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BOHNENBLUST–HILLE INEQUALITIES FOR LORENTZ SPACES VIA INTERPOLATION

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We prove that the Lorentz sequence space $\ell_{2m/(m+1),1}$ is, in a precise sense, optimal among all symmetric Banach sequence spaces satisfying a Bohnenblust–Hille-type inequality for m-linear forms or m-homogeneous polynomials on $\mathbb{C}^n$. Motivated by this result we develop methods for dealing with subtle Bohnenblust–Hille-type inequalities in the setting of Lorentz spaces. Based on an interpolation approach and the Blei–Fournier inequalities involving mixed-type spaces, we prove multilinear and polynomial Bohnenblust–Hille-type inequalities in Lorentz spaces with subpolynomial and subexponential constants. An application to the theory of Dirichlet series improves a deep result of Balasubramanian, Calado and Queffélec.

1. Introduction and classical results

In seminal work, Bohnenblust and Hille [1931] proved that there exists a positive function f on $\mathbb{N}$ such that, for each n and every m-homogeneous polynomial on $\mathbb{C}^n$, the ℓ_p-norm with $p = 2m/(m+1)$ of the set of its coefficients is bounded above by the constant $f(m)$ times the supremum norm of the polynomial on the unit polydisc $\mathbb{D}^n$. The primary interest of this result is that $f(m)$ is independent of the dimension n and, moreover, the exponent $2m/(m+1)$ is optimal. This result was a key point in the celebrated solution by Bohnenblust and Hille of Bohr's absolute convergence problem for Dirichlet series (see, e.g., [Bohnenblust and Hille 1931; Bohr 1913; Defant et al. 2016; Defant and Sevilla-Peris 2014]).

Recently, more sophisticated results were obtained and successfully applied to verify several long-standing conjectures in the convergence theory for Dirichlet series (and intimately related complex analysis in high dimensions). A striking improvement was given in [Defant et al. 2011], proving that $f(m)$ in fact grows at most exponentially in m, and a recent result even states that $f(m)$ is subexponential, in the sense that for every $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that $f(m) \le C(\varepsilon)(1+\varepsilon)^m$ for each $m \in \mathbb{N}$ [Bayart et al. 2014b]. Estimates of this type proved to be useful in many different areas of analysis, for example the modern $\mathcal{H}_p$-theory of Dirichlet series and (the intimately connected) infinite-dimensional holomorphy (see, e.g., [Bayart et al. 2014a; Defant and Sevilla-Peris 2014]), the study of summing polynomials in Banach spaces (see [Albuquerque et al. 2014; Defant et al. 2012; Dimant and Sevilla-Peris 2013], for example), and even in quantum information theory (see [Montanaro 2012]) and more generally in Fourier analysis of Boolean functions. A good general reference in this area is the recent book of O'Donnell [2014].

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Our aim is to prove multilinear and polynomial Bohnenblust–Hille inequalities in the setting of Lorentz spaces. In the remainder of this introduction we give more precise details on the state of the art of BH inequalities (multilinear and polynomial) and isolate the two natural problems that mainly concern us.

We will consider Banach sequence spaces $(X(I), \|\cdot\|_X)$ of $\mathbb{C}$-valued sequences $(x_i)_{i\in I}$, which are defined over arbitrary given (index) sets I. In what follows, Lorentz spaces will play an important role. Given $1 \le p < \infty$ and $1 \le q \le \infty$, the Lorentz space $\ell_{p,q}(I)$ ($\ell_{p,q}$ for short) on a nonempty set I consists of all $x = (x_i)_{i\in I}$ for which the expression

$$\|x\|_{\ell_{p,q}} = \begin{cases} \left(\sum_{k\in J} x_k^{*q}(k^{q/p} - (k-1)^{q/p})^q\right)^{1/q} & \text{if } q < \infty, \\ \sup_{k\in J} k^{1/p} x_k^* & \text{if } q = \infty, \end{cases} \tag{1}$$

is finite. Here, as usual, for a given $x = (x_i)_{i\in I} \in \ell_\infty(I)$, we denote by $x^* = (x_j^*)_{j\in J}$ the nonincreasing rearrangement of x, defined by

$$x_j^* = \inf\{\lambda > 0 : \operatorname{card}\{i \in I : |x_i| > \lambda\} \le j\}, \quad j \in J,$$

where $J = \{1, \dots, n\}$ whenever $\operatorname{card} I = n$, and $J = \mathbb{N}$ whenever I is infinite. The expression (1) is a norm if $q \le p$ and a quasinorm if $q > p$. In the second case, $\|\cdot\|_{\ell_{p,q}}$ is equivalent to a norm. Of course, $\ell_{p,p}$ is the Minkowski space ℓ_p, since the map $x \mapsto x^*$ is an isometry.

The following two finite index sets will be of special interest: for each $m, n \in \mathbb{N}$,

$$\mathcal{M}(m,n) = \{\boldsymbol{i} = (i_1, \dots, i_m) : i_k \in \mathbb{N},\ 1 \le i_k \le n\} \quad \text{and} \quad \mathcal{J}(m,n) = \{\boldsymbol{j} \in \mathcal{M}(m,n) : j_1 \le j_2 \le \dots \le j_m\}.$$

Below we explain the two inequalities we are interested in, the so-called multilinear and polynomial Bohnenblust–Hille inequalities, and we motivate the two problems we intend to handle.

The multilinear BH inequality. Given a Banach sequence space X (defined over arbitrary index sets) and $m \in \mathbb{N}$, we denote by

$$\mathrm{BH}_X^{\mathrm{mult}}(m) \in [1, \infty]$$

the best constant $C \ge 1$ such that for each n and every complex matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ we have

$$\|(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}\|_X \le C\|a\|_\infty, \tag{2}$$

where

$$\|a\|_\infty = \sup_{\substack{\|(x_i^k)_{i=1}^n\|_\infty \le 1 \\ 1\le k\le m}} \left| \sum_{\boldsymbol{i}=(i_1,\dots,i_m)\in\mathcal{M}(m,n)} a_{\boldsymbol{i}} x_{i_1}^1 \cdots x_{i_m}^m \right|.$$

For the sake of completeness we give a short review of the history of the inequalities of the form (2), emphasizing those results, old and very recent, which are of relevance to this article. (For more on that we once again refer to [Defant and Sevilla-Peris 2014].) The case $m = 2$ reflects a famous result of Littlewood [1930]:

$$\mathrm{BH}_{\ell_{4/3}}^{\mathrm{mult}}(2) < \infty.$$

Solving Bohr's so-called absolute convergence problem on Dirichlet series, Bohnenblust and Hille [1931] studied the case of arbitrary m and proved that

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1)}}(m) < \infty. \tag{3}$$

This result was improved by [Blei and Fournier 1989; Fournier 1987] showing that, even,

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m) < \infty. \tag{4}$$

In Section 4 we give a modified version of their proof from [Blei and Fournier 1989].

Finally, Bayart, Pellegrino and Seoane-Sepúlveda [Bayart et al. 2014b] showed that the constants in (3) are subpolynomial in the following sense: there is a constant $\kappa > 1$ such that for all m we have

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1)}}(m) \le \kappa m^{(1-\gamma)/2}, \tag{5}$$

where γ is the Euler–Masceroni constant. Note that there exists a uniform constant $C > 0$ such that, for any finite index set I,

$$\|\ell_p(I) \hookrightarrow \ell_{p,1}(I)\| \le C \log(\operatorname{card} I); \tag{6}$$

hence, by (5), there exists $\delta > 1$ such that, for each m, n and every matrix $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$,

$$\|(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}\|_{2m/(m+1),1} \le m^{\delta}(\log n)\|a\|_\infty.$$

In view of this, and comparing with (4) and (5), the following natural question appears:

Problem 1. *Does there exist a constant $\delta > 0$ such that for each m we have*

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m) \le m^{\delta}?$$

We provide far-reaching partial solutions extending all results mentioned before. The main contributions are given in Theorems 6 and 12.

The polynomial BH inequality. Every m-homogenous polynomial

$$P(z) = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|=m}} c_\alpha z^\alpha$$

in n complex variables $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ can be uniquely rewritten in the form

$$P(z) = \sum_{\boldsymbol{j}\in\mathcal{J}(m,n)} c_{\boldsymbol{j}} z_{j_1} \cdots z_{j_m}, \tag{7}$$

and we denote its supremum norm by

$$\|P\|_\infty = \sup_{\|(z_i)_{i=1}^n\|_\infty\le 1} \left| \sum_{\boldsymbol{j}=(i_1,\dots,i_n)\in\mathcal{J}(m,n)} c_{\boldsymbol{j}} z_{j_1}\cdots z_{j_m} \right|.$$

Given a Banach sequence space X (defined over an arbitrary index set) and $m \in \mathbb{N}$, we denote by

$$\mathrm{BH}^{\mathrm{pol}}_X(m) \in [1, \infty]$$

the best constant $C \geq 1$ such that, for each n and every m-homogeneous polynomial P as in (7), we have

$$\|(c_{\boldsymbol{j}}(P))_{\boldsymbol{j}\in\mathcal{J}(m,n)}\|_X \leq C\|P\|_\infty. \tag{8}$$

Let us again give a short review of the most important results on such inequalities (for more information, again see [Defant and Sevilla-Peris 2014]).

By inventing polarization, Bohnenblust and Hille [1931] deduced from (3) that

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1)}}(m) < \infty. \tag{9}$$

The fact that $p = 2m/(m+1)$ is optimal here was a crucial step in the solution of Bohr's so-called absolute convergence problem. Again, mainly motivated by problems on the general theory of Dirichlet series and holomorphic functions in high dimensions, the first qualitative improvement of the constants was done in [Defant et al. 2011]: for every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that, for all m,

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1)}}(m) \leq C(\varepsilon)(\sqrt{2}+\varepsilon)^m. \tag{10}$$

Bayart et al. [2014b] proved that these constants are even subexponential in the following sense:

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1)}}(m) \leq C(\varepsilon)(1+\varepsilon)^m. \tag{11}$$

We are going to see that a standard polarization argument extends (9) to Lorentz spaces:

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1),1}}(m) < \infty; \tag{12}$$

but the following problem will turn out to be much more challenging:

Problem 2. *To what extent do* (10) *and* (11) *hold when we replace* $\ell_{2m/(m+1)}$ *by the Lorentz sequence space* $\ell_{2m/(m+1),1}$?

