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MIURA MAPS AND INVERSE SCATTERING FOR THE NOVIKOV–VESELOV EQUATION

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We use the inverse scattering method to solve the zero-energy Novikov–Veselov (NV) equation for initial data of conductivity type, solving a problem posed by Lassas, Mueller, Siltanen, and Stahel. We exploit Bogdanov's Miura-type map which transforms solutions of the modified Novikov–Veselov (mNV) equation into solutions of the NV equation. We show that the Cauchy data of conductivity type considered by Lassas, Mueller, Siltanen, and Stahel lie in the range of Bogdanov's Miura-type map, so that it suffices to study the mNV equation. We solve the mNV equation using the scattering transform associated to the defocussing Davey–Stewartson II equation.

1. Introduction

In this paper we will use inverse scattering methods to solve the Novikov–Veselov (NV) equation, a completely integrable, dispersive nonlinear equation in two space and one time $(2+1)$ dimensions, for the class of *conductivity type* initial data that we define below. Our results solve a problem posed by Lassas, Mueller, Siltanen and Stahel [Lassas et al. 2012] in their analytical study of the inverse scattering method for the NV equation.

Denoting $z = x_1 + ix_2$, $\overline{\partial} = (1/2)(\partial_{x_1} + i\partial_{x_2})$, $\partial = (1/2)(\partial_{x_1} - i\partial_{x_2})$, the Cauchy problem for the NV equation is

$$q_t + \partial^3 q + \overline{\partial}^3 q - \tfrac{3}{4}\partial(q\overline{\partial}^{-1}\partial q) - \tfrac{3}{4}\overline{\partial}(q\partial^{-1}\overline{\partial} q) = 0, \qquad (1\text{-}1)$$
$$q|_{t=0} = q_0.$$

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where q_0 is a real-valued function that vanishes at infinity. The NV equation generalizes the celebrated KdV equation

$$q_t + q_{xxx} + 6qq_x = 0$$

in the sense that any solution of KdV (after rescaling) solves NV when regarded as a function of (x_1, x_2, t) with no x_2-dependence. As has recently been proved by Angelopoulos [2013], the Cauchy problem for the NV equation is locally well-posed in the Sobolev space $H^s(\mathbb{R}^2)$ for any $s > 1$. The inverse scattering method considered here yields solutions global in time, albeit for a more restrictive class of initial data.

The Novikov–Veselov equation is one of a hierarchy of dispersive nonlinear equations in $2+1$ dimensions discovered by Novikov and Veselov [1984; 1986]. Up to trivial scalings, our equation is the zero-energy ($E = 0$) case of the equation they studied, which reads

$$\begin{aligned} q_t &= 4\,\mathrm{Re}(4\partial^3 q + \partial(qw) - E\partial q), \\ \bar{\partial} w &= \partial q. \end{aligned} \tag{1-2}$$

In the papers cited, Novikov and Veselov constructed explicit solutions from the spectral data associated to a two-dimensional Schrödinger problem at a single energy. Novikov conjectured that the inverse problem for the two-dimensional Schrödinger operator at a fixed energy should be completely solvable (see the remarks in [Grinevich 2000]), and that inverse scattering for the Schrödinger equation at a fixed energy E could be used to solve the NV equation at the same energy E by inverse scattering. Subsequent studies [Grinevich 1986; Grinevich and Manakov 1986; Grinevich and Novikov 1985; 1986; 1988b; 1988a; 1995] further developed the inverse scattering method and constructed multisoliton solutions (see also [Kazeykina 2012a; 2012b; Kazeykina and Novikov 2011a; 2011b; 2011c] for further results). Independently, Boiti, Leon, Manna, and Pempinelli [Boiti et al. 1987] proposed an inverse scattering method to solve the NV equation at zero energy with data vanishing at infinity. We refer the reader to the recent survey [Croke et al. 2013] for further references and further information on the Novikov–Veselov equation. Recently, Angelopoulos [2013] has proved local well-posedness for the Novikov–Veselov equation in the space $H^s(\mathbb{R}^2)$ for $s > \frac{1}{2}$.

It has long been understood that the inverse Schrödinger scattering problem at zero energy poses special challenges (see, for example, the discussion in Part I of supplement 1 in [Grinevich and Novikov 1988a], and the comments in [Grinevich 2000, Section 7.3]). In particular, the scattering transform for the Schrödinger operator at zero energy is known to be well-behaved *only* for a special class of potentials, the potentials of "conductivity type", which may be thought of as follows.

Definition 1.1. A real-valued function $u \in C_0^\infty(\mathbb{R}^2)$ is called a *potential of conductivity type* if the equation $(-\Delta + q)\psi = 0$ admits a unique, strictly positive solution normalized so that $\psi(z) = 1$ in a neighborhood of infinity.

Remark 1.2. If q is a potential of conductivity type, it is not difficult to see that the corresponding Schrödinger operator has no eigenvalues (including no eigenvalues at zero energy), and that $q = \psi^{-1}(\Delta\psi)$ for a *unique* strictly positive function ψ with $\psi(z) = 1$ near infinity. See [Music et al. 2013] for further discussion.

The class of conductivity type potentials can also be defined for less regular q (see [Nachman 1996, Theorem 3]), but this definition will suffice for the present purpose. The terminology comes from the connection of the Schrödinger inverse problem at zero energy with Calderón's inverse conductivity problem [Calderón 1980] (see [Nachman 1996] for a solution for conductivities $\sigma \in W^{2,p}$ via the scattering transform, and see [Astala and Päivärinta 2006] for the solution to Calderón's inverse problem for general $\gamma \in L^\infty$, and for references to the literature). The problem is to reconstruct the conductivity γ of a conducting body $\Omega \subset \mathbb{R}^2$ from the Dirichlet to Neumann map, defined as follows. Let $f \in H^{1/2}(\partial\Omega)$ and let $u \in H^1(\Omega)$ solve the problem

$$\nabla \cdot (\gamma \nabla u) = 0, \quad u|_{\partial\Omega} = f.$$

This problem has a unique solution for conductivities $\gamma \in L^\infty(\Omega)$ with $\gamma(z) \geq c > 0$ for a.e. z. The Dirichlet to Neumann map is the mapping

$$\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad f \mapsto \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

Nachman [1996] exploited the fact that $v = \gamma^{1/2} u$ solves the Schrödinger equation at zero energy where

$$q = \gamma^{-1/2} \Delta(\gamma^{1/2}). \tag{1-3}$$

The Schrödinger problem also has a Dirichlet to Neumann map

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad f \mapsto \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega},$$

defined by the unique solution of

$$(-\Delta + q)v = 0, \quad v|_{\partial\Omega} = f.$$

The operator Λ_q determines and is determined by the scattering data for q of the form (1-3) at zero energy, and Λ_q determines Λ_γ. Note that q is of conductivity type if we take $\psi = \gamma^{1/2}$ and extend ψ to $\mathbb{R}^2 \backslash \Omega$ setting $\psi(z) = 1$. Nachman showed that the scattering transform at zero energy is well-defined *only* when q is of conductivity type (we give a precise statement below) and used the inverse scattering transform to reconstruct q from its scattering data.

The set of conductivity-type potentials is highly unstable, even under $C_0^\infty(\mathbb{R}^2)$ perturbations of arbitrarily small size. To explain this, we recall from [Murata 1986] (see also [Gesztesy and Zhao 1995] for more recent work and further references) that a Schrödinger operator is called

(i) *subcritical* if $-\Delta + q$ has a positive Green's function,

(ii) *critical* if $-\Delta + q$ does not have a positive Green's function, but the quadratic form

$$\mathfrak{q}(\varphi) = \int_{\mathbb{R}^2} \left(|(\nabla\varphi)(z)|^2 + q(z)|v(z)|^2 \right) dA(z)$$

on $C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$ is nonnegative, or

(iii) *supercritical* if the quadratic form $\mathfrak{q}$ is not nonnegative.

It follows from Theorem 3.1(iii) of [Murata 1986] that a conductivity-type potential is critical. From Theorem 2.4(i) of the same reference we may conclude that for any $w \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and any $\lambda > 0$, the potential $q_0 - \lambda w$ is subcritical and not of conductivity type. We refer the reader to Appendix B of [Music et al. 2013] for further details.

Thus, the set of conductivity-type potentials is nowhere dense in any reasonable function space! For this reason one expects the direct and inverse scattering maps for the Schrödinger operator at zero energy not to have good continuity properties as a function of the potential q.

Let us describe the direct scattering transform $\mathcal{T}$ and inverse scattering transform $\mathcal{Q}$ for the Schrödinger operator at zero energy in more detail (see [Nachman 1996] and [Lassas et al. 2012] for details and references). To define the direct scattering map $\mathcal{T}$ on potentials $q \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we seek complex geometric optics (CGO) solutions $\psi = \psi(z, k)$ of

$$(-\Delta + q)\psi = 0, \tag{1-4}$$

which satisfy the asymptotic condition

$$\lim_{|z|\to\infty} e^{-ikz}\psi(z, k) = 1 \tag{1-5}$$

for a fixed $k \in \mathbb{C}$. Let $m(z, k) = e^{-izk}\psi(z, k)$. Assuming that the problem (1-4)–(1-5) has a unique solution for all k, we define the scattering transform $\mathbf{t} = \mathcal{T}q$ via the formula

$$\mathbf{t}(k) = \int e^{i(\bar{k}\bar{z}+kz)} q(z)m(z, k)\, dA(z), \tag{1-6}$$

where $dA(z)$ is Lebesgue measure on $\mathbb{R}^2$. The surprising fact is that, if $\mathbf{t}$ is well-behaved, the solutions $\psi(z, k)$, and hence the potential q, may be recovered from $\mathbf{t}(k)$. This fact leads to an inverse scattering transform $q = \mathcal{Q}\mathbf{t}$ given by

$$q(z) = \frac{i}{\pi^2}\bar{\partial}_z\left(\int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e^{-i(kz+\bar{k}\bar{z})}\overline{m(z, k)}\, dA(k)\right). \tag{1-7}$$

Boiti, Leon, Manna and Pempinelli [Boiti et al. 1987], proposed an inverse scattering solution to the Novikov–Veselov equation using these maps:

$$q(t) = \mathcal{Q}\left(e^{it((\diamond)^3+(\bar{\diamond})^3)}(\mathcal{T}q_0)(\diamond)\right), \tag{1-8}$$

and gave formal arguments to justify it. The maps were further studied in [Tsai 1993]. Lassas, Mueller, Siltanen, and Stahel [Lassas et al. 2012], building on [Lassas et al. 2007], showed that the scattering transforms are well-defined for certain potentials of conductivity type. For conductivity-type potentials, they proved that $\mathcal{T}$ and $\mathcal{Q}$ are inverses, and that (1-8) defines a continuous $L^p(\mathcal{R}^2)$-valued function of t for $p \in (1, 2)$. They conjectured that $q(t)$ is in fact a classical solution of (1-1) if q_0 is a smooth, decreasing, real-valued potential of conductivity type but were unable to prove that this was the case.

The fact, already mentioned, that conductivity-type potentials are a nowhere dense set in the space of potentials, suggests that studying the NV equation using the maps $\mathcal{T}$ and $\mathcal{Q}$ is likely to be technically challenging. The following result of Nachman makes the difficulty clearer. For given q, let $\mathcal{E}_q$ be the set of all k for which the problem (1-4)–(1-5) does *not* have a unique solution. Let $L^p_\rho(\mathbb{R}^2)$ denote the

Banach space of real-valued measurable functions q with

$$\|q\|_{L^p_\rho} := \left[\int (1+|z|)^{p\rho} |q(z)|^p \, dA(z) \right]^{1/p} < \infty.$$

Theorem 1.3 [Nachman 1996, Theorem 3]. *Suppose that $q \in L^p_\rho(\mathbb{R}^2)$ for some $p \in (1,2)$, and $\rho > 1$: The following are equivalent*:

(i) *The set $\mathcal{E}_q$ is empty and $|\mathbf{t}(k)| \le C|k|^\varepsilon$ for some fixed $\varepsilon > 0$ and all sufficiently small k.*

(ii) *There is a real-valued function $\gamma \in L^\infty(\mathbb{R}^2)$ with $\gamma(z) \ge c > 0$ for a.e. z and a fixed constant c so that $q = \gamma^{-1/2}\Delta(\gamma^{1/2})$.*

One should think of γ as ψ^2 where ψ is the unique normalized positive solution of $(-\Delta+q)\psi = 0$ for a potential of conductivity type. Nachman's result suggests that non-conductivity type potentials will have singular scattering transforms: Music, Perry and Siltanen [Music et al. 2013] construct an explicit one-parameter deformation $\lambda \mapsto q_\lambda$ of a conductivity type potentials (q_0 is of conductivity type, but q_λ is not for $\lambda \neq 0$) for which the corresponding family $\lambda \mapsto \mathbf{t}_\lambda$ of scattering transforms has an essential singularity at $\lambda = 0$.

