ANALYSIS & PDE

Volume 6 No. 4 2013

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msp

ANALYSIS AND PDE
Vol. 6, No. 4, 2013
dx.doi.org/10.2140/apde.2013.6.859

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DISCRETE FOURIER RESTRICTION ASSOCIATED WITH KDV EQUATIONS

YI HU AND XIAOCHUN LI

In this paper, we consider a discrete restriction associated with KdV equations. Some new Strichartz estimates are obtained. We also establish the local well-posedness for the periodic generalized Korteweg–de Vries equation with nonlinear term $F(u)\partial_x u$ provided $F \in C^5$ and the initial data $\phi \in H^s$ with $s > 1/2$.

1. Introduction

The discrete restriction problem associated with KdV equations is a problem asking the best constant $A_{p,N}$ satisfying

$$\sum_{n=-N}^{N} |\hat{f}(n, n^3)|^2 \le A_{p,N} \|f\|_{p'}^2, \tag{1-1}$$

where f is a periodic function on $\mathbb{T}^2$, $\hat{f}$ is the Fourier transform of f on $\mathbb{T}^2$, $p \ge 2$, and $p' = p/(p-1)$. It is natural to pose a conjecture asserting that for any $\varepsilon > 0$, $A_{p,N}$ satisfies

$$A_{p,N} \le \begin{cases} C_p N^{1-8/p+\varepsilon} & \text{for } p \ge 8, \\ C_p & \text{for } 2 \le p < 8. \end{cases} \tag{1-2}$$

It was proved by Bourgain that $A_{6,N} \le N^{\varepsilon}$. The desired upper bound for $A_{8,N}$ is not yet obtained; however, we are able to establish an affirmative answer for large p.

Theorem 1.1. *Let $A_{p,N}$ be defined as in (1-1). If $p \ge 14$, for any $\varepsilon > 0$, there exists a constant C_p independent of N such that*

$$A_{p,N} \le C_p N^{1-8/p+\varepsilon}. \tag{1-3}$$

The periodic Strichartz inequality associated to KdV equations is the inequality seeking the best constant $K_{p,N}$ satisfying

$$\left\| \sum_{n=-N}^{N} a_n e^{2\pi i t n^3 + 2\pi i x n} \right\|_{L^p_{x,t}(\mathbb{T}\times\mathbb{T})} \le K_{p,N} \left(\sum_{n=-N}^{N} |a_n|^2 \right)^{1/2}. \tag{1-4}$$

By duality, we immediately see that

$$K_{p,N} \sim \sqrt{A_{p,N}}.$$

This work was partially supported by NSF grant DMS-0801154.
MSC2010: primary 35G25; secondary 42A16.
Keywords: discrete Fourier restriction, KdV, local well-posedness.

Hence, Theorem 1.1 is equivalent to Strichartz estimates,

$$K_{p,N} \leq C N^{1/2-4/p+\varepsilon}, \quad \text{for } p \geq 14. \tag{1-5}$$

It was observed by Bourgain that the periodic Strichartz inequalities (1-4) for $p = 4, 6$ are crucial for obtaining the local well-posedness of periodic KdV (mKdV or gKdV). The local (global) well-posedness of periodic KdV for $s \geq 0$ was first studied by Bourgain [1993b]. Via a bilinear estimate approach, Kenig, Ponce, and Vega [Kenig et al. 1996] established the local well-posedness of periodic KdV for $s > -1/2$. The sharp global well-posedness of the periodic KdV was proved by Colliander, Keel, Staffilani, Takaoka, and Tao [Colliander et al. 2003], by utilizing the I-method.

Inspired by Bourgain's work, we can obtain the following theorem on gKdV. Here the gKdV is the generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} u_t + u_{xxx} + u^k u_x = 0, \\ u(x,0) = \phi(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}, \end{cases} \tag{1-6}$$

where $k \in \mathbb{N}$ and $k \geq 3$.

Theorem 1.2. *The Cauchy problem (1-6) is locally well-posed if the initial data $\phi \in H^s$ for $s > 1/2$.*

Theorem 1.2 is not new. It was proved by Colliander, Keel, Staffilani, Takaoka, and Tao [Colliander et al. 2004], but our method is different. The method used by those authors is based on a rescaling argument and the bilinear estimates proved by Kenig, Ponce and Vega [Kenig et al. 1996]. Our method is more straightforward and does not need the rescaling argument, the bilinear estimates, or the multilinear estimates in the earlier papers. This allows us to extend Theorem 1.2 to a very general setting. More precisely, consider the Cauchy problem for periodic generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} u_t + u_{xxx} + F(u) u_x = 0, \\ u(x,0) = \phi(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}. \end{cases} \tag{1-7}$$

Here F is a suitable function. Then the following theorem can be established.

Theorem 1.3. *The Cauchy problem (1-7) is locally well-posed provided F is a C^5 function and the initial data $\phi \in H^s$ for $s > 1/2$.*

For sufficiently smooth F, say $F \in C^{15}$, the existence of a local solution of (1-7) for $s \geq 1$ and the global well-posedness of (1-7) for small data $\phi \in H^s$ with $s > 3/2$ were proved by Bourgain [1995]. The index $1/2$ is sharp because the ill-posedness of (1-6) for $s < 1/2$ is known; see [Colliander et al. 2004]. In order to make Theorem 1.3 well-posed for the initial data $\phi \in H^s$ with $s > 1/2$, the sharp regularity condition for F is perhaps C^4. But the method utilized in this paper, with a small modification, seems only to be able to reach an affirmative result for $F \in C^{(9/2)+}$ and $s > 1/2$. Moreover, the endpoint $s = 1/2$ case could possibly be done by combining the ideas from [Colliander et al. 2004] and this paper. We do not pursue this here.

