# Non-negative Legendrian isotopy in $ST^*M$

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It is shown that if the universal cover of a manifold $M$ is an open manifold, then two different fibres of the spherical cotangent bundle $ST^*M$ cannot be connected by a non-negative Legendrian isotopy. This result is applied to the study of causality in globally hyperbolic spacetimes. It is also used to strengthen a result of Eliashberg, Kim and Polterovich on the existence of a partial order on $\widetilde{\mathrm{Cont}}_0(ST^*M)$.

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## 1 Introduction

Let $M$ be a connected not necessarily orientable manifold of dimension $m \geq 2$ and let $\pi_M\colon ST^*M \to M$ be its spherical cotangent bundle. It is well-known that $ST^*M$ carries a canonical co-oriented contact structure. An isotopy $\{L_t\}_{t\in[0,1]}$ of Legendrian submanifolds in a co-oriented contact manifold is called *non-negative* if it can be parameterised in such a way that the tangent vectors of the trajectories of individual points lie in the non-negative tangent half-spaces defined by the contact structure, see Definition 2.1.

**Theorem 1.1** *Assume that the universal cover of $M$ is an open manifold. Then there does not exist a non-negative Legendrian isotopy connecting two different (nonoriented) fibres of $ST^*M$.*

In the special case when $M$ can be covered by an open subset of $\mathbb{R}^m$, this statement was proved by Colin, Ferrand and Pushkar [8]. Independently, a slightly stronger result was obtained by the present authors in the course of the proof of the so-called Legendrian Low Conjecture from Lorentz geometry [6]. Theorem 1.1 allows us to extend the results of [6] to a wider class of Lorentz manifolds, see Section 10.

A closely related notion of non-negative contact isotopy plays a key role in the orderability problem for contactomorphism groups, see the work of Eliashberg, Kim and Polterovich [11; 10] and Bhupal [4]. Theorem 1.1 can be applied to settle a question left open in [10], see Section 9.

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It is easy to see that the assertion of Theorem 1.1 is false if $M$ carries a Riemannian metric turning it into a $Y_\ell^x$–manifold, that is, such that all unit speed geodesics starting from a point $x \in M$ return to $x$ in time $\ell > 0$, see Example 8.3. In particular, it is false if $M$ is a metric quotient of the standard sphere. Hence, the hypothesis of Theorem 1.1 cannot be weakened for surfaces (this is obvious) and 3–manifolds (this follows from Perelman's work on the Poincaré conjecture [19; 20; 21]). On the other hand, it seems very likely that there exist simply connected compact manifolds of higher dimension which obey the conclusion of Theorem 1.1.

The proof of Theorem 1.1 is based on Viterbo's invariants of generating functions [22]. However, it is different from the arguments in [8] and [6] already for $M = \mathbb{R}^m$. In particular, no use is made of the identification $ST^*\mathbb{R}^m \cong \mathcal{J}^1(S^{m-1})$.

All manifolds, maps etc. are assumed to be smooth unless the opposite is explicitly stated, and the word *smooth* means $C^\infty$.

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## 2 Non-negative Legendrian isotopies

Let $(Y, \ker\alpha)$ be a contact manifold with a co-oriented contact structure defined by a contact form $\alpha$.

**Definition 2.1** A Legendrian isotopy $\{L_t\}_{t\in[0,1]}$ in $(Y, \ker\alpha)$ is called *non-negative* if it has a parameterisation $F\colon L_0 \times [0,1] \to Y$ such that $(F^*\alpha)\left(\frac{\partial}{\partial t}\right) \geq 0$. If the latter inequality is strict, the isotopy is called *positive*.

Clearly, this definition does not depend on the choice of the parameterisation $F$ of the Legendrian isotopy and on the choice of the contact form defining the co-oriented contact structure. It is also obvious that (co-orientation preserving) contactomorphisms preserve the property of being non-negative or positive.

**Lemma 2.2** *Let $\{L_t\}_{t\in[0,1]}$ be a non-negative Legendrian isotopy of compact submanifolds such that $L_0 \cap L_1 = \varnothing$. Then there exists a ($C^\infty$–close) positive Legendrian isotopy with the same ends.*

**Proof** Let $L'_t = \psi_{\epsilon t}(L_t)$, where $\psi_t$ is the Reeb flow on $(Y,\alpha)$ and $\epsilon > 0$. The isotopy $\{L'_t\}$ is positive. If $\epsilon$ is small enough, then $L'_1$ is Legendrian isotopic to $L_1$ in $Y \setminus L_0$. By the Legendrian isotopy extension theorem (see Geiges [13, Theorem 2.6.2]) there exists a contactomorphism $\phi$ supported in $Y \setminus L_0$ such that $\phi(L'_1) = L_1$. The image $L''_t := \phi(L'_t)$ of the positive isotopy $\{L'_t\}$ is a positive Legendrian isotopy connecting $L_0$ and $L_1$. □

An advantage of positive isotopies is that positivity is an open condition and hence one can make a positive Legendrian isotopy generic by a small perturbation.

**Definition 2.3** An isotopy $\{L_t\}_{t\in[0,1]}$ is in general position with respect to a submanifold $\Lambda \subset Y$ of codimension $\dim_{\mathbb{R}} L_0$ if it has a parameterisation $F\colon L_0 \times [0,1] \to Y$ such that

(a) $F^{-1}(\Lambda)$ is a 1–dimensional submanifold in $L_0 \times [0,1]$;

(b) the projection $F^{-1}(\Lambda) \to [0,1]$ has isolated critical points;

(c) $F^{-1}(\Lambda)$ is transverse to $L_0 \times \{0\}$ and $L_0 \times \{1\}$.