We will show that, nonetheless, the formula (1-8) does yield classical solutions of the NV equation for a much larger class of initial data than considered in [Lassas et al. 2012]. We achieve this result by circumventing the scattering maps studied in [Lassas et al. 2012]. Instead, we exploit Bogdanov's observation [1987] (see also [Dubrovsky and Gramolin 2008; 2009]) that the Miura-type map

$$\mathcal{M}(v) = 2\partial v + |v|^2 \tag{1-9}$$

takes solutions u of the modified Novikov–Veselov (mNV) equation

$$u_t + (\partial^3 + \bar\partial^3)u - NL(u) = 0, \tag{1-10}$$

where

$$NL(u) = \tfrac{3}{4}(\partial\bar u)\cdot(\bar\partial\partial^{-1}(|u|^2)) + \tfrac{3}{4}(\bar\partial u)\cdot(\bar\partial\partial^{-1}(|u|^2)) + \tfrac{3}{4}\bar u\bar\partial\partial^{-1}(\bar u\bar\partial u) + \tfrac{3}{4}u\partial^{-1}(\bar\partial(\bar u\bar\partial u)),$$

to solutions q of the NV equation. This map is an analogue of the celebrated Miura map $u \mapsto u_x + u^2$ which takes solutions of the modified Korteweg–de Vries equation to solutions of the Korteweg–de Vries equation [Miura 1968; Kappeler et al. 2005]. We remark that local well-posedness for the mNV equation in $H^s(\mathbb{R}^2)$ for any $s > 1$ was recently proved in [Angelopoulos 2013].

In (1-9), the domain of the Miura map is understood to be smooth functions v with $\partial v = \overline{\partial v}$. As we will show, the range of this Miura-type map consists exactly of initial data of conductivity type! In particular, we show that the range of $\mathcal{M}$ contains the conductivity-type potentials studied by in [Lassas et al. 2012].

Thus, to solve the NV equation for initial data of conductivity type, it suffices to solve the mNV equation and use the map $\mathcal{M}$ to obtain a solution of NV. The mNV equation is a member of the Davey–Stewartson II hierarchy, so the well-known scattering maps for the DS II hierarchy (see [Fokas and Ablowitz 1983; 1984; Beals and Coifman 1984; 1985; 1989; 1990; Brown 2001; Perry 2011; Sung 1994a; 1994b; 1994c]) can be used to solve the Cauchy problem for mNV. We denote by $\mathcal{R}$ and $\mathcal{I}$ respectively the scattering transform and inverse scattering transform associated to the defocusing DS II equation (see Section 3 for the definitions). We show in Appendix A that the function

$$u(t) = \mathcal{I}\big(\exp((\bar{\diamond}^3 - \diamond^3)t)(\mathcal{R}u_0)(\diamond)\big) \tag{1-11}$$

is a classical solution of the mNV equation (1-10) for initial data $u_0 \in \mathcal{S}(\mathbb{R}^2)$.

In order to obtain good mapping properties for the solution map $u_0 \mapsto u(t)$ defined by (1-11), we need local Lipschitz continuity of the maps $\mathcal{I}$ and $\mathcal{R}$ on spaces that are preserved under the flow (compare the treatment of the cubic NLS in one dimension in [Deift and Zhou 2003] and the Sobolev mapping properties for the scattering maps for NLS proven in [Zhou 1998]). In [Perry 2011] it was shown that $\mathcal{R}$ and $\mathcal{I}$ are mutually inverse mappings of $H^{1,1}(\mathbb{R}^2)$ into itself where

$$H^{m,n}(\mathbb{R}^2) = \big\{u \in L^2(\mathbb{R}^2) : (1-\Delta)^{m/2}u,\ (1+|\cdot|)^n u(\cdot) \in L^2(\mathbb{R}^2)\big\}.$$

In order to use (1-11), we need the following refined mapping property of $\mathcal{I}$ and $\mathcal{R}$.

Theorem 1.4. *The scattering maps $\mathcal{R}$ and $\mathcal{I}$ restrict to locally Lipschitz continuous maps*

$$\mathcal{R} : H^{2,1}(\mathbb{R}^2) \to H^{1,2}(\mathbb{R}^2), \quad \mathcal{I} : H^{1,2}(\mathbb{R}^2) \to H^{2,1}(\mathbb{R}^2).$$

This immediately implies that the solution formula (1-11) defines a continuous map

$$H^{2,1}(\mathbb{R}^2) \to C\big([0,T]; H^{2,1}(\mathbb{R}^2)\big), \quad t \mapsto u(t),$$

for any $T > 0$. We say that u is a weak solution of the mNV equation (see (5-1)) on $[0,T]$ if

$$(\varphi_t + \partial^3\varphi + \bar{\partial}^3\varphi, u) + (\varphi, NL(u)) = 0, \tag{1-12}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2 \times [0,T])$, where $(\cdot\,,\cdot)$ denotes the inner product on $L^2(\mathbb{R}^2 \times [0,T])$. We will show that (1-11) defines a weak solution in this sense and that, also, the flow (1-11) leaves the domain of $\mathcal{M}$ invariant. We will prove:

Theorem 1.5. *For $u_0 \in \mathcal{S}(\mathbb{R}^2)$, the solution formula (1-11) gives a classical solution of mNV. Moreover, if $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $\partial u_0 = \overline{\partial u_0}$, and $\int u_0(z)\, dA(z) = 0$, then $u(t)$ is a weak solution of mNV and the relations $(\partial u)(\cdot\,, t) = \overline{(\partial u)(\cdot\,, t)}$ and $\int u(z,t)\, dA(z) = 0$ hold for all t.*

Now we can solve the NV equation using the solution map for mNV and the Miura map (1-9). We say that q is a weak solution of the NV equation on $[0,T]$ if

$$(\varphi_t + \partial^3\varphi + \bar{\partial}^3\varphi, q) + \tfrac{3}{4}(\partial\varphi, q\bar{\partial}^{-1}\partial q) + \tfrac{3}{4}(\bar{\partial}\varphi, q\partial^{-1}\bar{\partial}q) = 0, \tag{1-13}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2 \times (0,T))$. Using Theorem 1.5, we will prove:

Theorem 1.6. *Suppose that $q_0 = 2\partial u_0 + |u_0|^2$ where $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $\partial u_0 = \overline{\partial u_0}$, and $\int u_0(z)\, dA(z) = 0$. Then*

$$q(t) = \mathcal{M}\big(\mathcal{I}\big(e^{2it((\diamond)^2 + (\bar{\diamond})^2)}(\mathcal{R}u_0)(\diamond)\big)\big) \tag{1-14}$$

is a weak solution the NV equation with initial data q_0. If $u_0 \in \mathcal{S}(\mathbb{R}^2)$, then $q(t)$ is a classical solution of the NV equation.

The class of initial data covered by Theorem 1.6 includes the conductivity-type potentials considered in [Lassas et al. 2012]. The connection between that work and ours is given in the following theorem.

Theorem 1.7. *Suppose that* $u_0 \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ *with* $\int u_0(z)\, dA(z) = 0$ *and* $\overline{\partial u_0} = \partial u_0$, *and let* $q_0 = 2\partial u_0 + |u_0|^2$. *Then, for any* t,

$$\mathcal{Q}\big(e^{it((\diamond)^3+(\bar\diamond)^3)}(\mathcal{T}q_0)(\diamond)\big) = \mathcal{M}\mathcal{I}\big(e^{t((\bar\diamond)^3-(\diamond)^3)}(\mathcal{R}u_0)(\diamond)\big),$$

and their common value is a classical solution to the Novikov–Veselov equation.

It should be noted that the solution formula (1-14) provides a solution which exists *globally* in time. On the other hand, Taimanov and Tsaryov [2007; 2008a; 2008b; 2010]] have used Moutard transformations to construct explicit, nonsingular Cauchy data q_0 with rapid decay at infinity and having the following properties: (i) the Schrödinger operator $-\Delta + q_0$ has nonzero eigenvalues at zero energy (and so is not of conductivity type) and (ii) the solution of (1-1) with Cauchy data q_0 blows up in finite time.

To close this introduction, we comment on the seemingly restrictive hypothesis in Theorems 1.6 and 1.7. In both theorems, we assume that $\int u_0 = 0$. To understand what this assumption means, we recall that if $\phi_0 = \bar\partial^{-1}u_0$, then the unique, positive, normalized zero-energy solution of the Schrödinger equation (1-4) is given by $\psi_0 = \exp(\phi_0)$. For $u_0 \in \mathcal{S}(\mathbb{R}^2)$ say, we have from the integral expression for $\bar\partial^{-1}$ that

$$\phi_0(z) = -\frac{1}{\pi}\frac{\int u_0(\zeta)\, d\zeta}{z} + \mathcal{O}(|z|^{-2}),$$

so that, to leading order

$$\psi_0 - 1 = -\frac{1}{\pi}\frac{\int u_0(\zeta)\, d\zeta}{z} + \mathcal{O}(|z|^{-2}).$$

Recalling that $\gamma^{1/2}(z) = \psi_0(z)$ we see that the vanishing of $\int u_0(z)\, dA(z)$ implies that $\gamma(z) - 1 = \mathcal{O}(|z|^{-2})$ as $|z| \to \infty$. In particular, for conductivities with $\gamma = 1$ outside a compact set, $\int u_0(z)\, dA(z) = 0$.

Indeed, suppose that $q = \gamma^{-1/2}\Delta(\gamma^{1/2})$ in distribution sense, where $\gamma \in L^\infty(\mathbb{R}^2)$, $\gamma(z) \geq c > 0$, and suppose further that $\Delta(\nabla\gamma)$ and $\gamma - 1$ belong to $L^2(\mathbb{R}^2)$. It follows that $\varphi = \log\gamma \in H^{3,1}(\mathbb{R}^2)$ and the function

$$u = 2\bar\partial\varphi$$

belongs to $H^{2,1}$. We then compute that $q = 2\partial u + |u|^2$. If we have stronger decay of $\gamma(z)$ as $|z| \to \infty$, this will imply additional decay of $\varphi(z)$ that can be used to check $\int u(z)\, dA(z) = 0$ by Green's formula $\int_\Omega \bar\partial\varphi\, dA(z) = \frac{1}{2}\int_{\partial\Omega}\varphi(\nu_{x_1} + i\nu_{x_2})\, d\sigma$.

The structure of this paper is as follows. In Section 2 we review some important linear and multilinear estimates which will be used to study the scattering maps $\mathcal{R}$ and $\mathcal{I}$. In Section 3 we recall how the scattering maps $\mathcal{R}$ and $\mathcal{I}$ for the Davey–Stewartson system are defined, while in Section 4 we prove that $\mathcal{R}: H^{2,1}(\mathbb{R}^2) \to H^{1,2}(\mathbb{R}^2)$ and $\mathcal{I}: H^{1,2}(\mathbb{R}^2) \to H^{2,1}(\mathbb{R}^2)$ are locally Lipschitz continuous. In Section 5 we solve the mNV equation using the inverse scattering method and prove that, for initial data $u_0 \in H^{2,1}(\mathbb{R}^2)$ with $\partial u_0 = \overline{\partial u_0}$ and $\int_{\mathbb{R}^2} u_0(z)\, dA(z) = 0$, the condition $\partial u = \overline{\partial u}$ holds for all $t > 0$. In Section 6 we prove Theorem 1.6. In Section 7 we show that our class of potentials extends the class of conductivity type potentials considered in [Lassas et al. 2012], and that our solution coincides with theirs where the two constructions overlap. Appendix A sketches the solution of the mNV equation by scattering theory for initial data in the Schwarz class.

2. Preliminaries

Notation. In what follows, $\|\cdot\|_p$ denotes the usual L^p-norm and $p' = p/(p-1)$ denotes the conjugate exponent. If f is a function of (z, k), $f(z, \diamond)$ (resp. $f(\cdot, k)$) denotes f with a generic argument in the z (resp. k) variable. We will write L^p_z or L^p_k for L^p-spaces with respect to the z or k variable, and $L^p_z(L^q_k)$ for the mixed spaces with norm

$$\|f\|_{L^p_z(L^q_k)} = \left(\int \|f(z, \diamond)\|^p_q \, dA(z) \right)^{1/p}.$$

If f is a function of z and k, $\|f\|_\infty$ denotes $\|f\|_{L^\infty(\mathbb{R}^2_z \times \mathbb{R}^2_k)}$.

In what follows, $\langle \cdot, \cdot \rangle$ denotes the pairing

$$\langle f, g \rangle = \frac{1}{\pi} \int \overline{f(z)} \, g(z) \, dA(z).$$

We will call a mapping f from a Banach space X to a Banach space Y a *locally Lipschitz continuous map* (LLCM) if, for any bounded subset B of X, there is a positive constant $C = C(B)$ such that, for all $x_1, x_2 \in B$,

$$\|f(x_1) - f(x_2)\|_Y \le C(B) \|x_1 - x_2\|_X.$$

For example, if $M : X^m \to Y$ is a continuous multilinear map, then

$$f \mapsto M(f, f, \dots, f)$$

is an LLCM from X to Y.

Cauchy transforms. The integral operators

$$P\psi = \frac{1}{\pi} \int \frac{1}{z - \zeta} f(\zeta) \, dm(\zeta), \qquad \overline{P}\psi = \frac{1}{\pi} \int \frac{1}{\bar{z} - \bar{\zeta}} f(\zeta) \, dm(\zeta)$$

are formal inverses respectively of $\bar{\partial}$ and ∂. We denote by P_k and $\overline{P}_k$ the corresponding formal inverses of $\bar{\partial}_k$ and ∂_k. The following estimates are standard (see, for example, [Astala et al. 2009, Section 4.3] or [Vekua 1959]).