Concerning the extension of (10), our main result is given in Theorem 14.

Why do Lorentz spaces play an essential role within the context of Bohnenblust–Hille inequalities? We prove (see Theorem 1) that, among all symmetric Banach sequence spaces X satisfying a multilinear or polynomial Bohnenblust–Hille inequality as in (2) or (8), the sequence space $X = \ell_{2m/(m+1),1}$ is the smallest one (and in this sense the "best").

2. Preliminaries

Throughout the paper, for a given finite set $\{X_i\}_{i\in I}$ of Banach spaces which are all contained in some linear space $\mathcal{X}$, we denote by $\bigoplus_{i\in I} X_i$ the Banach space of all $x \in \bigcap_{i\in I} X_i$ equipped with the norm

$$\|x\|_{\bigoplus_{i\in I} X_i} = \sum_{i\in I} \|x\|_{X_i}.$$

For each $m \in \mathbb{N}$ we denote by $\mathcal{M}(m)$ and $\mathcal{J}(m)$ the union of all $\mathcal{M}(m,n)$ and $\mathcal{J}(m,n)$, $n \in \mathbb{N}$, respectively. We define an equivalence relation in $\mathcal{M}(m,n)$ in the following way: $\boldsymbol{i} \sim \boldsymbol{j}$ if there is a permutation σ of $\{1,\ldots,m\}$ such that $(i_1,\ldots,i_m) = (j_{\sigma(1)},\ldots,j_{\sigma(m)})$, and denote by $[\boldsymbol{i}]$ the equivalence class of $\boldsymbol{i} \in \mathcal{M}(m,n)$. The following disjoint partition of $\mathcal{M}(m,n)$ will be very useful:

$$\mathcal{M}(m,n) = \bigcup_{\boldsymbol{j} \in \mathcal{J}(m,n)} [\boldsymbol{j}].$$

For $1 \le k \le m$, let $\mathcal{P}_k(m)$ denote the set of all subsets of $\{1, \dots, m\}$ with cardinality k. We denote the complement of $S \in \mathcal{P}_k(m)$ in $\{1, \dots, m\}$ by $\widehat{S}$. If $S \in \mathcal{P}_k(m)$, then let $\mathcal{M}(S,n)$ be the set of all indices $\boldsymbol{i} : S \to \{1, \dots, n\}$, so in the special case $S = \{1, \dots, k\}$ we clearly have that $\mathcal{M}(k,n) = \mathcal{M}(S,n)$. Finally, for $\boldsymbol{i} \in \mathcal{M}(S,n)$ and $\boldsymbol{j} \in \mathcal{M}(\widehat{S},n)$ we define $\boldsymbol{i} \oplus \boldsymbol{j} \in \mathcal{M}(m,n)$ through

$$\boldsymbol{i} \oplus \boldsymbol{j} = \begin{cases} \boldsymbol{i} & \text{on } S, \\ \boldsymbol{j} & \text{on } \widehat{S}. \end{cases}$$

Given m, n, $k \in \mathbb{N}$ with $1 \le k < m$ and $1 \le p, q \le \infty$, we define the norm $\|\cdot\|_{(m,n,k,p,q)}$ on the space $\mathbb{C}^{\mathcal{M}(m,n)}$ of all matrices $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ by

$$\|a\|_{(m,n,k,p,q)} = \sum_{S \in \mathcal{P}_k(m)} \Biggl(\sum_{\boldsymbol{i} \in \mathcal{M}(S,n)} \Biggl(\sum_{\boldsymbol{j} \in \mathcal{M}(\widehat{S},n)} |a_{\boldsymbol{i} \oplus \boldsymbol{j}}|^q \Biggr)^{p/q} \Biggr)^{1/p},$$

and denote the corresponding Banach space by

$$\bigoplus_{S \in \mathcal{P}_k(m)} \ell_p(S)[\ell_q(\widehat{S})].$$

Clearly, this is the ℓ_1-sum of all Banach spaces $\ell_p(S)[\ell_q(\widehat{S})]$, where $\ell_p(S)[\ell_q(\widehat{S})]$ is, by definition, $\mathbb{C}^{\mathcal{M}(m,n)}$ normed by

$$\|a\|_{\ell_p(S)[\ell_q(\widehat{S})]} = \Biggl(\sum_{\boldsymbol{i} \in \mathcal{M}(S,n)} \Biggl(\sum_{\boldsymbol{j} \in \mathcal{M}(\widehat{S},n)} |a_{\boldsymbol{i} \oplus \boldsymbol{j}}|^q \Biggr)^{p/q} \Biggr)^{1/p}.$$

We will consider (classes of) Banach lattices. Of particular importance are *symmetric* spaces. We recall that a Banach lattice E on a measure space (Ω, Σ, μ) is said to be symmetric if $g \in E$ and $\|f\|_E = \|g\|_E$ whenever $\mu_f = \mu_g$ and $f \in E$. Here μ_f denotes the distribution function of f, defined by $\mu_f(\lambda) = \mu\{t \in \Omega : |f(t)| > \lambda\}$ for $\lambda \ge 0$. Throughout the paper, by a Banach sequence lattice on a finite or countable set I we mean a real or complex Banach lattice E on the measure space $(I, 2^I, \mu)$ (on I, for short), where μ is the counting measure. In the case when E is symmetric, E is said to be a symmetric Banach (sequence) space.

A symmetric space E is called *fully symmetric* whenever it is an exact interpolation space between $L_1(\mu)$ and $L_\infty(\mu)$; that is, for any linear operator $T : L_1(\mu) + L_\infty(\mu) \to L_1(\mu) + L_\infty(\mu)$ such that $\|T\|_{L_1(\mu) \to L_1(\mu)} \le 1$ and $\|T\|_{L_\infty(\mu) \to L_\infty}(\mu) \le 1$ we have that T maps E into E and $\|T\|_{E \to E} \le 1$. It is well known that symmetric spaces that have the Fatou property or have order continuous norm are fully symmetric (see [Bennett and Sharpley 1988; Kreĭn et al. 1982], for example).

We will need the concept of discretization of a Banach lattice. Let (Ω, Σ, μ) be a measure space and let $d = \{\Omega_k\}_{k=1}^N \subset \Sigma$ be a measurable partition of Ω, i.e., $\Omega = \bigcup_{k=1}^N \Omega_k$, where $\Omega_i \cap \Omega_j = \varnothing$ for each $i, j \in \{1, \dots, N\}$ with $i \neq j$. Then, given a Banach lattice X on (Ω, Σ, μ), the discretization X^d is the Banach space of all simple functions $f \in X$ of the form $f = \sum_{k=1}^N \xi_k \chi_{\Omega_k} \in X$, equipped with the induced norm from X.

The notion of Lorentz spaces over arbitrary measure spaces will be essential in what follows. Given a measure space (Ω, Σ, μ) and $0 < p < \infty$, $0 < q \leq \infty$, the Lorentz space $L_{p,q}(\Omega, \mu)$ ($L_{p,q}(\Omega)$ or $L_{p,q}$, for short) is defined to be the space of all (equivalence classes of) measurable functions f on Ω, equipped with the quasinorm

$$\|f\|_{L_{p,q}} = \begin{cases} \big((q/p)\int_0^\infty f^*(t)^q t^{q/p-1}\,dt\big)^{1/q} & \text{if } q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty, \end{cases}$$

where f^* is the decreasing rearrangement of f, defined on $[0, \infty)$ by

$$f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\}.$$

(We adopt the convention $\inf \varnothing = \infty$.) In the case when $\Omega = I$ is a nonempty set with counting measure μ, the space $L_{p,q}(\Omega, \mu)$ in fact coincides with the Lorentz sequence space $\ell_{p,q}(I)$ already defined in (1). Indeed, in this case, given a function $f = x$ on $\Omega = I$ we have $x_k^* = f^*(t)$ for every $t \in [k-1, k)$, $k \in J$, where $J = \{1, \dots, \operatorname{card} I\}$ if I is finite and $J = \mathbb{N}$ if I is infinite. Thus $\|f\|_{L_{p,q}} = \|x\|_{\ell_{p,q}}$, where the latter norm is as defined by the formula (1).

We recall that the Köthe dual space $(\ell_{p,1})'$ of the Lorentz space $\ell_{p,1} = \ell_{p,1}(I)$ coincides with the Marcinkiewicz space m_p, which consists of all complex sequences $x = (x_i)_{i \in I}$ such that

$$\|x\|_{m_p} = \sup_{k \in J} \frac{1}{k^{1/p}} \sum_{j=1}^k x_j^* < \infty,$$

and which, with this norm, forms a Banach space. Moreover, we note that by standard comparison with the integral of t^α on $[1, N]$, we have for each $N \in \mathbb{N}$ and every $\alpha \in (0, 1)$,

$$\sum_{k=1}^N \frac{1}{k^\alpha} < \frac{1}{1-\alpha} N^{1-\alpha}. \tag{13}$$

Combining this inequality (for $\alpha = 1/p$) with $x_k^* \leq k^{-1/p}\|x\|_{\ell_{p,\infty}}$ for $k \in J$ yields

$$m_p = \ell_{p,\infty}$$

up to equivalent norms:

$$\frac{1}{p'}\|x\|_{m_p} \leq \|x\|_{\ell_{p,\infty}} \leq \|x\|_{m_p}, \quad x \in \ell_{p,\infty}.$$

(As usual we write $1/p' := 1 - 1/p$.) Many of our arguments will be based on interpolation theory. Here we recall some of its basic concepts and provide some special facts we are going to use. Recall that if

$\vec{A} = (A_0, A_1)$ is a quasinormed couple then, for any $a \in A_0 + A_1$, we define the K-functional

$$K(t, a; \vec{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a_0 + a_1 = a\}, \quad t > 0.$$

For $0 < \theta < 1$ and $0 < q < \infty$, the real interpolation space $(A_0, A_1)_{\theta,q}$ is the space of all $a \in A_0 + A_1$, equipped with the quasinorm

$$\|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(t, a; \vec{A}))^q \,\frac{dt}{t}\right)^{1/q},$$

with an obvious modification for $q = \infty$.