2. Proof of Theorem 1.1

Proof. To prove Theorem 1.1, we need to introduce a level set. Since $\sqrt{A_{p,N}} \sim K_{p,N}$, it suffices to prove the Strichartz estimates (1-4). Let F_N be a periodic function on $\mathbb{T}^2$ given by

$$F_N(x,t) = \sum_{n=-N}^{N} a_n e^{2\pi i n x} e^{2\pi i n^3 t}, \tag{2-1}$$

where $\{a_n\}$ is a sequence with $\sum_n |a_n|^2 = 1$ and $(x,t) \in \mathbb{T}^2$. For any $\lambda > 0$, set a level set E_λ to be

$$E_\lambda = \{(x,t) \in \mathbb{T}^2 : |F_N(x,t)| > \lambda\}. \tag{2-2}$$

To obtain the desired estimate for the level set, let us first state a lemma on Weyl's sums.

Lemma 2.1. *Suppose that $t \in \mathbb{T}$ satisfies $|t - a/q| \le 1/q^2$, where a and q are relatively prime. Then if $q \ge N^2$,*

$$\left| \sum_{n=1}^{N} e^{2\pi i (t n^3 + b n^2 + c n)} \right| \le C N^{1/4+\varepsilon} q^{1/4}. \tag{2-3}$$

Here b and c are real numbers, and the constant C is independent of b, c, t, a, q, and N.

The proof of Lemma 2.1 relies on Weyl's squaring method. See [Hua 1965] or [Montgomery 1994] for details. We also need the following lemma.

Lemma 2.2 [Bourgain 1993a]. *For any integer $Q \ge 1$ and any integer $n \ne 0$, and any $\varepsilon > 0$,*

$$\sum_{Q \le q < 2Q} \left| \sum_{a \in \mathcal{P}_q} e^{2\pi i (a/q) n} \right| \le C_\varepsilon d(n, Q) Q^{1+\varepsilon}.$$

Here $\mathcal{P}_q$ is given by

$$\mathcal{P}_q = \{a \in \mathbb{N} : 1 \le a \le q \text{ and } (a,q) = 1\}, \tag{2-4}$$

and $d(n, Q)$ denotes the number of divisors of n less than Q and C_ε is a constant independent of Q, n.

Lemma 2.2 can be proved by observing that the arithmetic function defined by $f(q) = \sum_{a \in \mathcal{P}_q} e^{2\pi i (a/q) n}$ is multiplicative, and then utilizing the prime factorization for q to conclude the lemma.

Proposition 2.3. *Let K_N be a kernel defined by*

$$K_N(x,t) = \sum_{n=-N}^{N} e^{2\pi i t n^3 + 2\pi i x n}. \tag{2-5}$$

For any given positive number Q with $N^2 \le Q \le N^3$, the kernel K_N can be decomposed into $K_{1,Q} + K_{2,Q}$ such that

$$\|K_{1,Q}\|_\infty \le C_1 N^{1/4+\varepsilon} Q^{1/4}. \tag{2-6}$$

and

$$\|\widehat{K_{2,Q}}\|_\infty \le \frac{C_2 N^\varepsilon}{Q}. \tag{2-7}$$

Here the constants C_1, C_2 are independent of Q and N.

Proof. We can assume that Q is an integer, since otherwise we can take the integer part of Q. For a standard bump function φ supported on $[1/200, 1/100]$, we set

$$\Phi(t) = \sum_{Q \le q \le 5Q} \sum_{a \in \mathcal{P}_q} \varphi\left(\frac{t - a/q}{1/q^2}\right). \tag{2-8}$$

Clearly Φ is supported on $[0, 1]$. We can extend Φ to other intervals periodically to obtain a periodic function on $\mathbb{T}$. This periodic function, generated by Φ, will also be denoted by Φ. It is easy to see that

$$\widehat{\Phi}(0) = \sum_{q \sim Q} \sum_{a \in \mathcal{P}_q} \frac{\mathcal{F}_{\mathbb{R}}\varphi(0)}{q^2} = \sum_{q \sim Q} \frac{\phi(q)}{q^2} \mathcal{F}_{\mathbb{R}}\varphi(0) \tag{2-9}$$

is a constant independent of Q. Here ϕ is Euler's phi function, and $\mathcal{F}_{\mathbb{R}}$ denotes the Fourier transform of a function on $\mathbb{R}$. Also we have

$$\widehat{\Phi}(k) = \sum_{q \sim Q} \sum_{a \in \mathcal{P}_q} \frac{1}{q^2} e^{-2\pi i (a/q) k} \mathcal{F}_{\mathbb{R}}\varphi(k/q^2). \tag{2-10}$$

Applying Lemma 2.2 and the fact that $Q \le N^3$, we obtain

$$|\widehat{\Phi}(k)| \le \frac{N^{\varepsilon}}{Q}, \tag{2-11}$$

if $k \ne 0$.

We now define

$$K_{1,Q}(x, t) = \frac{1}{\widehat{\Phi}(0)} K_N(x, t)\Phi(t) \quad \text{and} \quad K_{2,Q} = K_N - K_{1,Q}.$$

Equation (2-6) follows from Lemma 2.1 since the intervals $J_{a/q} = \left[\frac{a}{q} + \frac{1}{100q^2}, \frac{a}{q} + \frac{1}{50q^2}\right]$ are pairwise disjoint for all $Q \le q \le 5Q$ and $a \in \mathcal{P}_q$.