Lemma 2.1. (i) *For any $p \in (2, \infty)$ and $f \in L^{2p/(p+2)}$, $\|Pf\|_p \le C_p \|f\|_{2p/(p+2)}$.*

(ii) *For any p, q with $1 < q < 2 < p < \infty$ and any $f \in L^p \cap L^q$, $\|Pf\|_\infty \le C_{p,q} \|f\|_{L^p \cap L^q}$ and Pf is Hölder continuous of order $(p-2)/p$ with*

$$|(Pf)(z) - (Pf)(w)| \le C_p |z - w|^{(p-2)/p} \|f\|_p.$$

(iii) *For $2 < p, q$ and $u \in L^s$ for $q^{-1} + 1/2 = p^{-1} + s^{-1}$,*

$$\|P(u\psi)\|_q \le C_{p,q} \|u\|_s \|\psi\|_p.$$

Remark 2.2. If $p > 2$ and $u \in L^s$ for $s \in (1, \infty)$, then estimate (iii) holds true for any $q > 2$.

Beurling transform. The operator

$$(\mathcal{S}f)(z) = -\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|z-w|>\varepsilon} \frac{1}{(z-w)^2} f(w)\, dw, \tag{2-1}$$

defined as a Calderón–Zygmund type singular integral, has the property that for $f \in C_0^\infty(\mathbb{R}^2)$ we have $\mathcal{S}(\bar{\partial} f) = \partial f$. The operator $\mathcal{S}$ is a bounded operator on L^p for $p \in (1, \infty)$ (see, for example, [Astala et al. 2009, Section 4.5.2]). This fact allows us to obtain L^p-estimates on ∂-derivatives of functions of interest from L^p-estimates on $\bar{\partial}$-derivatives.

We will also need the following trivial estimate on the Beurling transform of a smooth, rapidly decreasing function.

Lemma 2.3. *Suppose that $M > 2$ and $\sup_{|\alpha| \le 2} |D^\alpha g(z)| \le C(1+|z|)^{-M}$. For any β with $0 \le \beta < M - 2$ and $\beta \le 2$, the estimate $|\mathcal{S}(g)| \le C(1+|z|)^{-\beta}$ holds.*

Proof. Compute

$$\int_{\varepsilon<|w|} \frac{1}{(z-w)^2} f(w)\, dw = \left(\int_{\varepsilon<|z-w|<1} + \int_{|z-w|\ge 1} \right) \frac{1}{(z-w)^2} f(w)\, dw.$$

In the first term we may Taylor-expand $f(w)$, note that $\int_{\varepsilon<|w|<1} (z-w)^{-2}\, dw = 0$, and conclude that the first term is estimated by

$$C \sup_{\substack{|\alpha| \le 2 \\ |z-w| \le 1}} |(D^\alpha f(w)|,$$

which is $\mathcal{O}(|z|^{-M})$ by hypothesis. The second term is estimated by a constant times

$$(1+|z|)^{-\beta} \int \frac{1}{(1+|z-w|)^{2-\beta}} \frac{1}{(1+|w|)^{M-\beta}}\, dw,$$

which gives the required decay. □

Brascamp–Lieb type estimates. A fundamental role is played by the following multilinear estimate due to Russell Brown [2001], who initiated their use in the analysis of the DS II scattering maps. See [Christ 2011] for a proof of these estimates using the methods of Bennett, Carbery, Christ and Tao [Bennett et al. 2008; 2010], and see [Nie and Brown 2011] for a different proof. Define

$$\Lambda_n(\rho, u_0, u_1, \dots, u_{2n}) = \int_{\mathbb{C}^{2n+1}} \frac{|\rho(\zeta)|\, |u_0(z_0)| \dots |u(z_{2n})|}{\prod_{j=1}^{2k} |z_{j-1} - z_j|}\, dA(z),$$

where $dA(z)$ is product measure on $\mathbb{C}^{2n+1}$, and set

$$\zeta = \sum_{j=0}^{2n} (-1)^j z_j. \tag{2-2}$$

Proposition 2.4 [Brown 2001]. *The estimate $\left|\Lambda_n(\rho, u_0, u_1, \dots, u_{2n})\right| \le C_n \|\rho\|_2 \prod_{j=0}^{2n} \|u_j\|_2$ holds.*

Remark 2.5. For $u_1, \dots, u_{2n} \in \mathcal{S}(\mathbb{R}^2)$, define operators W_j by $W_j \psi = P e_k u_j \overline{\psi}$. Proposition 2.4 implies that

$$F(k) = \langle e_k u_0, W_1 W_2 \dots W_{2n} 1 \rangle \tag{2-3}$$

is a multilinear $L^2_k(\mathbb{R}^2)$-valued function of $(u_0, \dots, u_{2n})$ with

$$\|F\|_2 \le C \prod_{j=0}^{2n} \|u_j\|_2.$$

Pseudodifferential operators. In Section 5 we will use pseudodifferential operators to prove key estimates on a third-order linear evolution equation. We recall that a function $p \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ belongs to the symbol class $S^m(\mathbb{R}^n)$ if for all multiindices α, β, the seminorms

$$\rho_{\alpha,\beta}(p) := \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} \left|(1+|\xi|)^{m-|\alpha|} p(x,\xi)\right|. \tag{2-4}$$

are finite. The corresponding pseudodifferential operator $P(x, D)$ is given by the Weyl quantization

$$(P(x,D)f)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p\left(\frac{x+y}{2}, \xi\right) e^{i(x-y)\cdot\xi} f(y)\, dy,$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, and we say that $P(x, D) \in \mathrm{OPS}^m(\mathbb{R}^n)$. We also write $\sigma(P)$ for p. For the Weyl quantization, if p is a real-valued symbol, then $p(x, D)$ is formally symmetric.

The celebrated Calderón–Vaillancourt theorem [1972] implies that if $p \in S^0(\mathbb{R}^n)$, then $p(x, D)$ extends to a bounded operator on $L^2(\mathbb{R}^n)$. If $\{p(x,\xi,t)\}_{t\in[0,T]}$ is a smooth family of symbols in $S^0(\mathbb{R}^n)$ with the seminorms (2-4) bounded uniformly in $t \in [0, T]$ for each fixed α, β, then $\|p(x, D, t)\|_{L^2}$ is bounded independently of $t \in [0, T]$.

We will also use a simple version of the sharp Gårding inequality: if $P \in \mathrm{OPS}^1(\mathbb{R}^n)$ and $p(x,\xi)$ is real-valued and nonnegative for $x \in \mathbb{R}^n$ and ξ outside a compact subset of $\mathbb{R}^n$, there is a constant C such that

$$(\varphi, P(x, D)\varphi) \ge -C\|\varphi\|^2 \tag{2-5}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. If $p(x,\xi,t)$ is a smooth family of symbols in $S^0(\mathbb{R}^n)$ such that

(i) the seminorms (2-4) are bounded uniformly in $t \in [0, T]$ for each fixed α, β, and

(ii) $p(x,\xi,t)$ is real-valued and nonnegative for $x \in \mathbb{R}^n$ and ξ outside a fixed compact subset of $\mathbb{R}^n$, independent of $t \in [0, T]$.

Then the lower bound (2-5) holds for a C independent of $t \in [0, T]$.

3. Scattering maps and an oscillatory $\overline{\partial}$-problem

First, we recall that the Davey–Stewartson scattering maps $\mathcal{R}$ and $\mathcal{I}$ are both defined by $\overline{\partial}$-problems; see [Perry 2011] for discussion. The inverse scattering method for the Davey–Stewartson II equation was developed by Ablowitz and Fokas [1983; 1984] and Beals and Coifman [1984; 1985; 1989; 1990]. Sung

[1994a; 1994b; 1994c] and Brown [2001] carried out detailed analytical studies of the direct and inverse scattering maps.

For a complex parameter k and for $z = x_1 + ix_2$, let

$$e_k = e^{\bar{k}\bar{z}-kz}.$$

Given $u \in H^{1,1}(\mathbb{R}^2)$ and $k \in \mathbb{C}$, there exists a unique bounded continuous solution of

$$\begin{aligned} \bar{\partial}\mu_1 &= \tfrac{1}{2} e_k u \overline{\mu_2}, \\ \bar{\partial}\mu_2 &= \tfrac{1}{2} e_k u \overline{\mu_1}, \\ \lim_{|z|\to\infty} (\mu_1(z,k), \mu_2(z,k)) &= (1,0). \end{aligned} \tag{3-1}$$

We then define $r = \mathcal{R}u$ by

$$r(k) = \frac{1}{\pi} \int e_k(z) u(z) \overline{\mu_1(zk)} \, dA(z). \tag{3-2}$$

On the other hand, it can be shown that

$$\nu_1 = \mu_1 \quad \text{and} \quad \nu_2 = e_k \overline{\mu_2} \tag{3-3}$$

solve a $\bar{\partial}$-problem in the k variable:

$$\begin{aligned} \bar{\partial}_k \nu_1 &= \tfrac{1}{2} e_k \overline{r \nu_2}, \\ \bar{\partial}_k \nu_2 &= \tfrac{1}{2} e_k \overline{r \nu_1}, \\ \lim_{|k|\to\infty} (\nu_1(z,k), \nu_2(z,k)) &= (1,0), \end{aligned} \tag{3-4}$$

and that this solution is unique within the space of bounded continuous functions. Given $r \in H^{1,1}(\mathbb{R}^2)$, we solve the $\bar{\partial}$-system (3-4) and define $u = \mathcal{I}r$ by

$$u(z) = \frac{1}{\pi} \int e_{-k}(z) r(k) \nu_1(z,k) \, dA(k). \tag{3-5}$$

Theorem 3.1 [Perry 2011]. *The maps $\mathcal{R}$ and $\mathcal{I}$, initially defined on $\mathcal{S}(\mathbb{R}^2)$, extend to LLCM's from $H^{1,1}(\mathbb{R}^2)$ to itself. Moreover $\mathcal{R} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{R} = I$, where I denotes the identity map on $H^{1,1}(\mathbb{R}^2)$.*

In what follows, we will study the restriction of the maps $\mathcal{R}$ and $\mathcal{I}$ respectively to $H^{2,1}(\mathbb{R}^2)$ and $H^{1,2}(\mathbb{R}^2)$, and obtain refined continuity results. To do so, we first describe three basic tools used in [Perry 2011] to analyze the generic system

$$\begin{aligned} \bar{\partial} w_1 &= \tfrac{1}{2} e_k u \overline{w_2}, \\ \partial w_2 &= \tfrac{1}{2} e_k u \overline{w_1}, \\ \lim_{|z|\to\infty} (w_1(z,k), w_2(z,k)) &= (1,0), \end{aligned} \tag{3-6}$$

for unknown functions $w_1(z,k)$ and $w_2(z,k)$, where k is a complex parameter, and $u \in H^{1,1}(\mathbb{R}^2)$. We refer the reader to [Perry 2011] for the proofs. We don't state the obvious analogues of the facts below when the roles of k and z are reversed, but use them freely in what follows.

1. *Finite L^p-expansions*. In [Perry 2011] it is shown that the system (3-6) has a unique solution in L^∞_z. This result, and further analysis of the solution, is a consequence of the following facts, which we recall from Section 3 of the same reference. Let T be the antilinear operator

$$T\psi = \tfrac{1}{2} P e_k u \overline{\psi},$$

which is a bounded operator from L^p to itself for $p \in (2, \infty]$ if $u \in H^{1,1}$ by Lemma 2.1(i). The system (3-6) is equivalent to the integral equation

$$w_1 = 1 + T^2 w_1$$

and the auxiliary formula $w_2 = T w_1$. The operator $I - T^2$ has trivial kernel as a map from $L^p(\mathbb{R}^2)$ to itself for any $p \in (2, \infty]$, and the estimate

$$\|T^2\|_{L^p \to L^p} \le C_p \|u\|^2_{H^{1,1}} (1 + |k|)^{-1}$$

holds for any $p \in (2, \infty)$. For any $p \in (2, \infty)$, the resolvent $(I - T^2)^{-1}$ is bounded uniformly in $k \in \mathbb{C}$ and u in bounded subsets of $H^{1,1}$ as an operator from L^p to itself. Note that if $u \in H^{1,1}$, the expression $T1 = \frac{1}{2} P e_k u$ is a well-defined element of L^p for all $p \in (2, \infty]$. The unique solution of (3-6) is given by

$$w_1 - 1 = (I - T^2)^{-1} T^2 1, \quad w_2 = T w_1.$$

From these facts, one has (see [Perry 2011, Section 3]):

Lemma 3.2 (finite L^p-expansions). *For any positive integer N, the expansions*

$$w_1 - 1 = \sum_{j=1}^{N} T^{2j} 1 + R_{1,N} \quad \textit{and} \quad w_2 = \sum_{j=1}^{N} T^{2j-1} 1 + R_{2,N}$$

hold, where the maps

$$u \mapsto (1 + |\diamond|)^N R_{1,N}(\,\cdot\,, \diamond), \quad u \mapsto (1 + |\diamond|)^N R_{2,N}(\,\cdot\,, \diamond)$$

are LLCMs from $H^{1,1}(\mathbb{R}^2)$ into $L^\infty_k(L^p_z)$.

2. *Multilinear estimates*. Substituting the expansions into the representation formulas (3-5) and (3-2) leads to expressions of the form

$$\langle e_* w, F_j \rangle,$$

where e_* denotes e_k or e_{-k}, w is a monomial in u and its derivatives, and F_j denotes $T^{2j} 1$ or $\overline{T^{2j} 1}$ for $j \ge 1$. We assume that w is bounded in L^2 norm by a power of $\|u\|_{H^{2,1}}$. The following fact is an immediate consequence of Remark 2.5.

Lemma 3.3. *The map $u \mapsto \langle e_* w, F_j \rangle$ is an LLCM from $H^{2,1}(\mathbb{R}^2)$ to $L^2_k(\mathbb{R}^2)$.*

3. *Large-parameter expansions*. Finally, the following large-z finite expansions for w_1 and w_2 will be useful. We omit the straightforward computational proof.