Note that a point $(x,\tau) \in F^{-1}(\Lambda)$ which is not critical for the projection $F^{-1}(\Lambda) \to [0,1]$ lies on the graph of a section of this projection over a non-trivial closed interval $[t',t''] \ni \tau$. In other words, there exists a curve $\gamma\colon [t',t''] \to L_0$ such that $\gamma(\tau) = x$ and $F(\gamma(t),t) \in \Lambda$ for all $t \in [t',t'']$.

## 3 Exact pre-Lagrangian submanifolds

Suppose that $\Lambda$ is an $m$–dimensional submanifold of a $(2m-1)$–dimensional contact manifold $(Y, \ker\alpha)$ such that

$$df = e^h \alpha|_\Lambda \tag{3.1}$$

for some functions $f,h\colon \Lambda \to \mathbb{R}$. Then $\Lambda$ is said to be *exact pre-Lagrangian* and $f$ is called a *contact potential* on $\Lambda$. (The terminology will be explained in Section 4.)

The following lemma shows that the contact potential is non-decreasing along any curve traced on $\Lambda$ by a non-negative Legendrian isotopy.

**Lemma 3.1** *Let $L_t = \phi_t(L_0)$, $t \in [0,1]$, be a non-negative Legendrian isotopy. Suppose that $\gamma\colon [0,1] \to L_0$ is a curve such that $\phi_t(\gamma(t)) \in \Lambda$, where $\Lambda$ is an exact pre-Lagrangian submanifold with contact potential $f$. Then the function $t \mapsto f(\phi_t(\gamma(t)))$ is non-decreasing.*

**Proof** This follows from the definitions and the chain rule. Indeed,

$$\begin{aligned}\frac{d}{dt} f(\phi_t(\gamma(t))) &= df\Big(\frac{d\phi_t}{dt}(\gamma(t)) + d\phi_t(\gamma(t))\,\frac{d\gamma}{dt}(t)\Big)\\ &= e^h\Big[\alpha\Big(\frac{d\phi_t}{dt}(\gamma(t))\Big) + \alpha\Big(d\phi_t(\gamma(t))\,\frac{d\gamma}{dt}(t)\Big)\Big].\end{aligned}$$

The first summand in square brackets is non-negative by the definition of non-negative Legendrian isotopy and the second one is zero because the isotopy is Legendrian and $\frac{d\gamma}{dt}$ is tangent to $L_0$. Hence, the derivative of our function is non-negative. □

## 4 Symplectisation

Let $(Y, \ker\alpha)$ be a contact manifold. Its symplectisation $Y^{\text{symp}}$ is the (exact) symplectic manifold $(Y \times \mathbb{R}, d(e^s\alpha))$.

**Example 4.1** Let $Y \subset T^*M$ be the unit sphere bundle with respect to a Riemannian metric on $M$. Then $\alpha := \lambda_{\text{can}}|_Y$ is a contact form defining the canonical contact structure on $Y \cong ST^*M$. The map

$$Y^{\text{symp}} \ni (\xi, s) \longmapsto e^s\xi \in T^*M$$

is a symplectomorphism onto the complement of the zero section of $T^*M$ such that the pull-back of the canonical 1–form $\lambda_{\text{can}}$ is precisely $e^s\alpha$.

A contactomorphism $\phi\colon Y \to Y$ lifts to a symplectomorphism $\widetilde{\phi}\colon Y^{\text{symp}} \to Y^{\text{symp}}$ defined by the formula

$$\widetilde{\phi}(x, s) := (\phi(x), s - \eta(x)),$$

where $\eta\colon Y \to \mathbb{R}$ is the function such that $\phi^*\alpha = e^\eta\alpha$. It follows from this definition that if $\{\phi_t\}_{t\in[0,1]}$ is a contact isotopy of $Y$, then $\{\widetilde{\phi}_t\}_{t\in[0,1]}$ is a Hamiltonian isotopy of $Y^{\text{symp}}$, see Eliashberg and Polterovich [11, Section 1.2].

An exact pre-Lagrangian submanifold $\Lambda$ with a contact potential $f$ such that $df = e^h\alpha|_\Lambda$ lifts to an exact Lagrangian submanifold

$$\widetilde{\Lambda} = \{(x, h(x)) \in Y^{\text{symp}} \mid x \in \Lambda\} \subset Y^{\text{symp}}.$$

Indeed, the function $\widetilde{f}\colon \widetilde{\Lambda} \to \mathbb{R}$, $\widetilde{f}(x, h(x)) = f(x)$, is a primitive for the 1–form $e^s\alpha|_{\widetilde{\Lambda}}$. Note that for any contactomorphism $\phi\colon Y \to Y$, the image $\phi(\Lambda)$ with the contact potential $f \circ \phi^{-1}$ is exact pre-Lagrangian and $\widetilde{\phi(\Lambda)} = \widetilde{\phi}(\widetilde{\Lambda})$.

## 5 Generating functions

Let $M$ be a manifold, which may be open or have boundary. Consider the product $M \times \mathbb{R}^N$ for some $N \geq 0$ and let $\pi\colon M \times \mathbb{R}^N \to M$ be the projection onto $M$. For a function $\mathcal{S}\colon M \times \mathbb{R}^N \to \mathbb{R}$, consider the set of its fibre critical points

$$\mathrm{FCrit}(\mathcal{S}) := \{z \in M \times \mathbb{R}^N \mid d\mathcal{S}(z)|_{\mathbb{R}^N} = 0\}.$$

Note that there is a natural fibrewise map

$$d_M\mathcal{S}\colon \mathrm{FCrit}(\mathcal{S}) \to T^*M$$

which associates to a point $z \in \mathrm{FCrit}(\mathcal{S})$ the linear form $v \mapsto d\mathcal{S}(z)(\widehat{v})$ on $T_{\pi(z)}M$, where $\widehat{v} \in T_z(M \times \mathbb{R}^N)$ is any tangent vector such that $d\pi(\widehat{v}) = v \in T_{\pi(z)}M$.