We now prove (2-7). In fact, represent Φ as its Fourier series to get

$$K_{2,Q}(x, t) = -\frac{1}{\widehat{\Phi}(0)} \sum_{k \ne 0} \widehat{\Phi}(k) e^{2\pi i k t} K_N(x, t).$$

Thus its Fourier coefficient is

$$\widehat{K_{2,Q}}(n_1, n_2) = -\frac{1}{\widehat{\Phi}(0)} \sum_{k \ne 0} \widehat{\Phi}(k) \mathbf{1}_{\{n_2 = n_1^3 + k\}}(k).$$

Here $(n_1, n_2) \in \mathbb{Z}^2$ and $\mathbf{1}_A$ is the indicator function of a set A. This implies that $\widehat{K_{2,Q}}(n_1, n_2) = 0$ if $n_2 = n_1^3$, and if $n_2 \ne n_1^3$,

$$\widehat{K_{2,Q}}(n_1, n_2) = -\frac{1}{\widehat{\Phi}(0)} \widehat{\Phi}(n_2 - n_1^3).$$

Applying (2-11), we estimate $\widehat{K_{2,Q}}(n_1, n_2)$ by

$$|\widehat{K_{2,Q}}(n_1, n_2)| \leq \frac{CN^{\varepsilon}}{Q},$$

since $N \leq Q \leq N^2$. Hence we obtain (2-7), completing the proof. □

Now we can state our theorem on the level set estimates.

Theorem 2.4. *For any positive numbers ε and $Q \geq N^2$, the level set defined as in (2-2) satisfies*

$$\lambda^2 |E_\lambda|^2 \leq C_1 N^{1/4+\varepsilon} Q^{1/4} |E_\lambda|^2 + \frac{C_2 N^{\varepsilon}}{Q} |E_\lambda| \tag{2-12}$$

for all $\lambda > 0$. Here C_1 and C_2 are constants independent of N and Q.

Proof. Notice that if $Q \geq N^3$, (2-12) becomes trivial, since $E_\lambda = \varnothing$ if $\lambda \geq CN^{1/2}$. So we can assume that $N^2 \leq Q \leq N^3$. For the function F_N and the level set E_λ given in (2-1) and (2-2), respectively, we define f to be

$$f(x,t) = \frac{F_N(x,t)}{|F_N(x,t)|} \mathbf{1}_{E_\lambda}(x,t)\,.$$

Clearly

$$\lambda |E_\lambda| \leq \int_{\mathbb{T}^2} \overline{F_N(x,t)}\, f(x,t)\, dx\, dt.$$

By the definition of F_N, we get

$$\lambda |E_\lambda| \leq \sum_{n=-N}^{N} \overline{a_n} \hat{f}(n, n^3).$$

Utilizing the Cauchy–Schwarz inequality, we have

$$\lambda^2 |E_\lambda|^2 \leq \sum_{n=-N}^{N} |\hat{f}(n, n^3)|^2.$$

The right hand side can be written as

$$\langle K_N * f, f \rangle. \tag{2-13}$$

For any Q with $N^2 \leq Q \leq N^3$, we employ Proposition 2.3 to decompose the kernel K_N. We then have

$$\lambda^2 |E_\lambda|^2 \leq |\langle K_{1,Q} * f, f \rangle| + |\langle K_{2,Q} * f, f \rangle|. \tag{2-14}$$

From (2-6) and (2-7), we then obtain

$$\lambda^2 |E_\lambda|^2 \leq C_1 N^{1/4+\varepsilon} Q^{1/4} \|f\|_1^2 + \frac{C_2 N^{\varepsilon}}{Q} \|f\|_2^2 \leq C_1 N^{1/4+\varepsilon} Q^{1/4} |E_\lambda|^2 + \frac{C_2 N^{\varepsilon}}{Q} |E_\lambda|.$$ □

Corollary 2.5. *If $\lambda \geq 2C_1 N^{3/8+\varepsilon}$,*

$$|E_\lambda| \leq \frac{CN^{1+\varepsilon}}{\lambda^{10}}. \tag{2-15}$$

Here C_1 is the constant C_1 in Theorem 2.4 and C is a constant independent of N and λ.

Proof. Since $\lambda \geq 2C_1 N^{3/8+\varepsilon}$, we simply take Q satisfying $2C_1 N^{1/4+\varepsilon} Q^{1/4} = \lambda^2$. Then Corollary 2.5 follows from Theorem 2.4. □

We are now ready to finish the proof of Theorem 1.1. In fact, let $p \geq 14$ and write $\|F\|_p^p$ as

$$p \int_0^{2C_1 N^{3/8+\varepsilon}} \lambda^{p-1} |E_\lambda| \, d\lambda + p \int_{2C_1 N^{3/8+\varepsilon}}^{2N^{1/2}} \lambda^{p-1} |E_\lambda| \, d\lambda. \tag{2-16}$$

Observe that $A_{6,N} \leq N^\varepsilon$ implies

$$|E_\lambda| \leq \frac{N^\varepsilon}{\lambda^6}. \tag{2-17}$$

Thus the first term in (2-16) is bounded by

$$CN^{3(p-6)/8+\varepsilon} \leq CN^{p/2-4+\varepsilon}, \tag{2-18}$$

since $p \geq 14$. From (2-15), the second term is majorized by

$$CN^{p/2-4+\varepsilon}. \tag{2-19}$$

Putting both estimates together, we complete the proof of Theorem 1.1. □

3. A Lower bound of $A_{p,N}$

In this section we show that $N^{1-8/p}$ is the best upper bound of $A_{p,N}$ if $p \geq 8$. Hence (1-3) can not be improved substantially, and it is sharp up to a factor of N^ε.

For $b \in \mathbb{N}$, let $J(N; b)$ be defined by

$$S(N; b) = \int_{\mathbb{T}^2} \left| \sum_{n=-N}^{N} e^{2\pi i t n^3 + 2\pi i x n} \right|^{2b} dx \, dt. \tag{3-1}$$

Proposition 3.1. *Let $S(N; b)$ be defined as in (3-1). Then*

$$S(N; b) \geq C(N^b + N^{2b-4}). \tag{3-2}$$

Here C is a constant independent of N.