The following well-known and easily verified interpolation property holds: if (A_0, A_1) and (B_0, B_1) are two quasinormed couples, T is a map from (A_0, A_1) to (B_0, B_1) (i.e., $T\colon A_0 + A_1 \to B_0 + B_1$ and the restrictions of T to A_j are bounded from A_j to B_j for each $j \in \{0, 1\}$) with the quasinorms $M_j = \|T : A_j \to B_j\|$, then $T\colon (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is also bounded and, for its quasinorm M, we have

$$M \le M_0^{1-\theta} M_1^{\theta}.$$

Lorentz spaces arise naturally in the real interpolation method since most of their important properties can be derived from real interpolation theorems. We briefly review some basic definitions. The pair (L_1, L_∞) is especially important for the understanding of the space $L_{p,q}$. It is well known that, for every $f \in L_1 + L_\infty$,

$$K(t, f; L_1, L_\infty) = \int_0^t f^*(s)\,ds = t f^{**}(t), \quad t > 0.$$

Hence, for each $\theta \in (0, 1)$,

$$\|f\|_{\theta,q} = \left(\int_0^\infty [t^{1-\theta} f^{**}(t)]^q \,\frac{dt}{t}\right)^{1/q}.$$

An immediate consequence of Hardy's inequality is the following well-known formula, which states that, for $1 < p < \infty,\ 1 \le q \le \infty$ and $\theta = 1 - 1/p$,

$$(L_1, L_\infty)_{\theta,q} = L_{p,q},$$

and, moreover,

$$\frac{1}{p'}\|f\|_{(L_1,L_\infty)_{\theta,q}} \le \|f\|_{L_{p,q}} \le \|f\|_{(L_1,L_\infty)_{\theta,p}}.$$

The following result will be used (which follows from the more general Theorem 4.3 of [Holmstedt 1970]): Let $1/p = (1-\theta)/p_0 + \theta/p_1,\ 0 < p_0,\ p_1 < \infty,\ p_0 \ne p_1$ and $0 < q \le \infty$. Then, up to equivalent norms, we have

$$(L_{p_0}, L_{p_1})_{\theta,q} = L_{p,q}.$$

More precisely,

$$\begin{aligned} C^{-1}\theta^{-\min(1/q,1/p_0)}(1-\theta)^{-\min(1/q,1/p_1)}\Big(\frac{p}{q}\Big)^{1/q}\|f\|_{L_{p,q}} \\ \le \|f\|_{(L_{p_0},L_{p_1})_{\theta,q}} \\ \le C\theta^{-\max(1/q,1/p_0)}(1-\theta)^{-\max(1/q,1/p_1)}\Big(\frac{p}{q}\Big)^{1/q}\|f\|_{L_{p,q}}, \end{aligned} \tag{14}$$

where $C > 0$ is a universal constant.

We will also make intensive use of complex interpolation, and denote by $[A_0, A_1]_\theta$ the complex interpolation spaces as defined, for example, in [Calderón 1964]. We recall that if X_0 and X_1 are two complex Banach lattices on a measure space (Ω, Σ, μ) then

$$[X_0, X_1]_\theta = X_0^{1-\theta} X_1^{\theta}, \tag{15}$$

with equality of norms provided one of the spaces has order continuous norm; here, following Calderón, we denote by $X_0^{1-\theta} X_1^{\theta}$ the Calderón space of all $x \in L^0(\mu)$ such that $|x| \le \lambda |x_0|^{1-\theta}|x_1|^{\theta}$ μ-a.e. on Ω for some constant $\lambda > 0$ and some $x_i \in X_i$ with $\|x_i\|_{X_i} \le 1$ for $i = 0, 1$. We put

$$\|x\|_{X_0^{1-\theta} X_1^{\theta}} = \inf \lambda.$$

3. The optimality of Lorentz spaces

The following theorem motivates our study; we show that, in the context of multilinear and polynomial Bohnenblust–Hille inequalities, Lorentz spaces are in a certain sense optimal. Before we state and prove these results we recall that, if X is a symmetric Banach sequence space on I and χ_A denotes the indicator function of a set $A \subset I$, clearly $\|\chi_A\|_X$ depends only on $\mathrm{card}(A)$. The function $\phi_X(k) = \|\chi_A\|_X$, where $A \subset I$ with $\mathrm{card}(A) = k$, is called the *fundamental function* of X. It is well known (see, e.g., [Kreĭn et al. 1982, Theorem 2.5.2]) that, if $1 \le p < \infty$ and X is a symmetric Banach sequence space on I such that $\|\chi_A\|_X = \mathrm{card}(A)^{1/p}$ for every indicator function χ_A (that is, $\phi_X(k) = k^{1/p}$ for every $A \subset I$ with $\mathrm{card}(A) = k$), then $\ell_{p,1} \hookrightarrow X$ with

$$\|x\|_X \le \|x\|_{\ell_{p,1}}, \quad x \in \ell_{p,1}.$$

Thus $\ell_{p,1}$ is the smallest symmetric Banach sequence space on I whose norm coincides with the ℓ_p-norm on indicator functions.

Theorem 1. *Fix a positive integer m. The Lorentz space $\ell_{2m/(m+1),1}$ is the smallest symmetric Banach sequence space X such that $\mathrm{BH}_X^{\mathrm{mult}}(m) < \infty$. Also, the Lorentz space $\ell_{2m/(m+1),1}$ is the smallest symmetric Banach sequence space X such that $\mathrm{BH}_X^{\mathrm{pol}}(m) < \infty$.*

Proof. We follow an argument inspired by [Bohnenblust and Hille 1931]. Assume that X is a symmetric Banach sequence space such that $\mathrm{BH}_X^{\mathrm{mult}}(m) < \infty$, i.e., for each $n \in \mathbb{N}$ and every complex matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ we have

$$\|a\|_X \le \mathrm{BH}_X^{\mathrm{mult}}(m)\|a\|_\infty. \tag{16}$$

It suffices to show that the fundamental function

$$\phi(n) := \Big\| \sum_{i=1}^{n} e_i \Big\|_X, \quad n \in \mathbb{N}, \tag{17}$$

satisfies

$$\phi(n) \le C(m) n^{(m+1)/(2m)} \tag{18}$$

for each $n \in \mathbb{N}$. For fixed N, choose some $N \times N$ matrix (a_{rs}) such for every r, s we have $|a_{rs}| = 1$ and $\sum_{k=1}^{N} a_{rk}\bar{a}_{sk} = N\delta_{rs}$ (e.g., $a_{rs} = e^{2\pi i rs/N}$ with $1 \le r, s \le N$), and define the matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ by

$$a_{i_1 \dots i_m} = a_{i_1 i_2} \cdots a_{i_{m-1} i_m}.$$

Since $|a_{i_1 \dots i_m}| = 1$, we have $\phi(N^m) = \|a\|_X$. We now estimate the norm $\|a\|_\infty$. We first do the trilinear case $m = 3$, where the argument becomes more transparent. We take $x, y, z \in \mathbb{C}^N$ with supremum norm at most 1; then, using the Cauchy–Schwarz inequality and the properties of the matrix, we have

$$\begin{aligned}
\Big| \sum_{i,j,k} a_{ij} a_{jk} x_i y_j z_k \Big| &\le \sum_k \Big| \sum_{i,j} a_{ij} a_{jk} x_i y_j \Big| |z_k| \\
&\le N^{1/2} \Big(\sum_k \Big| \sum_{i,j} a_{ij} a_{jk} x_i y_j \Big|^2 \Big)^{1/2} \\
&= N^{1/2} \Big(\sum_{\substack{i_1, i_2 \\ j_1, j_2}} a_{i_1 j_1} \bar{a}_{i_2 j_2} x_{i_1} \bar{x}_{i_2} y_{j_1} \bar{y}_{j_2} \sum_k a_{j_1 k} \bar{a}_{j_2 k} \Big)^{1/2} \\
&= N^{1/2} N^{1/2} \Big(\sum_{\substack{i_1, i_2 \\ j}} a_{i_1 j} \bar{a}_{i_2 j} x_{i_1} \bar{x}_{i_2} y_j \bar{y}_j \Big)^{1/2} = N \Big(\sum_j \Big| \sum_i a_{ij} x_i \Big|^2 |y_j|^2 \Big)^{1/2} \\
&\le N \Big(\sum_{i_1 i_2} \sum_j a_{i_1 j} \bar{a}_{i_2 j} x_{i_1} \bar{x}_{i_2} \Big)^{1/2} = N^{3/2} \Big(\sum_i |x_i|^2 \Big)^{1/2} \le N^{4/2}.
\end{aligned}$$

In the general case we take $z^{(1)}, \dots, z^{(m)} \in \mathbb{C}^N$, each with supremum norm at most 1, and repeat this procedure to get

$$\Big| \sum_{i_1, \dots, i_m = 1}^{N} a_{i_1 i_2} \cdots a_{i_{m-1} i_m} z^{(1)}_{i_1} \cdots z^{(m)}_{i_m} \Big| \le N^{m/2} \Big(\sum_{i_1} |z^{(1)}_{i_1}|^2 \Big)^{1/2} \le N^{m/2} N^{1/2}. \tag{19}$$

Hence $\|a\|_\infty \le N^{(m+1)/2}$ for each N, and by (16) we have $\phi(N^m) \le \mathrm{BH}^{\mathrm{mult}}_X(m)(N^m)^{(m+1)/(2m)}$. Since for each positive integer n there is N such that $N^m \le n < (N+1)^m$, we finally obtain (18).

