of the configuration is $\{y\} \times T_i^{-1} K_i$ or $\{y\} \times K_i$ for some $i \in \{1, \dots, m\}$. We regard this function as a random variable on Γ/Γ_i endowed with the equidistributed probability measure. Let

$$(5-10) \quad q_0 := \max \{ l(A_i^c), l(R_i^c) \mid i = 1, \dots, m \}$$

be the maximal level of the cylindrical sets appearing in \mathcal{V}_δ^c .

Lemma 5.10 For every $q \geq q_0$ and $j \in \mathbb{N}$ one has $\mathbb{E} \text{mult}^{\mathcal{V}_\delta^c}(j) = \mathbb{E} \text{mult}^{\mathcal{V}_\delta(q)}(j)$.

Proof For $x \in X$ and $y \in \Gamma/\Gamma_q$ let $C_j(x) \subset (\mathcal{V}_\delta^c)^j$ and $\tilde{C}_j(y) \subset \mathcal{V}_\delta(q)^j$ be the sets of j -configurations that contribute to $\text{mult}^{\mathcal{V}_\delta^c}(j)(x)$ and $\text{mult}^{\mathcal{V}_\delta(q)}(j)(y)$, respectively. So

$$\text{mult}^{\mathcal{V}_\delta^c}(j)(x) = |C(x)| \quad \text{and} \quad \text{mult}^{\mathcal{V}_\delta(q)}(j)(y) = |\tilde{C}(y)|.$$

The map $\phi(x): C_j(x) \rightarrow \tilde{C}_j(\pi_q(x))$ is defined component-wise: a j -configuration whose i^{th} set is $A \times U$ (thus $x \in A$) is sent to the j -configuration whose i^{th} set is $\{ \pi_q(x) \} \times U$. Injectivity of $\phi(x)$ is clear. Surjectivity is implied by the equivalences (cf (5-1))

$$\pi_q(x) \in \pi_q(A_i^c) \iff x \in A_i^c \quad \text{and} \quad \pi_q(x) \in \pi_q(R_i^c) \iff x \in R_i^c$$

provided q is at least the level of A_i^c and R_i^c . Hence

$$\text{mult}^{\mathcal{V}_\delta^c}(j)(x) = \text{mult}^{\mathcal{V}_\delta(q)}(j)(\pi_q(x))$$

for every $x \in X$. That the pushforward of the measure μ under the map π_q is the equidistributed probability measure on Γ/Γ_q finishes the proof. \square

5.6 Conclusion of the proof of Theorem 1.6

The reader is referred to [12, Chapter 2.1] for a discussion of the upcoming notion of Δ -complex. Its historical name is *semi-simplicial complex*. In brief, a Δ -complex is like a simplicial complex where one drops the requirement that a simplex is uniquely determined by its vertices. A 1-simplex in a Δ -complex, for instance, might be a loop.

Lemma 5.11 *For every $\delta > 0$ there is $q_0 \in \mathbb{N}$ with the following property. For every $q \geq q_0$ there is a Δ -complex $S(q)$ such that*

- (1) $S(q)$ has at most $2[\Gamma : \Gamma_q] \mathbb{E} \text{mult}^{\mathcal{V}_\delta}(n + 1)$ many n -simplices, and
- (2) there is a homotopy retract

$$\Gamma_q \backslash \tilde{M} \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} S(q) \quad \text{with } g \circ f \simeq \text{id}_{\Gamma_q \backslash \tilde{M}}.$$

Proof of Lemma 5.11 Let $\delta > 0$. We consider the Γ -equivariant measurable cover \mathcal{V}_δ^c from the previous subsection which we obtained by an application of Lemma 5.9 to \mathcal{V}_δ . It satisfies (5-8). Let q_0 be defined as in (5-10), as the maximal level of cylindrical subsets occurring in \mathcal{V}_δ^c . For every $q \in \mathbb{N}$ we obtain a Γ -equivariant cover $\mathcal{V}_\delta(q)$ of $\Gamma / \Gamma_q \times \tilde{M}$ from \mathcal{V}_δ^c by (5-9). By (5-8) and Lemma 5.10 we have

$$(5-11) \quad \mathbb{E} \text{mult}^{\mathcal{V}_\delta(q)}(n + 1) < 2 \mathbb{E} \text{mult}^{\mathcal{V}_\delta}(n + 1) \quad \text{for } q \geq q_0.$$