Proof. Clearly $S(N; b)$ is equal to the number of solutions of

$$\begin{cases} n_1 + \cdots + n_b = m_1 + \cdots + m_b, \\ n_1^3 + \cdots + n_b^3 = m_1^3 + \cdots + m_b^3, \end{cases} \tag{3-3}$$

with $n_j, m_j \in \{-N, \ldots, N\}$ for all $j \in \{1, \ldots, b\}$. For each $(m_1, \ldots, m_b)$, we may obtain a solution of (3-3) by taking $(n_1, \ldots, n_b) = (m_1, \ldots, m_b)$. Thus

$$S(N; b) \geq N^b. \tag{3-4}$$

To derive a further lower bound for $S(N; b)$, we set Ω to be

$$\Omega = \left\{ (x, t) : |x| \leq \frac{1}{60N}, \, |t| \leq \frac{1}{60N^3} \right\}. \tag{3-5}$$

If $(x,t)\in\Omega$ and $|n|\le N$,

$$|tn^3+xn|\le\frac{1}{30}. \tag{3-6}$$

Hence, if $(x,t)\in\Omega$,

$$\left|\sum_{n=-N}^{N}e^{2\pi itn^3+2\pi ixn}\right|\ge\left|\mathrm{Re}\sum_{n=-N}^{N}e^{2\pi itn^3+2\pi ixn}\right|\ge\sum_{n=-N}^{N}\cos(2\pi(tn^3+xn))\ge CN. \tag{3-7}$$

Consequently, we have

$$S(N;b)\ge\int_\Omega\left|\sum_{n=-N}^{N}e^{2\pi itn^3+2\pi ixn}\right|^{2b}dx\,dt\ge CN^{2b}|\Omega|\ge CN^{2b-4}. \qquad\square$$

Proposition 3.2. *Let $p\ge 2$ be even. Then $A_{p,N}$ satisfies*

$$A_{p,N}\ge C(1+N^{1-8/p}). \tag{3-8}$$

Here C is a constant independent of N.

Proof. Let $p=2b$ since p is even. Setting $a_n=1$ for all n in the definition of $K_{p,N}$, we get

$$S(N;b)\le K_{p,N}^p(2N)^b. \tag{3-9}$$

By Proposition 3.1, we have

$$K_{p,N}\ge C(1+N^{1/2-4/p}). \tag{3-10}$$

Consequently, we conclude (3-8) since $A_{p,N}\sim K_{p,N}^2$. $\square$

4. An estimate of Hua

The following theorem was proved by Hua [1965] by an arithmetic argument. We provide a different proof.

Theorem 4.1. *Let $S(N;b)$ be defined as in (3-1). Then*

$$S(N;5)\le CN^{6+\varepsilon}. \tag{4-1}$$

By Proposition 3.1, we see that the estimate (4-1) is (almost) sharp. $S(N;4)\le N^{4+\varepsilon}$ is still open.

Proof of Theorem 4.1. Let G_λ be the level set given by

$$G_\lambda=\{(x,t)\in\mathbb{T}^2:|K_N(x,t)|\ge\lambda\}. \tag{4-2}$$

Here K_N is the function defined as in (2-5).

Letting $f=\mathbf{1}_{G_\lambda}K_N/|K_N|$, we have

$$\lambda|G_\lambda|\le\sum_{n=-N}^{N}\hat f(n,n^3)=\langle f_N,K_N\rangle, \tag{4-3}$$

where f_N is a rectangular Fourier partial sum defined by

$$f_N(x,t) = \sum_{\substack{|n_1|\le N\\ |n_2|\le N^3}} \hat{f}(n_1,n_2) e^{2\pi n_1 x} e^{2\pi i n_2 t}. \tag{4-4}$$

Employing Proposition 2.3 for K_N, we estimate the level set G_λ by

$$\lambda |G_\lambda| \le |\langle f_N, K_{1,Q}\rangle| + |\langle f_N, K_{2,Q}\rangle| \tag{4-5}$$

for any $Q \ge N^2$. From (2-6) and (2-7), $\lambda|G_\lambda|$ can be bounded further by

$$C\Biggl(N^{1/4+\varepsilon} Q^{1/4} \|f_N\|_1 + \sum_{\substack{|n_1|\le N\\ |n_2|\le N^3}} |\widehat{K_{2,Q}}(n_1,n_2)\hat{f}(n_1,n_2)| \Biggr). \tag{4-6}$$

Thus, from the fact that the L^1 norm of Dirichlet kernel D_N is comparable to $\log N$, (2-7), and the Cauchy–Schwarz inequality, we have

$$\lambda |G_\lambda| \le C N^{1/4+\varepsilon} Q^{1/4} |G_\lambda| + \frac{C N^{2+\varepsilon}}{Q} |G_\lambda|^{1/2}, \tag{4-7}$$

for all $Q \ge N^2$. For $\lambda \ge 2CN^{3/4+\varepsilon}$, take Q to be a number satisfying

$$2CN^{1/4+\varepsilon} Q^{1/4} = \lambda,$$

and obtain

$$|G_\lambda| \le \frac{CN^{6+\varepsilon}}{\lambda^{10}}. \tag{4-8}$$

Notice that

$$\|K_N\|_6 \le N^{1/2} K_{6,p} \le N^{1/2+\varepsilon}. \tag{4-9}$$

Hence, by (4-3), we majorize $|G_\lambda|$ by

$$|G_\lambda| \le \frac{CN^{3+\varepsilon}}{\lambda^6}. \tag{4-10}$$

We now estimate $S(N;5)$ by

$$S(N;5) \le C \int_{2CN^{3/4+\varepsilon}}^{2N} \lambda^{10-1} |G_\lambda|\, d\lambda + C \int_0^{2CN^{3/4+\varepsilon}} \lambda^{10-1} |G_\lambda|\, d\lambda. \tag{4-11}$$

From (4-8), the first term in the right hand side of (4-11) can be bounded by $CN^{6+\varepsilon}$. From (4-10), the second term is clearly bounded by $N^{6+\varepsilon}$. Putting both estimates together,

$$S(N;5) \le CN^{6+\varepsilon}, \tag{4-12}$$

as desired. □

5. Estimates for the nonlinear term and Local well-posedness of (1-6)

For any integrable function u on $\mathbb{T} \times \mathbb{R}$, we define the space-time Fourier transform by

$$\hat{u}(n, \lambda) = \int_{\mathbb{R}} \int_{\mathbb{T}} u(x, t) e^{-inx} e^{-i\lambda t} \, dx \, dt \tag{5-1}$$

and set

$$\langle x \rangle := 1 + |x|.$$

We now introduce the $X_{s,b}$ space, initially used by Bourgain.