To prove the second statement, we assume that X is a symmetric Banach sequence space such that, for each n and every m-homogeneous polynomial $P(z) = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| = m} c_\alpha z^\alpha$, we have

$$\big\| (c_\alpha)_{\alpha \in \mathbb{N}_0^n, |\alpha| = m} \big\|_X \le \mathrm{BH}^{\mathrm{pol}}_X(m) \|P\|_\infty.$$

Following nontrivial ideas of Bohnenblust and Hille [1931] it is possible to modify the proof of the first statement, which leads to a sort of deterministic proof of the second statement. Here we give an alternative, probabilistic argument. As in (17) we consider the fundamental function $\phi(n)$, $n \in \mathbb{N}$, of X. Then, by the Kahane–Salem–Zygmund inequality (see [Kahane 1985], for example), there is a constant $C_{\mathrm{KSZ}} \geq 1$ such that for every choice of N there are signs $\varepsilon_\alpha = \pm 1$ for which

$$\sup_{z \in \mathbb{D}^N} \Big| \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| = m}} \varepsilon_\alpha z^\alpha \Big| \leq C_{\mathrm{KSZ}} \Big(N \binom{m+N-1}{m} \log m \Big)^{1/2}.$$

Since the sequence $(\phi(N)/N)$ is nonincreasing and for each N we have

$$\frac{N^m}{m!} \leq \binom{N+m-1}{m} \leq N^m,$$

it follows that $\phi(N^m) \leq m! \phi\big(\binom{N+m-1}{m}\big)$ for each N. Combining the above estimates we conclude that, for each N,

$$\phi(N^m) \leq \mathrm{BH}_X^{\mathrm{pol}}(m) C_{\mathrm{KSZ}} m! \sqrt{\log m}\, (N^m)^{(m+1)/(2m)}.$$

This easily implies that there exists a constant $C(m) > 0$ such that

$$\phi(n) \leq C(m) n^{(m+1)/(2m)}, \quad n \in \mathbb{N},$$

and the conclusion again follows. □

4. Multilinear BH inequalities for Lorentz spaces revisited

In this section we present a slightly modified proof of (4), which was first given in [Blei and Fournier 1989]. We need to prove four preliminary lemmas.

Lemma 2. *For each matrix* $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ *and each* $S \subset \mathcal{M}(m,n)$,

$$\frac{1}{E(S)} \sum_{\boldsymbol{i} \in S} |a_{\boldsymbol{i}}| \leq m \|a\|_{\ell_{m/(m-1),\infty}},$$

where

$$E(S) := \max_{1 \leq k \leq m} \operatorname{card}\{i_k : \boldsymbol{i} \in S\}.$$

Proof. Clearly

$$k^{(m-1)/m} a_k^* \leq \|a\|_{\ell_{m/(m-1),\infty}}, \quad 1 \leq k \leq n^m.$$

Now note that $\sum_{\boldsymbol{i} \in S} |a_{\boldsymbol{i}}|$ has not more that $E(S)^m$ summands and that $\sum_{k=1}^{E(S)^m} a^*(k)$ sums the first $E(S)^m$ many largest $|a_{\boldsymbol{i}}|$, $\boldsymbol{i} \in S$. As a consequence, we obtain by (13) (with $\alpha = 1 - 1/m$) that

$$\sum_{\boldsymbol{i} \in S} |a_{\boldsymbol{i}}| \leq \sum_{k=1}^{E(S)^m} a_k^* \leq \|a\|_{\ell_{m/(m-1),\infty}} \sum_{k=1}^{E(S)^m} k^{-(m-1)/m} \leq m \|a\|_{\ell_{m/(m-1),\infty}} E(S),$$

as desired. □

Lemma 3. *For each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ the index set $\mathcal{M}(m,n)$ splits into a union of m subsets S_k such that, for every $1 \le q < \infty$,*

$$\max_{1\le k\le m} \|a^{S_k}\|_{\ell_\infty(\{k\})[\ell_q(\widehat{\{k\}})]} \le m^{1/q} \|a\|_{\ell_{qm/(m-1),\infty}},$$

where, for $S \subset \mathcal{M}(m,n)$, we put $a^S = a_{\boldsymbol{i}}$ for $\boldsymbol{i} \in S$ and $a^S = 0$ for $\boldsymbol{i} \notin S$.

Proof. It suffices to show the desired inequality for $q = 1$: for arbitrary $1 < q < \infty$ apply the case $q = 1$ to $|a|^{1/q}$ instead of to a. In view of Lemma 2 we show that there are appropriate sets S_k for which

$$\max_{1\le k\le m} \|a^{S_k}\|_{\ell_\infty(\{k\})[\ell_1(\widehat{\{k\}})]} \le \sup_{S\subset\mathcal{M}(m,n)} \frac{1}{E(S)} \sum_{\boldsymbol{i}\in S} |a_{\boldsymbol{i}}|,$$

and without loss of generality we may assume that the supremum on the right side is at most 1. Given $1 \le k \le m$, observe that

$$\sum_{\ell=1}^{n} \sum_{\substack{\boldsymbol{i}\in\mathcal{M}(m,n)\\ i_k=\ell}} |a_{\boldsymbol{i}}| \le \sum_{\boldsymbol{i}\in\mathcal{M}(m,n)} |a_{\boldsymbol{i}}| \le E(\mathcal{M}(m,n)) = n.$$

Hence there is some $1 \le \ell(k) \le n$ such that for

$$T_k^1 = \{\boldsymbol{j} \in \mathcal{M}(m,n) : j_k = \ell(k)\}$$

we have

$$\sum_{\boldsymbol{i}\in T_k^1} |a_{\boldsymbol{i}}| \le 1.$$

Then, for

$$N_1 = \mathcal{M}(m,n) \setminus \bigcup_{k=1}^{m} T_k^1,$$

we obviously get $E(N_1) \le n - 1$. If we now repeat this procedure with N_1 instead of $\mathcal{M}(m,n)$, then we obtain m new index sets T_k^2, $1 \le k \le m$, in N_1, for which

$$\sum_{\boldsymbol{i}\in T_k^2} |a_{\boldsymbol{i}}| \le 1$$

and

$$E(N_2) \le n-2 \quad \text{with } N_2 = \Big(\mathcal{M}(m,n) \setminus \bigcup_{k=1}^{m} T_k^1\Big) \setminus \Big(\bigcup_{k=1}^{m} T_k^2\Big).$$

Continuing for $j \in \{3, \dots, n\}$, we find index sets T_k^j, $1 \le j \le n$, $1 \le k \le m$, such that

$$\sum_{\boldsymbol{i}\in T_k^j} |a_{\boldsymbol{i}}| \le 1, \quad 1 \le k \le m,\ 1 \le j \le n \tag{20}$$

and

$$E(N_n) = 0 \quad \text{with } N_n = \mathcal{M}(m,n) \setminus \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} T_k^j.$$

Define, for $1 \le k \le m$,

$$S_k = \bigcup_{j=1}^{n} T_k^j.$$

Obviously, we have that $N_n = \varnothing$ and hence

$$\mathcal{M}(m,n) = \bigcup_{k=1}^{m} S_k.$$

Finally, for any $1 \le k \le m$,

$$\|a^{S_k}\|_{\ell_\infty(\{k\})[\ell_q(\{\widehat{k}\})]} = \sup_{1\le j\le n} \sum_{\boldsymbol{i} \in \mathcal{M}(\{\widehat{k}\},n)} |a^{S_k}_{\boldsymbol{i}\oplus j}| \le \sup_{1\le j\le n} \sum_{\substack{\boldsymbol{i} \in \mathcal{M}(\{\widehat{k}\},n)\\ \boldsymbol{i}\oplus j \in \bigcup_{l=1}^{n} T_k^l}} |a_{\boldsymbol{i}\oplus j}| \le 1.$$

Let us comment on the argument for the last estimate: Assume without loss of generality that $n = 2$. Then, by construction, given $j = 1$ or $j = 2$ we have that either $\boldsymbol{i} \oplus j \in T_k^1$ for all $\boldsymbol{i} \in \mathcal{M}(\{\widehat{k}\},n)$ or $\boldsymbol{i} \oplus j \in T_k^2$ for all $\boldsymbol{i} \in \mathcal{M}(\{\widehat{k}\},n)$. The conclusion follows from (20). □

Lemma 4. *For each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ and every $1 \le q < \infty$,*

$$\|a\|_{\ell_{qm/((q-1)m+1),1}} \le m^{1/q} \sum_{1\le k\le m} \|a\|_{\ell_1(\{k\})[\ell_{q'}(\{\widehat{k}\})]}.$$

Proof. Since for every $1 < r < \infty$ we have $m_r = \ell_{r,\infty}$ with $\|\cdot\|_{\ell_{r,\infty}} \le \|\cdot\|_{m_r}$ and $(\ell_{r,1})' = m_r$ isometrically, the required inequality follows by Lemma 3 and a simple duality argument. Indeed, take a matrix a and sets S_k according to Lemma 3. Then

$$\begin{aligned}
\sum_{\boldsymbol{i}\in\mathcal{M}(m,n)} |a_{\boldsymbol{i}} b_{\boldsymbol{i}}| &\le \sum_{1\le k\le m} \sum_{\boldsymbol{i}\in\mathcal{M}(m,n)} |a_{\boldsymbol{i}} b^{S_k}_{\boldsymbol{i}}| \\
&\le \sum_{1\le k\le m} \|a\|_{\ell_1(\{k\})[\ell_{q'}(\{\widehat{k}\})]} \|b^{S_k}\|_{\ell_\infty(\{k\})[\ell_q(\{\widehat{k}\})]} \\
&\le \max_{1\le k\le m} \|b^{S_k}\|_{\ell_\infty(\{k\})[\ell_q(\{\widehat{k}\})]} \sum_{1\le k\le m} \|a\|_{\ell_1(\{k\})[\ell_{q'}(\{\widehat{k}\})]} \\
&\le m^{1/q} \|b\|_{\ell_{qm/(m-1),\infty}} \sum_{1\le k\le m} \|a\|_{\ell_1(\{k\})[\ell_{q'}(\{\widehat{k}\})]},
\end{aligned}$$

the desired conclusion. □

The last lemma needed is the following so-called mixed BH inequality (this is a simple consequence of the multilinear Khinchine inequality; see, e.g., [Bayart et al. 2014b; Bohnenblust and Hille 1931; Defant et al. 2016]).