A function $\mathcal{S}\colon M \times \mathbb{R}^N \to \mathbb{R}$ is called a *generating function* for a Lagrangian submanifold $L \subset T^*M$ if it satisfies the following two conditions:

(GF1) its set of fibre critical points is cut out transversely;

(GF2) the map $d_M\mathcal{S}\colon \mathrm{FCrit}(\mathcal{S}) \to T^*M$ is a diffeomorphism onto $L$.

Note that $\mathcal{S} \circ (d_M\mathcal{S})^{-1}\colon L \to \mathbb{R}$ is a primitive for $\lambda_{\mathrm{can}}|_L$. Hence, a Lagrangian submanifold of $T^*M$ admitting a generating function is exact.

A generating function $\mathcal{S}\colon M \times \mathbb{R}^N \to \mathbb{R}$ is called *quadratic at infinity* if furthermore

(GF3) $\mathcal{S}(y, \xi) = \sigma(y, \xi) + Q(\xi)$, where $Q$ is a non-degenerate quadratic form on $\mathbb{R}^N$ and the projection $\pi\colon \operatorname{supp} \sigma \to M$ is a proper map.

Note that a Lagrangian submanifold $L \subset T^*M$ admitting a quadratic at infinity generating function is *properly embedded*, that is, the projection $L \to M$ is a proper map.

**Proposition 5.1** *Let $\{L_t\}_{t\in[0,1]}$ be a compactly supported isotopy of properly embedded Lagrangian submanifolds in $T^*M$. Suppose that*

(a) *$L_0$ admits a quadratic at infinity generating function*;

(b) *there exists a family of functions $f_t\colon L_t \to \mathbb{R}$ such that $df_t = \lambda_{\mathrm{can}}|_{L_t}$.*

*Then there exists a family $\mathcal{S}_t\colon M \times \mathbb{R}^N \to \mathbb{R}$ of quadratic at infinity generating functions for $L_t$ such that $\mathcal{S}_t \circ (d_M\mathcal{S}_t)^{-1} = f_t$ for all $t \in [0,1]$.*

**Proof** This minor extension of the Laudenbach–Sikorav theorem [15] follows immediately from the generalisation of Chekanov's theorem [5] to properly embedded Legendrian submanifolds in 1–jet bundles of not necessarily compact manifolds, see Eliashberg and Gromov [9, Sec. 4] and Ferrand [12]. Indeed, consider the Legendrian isotopy

$$\widehat{L}_t := \{(x, f_t(x)) \in \mathcal{J}^1(M) \mid x \in L_t\}$$

in the 1–jet bundle of $M$. By the generalised Chekanov theorem, there exists a family $\mathcal{S}_t \colon M \times \mathbb{R}^N \to \mathbb{R}$ of quadratic at infinity generating functions for $\widehat{L}_t$. This means that

$$\mathrm{FCrit}(\mathcal{S}_t) \ni z \mapsto (d_M \mathcal{S}_t(z), \mathcal{S}_t(z)) \in \mathcal{J}^1(M)$$

is a diffeomorphism onto $\widehat{L}_t$ for every $t \in [0,1]$, which proves the proposition. □

## 6 Critical values of quadratic at infinity functions

Let $S \colon \mathbb{R}^N \to \mathbb{R}$ be a function quadratic at infinity in the sense that

$$S(z) = \sigma(z) + Q(z),$$

where $\sigma$ has compact support and $Q$ is a non-degenerate quadratic form on $\mathbb{R}^N$. (We will eventually take $S$ to be the restriction of a quadratic at infinity generating function $\mathcal{S} \colon M \times \mathbb{R}^N \to \mathbb{R}$ to the fibre $\{x\} \times \mathbb{R}^N$ over a point $x \in M$.) Following Viterbo [22, Section 2], let us define an invariant $c_-(S) \in \mathbb{R}$ of such a function.

Consider the sublevel sets

$$S^c := \{z \in \mathbb{R}^N \mid S(z) \le c\}$$

and denote by $S^{-\infty}$ the set $S^c$ for a sufficiently negative $c \ll 0$. Pick a $Q$–negative linear subspace $V \subset \mathbb{R}^N$ of maximal possible dimension $\kappa$. The relative homology class $[V] \in \mathrm{H}_\kappa(\mathbb{R}^N, S^{-\infty})$ does not depend on the choice of $V$. Set

$$c_-(S) := \inf\{c \in \mathbb{R} \mid [V] \in \iota_* \mathrm{H}_\kappa(S^c, S^{-\infty})\},$$

where $\iota_* \colon \mathrm{H}_\kappa(S^c, S^{-\infty}) \to \mathrm{H}_\kappa(\mathbb{R}^N, S^{-\infty})$ is the homomorphism of relative homology groups induced by the inclusion $\iota \colon S^c \to \mathbb{R}^N$.

By Morse theory, $c_-(S)$ is a critical value of $S$. In particular, if $S$ has a single critical point $z_0 \in \mathbb{R}^N$, then $c_-(S) = S(z_0)$.