Definition 5.1. Let I be a time interval in $\mathbb{R}$ and $s, b \in \mathbb{R}$. Let $X_{s,b}(I)$ be the space of functions u on $\mathbb{T} \times I$ that may be represented as

$$u(x, t) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \hat{u}(n, \lambda) e^{inx} e^{i\lambda t} d\lambda \quad \text{for } (x, t) \in \mathbb{T} \times I \tag{5-2}$$

with the space-time Fourier transform $\hat{u}$ satisfying

$$\|u\|_{X_{s,b}(I)} = \left(\sum_n \int \langle n \rangle^{2s} \langle \lambda - n^3 \rangle^{2b} |\hat{u}(n, \lambda)|^2 d\lambda \right)^{1/2} < \infty\,. \tag{5-3}$$

Here the norm should be understood as a restriction norm.

We take the time interval to be $[0, \delta]$ for a small positive number δ and abbreviate $\|u\|_{X_{s,b}(I)}$ as $\|u\|_{s,b}$ for any function u restricted to $\mathbb{T} \times [0, \delta]$. In this section, we always restrict the function u to $\mathbb{T} \times [0, \delta]$. Let w be the nonlinear function defined by

$$w = \left(u^k - \int u^k \, dx \right) u_x. \tag{5-4}$$

We also define

$$\|u\|_{Y_s} := \|u\|_{s,1/2} + \left(\sum_n \langle n \rangle^{2s} \left(\int |\hat{u}(n, \lambda)| \, d\lambda \right)^2 \right)^{1/2}. \tag{5-5}$$

We need the following estimate on the nonlinear function w, in order to establish a contraction on the space $\{u : \|u\|_{Y_s} \leq M\}$ for some $M > 0$.

Proposition 5.2. *For $s > 1/2$, there exists $\theta > 0$ such that, for the nonlinear function w given by (5-4),*

$$\|w\|_{s,-1/2} + \left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\widehat{w}(n, \lambda)|}{\langle \lambda - n^3 \rangle} \, d\lambda \right)^2 \right)^{1/2} \leq C \delta^{\theta} \|u\|_{Y_s}^{k+1}. \tag{5-6}$$

Here C is a constant independent of δ and u.

The proof of Proposition 5.2 will appear in Section 6, and is based on the idea applied by Bourgain [1993b] while proving the special case $k = 2$. In the proof, we write out the detailed treatment to some subcases, and omit the similar treatment of other subcases (but it is very easy to figure out). The main reason we include the proof of Proposition 5.2 in Section 6 is to provide the preparation so the readers can follow the (more technical) proof of the general case $F \in C^5$ more easily.

We now start to derive the local well-posedness of (1-6). For this purpose, we only need to consider the well-posedness of the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \left(u^k - \int_{\mathbb{T}} u^k dx\right) u_x = 0, \\ u(x,0) = \phi(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}. \end{cases} \tag{5-7}$$

This is because if v is a solution of (5-7), the gauge transform

$$u(x,t) := v\left(x - \int_0^t \int_{\mathbb{T}} v^k(y,\tau)\, dy d\tau, t\right) \tag{5-8}$$

is a solution of (1-6) with the same initial value ϕ. Notice that this transform is invertible and preserves the initial data ϕ. The inverse transform is

$$v(x,t) := u\left(x + \int_0^t \int_{\mathbb{T}} u^k(y,\tau)\, dy d\tau, t\right). \tag{5-9}$$

It is easy to see that for any solution u of (1-6), this inverse transform of u defines a solution of (5-7). Hence, to establish the well-posedness of (1-6), it suffices to obtain the well-posedness of (5-7). This gauge transform was used in [Colliander et al. 2004].

By Duhamel's principle, the corresponding integral equation associated to (5-7) is

$$u(x,t) = e^{-t\partial_x^3}\phi(x) - \int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)\, d\tau, \tag{5-10}$$

where w is defined as in (5-4).

Since we are only seeking the local well-posedness, we may use a bump function to truncate the time variable. Let ψ be a bump function supported in $[-2,2]$ with $\psi(t) = 1$, $|t| \le 1$, and let ψ_δ be

$$\psi_\delta(t) = \psi(t/\delta).$$

Then it suffices to find a local solution of

$$u(x,t) = \psi_\delta(t) e^{-t\partial_x^3}\phi(x) - \psi_\delta(t) \int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)\, d\tau.$$

Let T be an operator given by

$$Tu(x,t) := \psi_\delta(t) e^{-t\partial_x^3}\phi(x) - \psi_\delta(t) \int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)\, d\tau. \tag{5-11}$$

We denote the first term (the linear term) in (5-11) by $\mathcal{L}u$ and the second term (the nonlinear term) by $\mathcal{N}u$. Henceforth we represent Tu as $\mathcal{L}u + \mathcal{N}u$. The following two lemmas deal with $\mathcal{L}u$ and $\mathcal{N}u$ separately.

Lemma 5.3. *The linear term $\mathcal{L}$ satisfies*

$$\|\mathcal{L}u\|_{Y_s} \le C\|\phi\|_{H^s}. \tag{5-12}$$

Here C is a constant independent of δ.

Lemma 5.4. *The nonlinear term $\mathcal{N}$ satisfies*

$$\|\mathcal{N}u\|_{Y_s} \le C\biggl(\|w\|_{s,-1/2} + \biggl(\sum_n \langle n\rangle^{2s}\biggl(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle}\,d\lambda\biggr)^2\biggr)^{1/2}\biggr), \tag{5-13}$$

where C is a constant independent of δ.