Lemma 5. *For each n and each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ we have*

$$\sum_{j=1}^{n} \Bigl(\sum_{\boldsymbol{i}\in\mathcal{M}(\{\widehat{k}\},n)} |a_{\boldsymbol{i}\oplus j}|^2 \Bigr)^{1/2} \le \sqrt{2}^{\,m-1} \|a\|_\infty, \qquad 1 \le k \le m.$$

Combining Lemmas 4 (with $q = 2$) and 5 gives the proof of (4). As a byproduct we get the following estimate for the constant:

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m) \le m^{1/2}\sqrt{2}^{\,m-1}.$$

We note a disadvantage of this proof: it does not give polynomial growth of $\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m)$ in m as we obtained for $\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1)}}(m)$ in (5).

4.1. *Polynomial growth, part I.* We are going to give a first improvement of the result from (5). Our estimate shows that the symmetric Banach sequence space

$$X = \ell_{2m/(m+1),2(m-1)/m}$$

satisfies the BH inequality from (2) with a constant growing subpolynomially in m. It is important to note that X is strictly larger than the Lorentz space $\ell_{2m/(m+1),1}$; however, X has the same fundamental function as $\ell_{2m/(m+1),1}$, which of course fits with Theorem 1.

Theorem 6. *There exists a constant $\delta > 0$ such that, for each m,*

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),2(m-1)/m}}(m) \le m^{\delta}.$$

The proof combines ideas and tools from [Blei and Fournier 1989; Bohnenblust and Hille 1931; Littlewood 1930] with some more recent ones from [Bayart et al. 2014b]. The following lemma, the proof of which is explicitly included in the proof of [Bayart et al. 2014b, Proposition 3.1], is crucial. For $1 \le p \le 2$ we write $A_p \ge 1$ for the best constant in the Khinchine–Steinhaus inequality: for each choice of finitely many $\alpha_1, \dots, \alpha_N \in \mathbb{C}$,

$$\|(\alpha_k)_{k=1}^N\|_{\ell_2} \le A_p \left(\int_{\mathbb{T}^N} \left| \sum_{k=1}^N \alpha_k z_k \right|^p dz \right)^{1/p},$$

where dz stands for the normalized Lebesgue measure on the N-dimensional torus $\mathbb{T}^N$. Recall that $A_p \le \sqrt{2}$ for all $1 \le p \le 2$.

Lemma 7. *For each n, each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ and each $1 \le k < m$, we have*

$$\|a\|_{(m,n,k,2k/(k+1),2)} \le A^{m-k}_{2k/(k+1)}\, \mathrm{BH}^{\mathrm{mult}}_{\ell_{2k/(k+1)}}(k)\, \|a\|_\infty.$$

The second lemma needed is an immediate consequence of [Blei and Fournier 1989, Theorem 7.2]:

Lemma 8. *For each $1 \le q < \infty$ there is a constant $C_q \ge 1$ such that, for each $1 \le t < q$ and each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$,*

$$\|a\|_{\ell_{mqt/(mq+t-q),t}} \le C_q m \|a\|_{(m,n,m-1,t,q)}.$$

Proof of Theorem 6. For $q = 2$ and $t = 2(m-1)/m$ we have $mqt/(mq+t-q) = 2m/(m+1)$. Hence, given a matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$, Lemma 8 yields

$$\|a\|_{\ell_{2m/(m+1),2(m-1)/m}} \le C_2 m \|a\|_{(m,n,m-1,2(m-1)/m,2)}.$$

Moreover, by Lemma 7 we have

$$\|a\|_{(m,n,m-1,2(m-1)/m,2)} \le A_{2(m-1)/m}\, \mathrm{BH}^{\mathrm{mult}}_{\ell_{2(m-1)/m}}(m-1)\|a\|_\infty.$$

Combining with (5) we conclude (because $A_p \le \sqrt{2}$ for each $1 \le p \le 2$) that

$$\|a\|_{\ell_{2m/(m+1),2(m-1)/m}} \le C_2 m\sqrt{2}\kappa(m-1)^{(1-\gamma)/2}\|a\|_\infty,$$

as required. □

4.2. ***Polynomial growth, part II.*** In this section we use complex and real interpolation as well as results from [Fournier 1987] to improve Theorem 6 considerably (see Theorem 12). The starting point for what we intend to prove is the following result:

Lemma 9. *For each $m, n, k \in \mathbb{N}$ with $1 \le k \le m$ we have that*

$$\Big\| \bigoplus_{S \in \mathcal{P}_k(m)} \ell_1(S)[\ell_\infty(\hat{S})] \hookrightarrow \ell_{m/k,1}(\mathcal{M}(m,n)) \Big\| \le \binom{m}{k}^{-1}.$$

Proof. A variant of this result is mentioned without proof in [Fournier 1987, p. 69] — the special case $k = 1$ is given in Fournier's Theorem 4.1; for the general case, analyze the proof of that theorem and use in particular his Theorem 3.3 instead of Theorem 3.1, in combination with Cauchy's inequality. □

We will need the following obvious technical result; since we here are interested in precise norm estimates, we prefer to include a proof.

Lemma 10. *Let J be a finite set and let Y and X_j, $j \in J$, be Banach lattices on a measure space (Ω, Σ, μ). Then $\bigoplus_{j\in J}(X_j^{1-\theta}Y^\theta) = \big(\bigoplus_{j\in J} X_j\big)^{1-\theta} Y^\theta$ for every $\theta \in (0,1)$, with*

$$\Big\| \bigoplus_{j\in J}(X_j^{1-\theta}Y^\theta) \hookrightarrow \Big(\bigoplus_{j\in J} X_j\Big)^{1-\theta} Y^\theta \Big\| \le \operatorname{card} J,$$

$$\Big\| \Big(\bigoplus_{j\in J} X_j\Big)^{1-\theta} Y^\theta \hookrightarrow \bigoplus_{j\in J}(X_j^{1-\theta}Y^\theta) \Big\| \le \operatorname{card} J.$$

Proof. Choose $x \in \bigoplus_{j\in J}(X_j^{1-\theta}Y^\theta)$ with norm less than 1. Since $\|x\|_{X_j^{1-\theta}Y^\theta} < 1$ for each $j \in J$, there exist $y_j \in Y$ and $x_j \in X_j$ with $\|y_j\|_Y \le 1$ and $\|x_j\|_{X_j} \le 1$ for each $j \in J$ such that

$$|x| \le |x_j|^{1-\theta}|y_j|^\theta, \quad j \in J.$$

This implies

$$|x| \le \big(\min_{k\in J}|x_k|\big)^{1-\theta}\big(\max_{k\in J}|y_k|\big)^\theta.$$

Clearly, $\big\|\min_{k\in J}|x_k|\big\|_{\bigoplus_{j\in J}X_j} \le \sum_{j\in J}\|x_j\|_{X_j} \le \operatorname{card} J$ and $\big\|\max_{k\in J}|y_k|\big\|_Y \le \operatorname{card} J$ yield

$$x \in \Big(\bigoplus_{j\in J} X_j\Big)^{1-\theta} Y^\theta$$

with

$$\|x\|_{(\bigoplus_{j\in J} X_j)^{1-\theta} Y^\theta} \le \operatorname{card} J.$$

This shows the first estimate from our statement. The proof of the second statement is straightforward. □

Now we use real and complex interpolation to deduce, from Lemma 9, the following result:

Lemma 11. *For each $m, n, k \in \mathbb{N}$ with $1 \le k \le m$ we have*

$$\Big\| \bigoplus_{S\in\mathcal{P}_k(m)} \ell_{2k/(k+1)}(S)[\ell_2(\hat{S})] \hookrightarrow \ell_{2m/(m+1),2k/(k+1)}(\mathcal{M}(m,n)) \Big\| \le 2\binom{m}{k}^{3/2}.$$

Proof. We claim that the following norm estimate holds:

$$\Big\| \bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_2(\hat{S})] \hookrightarrow \ell_{2m/(m+k),1}(\mathcal{M}) \Big\| \le \sqrt{\binom{m}{k}}, \tag{21}$$

where $\mathcal{M} = \mathcal{M}(m,n)$. Indeed, combining complex interpolation first with Lemma 10 $\big($with norm $\binom{m}{k}\big)$ and then with Lemma 9 $\big($with norm $\binom{m}{k}^{-1/2}\big)$, we obtain

$$\begin{aligned}
\bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_2(\hat{S})] &= \bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)\big[[\ell_1(\hat{S}), \ell_\infty(\hat{S})]_{1/2}\big] \\
&= \bigoplus_{S\in\mathcal{P}_k(m)} \big[\ell_1(S)[\ell_1(\hat{S})], \ell_1(S)[\ell_\infty(\hat{S})]\big]_{1/2} \\
&= \bigoplus_{S\in\mathcal{P}_k(m)} \big[\ell_1(\mathcal{M}), \ell_1(S)[\ell_\infty(\hat{S})]\big]_{1/2} \\
&\hookrightarrow \Big[\ell_1(\mathcal{M}), \bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_\infty(\hat{S})]\Big]_{1/2} && \text{with norm } \le \binom{m}{k} \\
&\hookrightarrow [\ell_1(\mathcal{M}), \ell_{m/k,1}(\mathcal{M})]_{1/2} && \text{with norm } \le \binom{m}{k}^{-1/2} \\
&= \ell_{2m/(m+k),1}(\mathcal{M}).
\end{aligned}$$