We will need the following version of Viterbo's monotonicity lemma [22, Lemma 4.7].

**Lemma 6.1** *Let $\{S_t\}_{t\in[0,1]}$ be a family of quadratic at infinity functions on $\mathbb{R}^N$ and let*

$$C := \{(z,\tau)\in\mathbb{R}^N\times[0,1] \mid dS_\tau(z)=0\} = \bigcup_{\tau\in[0,1]} \mathrm{Crit}(S_\tau)\times\{\tau\}.$$

*Suppose that for any $(z,\tau)$ from a dense subset $C'\subseteq C$ there exists a non-trivial closed interval $[t',t'']\ni\tau$ and a smooth curve $\gamma\colon [t',t'']\to\mathbb{R}^N$ such that*

(a) $\gamma(\tau)=z$;

(b) $\gamma(t)\in\mathrm{Crit}(S_t)$ *for all* $t\in[t',t'']$;

(c) *the function $t\mapsto S_t(\gamma(t))$ is non-decreasing on $[t',t'']$.*

*Then $t\mapsto c_-(S_t)$ is a non-decreasing (continuous) function on $[0,1]$.*

**Proof** According to [22, Lemma 4.7], the claim will follow if we show that $\frac{\partial S_\tau}{\partial t}(z)\geq 0$ for all $(z,\tau)\in C$. If $(z,\tau)\in C'$, we have

$$0\leq \frac{d}{dt}S_t(\gamma(t))\Big|_{t=\tau} = \frac{\partial S_\tau}{\partial t}(z) + dS_\tau(z)\,\frac{d\gamma}{dt}(\tau) = \frac{\partial S_\tau}{\partial t}(z),$$

where $\gamma$ is a curve satisfying conditions (a)–(c) for $\tau$ and $z$. Since $C'$ is dense in $C$, this inequality is valid for all $(z,\tau)\in C$. □

In the proof of Theorem 1.1 in Section 8, the set $C$ will parameterise the intersection of a positive Legendrian isotopy with an exact pre-Lagrangian submanifold. The subset $C'$ will correspond to the generic part of that intersection, where Lemma 3.1 may be applied.

## 7 The Importance of Being Open

Let $M$ be an open manifold. We identify $ST^*M$ with the unit sphere bundle in $T^*M$ for some Riemannian metric on $M$ and view the complement to the zero section of $T^*M$ as the symplectisation of $ST^*M$, see Example 4.1. Let $\pi_M\colon T^*M\to M$ denote the bundle projection.

**Lemma 7.1** *There exists a function $\Phi\colon M\to\mathbb{R}$ without critical points.*

**Proof** This is well-known, see Godbillon [14, Lemma 1.15]. □

**Definition 7.2** Let $\Lambda_\Phi := \left\{ \dfrac{d\Phi(x)}{\|d\Phi(x)\|} \mid x\in M \right\}\subset ST^*M.$

It is clear from the definition of the canonical 1–form that $\Lambda_\Phi$ is an exact pre-Lagrangian submanifold of $ST^*M$ and the function

$$f_\Phi(\zeta) = \Phi(\pi_M(\zeta))$$

is a contact potential on $\Lambda_\Phi$. The associated Lagrangian lift

$$\widetilde{\Lambda}_\Phi = \{d\Phi(x) \mid x \in M\} \subset T^*M$$

is just the graph of the differential of $\Phi$. It has an obvious generating function

$$\mathcal{S}_\Phi \colon M \times \mathbb{R}^0 \to \mathbb{R}, \quad \mathcal{S}_\Phi(x \times \{\mathrm{pt}\}) := \Phi(x).$$

## 8 Proof of Theorem 1.1

Let $M$ be a manifold (universally) covered by an open manifold. Suppose that there exists a non-negative Legendrian isotopy $\{L_t\}_{t\in[0,1]}$ connecting two different fibres of $ST^*M$. Since such an isotopy lifts to the spherical cotangent bundle of the covering manifold, we may assume that $M$ is itself an open manifold. By Lemma 2.2, we may also assume that the Legendrian isotopy is positive.

Let $\Lambda_\Phi$ be the exact pre-Lagrangian submanifold with contact potential $f_\Phi = \Phi \circ \pi_M$ defined in Section 7. Applying a global contactomorphism induced by a suitable diffeomorphism of $M$, we can arrange that $L_0 = ST^*_{x_0}M$ and $L_1 = ST^*_{x_1}M$, where the points $x_0, x_1 \in M$ are such that $\Phi(x_0) > \Phi(x_1)$. Furthermore, we can put the isotopy in general position with respect to $\Lambda_\Phi$ in the sense of Definition 2.3, leaving $L_0$ and $L_1$ fixed (because they are already transversal to $\Lambda_\Phi$).

Let $\{\phi_t\}_{t\in[0,1]}$ be a compactly supported contact isotopy of $ST^*M$ such that $L_t = \phi_t(L_0)$ for all $t \in [0,1]$. (Such an isotopy exists by the Legendrian isotopy extension theorem, see Geiges [13, Theorem 2.6.2].) Consider the Hamiltonian isotopy of exact Lagrangian submanifolds $(\widetilde{\phi}_t)^{-1}(\widetilde{\Lambda}_\Phi) \subset T^*M$ and the functions $\widetilde{f}_\Phi \circ \widetilde{\phi}_t$ on these manifolds, see Section 4. By Proposition 5.1, there exists a family of quadratic at infinity generating functions

$$\mathcal{S}_t \colon M \times \mathbb{R}^N \to \mathbb{R}$$

for $(\widetilde{\phi}_t)^{-1}(\widetilde{\Lambda}_\Phi) \subset T^*M$ such that

$$\mathcal{S}_t \circ (d_M\mathcal{S}_t)^{-1} = \widetilde{f}_\Phi \circ \widetilde{\phi}_t. \tag{8.1}$$

Let

$$S_t := \mathcal{S}_t(x_0, \cdot) \colon \mathbb{R}^N \to \mathbb{R}$$

be the restrictions of $\mathcal{S}_t$ to the fibre $\{x_0\} \times \mathbb{R}^N$.