Let $q \geq q_0$, and let $N(q)$ be the nerve of $\mathcal{V}_\delta(q)$. Recall that the nerve of a cover is the simplicial complex whose vertices correspond to the subsets of the cover such that $(k + 1)$ subsets span a k -simplex if they have a non-empty intersection. The Γ -action on $\mathcal{V}_\delta(q)$ induces a simplicial Γ -action on $N(q)$. Since the sets in $\mathcal{V}_\delta(q)$ are relatively compact, the Γ -action on \tilde{M} is proper, and Γ is torsion-free, it follows that the Γ -action on $N(q)$ is free. Moreover, if a simplex is invariant under some $\gamma \in \Gamma$ as a set, then $\gamma = e$. Thus $N(q)$ is a free Γ -CW-complex [20, Proposition 1.15, page 101]. A free Γ -CW-complex is a Hausdorff space with a Γ -action that is built inductively by attaching equivariant cells $\Gamma \times D^i$ via equivariant attaching maps (see [20, page 98]). This will enable us below to construct equivariant maps with domain $N(q)$ by induction over skeleta. Let

$$S(q) := \Gamma \backslash N(q).$$

Whilst Γ acts simplicially on the simplicial complex $N(q)$, the simplicial structure of $N(q)$ does not necessarily induce a simplicial structure on $S(q)$. For instance, one might have a 1-simplex between a vertex v and a vertex γv for $\gamma \in \Gamma$ which yields a

loop in the quotient $S(q)$. But the simplicial structure on $N(q)$ induces the structure of a Δ -complex on $S(q)$. The number of (ordered) n -simplices of $S(q)$ is the number of $(n + 1)$ -configurations of sets in $\mathcal{V}_\delta(q)$ such that their intersection is non-empty and the first set of the configuration lies in a Γ -transversal of $\mathcal{V}_\delta(q)$, say, in

$$\{\{\pi_q(x)\} \times T_i^{-1} K_i \mid x \in A_i^c, i \in \{1, \dots, m\}\} \cup \{\{\pi_q(x)\} \times K_i \mid x \in R_i^c, i \in \{1, \dots, m\}\}.$$

On the other hand, $\text{mult}^{\mathcal{V}_\delta(q)}(n + 1)(y)$ with $y \in \Gamma / \Gamma_q$ is the number of $(n + 1)$ -configurations of sets in $\mathcal{V}_\delta(q)$ such that their intersection is non-empty and lies in the component $\{y\} \times \tilde{M}$ and the first set of the configuration lies in the above Γ -transversal. The probability measure on Γ / Γ_q is the normalized counting measure. Hence the number of n -simplices in $S(q)$ is bounded by $[\Gamma : \Gamma_q] \mathbb{E} \text{mult}^{\mathcal{V}_\delta(q)}(n + 1)$ and with (5-11) the first statement follows.

Finally, we construct equivariant maps

$$\tilde{f}: \Gamma / \Gamma_q \times \tilde{M} \rightarrow N(q) \quad \text{and} \quad \tilde{g}: N(q) \rightarrow \Gamma / \Gamma_q \times \tilde{M}$$

and an equivariant homotopy $\tilde{g} \circ \tilde{f} \simeq \text{id}$. The existence of the map \tilde{g} and the equivariant homotopy are ultimately a consequence of the general fact that \tilde{M} as a model of the classifying space $E\Gamma$ is a terminal object in the homotopy category of free Γ -CW complexes.

The maps f and g and the homotopy $g \circ f \simeq \text{id}$ in the statement of the lemma will be the induced maps on orbit spaces. This will finish the proof, since

$$\Gamma \backslash (\Gamma / \Gamma_q \times \tilde{M}) \cong \Gamma_q \backslash \tilde{M}.$$

By choosing an equivariant partition of unity subordinate to $\mathcal{V}_\delta(q)$ one obtains an equivariant map $\tilde{f}: \Gamma / \Gamma_q \times \tilde{M} \rightarrow N(q)$, called the nerve map. See [4, page 133] for a construction of the nerve map.

We construct the map \tilde{g} by an induction over the skeleta of $N(q)$. To this end, one chooses for every set V in $\mathcal{V}_\delta(q)$ a point $m_V \in V$ in an equivariant way. We define \tilde{g} on the 0-skeleton by mapping the vertex associated to V to m_V . The i -skeleton $N(q)^{(i)}$ of $N(q)$, $i \geq 1$, is built from the $(i - 1)$ -skeleton by attaching equivariant i -cells $\Gamma \times D^i$ along equivariant attaching maps from $\Gamma \times S^{i-1}$ to the $(i - 1)$ -skeleton. First let $i = 1$. If two subsets of $\mathcal{V}_\delta(q)$ intersect, they lie in the same path component of $\Gamma / \Gamma_q \times \tilde{M}$. Thus, if $\phi: \Gamma \times S^0 \rightarrow N(q)$ is the attaching map of an equivariant 1-cell, then

$$\{e\} \times S^0 \xrightarrow{\phi|_{\{e\} \times S^0}} N(q)^{(0)} \xrightarrow{\tilde{g}} \Gamma / \Gamma_q \times \tilde{M}$$

can be extended to $\{e\} \times D^1$, and the latter has a unique equivariant extension to $\Gamma \times D^1$. Next let $i \geq 2$. Since the homotopy group $\pi_{i-1}(\Gamma/\Gamma_q \times \tilde{M})$ vanishes with respect to arbitrary base points, any composition

$$\{e\} \times S^{i-1} \rightarrow N(q)^{(i-1)} \xrightarrow{\tilde{g}} \Gamma/\Gamma_q \times \tilde{M}$$

can be extended to $\{e\} \times D^i$, and, as before, the latter has a unique equivariant extension to $\Gamma \times D^i$.