Observe that the last equality here holds with equality of norms; to see this note that for every $1 < p < \infty$ and $0 < \theta < 1$ we have, by (15),

$$E := [\ell_1(\mathcal{M}), \ell_{p,1}(\mathcal{M})]_\theta = \ell_1(\mathcal{M})^{1-\theta} \ell_{p,1}(\mathcal{M})^\theta.$$

Taking Köthe duals we obtain $E' = \ell_\infty(\mathcal{M})^{1-\theta}(m_p(\mathcal{M}))^\theta = (m_p)^{1/\theta}$, which, for $\theta = \frac{1}{2}$ and $p = m/k$, gives $E' = m_{2m/(m-k)}(\mathcal{M})$, and by duality

$$E = \ell_{2m/(m+k),1}(\mathcal{M}).$$

This proves the claim from (21). Now, for $\theta_k = (k-1)/k$ we have

$$[\ell_1(S), \ell_2(S)]_{\theta_k} = \ell_{2k/(k+1)}(S).$$

Hence we deduce from (21) and, again, Lemma 10 that

$$\begin{aligned}\bigoplus_{S\in\mathcal{P}_k(m)} \ell_{2k/(k+1)}(S)[\ell_2(\widehat{S})] &= \bigoplus_{S\in\mathcal{P}_k(m)} [\ell_1(S),\ell_2(S)]_{\theta_k}[\ell_2(\widehat{S})]\\ &= \bigoplus_{S\in\mathcal{P}_k(m)} \big[\ell_1(S)[\ell_2(\widehat{S})],\ell_2(S)[\ell_2(\widehat{S})]\big]_{\theta_k}\\ &= \bigoplus_{S\in\mathcal{P}_k(m)} \big[\ell_1(S)[\ell_2(\widehat{S})],\ell_2(\mathcal{M})\big]_{\theta_k}\\ &\hookrightarrow \Big[\bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_2(\widehat{S})],\ell_2(\mathcal{M})\Big]_{\theta_k} \qquad \text{with norm } \le \binom{m}{k}\\ &\hookrightarrow [\ell_{2m/(m+k),1}(\mathcal{M}),\ell_2(\mathcal{M})]_{\theta_k} \qquad \text{with norm } \le \binom{m}{k}^{(1-\theta_k)/2}\end{aligned}$$

and so the norm of the inclusion map is less than or equal to

$$\binom{m}{k}\binom{m}{k}^{(1-\theta_k)/2} = \binom{m}{k}^{1+1/(2k)} \le \binom{m}{k}^{3/2}.$$

We now need the equality

$$[\ell_{2m/(m+k),1}(\mathcal{M}),\ell_2(\mathcal{M})]_{\theta_k} = \ell_{2m/(m+1),2k/(k+1)}$$

with

$$\big\|[\ell_{2m/(m+k),1}(\mathcal{M}),\ell_2(\mathcal{M})]_{\theta_k} \hookrightarrow \ell_{2m/(m+1),2k/(k+1)}(\mathcal{M})\big\| \le 2.$$

In fact, from (15) it follows that for $1 \le q_j \le p_j < \infty$ with $j = 0, 1$ and $\theta \in (0,1)$ we have

$$[\ell_{p_0,q_0},\ell_{p_1,q_1}]_\theta = (\ell_{p_0,q_0})^{1-\theta}(\ell_{p_1,q_1})^\theta.$$

And, further, for $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$ it can be shown, similarly to in the nonatomic case in [Grafakos and Mastyło 2014, Lemma 4.1], that in the atomic case we have

$$(\ell_{p_0,q_0})^{1-\theta}(\ell_{p_1,q_1})^\theta = \ell_{p,q}$$

with

$$\|(\ell_{p_0,q_0})^{1-\theta}(\ell_{p_1,q_1})^\theta \hookrightarrow \ell_{p,q}\| \le 2^{1/p}.$$

Thus, taking $\theta = (k-1)/k$, $q_0 = 1$, $p_0 = 2m/(m+k)$ and $p_1 = q_1 = 2$, we obtain the required embedding. Combining all together, we finally arrive at

$$\Big\|\bigoplus_{S\in\mathcal{P}_k(m)} \ell_{2k/(k+1)}(S)[\ell_2(\widehat{S})] \hookrightarrow \ell_{2m/(m+1),2k/(k+1)}\Big\| \le 2\binom{m}{k}^{3/2},$$

which completes the proof. □

A combination of (5) and Lemmas 7 and 11 leads to the following substantial improvement of Theorem 6:

Theorem 12. *For each $m, k \in \mathbb{N}$ with $1 \le k \le m$ we have*

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),2k/(k+1)}}(m) \le 2\binom{m}{k}^{3/2} A^{m-k}_{2k/(k+1)} \,\mathrm{BH}^{\mathrm{mult}}_{\ell_{2k/(k+1)}}(k).$$

In particular, for each k there is some $\delta(k) > 0$ such that, for each $m > k$,

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),2(m-k)/(m-k+1)}}(m) \le m^{\delta(k)}.$$

5. The polynomial BH inequality for Lorentz spaces

Let us start with a standard polarization argument, showing how the multilinear BH inequality in Lorentz spaces from (4) transfers to a polynomial BH inequality in Lorentz spaces (as already stated in (12)).

Theorem 13. *Given $m \in \mathbb{N}$, there is a constant $C > 0$ such that for every m-homogeneous polynomial $P = \sum_{\boldsymbol{j} \in \mathcal{J}(m,n)} c_{\boldsymbol{j}} z_{j_1} \cdots z_{j_m}$ in n complex variables we have*

$$\|(c_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}\|_{\ell_{2m/(m+1),1}} \le C\|P\|_\infty;$$

in other terms,

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1),1}}(m) < \infty.$$

Proof. Take some m-homogeneous polynomial P as above, and let $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ be the associated symmetric matrix. Then for every $\boldsymbol{j} \in \mathcal{J}(m,n)$ we have

$$c_{\boldsymbol{j}} = \mathrm{card}[\boldsymbol{j}]a_{\boldsymbol{j}}$$

and, by standard polarization,

$$\|a\|_\infty \le \frac{m^m}{m!}\|P\|_\infty.$$

Obviously,

$$\left\|\ell_{p,1}(\mathcal{M}(m,n)) \hookrightarrow \ell_{p,1}(\mathcal{J}(m,n)),\, (b_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)} \mapsto (b_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}\right\| \le 1.$$

Combining all this we obtain

$$\begin{aligned}\|(c_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}\|_{2m/(m+1),1} &= \|(\mathrm{card}[\boldsymbol{j}]a_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}\|_{\ell_{2m/(m+1),1}}\\ &\le \|(\mathrm{card}[\boldsymbol{i}]a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}\|_{\ell_{2m/(m+1),1}}\\ &\le m!\,\|(a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}\|_{\ell_{2m/(m+1),1}}\\ &\le m!\,\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m)\|a\|_\infty \le m^m\,\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m)\|P\|_\infty,\end{aligned}$$

which is the estimate we aimed for. □

5.1. *Hypercontractive growth.* We now improve the preceding theorem by showing for $X = \ell_{2m/(m+1),1}$ that the constant $\mathrm{BH}^{\mathrm{pol}}_X(m)$ in fact has hypercontractive growth in m; this extends (10) from Minkowski spaces $\ell_{2m/(m+1)}$ to Lorentz spaces $\ell_{2m/(m+1),1}$.

Theorem 14. *For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that, for each m,*

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1),1}}(m) \le C(\varepsilon)(\sqrt{2}+\varepsilon)^m.$$

Our proof needs four preliminary lemmas. The understanding of the diagonal operator

$$D(m,n)\colon \mathbb{C}^{\mathcal{M}(m,n),s} \hookrightarrow \mathbb{C}^{\mathcal{J}(m,n)}, \quad (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)} \mapsto (\operatorname{card}[\boldsymbol{j}]^{(m+1)/(2m)} a_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathcal{J}(m,n)},$$

will turn out to be crucial; here $\mathbb{C}^{\mathcal{M}(m,n),s}$ stands for all symmetric matrices in $\mathbb{C}^{\mathcal{M}(m,n)}$, namely all matrices $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ for which $a_{\boldsymbol{i}} = a_{\boldsymbol{j}}$ whenever $\boldsymbol{j} \in [\boldsymbol{i}]$. Moreover, for $1 < p < \infty$ denote by $\ell^s_{p,1}(\mathcal{M}(m,n))$ the subspace $\mathbb{C}^{\mathcal{M}(m,n),s}$ of $\ell_{p,1}(\mathcal{M}(m,n))$, and similarly define the subspace $\ell^s_p(\mathcal{M}(m,n))$ for $1 \le p < \infty$.