By the definition of a generating function, $z \in \{x_0\} \times \mathbb{R}^N$ is a critical point of $S_t$ if and only if $d_M \mathcal{S}_t(z)$ is an intersection point of $\widetilde{\phi}_t^{-1}(\widetilde{\Lambda}_\Phi)$ with the fibre $T^*_{x_0}M$ or, in other words, if and only if $\frac{d_M \mathcal{S}_t(z)}{\|d_M \mathcal{S}_t(z)\|}$ is an intersection point of $\phi_t^{-1}(\Lambda_\Phi)$ with $ST^*_{x_0}M$. Hence, the map

$$\mathrm{Crit}(S_t) \ni z \longmapsto \phi_t\left(\frac{d_M \mathcal{S}_t(z)}{\|d_M \mathcal{S}_t(z)\|}\right) \in L_t \cap \Lambda_\Phi$$

establishes a bijective correspondence between the set of critical points of $S_t$ and the intersection $L_t \cap \Lambda_\Phi$. Furthermore, it follows from formula (8.1) that the value of $S_t$ at a point $z \in \mathrm{Crit}\, S_t$ is equal to the value of $f_\Phi$ at the corresponding point in $\Lambda_\Phi$.

In particular, $S_0$ and $S_1$ each have a single critical point corresponding to the intersection of $\Lambda_\Phi$ with $L_0 = ST^*_{x_0}M$ and $L_1 = ST^*_{x_1}M$, respectively. Since $f_\Phi = \Phi \circ \pi_M$, we see that

$$c_-(S_0) = \Phi(x_0) \quad \text{and} \quad c_-(S_1) = \Phi(x_1).$$

Hence,

$$c_-(S_0) > c_-(S_1) \tag{8.2}$$

by our choice of the points $x_0$ and $x_1$.

On the other hand, it follows from Lemma 3.1 and the discussion after Definition 2.3 that the family of functions $\{S_t\}_{t\in[0,1]}$ satisfies the hypotheses of Lemma 6.1. Thus, $c_-(S_t)$ is a non-decreasing function of $t$ and therefore $c_-(S_0) \le c_-(S_1)$, which contradicts (8.2).

This contradiction shows that a non-negative Legendrian isotopy cannot connect two different fibres of $ST^*M$. □

**Corollary 8.1** *If the universal cover of $M$ is an open manifold, then there does not exist a positive Legendrian loop in the Legendrian isotopy class of the fibre of $ST^*M$.*

**Proof** Suppose that $\{L_t\}_{t\in[0,1]}$ is a positive Legendrian isotopy such that $L_0 = L_1 = ST^*_xM$. If $\{\phi_t\}_{t\in[0,1]}$ is a contact isotopy such that $\phi_0 = \mathrm{id}$, $\phi_1(ST^*_xM) = ST^*_yM$ for some $y \neq x$, and $\frac{d\phi_t}{dt}$ is sufficiently small, then the isotopy $\{\phi_t(L_t)\}_{t\in[0,1]}$ is positive and connects two different fibres of $ST^*M$, contradicting Theorem 1.1. □

**Remark 8.2** With a little more work, it can be shown that *if the universal cover of $M$ is non-compact, then any non-negative Legendrian loop in the Legendrian isotopy class of the fibre of $ST^*M$ is constant*, cf [6, Corollary 6.2]. Although we will not really need this result, we sketch the proof for completeness. In view of Theorem 1.1, we may assume that $M$ is itself non-compact. Suppose that $\{L_t\}_{t\in[0,1]}$ is a non-constant non-negative Legendrian isotopy such that $L_0 = L_1 = ST^*_{x_0}M$. Since the isotopy is non-constant, we can pick a $\tau \in (0,1)$ in such a way that $L_\tau$ differs from $L_0$ but is still close to it. Then there exists a function $\Phi\colon M \to \mathbb{R}$ without critical points such that $\Lambda_\Phi$ intersects $L_\tau$ at a single point $\xi \in ST^*M$ and $\Phi(\pi_M(\xi)) \neq \Phi(x_0)$. Using the Reeb flow as in the proof of Lemma 2.2, we can approximate $\{L_t\}$ by a positive Legendrian isotopy $\{L'_t\}$ with $L'_0 = L_0$ and put $\{L'_t\}$ in general position with respect to $\Lambda_\Phi$. If we now define the functions $S_t\colon \{x_0\}\times\mathbb{R}^N \to \mathbb{R}$ corresponding to the isotopy $\{L'_t\}$ as in the proof of Theorem 1.1, then Lemmas 3.1 and 6.1 apply to show that the function $t \mapsto c_-(S_t)$ is non-decreasing. On the other hand, $c_-(S_\tau) \approx \Phi(\pi_M(\xi))$ does not lie between $c_-(S_0) = \Phi(x_0)$ and $c_-(S_1) \approx \Phi(x_0)$, a contradiction.