Lemma 3.4. *For* $u \in H^{1,1}(\mathbb{R}^2)$,

$$w_1(z,k) - 1 = \frac{1}{2\pi z}\int e_k(z')u(z')\overline{w_2(z',k)}\,dm(z') + \frac{1}{2\pi z}\int \frac{e_k(z')}{z-z'}z'u(z')\overline{w_2(z',k)}\,dm(z'),$$

and similarly

$$w_2(z,k) = \frac{1}{2\pi z}\int e_k(z')u(z')\overline{w_1(z',k)}\,dm(z') + \frac{1}{2\pi z}\int \frac{e_k(z')}{z-z'}z'u(z')\overline{w_1(z',k)}\,dm(z').$$

Analogous expansions hold for the $\bar\partial$-problem in the k variables.

4. Restrictions of scattering maps

In this section we prove Theorem 1.4. By virtue of Theorem 3.1, it suffices to show that the maps $H^{2,1} \ni u \mapsto |\diamond|^2 r(\diamond)$ and $H^{1,2} \ni r \mapsto \Delta u \in L^2$ are LLCMs. First, we prove:

Lemma 4.1. *The map* $u \mapsto |\diamond|^2 r(\diamond)$ *is an LLCM from* $H^{2,1}(\mathbb{R}^2)$ *to* $L^2(\mathbb{R}^2)$.

Proof. We carry out all computations on $u \in C_0^\infty(\mathbb{R}^2)$ and extend by density to $H^{2,1}(\mathbb{R}^2)$. Note that $\|u\|_p \le C_p\|u\|_{H^{2,1}}$ for all $p \in (1,\infty)$ and $\|\partial u\|_p \le C_p\|u\|_{H^{2,1}}$ for $p \in [2,\infty)$. An integration by parts using (3-2) and the identity $\partial e_k = -k e_k$ shows that (up to trivial factors)

$$\begin{aligned}|k|^2 r(k) &= -\bar k\int e_k(\partial u) - \bar k\int e_k(\partial u)(\overline{\mu_1}-1) - \frac{\bar k}{2}\int |u|^2\mu_2\\ &= I_1 + I_2 + I_3,\end{aligned}$$

where in the last term we used

$$\bar\partial\mu_1 = \tfrac{1}{2}e_k u\overline{\mu_2}. \tag{4-1}$$

I_1: This term is the Fourier transform of $\partial\bar\partial u$ and hence defines a linear map from $H^{2,1}$ to L^2_k.

I_2: An integration by parts using (3-2), the identity $\partial(e_k) = -k e_k$, and (4-1) again shows that

$$\begin{aligned}I_2 &= \frac{\bar k}{k}\left(\int e_k(\partial^2 u)(\overline{\mu_1}-1) + \frac{1}{2}\int \bar u\,\partial u\,\mu_2\right)\\ &= I_{21} + I_{22}.\end{aligned}$$

In I_{21} we insert $1 = \chi + (1-\chi)$, where $\chi \in C_0^\infty(\mathbb{R}^2)$ satisfies $0 \le \chi(z) \le 1$, $\chi(z) = 1$ for $|z| \le 1$, and $\chi(z) = 0$ for $|z| \ge 2$. Drop the unimodular factor $\bar k/k$ and write $I_{21} = I_{21}^{\text{in}} + I_{21}^{\text{out}}$ corresponding to this decomposition. Since $\chi\partial^2 u \in L^{p'}$ for any $p > 2$, we may use Lemma 3.2 to get the expansion

$$I_{21}^{\text{in}} = \sum_{j=1}^{N}\int e_k(\partial^2 u)\chi(\overline{T^{2j}1}) + \int e_k(\partial^2 u)\chi\overline{(I-T^2)^{-1}T^{2j+2}1}.$$

By Lemmas 3.2 and 3.3 and the fact that $\chi\partial^2 u \in L^{p'}$, each right-hand term defines an LLCM from $H^{2,1}$ to L^2_k, hence $u \mapsto I_{21}^{\text{rm}}$ is an LLCM. In I_{21}^{out}, we use Lemma 3.4 to write

$$\int e_k(1-\chi)\partial^2 u(\overline{\mu_1}-1) \\ = -\frac{1}{2\pi}\left(\int e_k(1-\chi)(\partial^2 u)z^{-1}\right)\left(\int e_{-k}\bar{u}\mu_2\right) + \tfrac{1}{2}\left\langle e_{-k}(1-\chi)\overline{(\partial^2 u)z^{-1}},\, Pe_{-k}u_1(T\mu_1)\right\rangle. \quad (4\text{-}2)$$

The first term on the second line of (4-2) is the product of the Fourier transform of the L^2-function $(1-\chi(z))(\partial^2 u)(z)z^{-1}$ and the function $\int e_{-k}\bar{u}\mu_2$. Since $u \in L^{p'}$ for all $p > 2$ while $u \mapsto \mu_2$ is an LLCM from $H^{1,1}$ to $L_k^\infty(L_z^p)$, the map $u \mapsto \int e_{-k}\bar{u}\mu_2$ is an LLCM from $H^{2,1}$ to L_k^∞, so the first right-hand term in (4-2) defines an LLCM from $H^{2,1}$ to L_k^2. The second right-hand term in (4-2) may be controlled using Lemmas 3.2 and 3.3. This shows that $u \mapsto I_{21}^{\text{out}}$, and hence $u \mapsto I_{21}$, defines an LLCM from $H^{2,1}$ to L_k^2. Finally, to control I_{22}, we note that $\bar{u}\partial u \in L^{p'}$ for $p > 2$. Hence, using Lemma 3.2 we obtain

$$I_{22} = \sum_{j=0}^{N} \int \bar{u}\partial u\, T^{2j+1}1 + \int (\bar{u}\partial u)(I-T^2)T^{2j+1}1. \quad (4\text{-}3)$$

To control terms in the finite sum in (4-3), we write

$$\int \bar{u}\partial u\, T^{2j+1}1 = \left\langle u\partial\bar{u},\, P[e_k u(\overline{T^{2j}1})]\right\rangle \\ = -\left\langle e_{-k}\bar{u}\bar{P}(u\partial\bar{u}),\, \overline{T^{2j}1}\right\rangle.$$

and apply Lemma 3.3 since $\|u\bar{P}(u\partial\bar{u})\|_2' \le C\|u\|_{H^{2,1}}$. The second right-hand term in (4-3) defines an LLCM from $H^{2,1}$ to L_k^2 by Lemma 3.2. Hence, $u \mapsto I_2$ is a LLCM from $H^{2,1}$ to L_k^2.

I_3: Note that $|u|^2 \in L^{p'}$ for all $p > 2$ and use the expansion of μ_2 to write I_3 as

$$\sum_{j=1}^{N} -\frac{\bar{k}}{2}\int |u|^2 T^{2j+1}1 - \frac{\bar{k}}{2}\int |u|^2(I-T^2)^{-1}T^{2N+3}1.$$

The remainder is an LLCM from $H^{2,1}$ to L_k^2 by Lemma 3.2. A given term in the finite sum is written (up to constant factors)

$$\bar{k}\left\langle |u|^2,\, P[e_k u(\overline{T^{2j}1})]\right\rangle = \bar{k}\left\langle e_{-k}\bar{u}\bar{P}(|u|^2),\, \overline{T^{2j}1}\right\rangle \quad (4\text{-}4) \\ = -\left\langle \bar\partial(e_{-k}\bar{u}\bar{P}(|u|^2)),\, \overline{T^{2j}1}\right\rangle + \left\langle e_{-k}\bar\partial(\bar{u}\bar{P}(|u|^2)),\, \overline{T^{2j}1}\right\rangle,$$

where we integrated by parts to remove the factor of $\bar{k}$. The first term on the second line of (4-4) is

$$\left\langle e_{-k}\bar{u}\bar{P}(|u|^2),\, \partial(\overline{T^{2j}1})\right\rangle = \left\langle e_{-k}\bar{u}\bar{P}(|u|^2),\, e_{-k}\bar{u}P(e_k u\overline{T^{2j-2}1})\right\rangle \\ = \left\langle e_{-k}\bar{u}P(|u|^2P(|u|^2)),\, \overline{T^{2j-2}1}\right\rangle,$$

which defines an LLCM from $H^{2,1}$ to L_k^2 by Lemma 3.3 since $\bar{u}P(|u|^2P(|u|^2)) \in L^2$. The second right-hand term is treated similarly. Hence $u \mapsto I_3$ is an LLCM from $H^{2,1}$ to L_k^2.

Collecting these results, we conclude that $u \mapsto |\diamond|^2 r(\diamond)$ is an LLCM from $H^{2,1}$ to L_k^2. □

Lemma 4.2. *The map $r \mapsto \Delta u$ is an LLCM from $H^{2,1}(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.*

Proof. Since $r \in H^{1,2}$ we have $kr(k) \in L^p$ for all $p \in (1,2]$, $r \in L^p$ for all $p \in [1,\infty)$ and $\partial r \in L^p$ for all $p \in [2,\infty)$. A straightforward computation shows that

$$\begin{aligned}\partial\bar{\partial}u &= \int |k|^2 e_{-k} r + \int |k|^2 e_{-k} r(\nu_1 - 1) - \int \bar{k} e_{-k} r \partial \nu_1 + \int k e_{-k} r \bar{\partial} \nu_1 + \int e_{-k} r \partial\bar{\partial}\nu_1 \\ &= I_1 + I_2 + I_3 + I_4 + I_5,\end{aligned}$$

where all derivatives are taken with respect to z. We now show that each of I_1–I_5 defines a locally Lipschitz continuous map from $H^{2,1} \ni r$ into L^2_z.

I_1: This term is the Fourier transform of $\partial\bar{\partial}r$ and hence L^2.

I_2: Inserting $1 = \chi + (1-\chi)$ in I_2, where χ is as in the proof of Lemma 4.1 (except that, here, χ is a function of k, not z), we have $I_2 = I_{21} + I_{22}$, where

$$I_{21} = \int e_{-k} |k|^2 \chi r(\nu_1 - 1), \quad I_{22} = \int e_{-k} |k|^2 r(1-\chi)(\nu_1 - 1).$$

We will show that I_{21} and I_{22} are both LLCMs from $H^{1,2}$ to L^2_z. Since $|k|^2 \chi r \in L^{p'}$ for any $p > 2$, we can use Lemma 3.2 for $\nu_1 - 1$ together with Lemma 3.3 to conclude that $r \mapsto I_{21}$ is an LLCM from $H^{1,2}$ to L^2_z. For I_{22} we use the one-step large-k expansion of $\nu_1 - 1$ (Lemma 3.4):

$$\nu_1(z,k) - 1 = -\frac{1}{2\pi k}\int e_{k'}(z)\overline{r(k')}\,\overline{\nu_2(z,k')}\,dm(k') - \frac{1}{2\pi k}\int \frac{e_{k'}(z)}{k-k'} k' r(\overline{k'})\,\overline{\nu_2(z,k')}\,dm(k').$$

We then have

$$I_{22} = \int e_{-k} \bar{k} r(1-\chi)(F_1 + F_2),$$

where

$$\begin{aligned} F_1(z) &= -\frac{1}{2\pi}\int e_{k'}\overline{r(k')}\,\overline{\nu_2(z,k')}\,dm(k'), \\ F_2(z,k) &= -\frac{1}{2\pi}\int \frac{e_{k'}(z)}{k-k'} k' r(\overline{k'})\,\overline{\nu_2(z,k')}\,dm(k'). \end{aligned}$$

It is easy to see that $\|F_1\|_{L^\infty_z} \le \|r\|_1 \|\nu_2\|_\infty$, so that $r \mapsto F_1$ is an LLCM from $H^{1,2}$ to L^∞_z. Moreover, $\int e_{-k}\bar{k} r(1-\chi)$ is the inverse Fourier transform of the L^2 function $(\overline{\diamond}) r(\diamond)(1-\chi(\diamond))$. Hence, the map $r \mapsto \int e_{-k}\bar{k} r(1-\chi) F_1$ is an LLCM from $H^{1,2}$ to L^2_z. Next, we may use Lemma 3.2 in F_2 to conclude that

$$F_2 = -\frac{1}{2}\sum_{j=1}^{N} P_k\big(e_k k \bar{r}\, \overline{T^{2j+1} 1}\big) - \frac{1}{2} P_k\big(e_k k \bar{r}\, \overline{((I-T^2)^{-1} T^{2N+3} 1}\big). \tag{4-5}$$

The corresponding contributions to I_{22} from terms in the finite sum from (4-5) define LLCMs from $H^{1,2}$ to L^2_z by Lemma 3.3, while by the remainder estimate in Lemma 3.2, the mapping

$$r \mapsto P e_k k \bar{r} (I - T^2)^{-1} T^{2N+3} 1$$

is an LLCM from $H^{1,2}$ to $L^2_z(L^p_k)$ for $p>2$. Using these estimates we may conclude that

$$r \mapsto \int e_{-k}\bar{k} r(1-\chi)F_2$$

is an LLCM from $H^{1,2}$ to L^2_z.