Lemmas 5.3 and 5.4 are considered classical and their proofs can be found in many references, such as [Colliander et al. 2004].

Proposition 5.5. *Let $s > 1/2$ and T be the operator defined as in (5-11). Then there exists a positive number θ such that*

$$\|Tu\|_{Y_s} \le C(\|\phi\|_{H^s} + \delta^{\theta}\|u\|_{Y_s}^{k+1}). \tag{5-14}$$

Here C is a constant independent of δ.

Proof. Since $Tu = \mathcal{L}u + \mathcal{N}u$, Proposition 5.5 follows from Lemmas 5.3, 5.4, and Proposition 5.2. □

Proposition 5.5 yields that for δ sufficiently small, T maps a ball in Y_s into itself. Moreover, we write

$$\biggl(u^k - \int_{\mathbb{T}} u^k dx\biggr)u_x - \biggl(v^k - \int_{\mathbb{T}} v^k dx\biggr)v_x = \biggl(u^k - \int_{\mathbb{T}} u^k dx\biggr)(u-v)_x + \biggl((u^k - v^k) - \int_{\mathbb{T}} (u^k - v^k)\,dx\biggr)v_x$$

which equals

$$\biggl(u^k - \int_{\mathbb{T}} u^k dx\biggr)(u-v)_x + \sum_{j=0}^{k-1}\biggl((u-v)u^{k-1-j}v^j - \int_{\mathbb{T}} (u-v)u^{k-1-j}v^j\,dx\biggr)v_x. \tag{5-15}$$

For $k+1$ terms in (5-15), repeating similar argument as in the proof of Proposition 5.2, one obtains, for $s > 1/2$,

$$\|Tu - Tv\|_{Y_s} \le C\delta^{\theta}\biggl(\|u\|_{Y_s}^{k} + \sum_{j=1}^{k-1}\|u\|_{Y_s}^{k-1-j}\|v\|_{Y_s}^{j+1}\biggr)\|u-v\|_{Y_s}. \tag{5-16}$$

Hence, for $\delta > 0$ small enough, T is a contraction and the local well-posedness follows from Picard's fixed-point theorem.

6. Proof of Proposition 5.2

Proof. From the definition of w in (5-4), we may write $\hat{w}(n,\lambda)$ as

$$\sum_{\substack{m+n_1+\cdots+n_k=n\\ n_1+\cdots+n_k\neq 0}} m\int \hat{u}(m,\lambda-\lambda_1-\cdots-\lambda_k)\hat{u}(n_1,\lambda_1)\cdots\hat{u}(n_k,\lambda_k)\,d\lambda_1\cdots d\lambda_k. \tag{6-1}$$

By duality, there exists a sequence $\{A_{n,\lambda}\}$ satisfying

$$\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} |A_{n,\lambda}|^2 d\lambda \le 1, \tag{6-2}$$

and $\|w\|_{s,-1/2}$ is bounded by

$$\sum_{\substack{m+n_1+\cdots+n_k=n\\ n_1+\cdots+n_k\neq 0}} \int \frac{\langle n\rangle^s|m|}{\langle\lambda-n^3\rangle^{1/2}}|\hat{u}(m,\lambda-\lambda_1-\cdots-\lambda_k)||\hat{u}(n_1,\lambda_1)|\cdots|\hat{u}(n_k,\lambda_k)||A_{n,\lambda}|\,d\lambda_1\cdots d\lambda_k\,d\lambda. \tag{6-3}$$

Since the $X_{s,b}$ is a restriction norm, we may assume that u is supported in $\mathbb{T}\times[0,\delta]$. However, the inverse space-time Fourier transform $|\hat{u}|^\vee$ in general may not be a function with compact support. The following standard trick allows us to assume $|\hat{u}|^\vee$ has a compact support too. In fact, let η be a bump function supported on $[-2\delta, 2\delta]$ and with $\eta(t)=1$ in $|t|\le\delta$. Also $\hat{\eta}$ is positive. Then $u=u\eta$ and $\hat{u}=\hat{u}*\hat{\eta}$. Thus $|\hat{u}|\le|\hat{u}|*\hat{\eta}=(|\hat{u}|^\vee\eta)^\wedge$. Whenever we need to make $|\hat{u}|^\vee$ supported in a small time interval, we replace $|\hat{u}|$ by $(|\hat{u}|^\vee\eta)^\wedge$ since $|\hat{u}|^\vee\eta$ clearly is supported on $\mathbb{T}\times[-2\delta,2\delta]$. This will help us gain a positive power of δ in our estimates. Moreover, without loss of generality we can assume $|n_1|\ge|n_2|\ge\cdots\ge|n_k|$.

The trouble occurs mainly because of the factor $|m|$ resulting from $\partial_x u$. The idea (inspired by Bourgain [1993b]) is that either the factor $\langle\lambda-n^3\rangle^{-1/2}$ can be used to cancel $|m|$, or $|m|$ can be distributed to some of the $\hat{u}$. More precisely, we consider three cases:

$$|m| < 1000k^2|n_2|, \tag{6-4}$$

$$1000k^2|n_2| \le |m| \le 100k|n_1|, \tag{6-5}$$

$$|m| > 100k|n_1|. \tag{6-6}$$

<u>Case 1</u>: $|m|<1000k^2|n_2|$. This is the simplest case. In fact, in this case, it is easy to see that

$$\langle n\rangle^s|m| \le C\langle n_1\rangle^s\langle n_2\rangle^1/2\langle m\rangle^1/2. \tag{6-7}$$