In Lemma 16 we will use interpolation in order to establish norm estimates for these diagonal operators in Lorentz sequence spaces. In order to do so, we need another technical lemma on real interpolation:

Lemma 15. *Let X_0 and X_1 be fully symmetric spaces on a measure space (Ω, Σ, μ). If X_0^d and X_1^d are discretizations of X_0 and X_1 generated by the same measurable partition of Ω, then for every $\theta \in (0,1)$ and $1 \le q \le \infty$ the inclusion map* id: $(X_0^d, X_1^d)_{\theta,q} \to (X_0, X_1)_{\theta,q}$ *is an isometric isomorphism, i.e.,*

$$\|f\|_{(X_0^d, X_1^d)_{\theta,q}} = \|f\|_{(X_0,X_1)_{\theta,q}} \quad \textit{for } f \in (X_0^d, X_1^d)_{\theta,q}.$$

Proof. Let $\{\Omega_k\}_{k=1}^N \subset \Sigma$ be a given measurable partition of Ω. Define the linear map

$$P\colon L_1(\mu) + L_\infty(\mu) \to L_1(\mu) + L_\infty(\mu), \quad f \mapsto \sum_{k=1}^N \left(\frac{1}{\mu(\Omega_k)} \int_{\Omega_k} f\, d\mu\right) \chi_{\Omega_k}.$$

Since $P\colon (L_1(\mu), L_\infty(\mu)) \to (L_1(\mu), L_\infty(\mu))$ with $\|P\|_{L_1(\mu)\to L_1(\mu)} \le 1$ and $\|P\|_{L_\infty(\mu)\to L_\infty(\mu)} \le 1$, and X_0 and X_1 are fully symmetric, it follows that

$$P\colon (X_0, X_1) \to (X_0^d, X_1^d)$$

with $\|P\|_{X_j \to X_j^d} \le 1$ for $j \in \{0,1\}$. This implies that, for every $f \in X_0^d + X_1^d$, we have, since $P(f) = f$,

$$K(t, f; X_0^d, X_1^d) = K(t, Pf; X_0, X_1) \le K(t, f; X_0, X_1), \quad t > 0.$$

Since the opposite inequality is obvious, the required statement follows. □

The next result will be essential:

Lemma 16. *There is a uniform constant $L > 0$ such that, for each m and n,*

$$\left\|D(m,n)\colon \ell^s_{2m/(m+1),1}(\mathcal{M}(m,n)) \hookrightarrow \ell_{2m/(m+1),1}(\mathcal{J}(m,n))\right\| \le Lm.$$

Proof. The proof is based on interpolation, and the abbreviations $\mathcal{M} = \mathcal{M}(m,n)$ and $\mathcal{J} = \mathcal{J}(m,n)$ will be used. We claim that

$$\|D(m,n)\colon \ell^s_1(\mathcal{M}) \to \ell_1(\mathcal{J})\| \le 1, \quad \|D(m,n)\colon \ell^s_2(\mathcal{M}) \to \ell_2(\mathcal{J})\| \le \sqrt{m}. \tag{22}$$

Indeed, for every $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$ we have

$$\begin{aligned}\|D(m,n)a\|_{\ell_1(\mathcal{J})} &= \sum_{\boldsymbol{j}\in\mathcal{J}} \operatorname{card}[\boldsymbol{j}]^{(m+1)/(2m)} |a_{\boldsymbol{j}}| = \sum_{\boldsymbol{j}\in\mathcal{J}} \operatorname{card}[\boldsymbol{j}]^{(m+1)/(2m)-1} \operatorname{card}[\boldsymbol{j}]\,|a_{\boldsymbol{j}}| \\ &\le \sum_{\boldsymbol{j}\in\mathcal{J}} \operatorname{card}[\boldsymbol{j}]\,|a_{\boldsymbol{j}}| = \sum_{\boldsymbol{i}\in\mathcal{M}} |a_{\boldsymbol{i}}| = \|a\|_{\ell^s_1(\mathcal{M})}\end{aligned}$$

and

$$\|D(m,n)a\|_{\ell_2(\mathcal{J})} = \Big(\sum_{\boldsymbol{j}\in\mathcal{J}} \operatorname{card}[\boldsymbol{j}]^{(m+1)/m}|a_{\boldsymbol{j}}|^2\Big)^{1/2} = \Big(\sum_{\boldsymbol{j}\in\mathcal{J}} \operatorname{card}[\boldsymbol{j}]^{(m+1)/m-1}\operatorname{card}[\boldsymbol{j}]|a_{\boldsymbol{j}}|^2\Big)^{1/2}$$
$$= (m!)^{1/(2m)}\Big(\sum_{\boldsymbol{j}\in\mathcal{J}} \operatorname{card}[\boldsymbol{j}]|a_{\boldsymbol{j}}|^2\Big)^{1/2} \le \sqrt{m}\Big(\sum_{\boldsymbol{i}\in\mathcal{M}}|a_{\boldsymbol{i}}|^2\Big)^{1/2} = \sqrt{m}\,\|a\|_{\ell_2^s(\mathcal{M})},$$

which proves (22). We now apply the two-sided norm estimate from (14). In the special case when $p_0 = q_0 = 1$, $p_1 = q_1 = 2$, $q = 1$ and $\theta = (m-1)/m$, we have $p = 2m/(m+1)$ and, in particular, $1 \le (p/q)^{1/q} = 2m/(m+1) < 2$. Then, for $I = \mathcal{M}(m,n)$ or $I = \mathcal{J}(m,n)$,

$$(\ell_1(I), \ell_2(I))_{(m-1)/m,1} = \ell_{2m/(m+1),1}(I),$$

and there is $C > 0$ such that, for all $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$,

$$\frac{m^{3/2}}{C(m-1)}\|a\|_{\ell_{2m/(m+1),1}(I)} \le \|a\|_{(\ell_1(I),\ell_2(I))_{(m-1)/m,1}} \le \frac{Cm^2}{m-1}\|a\|_{\ell_{2m/(m+1),1}(I)}. \tag{23}$$

It follows from Lemma 15 that

$$\|a\|_{(\ell_1^s(\mathcal{M}),\ell_2^s(\mathcal{M}))_{(m-1)/m,1}} = \|a\|_{(\ell_1(\mathcal{M}),\ell_2(\mathcal{M}))_{(m-1)/m,1}} \quad \text{for } a \in \mathbb{C}^{\mathcal{M}(m,n),s}. \tag{24}$$

Now we interpolate; we recall that, for every operator T between interpolation pairs (A_0, A_1) and (B_0, B_1) and every $0 < \theta < 1$, we have

$$\|T\colon (A_0, A_1)_{\theta,1} \to (B_0, B_1)_{\theta,1}\| \le \|T\colon A_0 \to B_0\|^{1-\theta}\|T\colon A_1 \to B_1\|^{\theta}.$$

In particular,

$$\big\|D(m,n)\colon (\ell_1^s(\mathcal{M}), \ell_2^s(\mathcal{M}))_{(m-1)/m,1} \to (\ell_1(\mathcal{J}), \ell_2(\mathcal{J}))_{(m-1)/m,1}\big\|$$
$$\le \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{1/m}\|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{(m-1)/m}.$$

As a consequence we obtain that, for every $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$,

$$\frac{m^{3/2}}{C(m-1)}\|D(m,n)a\|_{\ell_{2m/(m+1),1}(\mathcal{M})}$$
$$\overset{(23)}{\le} \|D(m,n)a\|_{(\ell_1(\mathcal{J}),\ell_2(\mathcal{J}))_{(m-1)/m,1}}$$
$$\le \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{1/m}\|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{(m-1)/m}\|a\|_{(\ell_1^s(\mathcal{M}),\ell_2^s(\mathcal{M}))_{(m-1)/m,1}}$$
$$\overset{(24)}{=} \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{1/m}\|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{(m-1)/m}\|a\|_{(\ell_1(\mathcal{M}),\ell_2(\mathcal{M}))_{(m-1)/m,1}}$$
$$\overset{(23)}{\le} \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{1/m}\|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{(m-1)/m}\frac{Cm^2}{m-1}\|a\|_{\ell_{2m/(m+1),1}(\mathcal{J})}.$$

Combining the above estimates with (22), we conclude that, for every $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$,

$$\|D(m,n)a\|_{\ell_{2m/(m+1),1}(\mathcal{M})} \le C^2\sqrt{m}\sqrt{m}^{\,(m-1)/m}\|a\|_{\ell_{2m/(m+1),1}(\mathcal{J})} \le C^2 m\|a\|_{\ell_{2m/(m+1),1}(\mathcal{J})},$$

and this completes the proof. □

In what follows we will need the Khinchine–Steinhaus inequality for homogeneous polynomials due to [Bayart 2002]: given $0 < p < q < \infty$, for every m-homogeneous polynomial P on $\mathbb{C}^n$ we have

$$\left(\int_{\mathbb{T}^n} |P(z)|^q \, dz\right)^{1/q} \le \sqrt{\frac{q}{p}}^{\,m} \left(\int_{\mathbb{T}^n} |P(z)|^p \, dz\right)^{1/p}; \tag{25}$$

note that it is shown in [Defant and Mastyło 2015, Theorem 2.1] that the constant $\sqrt{q/p}$ that appears is optimal. For the proof of Theorem 14, this fact will only be used for the case $p = 1$ and $q = 2$.

Next, we also require a lemma — which is (implicitly) in [Bayart et al. 2014b] and (explicitly) in [Defant et al. 2016, Section 9] — however only in the case $k = 1$.

Lemma 17. *Let $P = \sum_{\boldsymbol{j} \in \mathcal{J}(m,n)} c_{\boldsymbol{j}} z_{j_1} \cdots z_{j_m}$ be an m-homogeneous polynomial in n variables and let $a = (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$ be its associated symmetric matrix. Then for every $S \in \mathcal{P}_k(m)$, $1 \le k \le m$, we have*

$$\left(\sum_{\boldsymbol{i} \in \mathcal{M}(S,n)} \left(\sum_{\boldsymbol{j} \in \mathcal{M}(\widehat{S},n)} \operatorname{card}[\boldsymbol{j}]\, |a_{\boldsymbol{i} \oplus \boldsymbol{j}}|^2\right)^{\frac{1}{2}\frac{2k}{k+1}}\right)^{\frac{k+1}{2k}} \le \sqrt{\frac{k+1}{k}}^{\,m-k} \frac{(m-k)!\, m^m}{(m-k)^{m-k} m!} B^{\mathrm{mult}}_{\ell_{2k/(k+1)}}(k) \|P\|_\infty.$$

The fourth lemma is an immediate consequence of [Blei and Fournier 1989, Theorem 3.3]; here we will use only the case $q = 2$.

Lemma 18. *Given $1 \le q < \infty$, there is a constant $C_q \ge 1$ such that, for every matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$,*

$$\|a\|_{\ell_{mq/(m+q-1),1}} \le C_q m \|a\|_{(m,n,1,1,q)}.$$

We are now ready to give the proof of Theorem 14.