I_3: Since $\mu_1=\nu_1$, we conclude from (4-1) and (3-3) that

$$\bar{\partial}_z \nu_1 = \tfrac{1}{2} e_k u \overline{\mu_2} = \tfrac{1}{2} u \nu_2, \tag{4-6}$$

so that

$$I_3 = -\int \bar{k} e_{-k} r (\partial \bar{\partial}^{-1})(\bar{\partial}\nu_1) = -\frac{1}{2}\int \bar{k} e_{-k} r (\partial\bar{\partial}^{-1})(u\nu_2).$$

Proceeding as in the analysis of I_{22} in Lemma 4.1, we use the one-step large-k expansion (Lemma 3.4) to obtain

$$\begin{aligned} \nu_2(z,k) &= -\frac{1}{2\pi k}\int e_{k'}(z)\overline{r(k')}\,\overline{\nu_2(z,k')}\,dm(k') - \frac{1}{2\pi k}\int \frac{e_{k'}(z)}{k-k'} k' r(\overline{k'})\,\overline{\nu_2(z,k')}\,dm(k') \\ &= F_1 + F_2. \end{aligned}$$

Hence, up to trivial factors,

$$I_3 = \int e_{-k} r (\partial\bar{\partial}^{-1})[u(F_1+F_2)].$$

By Minkowski's inequality,

$$\|I_3\|_{L^2_z} \le \frac{1}{2}\int |r| \big\| \partial\bar{\partial}^{-1}(u(F_1+F_2))\big\|_{L^2_z}.$$

Observe that $\|\partial\bar{\partial}^{-1}(uF_1)\|_{L^2_z} \le C\|uF_1\|_{L^2_z}$, while

$$\|\partial\bar{\partial}^{-1}(uF_2)\|_{L^p_k(L^2_z)} \le C_p \|u\|_2 \|F_2\|_{L^p_k(L^\infty_z)} \le C_p \|u\|_2 \|(\diamond) r(\diamond)\|_{2p/(p+2)} \|\nu_2\|_\infty$$

(where $\|\nu_2\|_\infty$ means $\|\nu_2\|_{L^\infty(\mathbb{R}^2_z\times\mathbb{R}^2_k)}$), so that altogether

$$\|I_3\|_{L^2_z} \le C\|u\|_2 \|r\|_{H^{1,2}}(1+\|\nu_2\|_\infty).$$

Thus $I_3 \in L^2_z$. The local Lipschitz continuity of I_3 follows from that of $r\mapsto u$ and $r\mapsto \nu_2$.

I_4: Using (4-6) again, we compute

$$\int k e_{-k} r \bar{\partial}\nu_1 = \frac{u}{2}\int e_{-k} k r \nu_2,$$

so it suffices to show that $r\mapsto \int e_{-k} k r\nu_2$ is an LLCM from $H^{1,2}$ to L^∞_z. Since $kr \in L^{p'}$ for $p>2$, and $r\mapsto \nu_2$ is an LLCM from $H^{1,1}$ to L^∞, the result follows.

I_5: Compute

$$I_5 = \int e_{-k} r \partial(u\nu_2) = \partial u \int e_{-k} r\nu_2 + u\int e_{-k} r(\partial\nu_2). \tag{4-7}$$

The first right-hand term in (4-7) defines an LLCM from $H^{1,2}$ to L^2_z since $r \mapsto \partial u$ has this property. Thus, to control the first right-hand term, it suffices to show that $r \mapsto \int e_{-k} r \nu_2$ defines an LLCM from $H^{1,2}$ to L^∞_z. To see this, note that $r \in L^{p'}$ for $p > 2$, and $r \mapsto \nu_2$ is an LLCM from $H^{1,1}$ to $L^\infty_z(L^p_z)$. To control the second right-hand term in (4-7), recall that $\nu_2 = e_k \overline{\mu_2}$, so that the second term is written

$$-u \int k r e_k \overline{\nu_2} + \frac{|u|^2}{2} \int e_{-k} r \nu_1. \tag{4-8}$$

Since u and $|u|^2$ belong to L^2 it is enough to show that the two integrals in (4-8) define LLCMs from $r \in H^{2,1}$ to L^∞_z. Since $kr \in L^{p'}$ for $p > 2$ and ν_2 is an LLCM from $H^{1,2}$ to $L^\infty_z(L^p_k)$, the first term in (4-8) clearly has this property. Since $r \in L^1$ and ν_1 is an LLCM from $r \in H^{2,1}$ to $L^\infty_z(L^\infty_k)$, we conclude that the second term also has this property. □

5. Solving the mNV equation

In this section we prove Theorem 1.5. Recall that the modified Novikov–Veselov (mNV) equation [Bogdanov 1987] is

$$u_t + (\partial^3 + \bar\partial^3)u - NL(u) = 0, \tag{5-1}$$

where

$$NL(u) = \tfrac{3}{4}(\partial \bar u)\cdot(\bar\partial \partial^{-1}(|u|^2)) + \tfrac{3}{4}(\bar\partial u)\cdot(\bar\partial\partial^{-1}(|u|^2)) + \tfrac{3}{4}\bar u \bar\partial \partial^{-1}(\bar u \bar\partial u) + \tfrac{3}{4} u \partial^{-1}(\bar\partial(\bar u \bar\partial u)).$$

By Theorem A, for $u_0 \in \mathcal{S}(\mathbb{R}^2)$, the formula

$$u(z,t) = \mathcal{I}\big(\exp((\bar\diamond^3 - \diamond^3)t)\mathcal{R}u_0(\diamond)\big)(z) \tag{5-2}$$

gives a classical solution of the mNV equation.

Proposition 5.1. *Suppose that $u_0 \in H^{2,1}(\mathbb{R}^2)$. Then (5-2) defines a weak solution of the mNV equation in the sense of (1-12) with $\lim_{t\to 0} u(t) = u_0$ in $L^2(\mathbb{R}^2)$.*

Proof. Let $r_0 = \mathcal{R}u_0$. By continuity of the maps $\mathcal{R}$, $r_0 \mapsto \exp((\bar\diamond^3 - \diamond^3)t) r_0(\diamond)$, and $\mathcal{I}$, the formula (5-2) extends to $u_0 \in H^{2,1}$, and exhibits the solution as a continuous curve in $H^{2,1}$ that depends continuously on the initial data. Since, for any $u_0 \in \mathcal{S}(\mathbb{R}^2)$, the function u given by (5-2) is a classical solution, it follows that u trivially satisfies (1-12). The same fact for $u(t)$ with $u_0 \in H^{2,1}$ follows from the density of $\mathcal{S}(\mathbb{R}^2)$ in $H^{2,1}$, the continuity of the map (5-2) in u_0, and an easy approximation argument. □

It remains to show:

Proposition 5.2. *Suppose that $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and that, also,*

$$\int u_0 \, dA(z) = 0, \qquad \partial u_0 = \overline{\partial \mu_0}. \tag{5-3}$$

Define $u(t)$ by (5-2). Then

$$\partial u = \overline{\partial u}, \tag{5-4}$$

for all t.

We will prove Proposition 5.2 by first showing that the relation (5-4) holds for initial data $u_0 \in S(\mathbb{R}^2)$ with the stated properties. We will then use Lipschitz continuity of the map $u_0 \to u(t)$ defined by (5-2) to extend to all $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ so that the conditions (5-3) hold.

First, we consider $u_0 \in \mathcal{S}(\mathbb{R}^2)$. It will be useful to consider the function

$$\varphi = \bar{\partial}^{-1} u,$$

which solves the Cauchy problem

$$\begin{aligned} \varphi_t &= -\partial^3\varphi - \bar{\partial}^3\varphi - \tfrac{1}{4}(\partial\varphi)^3 - \tfrac{1}{4}(\bar{\partial}\varphi)^3 + \tfrac{3}{4}\partial\varphi \cdot \bar{\partial}^{-1}\partial(|\partial\varphi|^2) + \tfrac{3}{4}\bar{\partial}\varphi \cdot \bar{\partial}^{-1}\partial(|\partial\varphi|^2), \\ \varphi|_{t=0} &= \varphi_0. \end{aligned} \tag{5-5}$$

The condition $\partial u_0 = \overline{\partial \mu_0}$ implies that φ_0 is real. On the other hand, to show that $\partial u = \overline{\partial u}$, it suffices to show that φ is real for $t > 0$. To this end, we consider the function

$$w = \varphi - \bar{\varphi},$$

and derive a linear Cauchy problem satisfied by w. We will need to know that w is L^2 in the space variables.

Lemma 5.3. *Suppose that $u_0 \in \mathcal{S}(\mathbb{R}^2)$, that $u(t)$ solves the mNV equation, and $\varphi(z,t) = (\bar{\partial}^{-1}u)(t)$. Then for each t,*

$$\varphi(z,t) = \frac{c_0}{z} + \mathcal{O}_t(|z|^{-2}),$$

where $c_0 = \int u(z,t)\,dm(z)$ is independent of t. If $c_0 = 0$, then $\varphi(\,\cdot\,,t) \in L^2(\mathbb{R}^2)$ for all $t > 0$.

Proof. To see that φ has the stated form if $u_0 \in \mathcal{S}(\mathbb{R}^2)$, we note that $u(t) \in \mathcal{S}(\mathbb{R}^2)$ by the mapping properties of the scattering transform (see [Sung 1994a; 1994b; 1994c]) and that

$$\varphi(z,t) = -\frac{1}{\pi z}\int u(z,t)\,dt + \mathcal{O}_t(|z|^{-2})$$

differentiably in z, t. Let $c_0(t) = \int u(z,t)\,dm(z)$. Substituting in (5-5) we easily conclude that $c_0'(t) = 0$. It now follows that $\varphi(\diamond, t) \in L^2(\mathbb{R}^2)$ for each t as claimed. □

Next, we derive a linear Cauchy problem obeyed by w and show that, if $w|_{t=0} = 0$, then $w(t) = 0$ identically. If so, it follows that φ is real, and hence $\partial u = \overline{\partial u}$ for all $t > 0$.

Using (5-5) and its complex conjugate, we see that

$$w_t = Lw, \tag{5-6}$$

where

$$Lw = L_0 + A\partial w + \overline{A\partial} w$$

with

$$L_0 w = -\partial^3 w - \bar{\partial}^3 w$$

and

$$A = \tfrac{1}{4}[(\partial\varphi)^2 + (\partial\varphi)\cdot(\partial\bar{\varphi}) + (\partial\bar{\varphi})^2] + \tfrac{3}{4}\bar{\partial}^{-1}\partial(|\partial\varphi|^2). \tag{5-7}$$

We will need the following property of A. We say that $g(z)$ is *integrable along lines* if $\int_{-\infty}^{\infty} |g(\gamma(t))|\,dt$ is finite for any path $\gamma(t) = z_0 + z_1 t$. We say that g is *uniformly integrable along lines* if

$$\sup_{\substack{z_0\in\mathbb{C}\\ |z_1|=1}} \int |g(\gamma(t))|\,dt < \infty.$$

Lemma 5.4. *Suppose that $\varphi = \bar\partial^{-1}u$, where $u \in C([0,T]; \mathcal{S}(\mathbb{R}^2))$ and*

$$\int u(z,t)\,dm(z) = 0$$

for all t. Then, the function $A(z,t)$ is uniformly integrable along lines in $\mathbb{R}^2$, with estimates uniform in $t \in [0,T]$.

Proof. Recall that if $f \in H^s(\mathbb{R}^2)$ then the restriction of f to a line belongs to $H^{s-1/2-\varepsilon}(\mathbb{R}^2)$ for any $\varepsilon > 0$. In particular, if $f \in H^1(\mathbb{R}^2)$, then f is square-integrable along lines. Note that $\partial\varphi = \partial\bar\partial^{-1}u$ and $\partial\bar\varphi = \bar u$ belong to $H^s(\mathbb{R}^2)$ for all $s > 0$ and each fixed $t \in [0,T]$ since $\partial\bar\partial^{-1}$ is a Fourier multiplier on H^s and $u \in H^s(\mathbb{R}^2)$ for all such s, uniformly in $t \in [0,T]$. In particular, $\partial\varphi$ and $\partial\bar\varphi$ restrict to square-integrable functions along lines in $\mathbb{R}^2$, so the first three terms in (5-7) are all integrable along lines with estimates bounded seminorms of u.

To handle the last term in (5-7), we note that $\partial\phi = \partial\bar\partial^{-1}u$. Hence, by Lemma 2.3 and the fact that differentiation commutes with the Beurling transform, we conclude that

$$\sup_{|\alpha|\le 2} |D^\alpha(|\partial\phi|^2)| \le C(1+|z|)^{-4}.$$

It now follows from Lemma 2.3 that again $\bar\partial^{-1}\partial(|\partial\phi|)^2$ is $\mathcal{O}(|z|^{2-\varepsilon})$ for any $\varepsilon > 0$, and hence is integrable along lines with appropriate uniform estimates. □

We wish to prove an a priori estimate for the problem (5-6) that bounds $\|w(t)\|$ in terms of $\|w(0)\|$, proving uniqueness of the initial value problem. A formal computation of $\frac{d}{dt}\|w(t)\|^2$ leads to uncontrolled derivatives since the principal part of L is skew-adjoint. Instead, following the multiplier method of [Chihara 2004] (applied to third-order dispersive nonlinear equations; see [Doi 1994] for a similar pseudodifferential multiplier method applied to Schrödinger-type equations), we find a family of invertible pseudodifferential operators $K(t)$ such that

(1) $\|K(t)w(t)\|$ controls $\|w(t)\|$, and

(2) $\dfrac{d}{dt}\|K(t)w(t)\|^2$ is bounded above.

A formal computation shows that

$$\frac{d}{dt}\|K(t)w(t)\|^2 = \big(K(t)w(t), C(t)K(t)w(t)\big), \tag{5-8}$$

where

$$\begin{aligned} C(t) &= 2\,\mathrm{Re}\big\{K'(t)K(t)^{-1} + K(t)L(t)K(t)^{-1}\big\} \\ &= 2\,\mathrm{Re}\big\{K'(t)K(t)^{-1} + K(t)(A\partial + \overline{A\partial})K(t)^{-1} + [K(t), L_0]K(t)^{-1}\big\}. \end{aligned} \tag{5-9}$$

We will choose $K(t)$ so that $C(t)$ is the sum of a negative definite operator and a bounded operator.