**Example 8.3** Suppose that there exists a Riemannian metric $g$ on $M$ such that $(M,g)$ is a $Y^x_\ell$–manifold for some $x \in M$ and $\ell > 0$, that is, such that all unit speed $g$–geodesics starting from $x$ return to $x$ in time $\ell$, see Besse [3, Definition 7.7(c)]. Then moving the fibre $ST^*_xM$ along the (co-)geodesic flow on $ST^*M$ defines a positive Legendrian loop based at $ST^*_xM$. Thus, Corollary 8.1 and Theorem 1.1 do not hold for such a manifold $M$.

Note that if $\dim M = 2$ or $3$, then either the universal cover of $M$ is open or $M$ admits a Riemannian metric turning it into a $Y^x_\ell$–manifold. For $\dim M = 2$, this statement follows immediately from the classification of surfaces. For $\dim M = 3$, the Poincaré conjecture proved by Perelman [19; 20; 21] implies that the universal cover of $M$ is either non-compact or diffeomorphic to $S^3$. In the latter case, the elliptisation conjecture also proved by Perelman guarantees that $M$ is diffeomorphic to a quotient of the standard round $S^3$ by the action of a finite group of isometries and the quotient metric turns $M$ into a $Y^x_\ell$–manifold. Thus, Theorem 1.1 fails for every surface or 3–manifold such that its universal cover is not open.

The weak form of the Bott–Samelson theorem proved by Bérard-Bergery, see [2] and [3, Theorem 7.37], says that if $(M,g)$ is a $Y^x_\ell$–manifold, then the universal cover of $M$ is compact and the rational cohomology ring $H^*(M,\mathbb{Q})$ is generated by one element. In view of the preceding discussion, it seems natural to ask whether the latter property is also shared by all manifolds $M$ such that there exists a positive Legendrian loop in the Legendrian isotopy class of the fibre of $ST^*M$.

**Remark 8.4** Let $p\colon Z \to N$ be a Legendrian fibration of a co-oriented contact manifold. Suppose that there exists a contact covering $ST^*M \to Z$ such that $M$ is open and the pre-image of any fibre of $p$ is a union of fibres of $ST^*M$. (Note that $p$ does not have to be locally trivial and its fibres do not have to be spheres.) Then Theorem 1.1 and Corollary 8.1 hold for the fibres of $p$.

## 9 Orderability of $ST^*M$

Let $(Y, \ker\alpha)$ be a connected contact manifold. Consider the identity component $\mathrm{Cont}_0(Y)$ of the group of compactly supported contactomorphisms of $(Y, \ker\alpha)$ and let $\widetilde{\mathrm{Cont}}_0(Y)$ denote the universal cover of this group corresponding to the base point $\mathrm{id}_Y \in \mathrm{Cont}_0(Y)$. For $f, g \in \widetilde{\mathrm{Cont}}_0(Y)$, write $f \preceq g$ if the element $gf^{-1}$ can be represented by a path $\phi_t \in \mathrm{Cont}_0(Y)$ such that the contact Hamiltonian $H := \alpha\big(\frac{d\phi_t}{dt}\big)$ is non-negative. Following Eliashberg and Polterovich [11], we say that the contact manifold $(Y, \ker\alpha)$ is *orderable* if the relation $\preceq$ defines a genuine partial order on $\widetilde{\mathrm{Cont}}_0(Y)$.

Eliashberg, Kim and Polterovich [10, Theorem 1.18] used contact homology to prove that $ST^*M$ is orderable for a closed manifold $M$ such that its fundamental group $\pi_1(M)$ is either finite or has infinitely many conjugacy classes. It is an open problem whether an infinite finitely presented group can have finitely many conjugacy classes, see Baumslag, Myasnikov and Shpilrain [1, Problem (FP19)]. The following result shows that the orderability of $ST^*M$ does not depend on the solution of that problem.

**Corollary 9.1** *$ST^*M$ is orderable for any closed manifold $M$.*

**Proof** By [11, Criterion 1.2.C], a closed contact manifold $(Y, \ker\alpha)$ is orderable if and only if there does not exist a contractible loop of contactomorphisms $\phi_t \in \mathrm{Cont}_0(Y)$, $t \in [0,1]$, such that $\phi_0 = \phi_1 = \mathrm{id}_Y$ and the corresponding contact Hamiltonian is everywhere positive. It is clear that applying a contact isotopy of $\mathrm{id}_Y$ generated by a positive contact Hamiltonian to any Legendrian submanifold $L \subset Y$, we obtain a positive Legendrian isotopy of $L$. Thus, if $Y$ is not orderable, then every Legendrian isotopy class contains a positive (contractible) Legendrian loop.

Suppose that $ST^*M$ is not orderable. By [10, Theorem 1.18], the fundamental group of $M$ is infinite and hence the universal cover of $M$ is open. In that case, however, the Legendrian isotopy class of the fibre of $ST^*M$ does not contain positive Legendrian loops by Corollary 8.1, a contradiction. □

**Example 9.2** The proof of Corollary 9.1 shows that if $\pi_1(M)$ is infinite, then there are no positive loops in $\mathrm{Cont}_0(ST^*M)$, contractible or not. On the other hand, the (co-)geodesic flow of the standard round metric on the $m$–sphere $S^m$ defines a *non-contractible* positive loop in $\mathrm{Cont}_0(ST^*S^m)$, cf Example 8.3.