Proof of Theorem 14. Assume that P is an m-homogeneous polynomial on $\mathbb{C}^n$ with coefficients $(c_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}$ and denote by $(a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ the coefficients of the associated symmetric m-linear form A. We have the simple fact that, for all $\boldsymbol{i} \in \mathcal{M}(\{1\},n)$ and $\boldsymbol{j} \in \mathcal{M}(\widehat{\{1\}},n)$,

$$\operatorname{card}[\boldsymbol{i} \oplus \boldsymbol{j}] \le m \operatorname{card}[\boldsymbol{j}].$$

Hence we deduce from Lemmas 16, 18 (with $q = 2$) and 17 (with $k = 1$) that, for each m and n,

$$\begin{aligned}
&\|(c_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}\|_{2m/(m+1),1} \\
&\quad = \|(\operatorname{card}[\boldsymbol{i}]\, a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{J}(m,n)}\|_{2m/(m+1),1} \\
&\quad \le Lm \|(\operatorname{card}[\boldsymbol{i}]^{1-(m+1)/2m} a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}\|_{2m/(m+1),1} \\
&\quad \le LmC_2 m \|(\operatorname{card}[\boldsymbol{i}]^{1-(m+1)/(2m)} a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}\|_{(m,n,1,1,2)} \\
&\quad = LmC_2 m \max_{S \in \mathcal{P}_1(m)} \sum_{\boldsymbol{i} \in \mathcal{M}(\{1\},n)} \left(\sum_{\boldsymbol{j} \in \mathcal{M}(\widehat{\{1\}},n)} \left|\operatorname{card}[\boldsymbol{i} \oplus \boldsymbol{j}]^{(m-1)/(2m)} a_{\boldsymbol{i} \oplus \boldsymbol{j}}\right|^2\right)^{1/2} \\
&\quad \le LmC_2 m \max_{S \in \mathcal{P}_1(m)} \sum_{\boldsymbol{i} \in \mathcal{M}(\{1\},n)} \left(\sum_{\boldsymbol{j} \in \mathcal{M}(\widehat{\{1\}},n)} \left|(m \operatorname{card}[\boldsymbol{j}])^{(m-1)/(2m)} a_{\boldsymbol{i} \oplus \boldsymbol{j}}\right|^2\right)^{1/2}
\end{aligned}$$

$$\leq LmC_2 m m^{(m-1)/(2m)} \max_{S\in\mathcal{P}_1(m)} \sum_{\boldsymbol{i}\in\mathcal{M}(\{1\},n)} \Bigg(\sum_{\boldsymbol{j}\in\mathcal{M}(\widehat{\{1\}},n)} \mathrm{card}[\boldsymbol{j}]^{(m-1)/m} |a_{\boldsymbol{j}}|^2 \Bigg)^{1/2}$$

$$\leq LmC_2 m m^{(m-1)/(2m)} \max_{S\in\mathcal{P}_1(m)} \sum_{\boldsymbol{i}\in\mathcal{M}(\{1\},n)} \Bigg(\sum_{\boldsymbol{j}\in\mathcal{M}(\widehat{\{1\}},n)} \mathrm{card}[\boldsymbol{j}] |a_{\boldsymbol{j}}|^2 \Bigg)^{1/2}$$

$$\leq LmC_2 m m^{(m-1)/(2m)} \sqrt{2}^{m-1} \times \frac{(m-1)!m^m}{(m-1)^{m-1}m!} \times B^{\mathrm{mult}}_{\ell_1}(1) \times \|P\|_\infty.$$

This completes the argument. □

We conclude with the following remark: The estimate (11) suggests that the constant $\sqrt{2}$ in Theorem 14 could be improved. Here $\sqrt{2}$ appears since our proof applies (25) for $p=1$ and $q=2$, which is an inequality on homogeneous polynomials of arbitrary degree m. We have already indicated that the constant $\sqrt{2}$ in the inequality (25) is optimal (note that, in contrast to this, the best constant in (25) for polynomials of degree only $m=1$ equals $\sqrt{\pi}/2$; see [Sawa 1985; König 2014]).

5.2. ***The Balasubramanian–Calado–Queffélec result revisited.*** In this section we improve a remarkable result by Balasubramanian, Calado and Queffélec [Balasubramanian et al. 2006]. By $\mathcal{P}(^m c_0)$ we denote the linear space of all m-homogeneous continuous polynomials on c_0, which, together with the supremum norm on the open unit ball in c_0, forms a Banach space. On the subspace c_{00} of all finite sequences in c_0, each such polynomial has a unique monomial series decomposition $P(z)=\sum_{|\alpha|=m} c_\alpha(P) z^\alpha$, $z\in c_{00}$, (or, in different notation, $P(z)=\sum_{\boldsymbol{j}\in\mathcal{J}(m)} c_{\boldsymbol{j}} z_{\boldsymbol{j}}$, $z\in c_{00}$). A Dirichlet series $D=\sum_n a_n n^{-s}$ is said to be m-homogeneous whenever $a_n\neq 0$ implies $n=\mathfrak{p}^\alpha$ and $|\alpha|=m$ (where $\mathfrak{p}$ is the sequence of primes). All m-homogeneous Dirichlet series $D=\sum_n a_n n^{-s}$ which converge on $\{s:\mathrm{Re}\, s>0\}$ and are such that the holomorphic function $D(s)=\sum_{n=1}^\infty a_n n^{-s}$ for $\mathrm{Re}\, s>0$ is bounded form (together with the supremum norm on $\{s:\mathrm{Re}\, s>0\}$) the Banach space $\mathcal{H}^m_\infty$.

It is remarkable that there is a unique isometric isomorphism

$$\mathfrak{B}\colon \mathcal{P}(^m c_0)\to\mathcal{H}^m_\infty,\quad P=\sum_{|\alpha|=m} c_\alpha(P) z^\alpha \mapsto D=\sum_n a_n n^{-s},$$

such that $c_\alpha=a_n$ whenever $n=\mathfrak{p}^\alpha$. (For more information see [Defant et al. 2016; Defant and Sevilla-Peris 2014; Queffélec and Queffélec 2013].) Then the following theorem is an immediate consequence of this identification and Theorem 14:

Theorem 19. *For every Dirichlet series* $D=\sum_n a_n n^{-s}\in\mathcal{H}^m_\infty$ *we have* $(a_n^*)\in\ell_{2m/(m+1),1}$. *More precisely, for every* $\varepsilon>0$ *there is* $C(\varepsilon)>0$ *such that, for every* $D\in\mathcal{H}^m_\infty$,

$$\sum_{n=1}^\infty a_n^* \frac{1}{n^{(m-1)/(2m)}} \leq C(\varepsilon)(\sqrt{2}+\varepsilon)^m \|D\|_\infty. \tag{26}$$

At the end of the previous section we discuss in some detail why our proof of Theorem 14 and then also (26) leads to the constant $\sqrt{2}$.

Note that for every sequence $a = (a_n) \in \ell_{2m/(m+1),1}$ we have

$$\sum_{n=1}^{\infty} |a_n| \frac{1}{n^{(m-1)/(2m)}} \leq \sum_{n=1}^{\infty} a_n^* \frac{1}{n^{(m-1)/(2m)}} \asymp \|a\|_{\ell_{2m/(m+1),1}} < \infty.$$

Balasubramanian et al. [2006] proved that there is a constant $c(m) > 0$ such that, for every Dirichlet series $D = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}_\infty^m$,

$$\sum_{n=1}^{\infty} |a_n| \frac{(\log n)^{(m-1)/2}}{n^{(m-1)/(2m)}} \leq c(m) \|D\|_\infty, \tag{27}$$

and in addition it is shown that the exponent in the log term is optimal. In contrast to (26), it is unknown whether the best constant in (27) has exponential growth.

A natural question appears: how is this result related to the estimate from Theorem 19? To see this, let $\ell_1(\omega)$ be the weighted ℓ_1-space with weight $\omega = (\omega_n)$ given by

$$\omega_n = \frac{(\log n)^{(m-1)/2}}{n^{(m-1)/(2m)}}, \quad n \in \mathbb{N}. \tag{28}$$

We observe that $\ell_1(\omega)$ is different from $\ell_{2m/(m+1),1}$; in fact, if we would have $\ell_1(\omega) \subset \ell_{2m/(m+1),1}$, or equivalently $\ell_1 \subset \ell_{2m/(m+1),1}(\omega^{-1})$, then by the closed graph theorem

$$\sup_{n \in \mathbb{N}} \|e_n\|_{\ell_{2m/(m+1),1}(\omega^{-1})} < \infty.$$

But since, for each $n \in \mathbb{N}$,

$$\|e_n\|_{\ell_{2m/(m+1),1}(\omega^{-1})} = \left\| \frac{e_n}{\omega_n} \right\|_{\ell_{2m/(m+1),1}} = \frac{n^{(m-1)/(2m)}}{(\log n)^{(m-1)/m}},$$

we get a contradiction. Similarly, if $\ell_{2m/(m+1),1} \subset \ell_1(\omega)$ then there would exist a constant $C > 0$ such that, for each $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \frac{(\log n)^{(m-1)/2}}{n^{(m-1)/(2m)}} = \left\| \sum_{n=1}^{N} e_n \right\|_{\ell_1(\omega)} \leq C \left\| \sum_{n=1}^{N} e_n \right\|_{\ell_{2m/(m+1),1}} = C N^{(m+1)/(2m)},$$

which is again impossible. We conclude the paper with the following formal improvement of Theorem 19 and the Balasubramanian–Calado–Queffélec result (27):

Corollary 20. *For each $m \in \mathbb{N}$ and every Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}_\infty^m$,*

$$(a_n)_n \in \ell_1(\omega) \cap \ell_{2m/(m+1),1},$$

where the weight ω is given by the formula (28).

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