The following lemma obtains the desired estimate. Note that Lemma 5.4 implies the existence of a function $\eta(z,t)$ satisfying the hypotheses of Lemma 5.5 if A is given by (5-7).

Lemma 5.5. *Suppose that $A(z,t)$ is a bounded smooth function on $\mathbb{R}^2\times[0,T]$ and that $\eta(z,t)$ is a bounded smooth nonnegative function with $|A(z,t)|\le\eta(z,t)$ for $z\in\mathbb{C}$ and $t\in[0,T]$. Writing $\eta(z,t)=\eta(x_1,x_2,t)$, suppose that there is a constant c such that $\int|\eta(y,x_2,t)|\,dy\le c$ and $\int|\eta(x_1,y,t)|\,dy\le c$ uniformly in $(x_1,x_2)\in\mathbb{R}^2$ and $t\in[0,T]$. Finally, let w be a smooth solution of (5-6) with $w(\diamond,t)\in L^2(\mathbb{R}^2)$ for each $t>0$. Then, there is a constant C such that*

$$\sup_{t\in[0,T]}\|w(t)\|\le e^{CT}\|w(0)\|.$$

Proof. Let η be a function with

$$2|A(z,t)|\le\eta(z,t),$$

and set

$$p_0(\xi)=\frac{i}{4}(\xi_1^3-3\xi_1\xi_2^2),$$

the symbol of the operator $-\partial^3-\overline{\partial}^3$. With $z=x_1+ix_2$ and $\lambda>0$ to be chosen, let

$$b(t,x,\xi)=i\left(\int_{-\infty}^{x_1}\eta(y,x_2,t)\,dy\right)\times\frac{\partial p_0(\xi)}{\partial\xi_1}\frac{|\xi|}{|\nabla p_0(\xi)|^2}\chi\left(\frac{|\xi|}{\lambda}\right)+i\left(\int_{-\infty}^{x_2}\eta(x_1,y,t)\,dy\right)\times\frac{\partial p_0(\xi)}{\partial\xi_2}\frac{|\xi|}{|\nabla p_0(\xi)|^2}\chi\left(\frac{|\xi|}{\lambda}\right),\tag{5-10}$$

where $\chi\in\mathcal{C}_0^\infty([0,\infty))$ is a nonnegative function with $\chi(t)=0$ for $0\le t<1/2$ and $\chi(t)=1$ for $t\ge1$. By the usual quantization, the pseudodifferential operator $b(t,x,D)$ belongs to the class $\mathrm{OPS}^{-1}(\mathbb{R}^n)$. It is easy to see that, also, the symbols

$$k(t,x,\xi)=e^{b(t,x,\xi)}\quad\text{and}\quad\tilde{k}(t,x,\xi)=e^{-b(t,x,\xi)}$$

define pseudodifferential operators $K(t):=K(t,x,D)$ and $\widetilde{K}(t):=\widetilde{K}(t,x,D)$ in $\mathrm{OPS}^0(\mathbb{R}^n)$ with

$$K(t)\widetilde{K}(t)-I\in\mathrm{OPS}^{-1}(\mathbb{R}^n)\quad\text{and}\quad\lim_{\lambda\to\infty}\sup_{t\in[0,T]}\|K(t)\widetilde{K}(t)-I\|=0.$$

Thus, there is a $\lambda_0>0$ such that $K(t)$ is invertible for all $|\lambda|\ge\lambda_0$. We take $|\lambda|\ge\lambda_0$ from now on.

We claim that, if $w(t)$ is a solution of the evolution equation (5-6) belonging to $L^2(\mathbb{R}^2)$, the inequality

$$\|K(t)w(t)\|\le\|K(0)w(0)\|e^{CT}\tag{5-11}$$

holds for $t\in[0,T]$ and a constant C. Since $K(t)$ is invertible for λ sufficiently large and $t\in[0,T]$, this implies that $w(t)=0$ for all t if $w(0)=0$.

To prove the inequality (5-11), we use (5-8). We will show that

$$2\,\mathrm{Re}\left\{A\partial+\overline{A\partial}+[K(t),L_0]K(t)^{-1}\right\}=-Q_1(t)+Q_2(t),\tag{5-12}$$

where $Q_1(t) \in \mathrm{OPS}^{1,0}(\mathbb{R}^2)$ with $q_1(x,\xi) := \sigma(Q_1(t))$ nonnegative for $|\xi| \geq 2\lambda$, and $Q_2(t) \in \mathrm{OPS}^0(\mathbb{R}^2)$. If so, then by the Gårding inequality (2-5),

$$\mathrm{Re}(v, Q_1(t)v) \geq -C_1\|v\|^2, \tag{5-13}$$

with C_1 uniform in $t \in [0, T]$. Hence

$$\frac{d}{dt}\|K(t)w(t)\|^2 \leq C_3\|K(t)w(t)\|^2,$$

where C_3 majorizes $C_1 + \sup_{t\in[0,T]}\big(\|Q_2(t)\| + \|K'(t)K^{-1}(t)\|\big)$. The desired result now follows from Gronwall's inequality.

Thus, to finish the proof of (5-11), we need only prove that (5-12) holds. From the computation

$$\sigma([K(t), L_0]) = -\frac{1}{i}\nabla_x(e^{\gamma(t,x,\zeta)}) \cdot (\nabla_\xi p_0)(\xi),$$

it follows that the left-side of (5-12) has leading symbol $-q_1(x_1, x_2, \xi, t)$ where

$$q_1(x_1, x_2, \xi, t) = \frac{1}{i}\nabla_\xi p_0(\xi) \cdot \nabla_x \gamma(t, x_1, x_2, \xi) + \mathrm{Re}[A(x_1, x_2, t)(\xi_1 - i\xi_2)],$$

which is nonnegative for $|\xi| \geq 2\lambda$ since $|A(x_1, x_2, t)| \leq \eta(x_1, x_2, t)$. This completes the proof. □

Proof of Proposition 5.2. First, suppose that $u_0 \in \mathcal{S}(\mathbb{R}^2)$, $\partial u_0 = \overline{\partial u_0}$, and $\int u_0(z)\, dm(z) = 0$. The function $\varphi_0 = \bar\partial^{-1} u_0$ is real-valued and if $u(t)$ solves the mNV equation with Cauchy data u_0, the function $\varphi(t) = (\bar\partial^{-1}u)(t)$ belongs to $L^2(\mathbb{R}^2)$ for all t. The same is true of $w(t) = \varphi(t) - \overline{\varphi(t)}$, and $w(0) = 0$. It now follows from Lemma 5.5 that $w(t) = 0$ and $\varphi(t)$ is real-valued for all t. This implies that $\partial u = \overline{\partial u}$ for all t.

To conclude that the proposition holds for $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, we first observe that there is a sequence $\{v_{n,0}\}$ from $\mathcal{S}(\mathbb{R}^2)$ with $v_{n,0} \to u_0$ in $H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Let f be a nonnegative C_0^∞ function with $\int f = 1$, and let $u_{n,0} = v_{n,0} - (\int u_{n,0}) f$. It is easy to see that $\int v_{n,0} = 0$ and $v_{n,0} \to u_0$ in $H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Since

$$u_n(t) := \mathcal{I}\big(\exp((\bar\diamond^3 - \diamond^3)t)\mathcal{R}u_{0,n}(\diamond)\big)$$

converges to

$$u(t) = \mathcal{I}\big(\exp((\bar\diamond^3 - \diamond^3)t)\mathcal{R}u_{0,n}(\diamond)\big)$$

in $C([0, T], H^{2,1})$, it now follows that $\partial u = \overline{\partial u}$, as claimed. □

Proof of Theorem 1.5. An immediate consequence of Propositions 5.1 and 5.2. □

6. Solving the NV equation

In this section we prove Theorem 1.6. The key observation is due to Bogdanov [1987] and can be checked by straightforward computation. Recall the Miura map $\mathcal{M}$, defined in (1-9).

Lemma 6.1. *Suppose that $u(z,t)$ is a smooth classical solution of (5-1) with*

$$(\partial_z u)(z, t) = \overline{(\partial_z u)(z, t)},$$

and $\int u(z,t)\,dm(z) = 0$ *for all* t. *Then, the function*

$$q(z,t) = \mathcal{M}(u(\,\cdot\,,t))(z)$$

is a smooth classical solution of (1-1).

Remark 6.2. In [Bogdanov 1987], the mNV and NV are shown to be gauge-equivalent, and the Miura map is computed from the gauge equivalence. Note that our conventions differ slightly from those of Bogdanov in order to insure that the range of the Miura map consists of real-valued functions.

Proof of Theorem 1.6. Pick $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ so that $\partial u_0 = \overline{\partial u_0}$ and $\int u_0(z)\,dm(z) = 0$. Let $\{u_{0,n}\}$ be a sequence from $\mathcal{S}(\mathbb{R}^2)$ with $u_{n,0} \to u_0$ in $H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. By local Lipschitz continuity of the scattering maps, for any $T > 0$, the sequence $\{u_n\}$ from $C([0,T]; H^{2,1}(\mathbb{R}^2))$ given by

$$u_n(z,t) = \mathcal{I}\big(e^{t((\diamond)^3 - (\bar{\diamond})^3)}(\mathcal{R}u_{0,n})(\diamond)\big)(z)$$

converges in $C([0,T]; H^{2,1}(\mathbb{R}^2))$ to

$$u(z,t) := \mathcal{I}\big(e^{t((\diamond)^3 - (\bar{\diamond})^3)}(\mathcal{R}u_0)(\diamond)\big)(z).$$

This convergence implies that $q_n(z,t) := \mathcal{M}(u_n(\diamond,t))(z)$ converges in $L^2(\mathbb{R}^2)$.

Recall (1-13). Since $q_n \to q$ in $C([0,T]; L^2(\mathbb{R}^2))$ it follows from the L^2-boundedness of $\mathcal{S} = \partial\bar{\partial}^{-1}$ that the two nonlinear terms converge in L^1; i.e., $q_n\bar{\partial}^{-1}\partial q_n \to q\bar{\partial}^{-1}\partial q$ and $q_n\partial^{-1}\bar{\partial}q_n \to q\partial^{-1}\bar{\partial}q$ in $C([0,T], L^1(\mathbb{R}^2))$. We conclude that q is a weak solution of the NV equation. □

7. Conductivity-type potentials

In this section we show that our solution of NV coincides with that of [Lassas et al. 2012] in the cases they consider, proving Theorem 1.7.

We briefly recall some of the notation and results of [Lassas et al. 2007]. Assume first that $q \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and is of conductivity type. We denote by $\psi(x,\zeta)$ the unique solution of the problem

$$(-\Delta + q)\psi = 0, \qquad (7\text{-}1)$$

$$\lim_{|z|\to\infty} (e^{-i(x\cdot\zeta)}\psi(x,\zeta) - 1) = 0,$$

where $x = (x_1, x_2)$ and $\zeta \in \mathbb{C}^2$ satisfies $\zeta \cdot \zeta = 0$. Here $a \cdot b$ denotes the Euclidean inner product without complex conjugation. Henceforth, we set $\zeta = (k, ik)$ for $k \in \mathbb{C}$, which amounts to choosing a branch of the variety $\mathcal{V} = \{\zeta \in \mathbb{C}^2 : \zeta\cdot\zeta = 0\}$. Since q is of conductivity type, it follows from Theorem 3 in [Nachman 1996] that the problem (7-1) admits a unique solution for each $k \in \mathbb{C}$. We set $z = x_1 + ix_2$ and define

$$m(z,k) = e^{-ikz}\psi(x,\zeta), \qquad (7\text{-}2)$$

for $\zeta = (k, ik)$.

The direct scattering map

$$\mathcal{T} : q \to \mathbf{t} \qquad (7\text{-}3)$$

is defined by

$$\mathbf{t}(k) = \int e^{i(\bar{k}\bar{z}+kz)} q(z) m(z,k)\, dm(z). \tag{7-4}$$

The inverse map

$$\mathcal{Q} : \mathbf{t} \to q \tag{7-5}$$

is defined by

$$q(z) = \frac{i}{\pi^2} \bar{\partial}_z \left(\int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e^{-i(kz+\bar{k}\bar{z})} \overline{m(z,k)}\, dm(k) \right), \tag{7-6}$$

where $m(z,k)$ is reconstructed from t via the $\bar{\partial}$-problem

$$\bar{\partial}_k m(x,k) = \frac{\mathbf{t}(k)}{4\pi k} e^{-i(kz+\bar{k}\bar{z})}(z) \overline{m(x,k)}. \tag{7-7}$$

Let

$$\mathbf{m}_t^n(k) = \exp(-i^n(k^n + \bar{k}^n)t),$$

for an odd positive integer n. Lassas, Mueller, Siltanen and Stahel proved:

Theorem 7.1 [Lassas et al. 2007, Theorem 1.1; 2012, Theorem 4.1]. *For $q_0 \in C_0^\infty(\mathbb{R}^2)$ radial and of conductivity type, $\mathcal{Q}\mathcal{T}(q_0) = q_0$. Moreover, if*

$$q(t) := \mathcal{Q}(\mathbf{m}_t^n \mathcal{T} q_0), \tag{7-8}$$

then $q(t)$ is a continuous, real-valued potential with $q(t) \in L^p(\mathbb{R}^2)$ for $p \in (1,2)$.