**Corollary 9.3** *Let $p\colon ST^*M \to Z$ be a contact covering of a closed contact manifold $Z$. Then $Z$ is orderable.*

**Proof** Suppose that $Z$ is not orderable and argue by contradiction. By [11, Criterion 1.2.C], there exists a contractible loop of contactomorphisms of $Z$ based at $\mathrm{id}_Z$ and generated by a positive contact Hamiltonian. Since this loop is contractible, it lifts to a loop of contactomorphisms of $ST^*M$ with the same properties. If $M$ is closed, we conclude that $ST^*M$ is not orderable by [11, Criterion 1.2.C], which contradicts Corollary 9.1. If $M$ is open, then the argument from the proof of that corollary shows that there exists a positive Legendrian loop based at a fibre of $ST^*M$, which contradicts Corollary 8.1. □

**Remark 9.4** Let $\mathcal{L}$ be a connected component of the space of Legendrian submanifolds of a contact manifold $(Y, \ker\alpha)$. Note that $\mathcal{L}$ is a homogeneous space of $\mathrm{Cont}_0(Y)$ and its universal cover $\widetilde{\mathcal{L}}$ is a homogeneous space of $\widetilde{\mathrm{Cont}}_0(Y)$. It was pointed out in [11, Section 1.9] that the relation $\preceq$ admits a natural extension to $\mathcal{L}$ and $\widetilde{\mathcal{L}}$. Namely, write $L_1 \preceq L_2$ for $L_1, L_2 \in \mathcal{L}$ if there exists a non-negative Legendrian isotopy connecting $L_1$ to $L_2$. Similarly, write $\widetilde{L}_1 \preceq \widetilde{L}_2$ for $\widetilde{L}_1, \widetilde{L}_2 \in \widetilde{\mathcal{L}}$ if there exists a path in $\widetilde{\mathcal{L}}$ connecting $\widetilde{L}_1$ to $\widetilde{L}_2$ such that its projection to $\mathcal{L}$ is a non-negative Legendrian isotopy.

From this point of view, Remark 8.2 says that $\preceq$ is a genuine partial order on the component $\mathcal{L}$ containing the fibre of $ST^*M$ if the universal cover of $M$ is non-compact. Furthermore, [6, Corollary 5.5] shows that $\preceq$ is a genuine partial order on the component $\mathcal{L}$ containing the zero section of the 1–jet bundle $\mathcal{J}^1(N)$ of a compact manifold $N$. On the other hand, $\preceq$ is *not* a genuine partial order on the component containing the fibre of $ST^*M$ for a $Y^x_\ell$–manifold $M$ because there is a positive Legendrian loop in that component. Similarly, the example of a positive Legendrian loop in $\mathcal{J}^1(S^1) \cong ST^*\mathbb{R}^2$ given in [8, Section 5.1] shows that $\preceq$ is *not* a genuine partial order on the corresponding component of the space of Legendrian knots.

As in the case of contactomorphisms, the behaviour of $\preceq$ seems to 'improve' on $\widetilde{\mathcal{L}}$. For instance, one might hope that it is a genuine partial order on the universal cover of the component containing the fibre of $ST^*M$ for *any* manifold $M$.

## 10 Legendrian Low Conjecture

Here we give a very brief exposition of the relevant material from Lorentz geometry. Further details and references may be found in [6].

Let $(X, g)$ be a connected Lorentz manifold of signature $(\overbrace{+,\dots,+}^{m},-)$, $m \geq 2$. A non-zero vector $v \in T_xX$ is called *timelike, non-spacelike, null*, or *spacelike* if $g(v,v)$ is respectively negative, non-positive, zero, or positive. A submanifold in $X$ is called *spacelike* if the restriction of $g$ to it is a Riemannian metric.

Assume that $X$ is *time-oriented*, that is, equipped with a continuous choice of the future and past hemicones $C_x^{\uparrow}$ and $C_x^{\downarrow}$ in the non-spacelike cone in each $T_xX$. A piecewise smooth curve $\gamma = \gamma(t)$ in $X$ is said to be *future directed* if $\dot\gamma(t) \in C_{\gamma(t)}^{\uparrow}$ and *past directed* if $\dot\gamma(t) \in C_{\gamma(t)}^{\downarrow}$ for all $t$.

**Definition 10.1** Two points $x, y \in X$ are called *causally related* if they can be connected by a future or past directed curve.

Assume further that there is a smooth spacelike hypersurface $M \subset X$ such that every endless future directed curve in $X$ meets $M$ exactly once. Then $(X, g)$ is called a *globally hyperbolic spacetime* with a *Cauchy surface* $M \subset X$. The simplest examples are direct products $(X, g) = (M \times \mathbb{R}, \overline{g} \oplus -dt^2)$, where $\overline{g}$ is a complete Riemannian metric on $M$. In this case, a time orientation is just an orientation on the $\mathbb{R}$–factor and each slice $M \times \{t\}$ is a Cauchy surface in $(X, g)$.

Let $\mathfrak{N}$ be the set of all future directed non-parameterised null geodesics in $(X, g)$ or, in other words, the set of all light rays of our spacetime. $\mathfrak{N}$ has a canonical structure of a contact manifold, see Low [17, Section 2] or Natário and Tod [18, pages 252–253]. There is a contactomorphism

$$\rho_M\colon \mathfrak{N} \xrightarrow{\simeq} ST^*M$$

that associates to a null geodesic $\gamma \in \mathfrak{N}$ the equivalence class of the (non-zero) linear form $v \mapsto g(\dot\gamma, v)$ on $T_{\gamma\cap M}M$, where $\dot\gamma$ is a future pointing tangent vector to $\gamma$ at $\gamma \cap M$.

The set $\mathfrak{S}_x$ of all null geodesics passing through a point $x \in X$ is a Legendrian sphere in $\mathfrak{N}$ called the *sky* of that point. Note that two skies intersect if and only if the corresponding points lie on the same null geodesic. Note also that $\rho_M(\mathfrak{S}_x) = ST_x^*M$ for any $x \in M$.