Let

$$F_1(x,t)=\sum_n\int\frac{|A_{n,\lambda}|}{\langle\lambda-n^3\rangle^{1/2}}e^{i\lambda t}e^{inx}\,d\lambda; \tag{6-8}$$

$$G(x,t)=\sum_n\int\langle n\rangle^1/2|\hat{u}(n,\lambda)|e^{i\lambda t}e^{inx}\,d\lambda; \tag{6-9}$$

$$H(x,t)=\sum_n\int\langle n\rangle^s|\hat{u}(n,\lambda)|e^{i\lambda t}e^{inx}\,d\lambda; \tag{6-10}$$

$$U(x,t)=\sum_n\int|\hat{u}(n,\lambda)|e^{i\lambda t}e^{inx}\,d\lambda. \tag{6-11}$$

Using (6-7), we can estimate (6-3) by

$$C\sum_{m+n_1+\cdots+n_k=n}\int\widehat{F}(n,\lambda)\widehat{G}(m,\lambda-\lambda_1-\cdots-\lambda_k)\widehat{H}(n_1,\lambda_1)\widehat{G}(n_2,\lambda_2)\prod_{j=3}^k\widehat{U}(n_j,\lambda_j)\,d\lambda_1\cdots d\lambda_k\,d\lambda,$$

which clearly equals

$$C\left|\int_{\mathbb{T}\times\mathbb{R}}F_1(x,t)G(x,t)^2H(x,t)U(x,t)^{k-2}\,dx\,dt\right|. \tag{6-12}$$

Apply Hölder's inequality to majorize it by

$$C\|F_1\|_4\,\|G\|_{6+}^2\,\|H\|_4\,\|U\|_{6(k-2)-}^{k-2}.$$

Since U is supported on $\mathbb{T}\times[-2\delta,2\delta]$, one more use of Hölder inequality yields

$$(6\text{-}3)\le C\delta^{\theta}\|F_1\|_4\,\|G\|_{6+}^2\,\|H\|_4\,\|U\|_{6(k-2)}^{k-2}. \tag{6-13}$$

Let us recall some useful local embedding facts on $X_{s,b}$.

$$X_{0,1/3}\subseteq L^4_{x,t},\qquad X_{0+,1/2+}\subseteq L^6_{x,t}\qquad (t\text{ local}), \tag{6-14}$$

$$X_{\alpha,1/2}\subseteq L^q_{x,t},\qquad 0<\alpha<\tfrac{1}{2},\ 2\le q<\frac{6}{1-2\alpha}\qquad (t\text{ local}), \tag{6-15}$$

$$X_{1/2-\alpha,1/2-\alpha}\subseteq L^q_tL^r_x,\qquad 0<\alpha<\tfrac{1}{2},\ 2\le q,\ r<1/\alpha. \tag{6-16}$$

The two embedding results in (6-14) are consequences of the discrete restriction estimates on L^4 and L^6, respectively (see [Bourgain 1993b] for details). (6-15) and (6-16) follow by interpolation (see [Colliander et al. 2004] for details). (6-14) yields

$$\|F_1\|_4\le C\|F_1\|_{0,\frac{1}{3}}\le C\Bigl(\sum_n\int|A_{n,\lambda}|^2d\lambda\Bigr)^{1/2}\le C,$$

and

$$\|H\|_4\le C\|H\|_{0,1/3}\le C\|u\|_{s,1/2}\le C\|u\|_{Y_s}.$$

From (6-15) we have

$$\|G\|_{6+}\le C\|G\|_{0+,1/2}\le C\|u\|_{s,1/2}\le C\|u\|_{Y_s}.$$

Using (6-16), we get

$$\|U\|_{6(k-2)}\le C\|U\|_{1/2-,1/2-}\le C\|u\|_{s,1/2}\le C\|u\|_{Y_s}.$$

Hence, for Case 1, we have

$$(6\text{-}3)\le C\delta^{\theta}\|u\|_{Y_s}^{k+1}. \tag{6-17}$$

Case 2: $1000k^2|n_2|\le|m|\le 100k|n_1|$. In this case, we further consider two subcases:

$$|m+n_1|\le 1000k^2|n_2|, \tag{6-18}$$

$$|m+n_1|>1000k^2|n_2|. \tag{6-19}$$

If $|m+n_1|\le 1000k^2|n_2|$, we use the triangle inequality to get

$$|n|=|m+n_1+n_2+\cdots+n_k|\le C|n_2|. \tag{6-20}$$

Hence we have

$$\langle n\rangle^s|m|\le C\langle n_2\rangle^s\langle m\rangle^{1/2}\langle n_1\rangle^{1/2}. \tag{6-21}$$

Thus this subcase can be treated exactly the same as Case 1. We omit the details.

In the second subcase, $|m+n_1| > 1000k^2|n_2|$, the crucial arithmetic observation is

$$n^3 - (m^3 + n_1^3 + \cdots + n_k^3) = 3(m+n_1)(m+a)(n_1+a) + a^3 - (n_2^3 + \cdots + n_k^3), \tag{6-22}$$

where $a = n_2 + \cdots + n_k$. This observation can be easily verified since $n = m + n_1 + \cdots + n_k$. From (6-5) and (6-19), we get

$$|n^3 - (m^3 + n_1^3 + \cdots + n_k^3)| \geq Ck^2 \langle n_2 \rangle |m|\,|n_1| \geq Ck|m|^2. \tag{6-23}$$

This implies that at least one of following statements holds:

$$|\lambda - n^3| \geq C|m|^2, \tag{6-24}$$

$$|(\lambda - \lambda_1 - \cdots - \lambda_k) - m^3| \geq C|m|^2, \tag{6-25}$$

$$\text{there exists an } i \in \{1, \ldots, k\} \text{ such that } |\lambda_i - n_i^3| \geq C|m|^2. \tag{6-26}$$

For (6-24), (6-3) can be bounded by

$$\sum_{m+n_1+\cdots+n_k=n} \int \langle n_1 \rangle^s |\hat{u}(m, \lambda - \lambda_1 - \cdots - \lambda_k)|\,|\hat{u}(n_1, \lambda_1)| \cdots |\hat{u}(n_k, \lambda_k)|\,|A_{n,\lambda}|\,d\lambda_1 \cdots d\lambda_k\,d\lambda. \tag{6-27}$$