They conjecture that for $n = 3$, $q(t)$ given by (7-8) solves the NV equation, provided that q_0 obeys the hypotheses of Theorem 7.1. We will prove that this is the case (for a larger class of q_0) by proving Theorem 1.7.

We will prove Theorem 1.7 in two steps. First, we show that for $u_0 \in \mathcal{S}(\mathbb{R}^2)$ with $\partial u_0 = \overline{\partial u_0}$ and $\int u_0(z)\, dm(z) = 0$, the scattering data $r = \mathcal{R}u$ is related to the scattering transform $\mathbf{t} = \mathcal{T}q$ for $q = 2\partial u + |u|^2$ by the identity

$$\mathbf{t}(k) = -2\pi i \bar{k}\, \overline{r(ik)}.$$

Next, we show that for $\mathbf{t}$ of the above form with $r = \mathcal{R}u$, the identity

$$(\mathcal{Q}\mathbf{t})(z) = 2(\partial u)(z) + |u(z)|^2.$$

Theorem 1.7 is an easy consequence of these two identities.

The key to both computations is the following construction of complex geometric optics solutions for the potential $q = 2\partial u + |u|^2$ from the solutions $\mu = (\mu_1, \mu_2)^T$ of (3-1). First, suppose that $\Phi = (\Phi_1, \Phi_2)^T$ is a vector-valued solution of the linear system

$$\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \Phi = \frac{1}{2} \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \Phi. \tag{7-9}$$

A straightforward calculation shows that the function

$$\widetilde{\psi} = \Phi_1 + \Phi_2$$

solves the zero-energy Schrödinger equation

$$(-\Delta+q)\widetilde{\psi}=0 \tag{7-10}$$

for $q=2\partial u+|u|^2$.

Recall that matrix-valued solutions of (7-9) are related to the solutions μ of (3-1) by

$$\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}e^{-kz},$$

so that

$$\Phi_1+\Phi_2=e^{kz}\mu_1(z,k)+e^{\bar{k}\bar{z}}\,\overline{\mu_2(z,k)} \tag{7-11}$$

solves (7-10). To compute its asymptotic behavior, using $(\mu_1,\mu_2)\to(1,0)$ as $|z|\to\infty$ we conclude that $e^{-kz}\widetilde{\psi}(z,k)\to 1$ as $|z|\to\infty$. Hence, denoting by ψ the solution of the problem (7-10) with $\zeta=(k,ik)$ for $k\in\mathbb{C}$, we have

$$\psi(z,k)=\widetilde{\psi}(z,ik)=e^{ikz}\mu_1(z,ik)+e^{-i\bar{k}\bar{z}}\,\overline{\mu_2(z,ik)}, \tag{7-12}$$

so

$$m(z,k)=\mu_1(z,k)+e^{-i(kz+\bar{k}\bar{z})}\,\overline{\mu_2(z,ik)}.$$

Lemma 7.2. *Let $u\in C_0^\infty(\mathbb{R}^2)$ with $\partial u=\overline{\partial u}$, suppose $\int u(z)\,dm(z)=0$, and let $q=2\partial u+|u|^2$. Then*

$$(\mathcal{T}q)(k)=-2\pi i\bar{k}\,\overline{(\mathcal{R}u)(ik)}. \tag{7-13}$$

Proof. We compute

$$\begin{aligned}(\mathcal{T}q)(k)&=\int q(z)e^{i\bar{k}\bar{z}}\psi(z,k)\,dm(z)\\ &=\int 2(\overline{\partial u})(z)e^{i(\bar{k}\bar{z}+kz)}\mu_1(z,ik)\,dm(z)\\ &\qquad+\int 2(\partial u)(z)\overline{\mu_2(z,ik)}\,dm(z)\\ &\qquad+\int |u(z)|^2\big(e^{i(\bar{k}\bar{z}+kz)}\mu_1(z,ik)+\overline{\mu_2(z,ik)}\big)\,dm(z)\\ &=I_1+I_2+I_3,\end{aligned}$$

where in the first right-hand term we used $\partial u=\overline{\partial u}$. We can integrate by parts in each of the first two right-hand terms and use (3-1) to obtain

$$\begin{aligned}I_1&=-2i\bar{k}\int\overline{u(z)}\,e^{i(kz+\bar{k}\bar{z})}\mu_1(z,ik)\,dm(z)-\int|u(z)|^2\,\overline{\mu_2(z,ik)}\,dm(z),\\ I_2&=-\int|u(z)|^2e^{i(kz+\bar{k}\bar{z})}\mu_1(z,ik)\,dm(z).\end{aligned}$$

Using the relation (3-2), we recover (7-13). □

Next, we analyze the inverse scattering transform $\mathcal{Q}$ defined by (1-7).

Lemma 7.3. *Let $u \in \mathcal{S}(\mathbb{R}^2)$ with $\partial u = \overline{\partial u}$, and suppose that $\int u(z)\, dm(z) = 0$. Let $r = \mathcal{R}u$ and suppose that t is given by (7-13). Then*

$$(\mathcal{Q}t)(z) = 2(\partial u)(z) + |u(z)|^2.$$

Proof. We compute from (1-7), (7-13), and (7-12) that

$$\begin{aligned}(\mathcal{Q}t)(z) &= \frac{2}{\pi}\bar{\partial}_z\Big(\int \overline{r(ik)}\, e^{-i(kz+\bar{k}\bar{z})}\,\overline{\mu_1(z,ik)}\, dm(k)\Big) + \frac{2}{\pi}\bar{\partial}_z\Big(\int \overline{r(ik)}\,\mu_2(z,ik)\, dm(k)\Big)\\ &= T_1 + T_2.\end{aligned}$$

Changing variables to $\zeta = ik$ in T_1 we recover

$$T_1 = \frac{2}{\pi}\bar{\partial}_z\Big(\int \overline{r(\zeta)}\, e^{\bar{\zeta}\bar{z}-\zeta z}\,\overline{\mu_1(z,\zeta)}\, dm(\zeta)\Big) = 2(\overline{\partial u})(z)[5pt] = 2(\partial u)(z),$$

where we have used (3-5). Using (3-1) in T_2 we have

$$T_2 = \frac{1}{\pi}\int \overline{r(ik)}\, u(z)\, e^{-i(kz+\bar{k}\bar{z})}\,\overline{\mu_1(z,ik)}\, dm(k) = \frac{1}{\pi}u(z)\int r(\zeta)\, e^{\bar{\zeta}\bar{z}-\zeta z}\,\overline{\mu_1(z,\zeta)}\, dm(\zeta)[2pt] = |u(z)|^2.$$

Combining these computations gives the desired result. □

Proof of Theorem 1.7. For u_0 satisfying the hypotheses and $q = 2\partial u_0 + |u_0|^2$, we have by Lemma 7.2 that

$$(\mathcal{T}q_0)(k) = -2\pi i\,\bar{k}\,\overline{r(ik)},$$

where $r = \mathcal{R}(u_0)$, and hence

$$e^{-it(k^3+\bar{k}^3)}(\mathcal{T}q_0)(k) = -2\pi i\,\bar{k}\,\overline{\big(e^{t((\bar{\diamond})^3-(\diamond)^3)}r(\diamond)\big)(ik)}.$$

We can now apply Lemma 7.3 to conclude that

$$\mathcal{Q}\big(e^{-it((\diamond)^3+(\bar{\diamond})^3)}(\mathcal{T}q_0)(\diamond)\big) = \mathcal{M}\mathcal{I}\big(e^{t((\bar{\diamond})^3-(\diamond)^3)}r(\diamond)\big),$$

as claimed. □

Appendix: Schwarz class inverse scattering for the mNV equation

In this appendix we develop the Schwarz class inverse theory for the mNV equation, using freely the results and notation of [Perry 2011]. Our main result is this:

Theorem A. *Suppose that $u_0 \in \mathcal{S}(\mathbb{R}^2)$, and let $\mathcal{R}$ and $\mathcal{I}$ be the scattering maps defined respectively by (3-2) and (3-5). Finally, define*

$$u(t) = \mathcal{I}\big(e^{t((\diamond)^3-(\bar{\diamond})^3)}(\mathcal{R}u_0)(\diamond)\big).$$

Then $u(t)$ is a classical solution of the modified Novikov–Veselov equation (5-1).

The proof follows the method of [Beals and Coifman 1985; 1989; 1990; Sung 1994a; 1994b; 1994c] but necessitates some long computations.

A.1. ***Scattering solutions and tangent maps.*** First we recall the solutions ν and $\tilde{\nu}$ of the $\bar{\partial}$ problem with $\bar{\partial}$-data determined by the time-dependent coefficient r and the formulas from [Perry 2011] for the tangent maps.

We recall that $\nu = (\nu_1, \nu_2)^T$ is the unique solution of the $\bar{\partial}$ problem

$$\begin{aligned} \bar{\partial}_k \nu_1 &= \tfrac{1}{2} e_k \bar{r} \overline{\nu_2}, \\ \bar{\partial}_k \nu_2 &= \tfrac{1}{2} e_k \bar{r} \overline{\nu_1}, \\ \lim_{|k|\to\infty} \nu(z,k) &= (1,0), \end{aligned} \tag{A-1}$$

where $r = \mathcal{R}(u)$. Here

$$e_k(z) = e^{\bar{k}\bar{z} - kz}.$$

The function $\nu^\# = (\nu_1^\#, \nu_2^\#)$ solves the same problem but for $u^\#(\,\cdot\,) = -\bar{u}(-\,\cdot\,)$ and $r^\# = \mathcal{R}(u^\#) = -\bar{r}$ (see [Perry 2011, Lemma B.1]). Thus

$$\begin{aligned} \bar{\partial}_k \nu_1^\# &= -\tfrac{1}{2} e_k r \overline{\nu_2^\#}, \\ \bar{\partial}_k \nu_2^\# &= -\tfrac{1}{2} e_k \bar{r} \overline{\nu_1^\#}, \\ \lim_{|k|\to\infty} \nu^\#(z,k) &= (1,0). \end{aligned} \tag{A-2}$$

The tangent map formula gives an expression for u if $u = \mathcal{R}(r)$ where r is a C^1-curve in $\mathcal{S}(\mathbb{R}^2)$. Assuming the law of evolution

$$\dot{r} = (\bar{k}^3 - k^3) r,$$

and following the calculations in Appendix B of [Perry 2011], we find that

$$u = 2i(I_1 + \overline{I_2}), \tag{A-3}$$

where

$$I_1 = \frac{1}{\pi} \int k^3 \bar{\partial}_k [\nu_2^\#(-z,k) \nu_1(z,k)]\, dm(k), \tag{A-4}$$

$$I_2 = -\frac{1}{\pi} \int k^3 \bar{\partial}_k [\nu_1^\#(-z,k) \nu_2(z,k)]\, dm(k). \tag{A-5}$$

As in [Perry 2011, Appendix B], we evaluate these integrals using the following fact: if g is a $\mathcal{C}^\infty$ function with asymptotic expansion

$$g(k, \bar{k}) \sim 1 + \sum_{\ell \ge 0} \frac{g_\ell}{k^{\ell+1}}, \tag{A-6}$$

as $|k| \to \infty$ then

$$\lim_{R\to\infty} \left(-\frac{1}{\pi} \int_{|k|\le R} k^n (\bar{\partial}_k g)(k)\, dm(k) \right) = g_n. \tag{A-7}$$

Using (A-7) we get (noting the $-$ sign in (A-5))

$$I_1 = 2[\nu_1(z,\diamond)\nu_2^\#(-z,\diamond)]_3 \quad \text{and} \quad \overline{I_2} = 2[\nu_2(z,\diamond)\nu_1^\#(-z,\diamond)]_3,$$

so that

$$\dot{u} = 2\left\{[\nu_1(z,\diamond)\nu_2^\#(-z,\diamond)]_3 + \overline{[\nu_2(z,\diamond)\nu_1^\#(-z,\diamond)_3]}\right\} \tag{A-8}$$

Here $[\,\diamond\,]_n$ denotes the coefficient of k^{-n-1} in an asymptotic expansion of the form (A-6). The formulas

$$[\nu_1(z,\diamond)\nu_2^\#(-z,\diamond)]_n = (\nu_n^\#)_{21} + \sum_{j=0}^{n-1}(\nu_{n-j-1}^\#)_{21}(\nu_j)_{11},$$

$$[\nu_2(z,\diamond)\nu_1^\#(-z,\diamond)]_n = (\nu_n)_{12} + \sum_{j=0}^{n-1}(\nu_{n-1-j})_{12}(\nu_j^\#)_{22}$$

will be used in concert with the residue formulae below to obtain the equation of motion.

A.2. ***Expansion coefficients for ν.*** Following the method of Appendix C in [Perry 2011], we can compute the additional coefficients in the asymptotic expansion

$$\nu \sim (1,0) + \sum_{\ell\geq 0} k^{-(\ell+1)}\nu^{(\ell)} \tag{A-9}$$

needed to compute $\dot{u}$ from the formula (A-8). Let us set $\nu^{(\ell)} = (\nu_{1,\ell},\, \nu_{2,\ell})^T$. We recall from [Perry 2011] the “initial data”

$$\nu_{1,0} = \tfrac{1}{4}\bar{\partial}^{-1}(|u|^2), \qquad \nu_{2,0} = \tfrac{1}{2}\bar{u}, \tag{A-10}$$

and the recurrence relations

$$\nu_{2,\ell} = \tfrac{1}{2}\bar{u}\nu_{1,\ell-1} - \partial\nu_{2,\ell-1}, \qquad \nu_{1,\ell} = \tfrac{1}{2}P(u\nu_{2,\ell}).$$

The following formulas are a straightforward consequence.