Similarly, the equivariant homotopy between $\tilde{g} \circ \tilde{f}$ and id is constructed by an induction over the skeleta using that $\tilde{f} \circ \tilde{g}(z)$ and z lie in the same path component for every $z \in \Gamma/\Gamma_q \times \tilde{M}$ and the vanishing of homotopy groups of $\Gamma/\Gamma_q \times \tilde{M}$ in degrees ≥ 1 . \square

End of the proof of Theorem 1.6 According to Section 2.2 we may and will assume that M is oriented. Let $\epsilon > 0$. By Theorem 5.8 there is $\delta > 0$ such that

$$\mathbb{E} \text{mult}^{\nu_\delta}(n+1) < \frac{\epsilon}{2}.$$

For this δ we take q_0 and f, g and $S(q)$ as in the preceding lemma. For $q \geq q_0$, the number of n -simplices of $S(q)$ is at most $[\Gamma : \Gamma_q]\epsilon$. By the isomorphism between simplicial and singular homology for Δ -complexes [12, Theorem 2.27, page 128] the homology class $H_n(f)([\Gamma_q \backslash \tilde{M}])$ has a representative that is an integral linear combination of at most $[\Gamma : \Gamma_q]\epsilon$ -many singular n -simplices. Hence the fundamental class

$$[\Gamma_q \backslash \tilde{M}] = H_n(g) \circ H_n(f)([\Gamma_q \backslash \tilde{M}])$$

of $\Gamma_q \backslash \tilde{M}$ can also be written as an integral linear combination of at most $[\Gamma : \Gamma_q]\epsilon$ -many singular n -simplices. Since $\epsilon > 0$ was arbitrary, Theorem 1.6 finally follows from Theorems 3.2 and 3.5. \square

References

- [1] M Abért, N Nikolov, *Rank gradient, cost of groups and the rank versus Heegaard genus problem*, J. Eur. Math. Soc. (JEMS) 14 (2012) 1657–1677 MR2966663
- [2] W Ballmann, M Gromov, V Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics 61, Birkhäuser, Boston (1985) MR823981
- [3] N Bergeron, A Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, J. Inst. Math. Jussieu 12 (2013) 391–447 MR3028790
- [4] G E Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics 46, Academic Press, New York (1972) MR0413144

- [5] **C W Curtis, I Reiner**, *Methods of representation theory, I*, Wiley, New York (1981) [MR632548](#)
- [6] **V Emery**, *Torsion homology of arithmetic lattices and K_2 of imaginary fields*, *Math. Z.* 277 (2014) 1155–1164 [MR3229985](#)
- [7] **T Glander**, *Homotopy type and volume of locally symmetric manifolds*, *Duke Math. J.* 124 (2004) 459–515 [MR2084613](#)
- [8] **M Gromov**, *Volume and bounded cohomology*, *Inst. Hautes Études Sci. Publ. Math.* (1982) 5–99 [MR686042](#)
- [9] **M Gromov**, *Large Riemannian manifolds*, from: “Curvature and topology of Riemannian manifolds”, (K Shiohama, T Sakai, T Sunada, editors), *Lecture Notes in Math.* 1201, Springer, Berlin (1986) 108–121 [MR859578](#)
- [10] **M Gromov**, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser, Boston (2007) [MR2307192](#)
- [11] **L Guth**, *Volumes of balls in large Riemannian manifolds*, *Ann. of Math.* 173 (2011) 51–76 [MR2753599](#)
- [12] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) [MR1867354](#)
- [13] **M Lackenby**, *Large groups, property (τ) and the homology growth of subgroups*, *Math. Proc. Cambridge Philos. Soc.* 146 (2009) 625–648 [MR2496348](#)
- [14] **M Lackenby**, *Detecting large groups*, *J. Algebra* 324 (2010) 2636–2657 [MR2725193](#)
- [15] **W Lück**, *L^2 -invariants: theory and applications to geometry and K -theory*, *Ergeb. Math. Grenzgeb.* 44, Springer, Berlin (2002) [MR1926649](#)
- [16] **W Lück**, *Approximating L^2 -invariants and homology growth*, *Geom. Funct. Anal.* 23 (2013) 622–663 [MR3053758](#)
- [17] **D S Ornstein, B Weiss**, *Ergodic theory of amenable group actions, I: The Rohlin lemma*, *Bull. Amer. Math. Soc.* 2 (1980) 161–164 [MR551753](#)
- [18] **R Sauer**, *Amenable covers, volume and L^2 -Betti numbers of aspherical manifolds*, *J. Reine Angew. Math.* 636 (2009) 47–92 [MR2572246](#)
- [19] **C Soulé**, *Perfect forms and the Vandiver conjecture*, *J. Reine Angew. Math.* 517 (1999) 209–221 [MR1728540](#)
- [20] **T tom Dieck**, *Transformation groups*, de Gruyter Studies in Mathematics 8, de Gruyter, Berlin (1987) [MR889050](#)
- [21] **B Weiss**, *Monotileable amenable groups*, from: “Topology, ergodic theory, real algebraic geometry”, (V Turaev, A Vershik, editors), *Amer. Math. Soc. Transl. Ser. 2* 202, Amer. Math. Soc. (2001) 257–262 [MR1819193](#)

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