Since $X$ is connected, the skies of any two points are Legendrian isotopic in $\mathfrak{N}$. However, Legendrian links formed by unions of disjoint skies may be quite different.

A basic observation is that all Legendrian links $\mathfrak{S}_x \sqcup \mathfrak{S}_y$ corresponding to causally *un*related points $x, y \in X$ belong to the same Legendrian isotopy class, see [7, Theorem 8] or [6, Lemma 4.3]. Let us denote this isotopy class of Legendrian links by $\mathcal{U}$ (as in *u*nrelated and *u*nlinked). A natural way to represent $\mathcal{U}$ is to pick the points $x$ and $y$ on the Cauchy surface $M$ so that $\rho_M$ identifies $\mathfrak{S}_x \sqcup \mathfrak{S}_y$ with $ST_x^*M \sqcup ST_y^*M \subset ST^*M$.

Two skies $\mathfrak{S}_x, \mathfrak{S}_y \subset \mathfrak{N}$ are said to be *Legendrian linked* if either $\mathfrak{S}_x \cap \mathfrak{S}_y \neq \varnothing$ or the Legendrian link $\mathfrak{S}_x \sqcup \mathfrak{S}_y$ does *not* belong to $\mathcal{U}$. We have just seen that if $\mathfrak{S}_x$ and $\mathfrak{S}_y$ are Legendrian linked, then the points $x$ and $y$ are causally related.

**Definition 10.2** The *Legendrian Low Conjecture* holds for a globally hyperbolic spacetime if two points in it are causally related if and *only if* their skies are Legendrian linked.

**Remark 10.3** The general problem of describing causal relations in terms of linking in the space of null geodesics originates from the work of Robert Low that was apparently inspired by a question raised by Penrose, see, for example, Low [16; 17]. The Legendrian Low Conjecture was explicitly stated by Natário and Tod [18, Conjecture 6.4] in the case when the Cauchy surface $M$ is diffeomorphic to an open subset of $\mathbb{R}^3$.

It was shown in our paper [6] that the Legendrian Low Conjecture holds for any globally hyperbolic spacetime such that its Cauchy surface has a cover diffeomorphic to an open subset of $\mathbb{R}^m$, $m \geq 2$. Using Theorem 1.1 instead of [6, Corollary 6.2], we can now extend our result to a wider class of spacetimes.

**Theorem 10.4** *The Legendrian Low Conjecture holds for any globally hyperbolic spacetime such that the universal cover of its Cauchy surface is not compact.*

**Proof** Let $x, y \in X$ be two points such that their skies are disjoint and there exists a future directed curve connecting $x$ to $y$. By [6, Proposition 4.2], there exists a non-negative Legendrian isotopy connecting $\mathfrak{S}_y$ to $\mathfrak{S}_x$. Suppose that the Legendrian link $\mathfrak{S}_x \sqcup \mathfrak{S}_y$ belongs to $\mathcal{U}$. Then the link $\rho_M(\mathfrak{S}_x \sqcup \mathfrak{S}_y) \subset ST^*M$ is Legendrian isotopic to a link formed by a pair of fibres of $ST^*M$. Since Legendrian isotopic links are ambiently contactomorphic, we obtain a non-negative Legendrian isotopy connecting two different fibres of $ST^*M$, which contradicts Theorem 1.1. Thus, $\mathfrak{S}_x$ and $\mathfrak{S}_y$ are Legendrian linked. $\square$

Combining this theorem with Perelman's proof of the Poincaré conjecture, we see that the Legendrian Low Conjecture holds for any $(3+1)$–dimensional globally hyperbolic spacetime such that the universal cover of its Cauchy surface is not diffeomorphic to $S^3$. (This result was obtained in [6] by a more involved argument using the full strength of the geometrisation conjecture.) On the other hand, if the Cauchy surface is a quotient of $S^3$, then the Legendrian Low Conjecture may fail because of the following general construction, cf [7, Example 3].

**Example 10.5** If $(M, \overline{g})$ is a Riemannian $Y^x_\ell$–manifold (see Example 8.3), then the Legendrian Low Conjecture is false for the globally hyperbolic spacetime $(M \times \mathbb{R}, \overline{g} \oplus -dt^2)$. Indeed, null geodesics in this spacetime have the form $\gamma(s) = (\overline{\gamma}(s), s)$, where $\bar{\gamma}$ is a $\bar{g}$–geodesic on $M$ and $s$ is the natural parameter on $\bar{\gamma}$. In particular, $\rho_{M\times\{0\}}(\mathfrak{S}_{(x,\ell)}) = ST^*_x M$ by the definition of a $Y^x_\ell$–manifold. Thus, the skies $\mathfrak{S}_{(x',0)}$ and $\mathfrak{S}_{(x,\ell)}$ are *not* Legendrian linked if $x' \neq x \in M$. However, the points $(x',0)$ and $(x,\ell)$ in $M\times\mathbb{R}$ are causally related if $x'$ is sufficiently close to $x$ in $M$.

**Remark 10.6** One can use Remark 8.2 and the proof of [6, Theorem C] to show that if the universal cover of the Cauchy surface of a globally hyperbolic spacetime $X$ is non-compact, then the Legendrian links $\mathfrak{S}_x \sqcup \mathfrak{S}_y$ and $\mathfrak{S}_y \sqcup \mathfrak{S}_x$ are different for any pair of causally related points $x, y \in X$ with disjoint skies.

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