$\ell = 0$:

$$\nu_{1,0} = \tfrac{1}{4}\bar{\partial}^{-1}(|u|^2), \tag{A-11}$$

$$\nu_{2,0} = \tfrac{1}{2}\bar{u}. \tag{A-12}$$

$\ell = 1$:

$$\nu_{1,1} = \tfrac{1}{16}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2)) - \tfrac{1}{4}\bar{\partial}^{-1}(u\partial\bar{u}), \tag{A-13}$$

$$\nu_{2,1} = \tfrac{1}{8}\bar{u}\bar{\partial}^{-1}(|u|^2) - \tfrac{1}{2}\partial\bar{u}. \tag{A-14}$$

$\ell = 2$:

$$\begin{aligned}\nu_{1,2} = \tfrac{1}{64}\,&\bar{\partial}^{-1}\big(|u|^2\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big)\\ &- \tfrac{1}{16}\left\{\bar{\partial}^{-1}\big(u\partial(\bar{u}\bar{\partial}^{-1}(|u|^2))\big) + \bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(u\partial\bar{u}))\right\} + \tfrac{1}{4}\bar{\partial}^{-1}(u\partial^2\bar{u}),\end{aligned} \tag{A-15}$$

$$\nu_{2,2} = \tfrac{1}{32}\bar{u}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2)) - \tfrac{1}{8}\left\{\partial(\bar{u}\bar{\partial}^{-1}(|u|^2)) + \bar{u}\bar{\partial}^{-1}(u\partial\bar{u})\right\} + \tfrac{1}{2}\partial^2\bar{u}. \tag{A-16}$$

$\ell = 3$:

$$\begin{aligned}\nu_{2,3} = {} & \tfrac{1}{128}\bar{u}\bar{\partial}^{-1}\big(|u|^2\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big) \\ & - \tfrac{1}{32}\left\{\bar{u}\bar{\partial}^{-1}\big(u\partial(\bar{u}\bar{\partial}^{-1}(|u|^2))\big) + \bar{u}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(u\partial\bar{u})) + \partial\big(\bar{u}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big)\right\} \\ & + \tfrac{1}{8}\left\{\bar{u}\bar{\partial}^{-1}(u\partial^2\bar{u}) + \partial^2(\bar{u}\bar{\partial}^{-1}(|u|^2)) + \partial(\bar{u}\bar{\partial}^{-1}(u\partial\bar{u}))\right\} \\ & - \tfrac{1}{2}\partial^3\bar{u}.\end{aligned} \tag{A-17}$$

A.3. ***Expansion coefficients for $\nu^\#$.*** The solution $\nu^\#$ corresponds to the potential $-\overline{u}(-z)$. To compute the corresponding residues for $\nu^\#(-z, k)$ we therefore make the following substitutions in the formulas above:

$$\bar{\partial}^{-1} \to -\bar{\partial}^{-1}, \quad \partial \to -\partial,$$
$$u \to -\lambda\bar{u}, \quad \bar{u} \to -\lambda u,$$

Thus the overall sign change is $(-1)^{n_u+n_\partial}$ where n_u is the number of factors of u and $\bar{u}$, while n_∂ is the number of factors of ∂ and $\bar{\partial}^{-1}$. There is also an overall factor of $(\lambda)^{n_u}$, that is, λ if n_u is odd, or 1 if n_u is even. Applying these rules we obtain:

$\ell = 0$:

$$\nu^\#_{1,0} = -\tfrac{1}{4}\bar{\partial}^{-1}(|u|^2), \tag{A-18}$$
$$\nu^\#_{2,0} = -\tfrac{1}{2}u. \tag{A-19}$$

$\ell = 1$:

$$\nu^\#_{1,1} = \tfrac{1}{16}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2)) - \tfrac{1}{4}\bar{\partial}^{-1}(\bar{u}\partial u), \tag{A-20}$$
$$\nu^\#_{2,1} = \tfrac{1}{8}u\bar{\partial}^{-1}(|u|^2) - \tfrac{1}{2}\partial u. \tag{A-21}$$

$\ell = 2$:

$$\begin{aligned}\nu^\#_{1,2} = {} & -\tfrac{1}{64}\bar{\partial}^{-1}\big(|u|^2\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big) \\ & + \tfrac{1}{16}\left\{\bar{\partial}^{-1}\big(\bar{u}\partial(u\bar{\partial}^{-1}(|u|^2))\big) + \bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(\bar{u}\partial u))\right\} - \tfrac{1}{4}\bar{\partial}^{-1}(\bar{u}\partial^2 u),\end{aligned} \tag{A-22}$$

$$\nu^\#_{2,2} = -\tfrac{1}{32}u\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2)) + \tfrac{1}{8}\left\{\partial(u\bar{\partial}^{-1}(|u|^2)) + u\bar{\partial}^{-1}(\bar{u}\partial u)\right\} - \tfrac{1}{2}\partial^2 u. \tag{A-23}$$

$\ell = 3$:

$$\begin{aligned}\nu^\#_{2,3} = {} & \tfrac{1}{128}u\bar{\partial}^{-1}\big(|u|^2\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big) \\ & - \tfrac{1}{32}\left\{u\bar{\partial}^{-1}\big(\bar{u}\partial(u\bar{\partial}^{-1}(|u|^2))\big) + u\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(\bar{u}\partial u)) + \partial\big(u\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big)\right\} \\ & + \tfrac{1}{8}\left\{u\bar{\partial}^{-1}(\bar{u}\partial^2 u) + \partial^2(u\bar{\partial}^{-1}(|u|^2)) + \partial(u\bar{\partial}^{-1}(\bar{u}\partial u))\right\} \\ & - \tfrac{1}{2}\partial^3 u.\end{aligned} \tag{A-24}$$

A.4. ***Inverse scattering method for mNV.*** We now compute the motion of the putative solution

$$u = \mathcal{I}r$$

if the reflection coefficient evolves according to the law

$$\dot{r} = -(k^3 - \overline{k}^3)r, \quad r|_{t=0} = \mathcal{R}u_0.$$

To use (A-8), we compute $[\nu_1(z,\diamond)\nu_2^\#(-z,\diamond)]_3$ and $[\nu_2(z,\diamond)\nu_1^\#(-z,\diamond)]_3$.

First, we have

$$[\nu_1(z,\diamond)\nu_2^\#(-z,\diamond)]_3 = \nu_{2,3}^\# + \nu_{2,2}^\#\nu_{1,0} + \nu_{2,1}^\#\nu_{1,1} + \nu_{2,0}^\#\nu_{1,2}. \tag{A-25}$$

From the formulas above we have

$$\begin{aligned}\nu_{2,2}^\#\nu_{1,0} = &-\tfrac{1}{128}u\bar\partial^{-1}(|u|^2\bar\partial^{-1}(|u|^2))\cdot(\bar\partial^{-1}|u|^2)\\ &+\tfrac{1}{32}\big\{\partial(u\bar\partial^{-1}(|u|^2))\cdot(\bar\partial^{-1}(|u|^2)) + u\bar\partial^{-1}(\bar u\partial u)\cdot(\bar\partial^{-1}(|u|^2))\big\}\\ &-\tfrac{1}{8}\partial^2 u\cdot\bar\partial^{-1}(|u|^2),\end{aligned} \tag{A-26}$$

$$\begin{aligned}\nu_{2,1}^\#\nu_{1,1} = &\tfrac{1}{128}u\bar\partial^{-1}(|u|^2)\cdot\bar\partial^{-1}(|u|^2\bar\partial^{-1}(|u|^2))\\ &-\tfrac{1}{32}\big\{u\bar\partial^{-1}(|u|^2)\cdot\bar\partial^{-1}(u\partial\bar u) + \partial u\cdot\big(\bar\partial^{-1}(|u|^2\bar\partial^{-1}(|u|^2))\big)\big\}\\ &+\tfrac{1}{8}\partial u\cdot\bar\partial^{-1}(u\partial\bar u),\end{aligned} \tag{A-27}$$

$$\begin{aligned}\nu_{2,0}^\#\nu_{1,2} = &-\tfrac{1}{128}u\bar\partial^{-1}\big(|u|^2\bar\partial^{-1}(|u|^2\bar\partial^{-1}(|u|^2))\big)\\ &+\tfrac{1}{32}\big\{u\bar\partial^{-1}\big(u\partial(\bar u\bar\partial^{-1}(|u|^2))\big) + u\bar\partial^{-1}(|u|^2\bar\partial^{-1}(u\partial\bar u))\big\}\\ &-\tfrac{1}{8}u\bar\partial^{-1}(u\partial^2\bar u).\end{aligned} \tag{A-28}$$

Using (A-24) and (A-26)–(A-28) in (A-25) we see that seventh-order terms cancel, while fifth-order terms sum to zero, as may be shown using the identity

$$\bar\partial^{-1}f\cdot\bar\partial^{-1}g = \bar\partial^{-1}(f\bar\partial^{-1}g + g\bar\partial^{-1}f), \tag{A-29}$$

while third-order terms may be simplified using the same identity with $f = g$. The result is

$$\big[\nu_{11}(z,\diamond)\tilde\nu_{21}(-z,\diamond)\big]_3 = \tfrac{3}{8}\big[(\partial u)\cdot\big(\bar\partial^{-1}(\partial(|u|^2))\big)\big] + \tfrac{3}{8}\big[u\bar\partial^{-1}(\bar u\bar\partial u)\big] - \tfrac{1}{2}\partial^3 u. \tag{A-30}$$

Next, we compute

$$\big[\nu_2(z,\diamond)\nu_1^\#(-z,\diamond)\big]_3 = \nu_{2,3} + \nu_{2,2}\nu_{1,0}^\# + \nu_{2,1}\nu_{1,1}^\# + \nu_{2,0}\nu_{1,2}^\#. \tag{A-31}$$

From the formulas above we have

$$\begin{aligned}\nu_{2,2}\nu^{\#}_{1,0} &= -\tfrac{1}{128}\lambda\bar{u}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\cdot\bar{\partial}^{-1}(|u|^2)\\ &\quad+\tfrac{1}{32}\bigl\{\partial(\bar{u}\bar{\partial}^{-1}(|u|^2))\cdot\bar{\partial}^{-1}(|u|^2)+\bar{u}\bar{\partial}^{-1}(u\partial\bar{u})\cdot(\bar{\partial}^{-1}(|u|^2))\bigr\}\\ &\quad-\tfrac{1}{8}\lambda\partial^2\bar{u}\cdot\bar{\partial}^{-1}(|u|^2),\end{aligned}\tag{A-32}$$

$$\begin{aligned}\nu_{2,1}\nu^{\#}_{1,1} &= \tfrac{1}{128}\bar{u}\bar{\partial}^{-1}(|u|^2)\cdot\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\\ &\quad-\tfrac{1}{32}\bigl\{\bar{u}\bar{\partial}^{-1}(|u|^2)\cdot\bar{\partial}^{-1}(\bar{u}\partial u)+\partial\bar{u}\cdot\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\bigr\}\\ &\quad+\tfrac{1}{8}\partial\bar{u}\cdot\bar{\partial}^{-1}(\bar{u}\partial u),\end{aligned}\tag{A-33}$$

$$\begin{aligned}\nu_{2,0}\nu^{\#}_{1,2} &= -\tfrac{1}{128}\bar{u}\bar{\partial}^{-1}\bigl(|u|^2\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\bigr)\\ &\quad+\tfrac{1}{32}\bigl\{\bar{u}\bar{\partial}^{-1}(\bar{u}\partial(u\bar{\partial}^{-1}(|u|^2)))+\bar{u}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(\bar{u}\partial u))\bigr\}\\ &\quad-\tfrac{1}{8}\bar{u}\bar{\partial}^{-1}(\bar{u}\partial^2 u).\end{aligned}\tag{A-34}$$

Using (A-17) and (A-32)–(A-34) in (A-31), noting the cancellation of fifth-order terms, we obtain

$$[\nu_2(z,\diamond)\nu_1^{\#}(-z,\diamond)]_3=\tfrac{3}{8}\bigl[\bar{u}\bar{\partial}^{-1}(\partial(u\partial\bar{u}))\bigr]+\tfrac{3}{8}(\partial\bar{u})\cdot\partial\bar{\partial}^{-1}(|u|^2)-\tfrac{1}{2}\partial^3\bar{u},\tag{A-35}$$

or upon complex conjugation

$$[\overline{\nu_2(z,\diamond)\nu_1^{\#}(-z,\diamond)}_3]=\tfrac{3}{8}u\partial^{-1}(\bar{\partial}(\bar{u}\bar{\partial}u))+\tfrac{3}{8}(\bar{\partial}u)\cdot\partial^{-1}(\bar{\partial}(|u|^2))-\tfrac{1}{2}\bar{\partial}^3 u.\tag{A-36}$$

Using these equations in (A-8), we obtain the mNV equation:

$$\frac{\partial u}{\partial t}=-\partial^3 u-\bar{\partial}^3 u+\tfrac{3}{4}(\partial\bar{u})\cdot(\bar{\partial}\partial^{-1}(|u|^2))+\tfrac{3}{4}(\bar{\partial}u)\cdot(\bar{\partial}\partial^{-1}(|u|^2))+\tfrac{3}{4}\bar{u}\bar{\partial}\partial^{-1}(\bar{u}\bar{\partial}u)+\tfrac{3}{4}u\partial^{-1}(\bar{\partial}(\bar{u}\bar{\partial}u)).$$

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