Let F_2 be defined by

$$F_2(x,t) = \sum_n \int |A_{n,\lambda}| e^{i\lambda t} e^{inx}\,d\lambda. \tag{6-28}$$

Then we represent (6-27) as

$$\sum_{m+n_1+\cdots+n_k=n} \int \widehat{F_2}(n,\lambda) \widehat{U}(m, \lambda - \lambda_1 - \cdots - \lambda_k) \widehat{H}(n_1, \lambda_1) \prod_{j=2}^{k} \widehat{U}(n_j, \lambda_j)\,d\lambda_1 \cdots d\lambda_k\,d\lambda. \tag{6-29}$$

Here H and U are the functions defined in (6-10) and (6-11). Clearly (6-29) equals

$$\int_{\mathbb{T}\times\mathbb{R}} F_2(x,t) H(x,t) U(x,t)^k\,dx\,dt. \tag{6-30}$$

Utilizing Hölder's inequality, we estimate it further by

$$\|F_2\|_2\,\|H\|_4\,\|U\|_{4k}^k \leq C\delta^\theta \|u\|_{Y_s}^{k+1}. \tag{6-31}$$

This yields the desired estimate for subcase (6-24).

One can similarly complete the proofs of subcases (6-25) and (6-26), and hence the proof of Case 2.

Case 3: $|m| > 100k|n_1|$. The arithmetic observation (6-22) again plays an important role. In this case, let us further consider two subcases:

$$|m|^2 \leq 1000k^2|n_2|^2|n_3|, \tag{6-32}$$

$$|m|^2 > 1000k^2|n_2|^2|n_3|. \tag{6-33}$$

For the first subcase, we observe that, from (6-32),

$$|m|^2 \le C|n_1||n_2||n_3|,$$

since $|n_2| \le |n_1|$. Hence we have

$$|m| = |m|^{1/2}|m|^{2/3} \le C|m|^{1/3}|n_1|^{1/3}|n_2|^{1/3}|n_3|^{1/3}. \tag{6-34}$$

This immediately implies

$$\langle n\rangle^s|m| \le C|m|^{s+1} \le \langle m\rangle^{(s+1)/3}\langle n_1\rangle^{(s+1)/3}\langle n_2\rangle^{(s+1)/3}\langle n_3\rangle^{(s+1)/3}. \tag{6-35}$$

Note that $(s+1)/3 < s$ for $s > 1/2$. By distributing the four factors to the corresponding functions, one can mimic the proof of Case 1 to finish subcase (6-32).

We now turn to the contribution of (6-33). Clearly we have

$$|(n_2+\cdots+n_k)^3 - (n_2^3+\cdots+n_k^3)| \le 10k|n_2|^2|n_3|, \tag{6-36}$$

since $|n_2| \ge |n_3| \ge \cdots \ge |n_k|$. From the crucial arithmetic observation (6-22), (6-36), and (6-33), we have

$$|n^3 - (m^3+n_1^3+\cdots+n_k^3)| \ge Ck|m|^2. \tag{6-37}$$

This is the same as (6-23). Hence we again reduce the problems to (6-24), (6-25), and (6-26), which were all done in Case 2. Therefore Case 3 is finished.

Putting all the cases together, we obtain

$$\|w\|_{s,-1/2} \le C\delta^\theta\|u\|_{Y_s}^{k+1}. \tag{6-38}$$

Finally we need to estimate

$$\left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle\lambda-n^3\rangle}\,d\lambda\right)^2\right)^{1/2}. \tag{6-39}$$

Let $\{A_n\}$ be a sequence with

$$\left(\sum_n |A_n|^2\right)^{1/2} \le 1.$$

By duality, it suffices to estimate

$$\sum_{\substack{m+n_1+\cdots+n_k=n\\ n_1+\cdots+n_k\neq 0}} \int \frac{\langle n\rangle^s|m|}{\langle\lambda-n^3\rangle}|\hat{u}(m,\lambda-\lambda_1-\cdots-\lambda_k)||\hat{u}(n_1,\lambda_1)|\cdots|\hat{u}(n_k,\lambda_k)||A_n|\,d\lambda_1\cdots d\lambda_k\,d\lambda. \tag{6-40}$$

By the same idea and similar techniques, one can bound (6-40) by mimicking the treatment of (6-3) and get

$$\left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle\lambda-n^3\rangle}\,d\lambda\right)^2\right)^{1/2} \le C\delta^\theta\|u\|_{Y_s}^{k+1}. \tag{6-41}$$

We complete the proof of Proposition 5.2 by combining (6-38) and (6-41). $\square$

7. Proof of Theorem 1.3

The argument is similar to that in Section 5. By using a gauge transform as in (5-8) with v^k replaced by $F(v)$, the well-posedness of (1-7) is equivalent to the well-posedness of the following equation:

$$\begin{cases} u_t + u_{xxx} + (F(u) - \int_{\mathbb{T}} F(u)\,dx)u_x = 0, \\ u(x,0) = \phi(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}. \end{cases} \tag{7-1}$$

Now the nonlinear function w is defined by

$$w = \partial_x u\Big(F(u) - \int_{\mathbb{T}} F(u)\,dx\Big). \tag{7-2}$$

Let T_F be an operator given by

$$T_F u(x,t) := \psi_\delta(t)e^{-t\partial_x^3}\phi(x) - \psi_\delta(t)\int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)\,d\tau. \tag{7-3}$$

As in Section 5, the local well-posedness is a consequence of the following proposition.

Proposition 7.1. *Let $s > 1/2$. There exists $\theta > 0$ such that, for the nonlinear function w given by (7-2) and any u satisfying $\|u\|_{Y_s} \le C_0\|\phi\|_{H^s}$,*

$$\|w\|_{s,-1/2} + \bigg(\sum_n \langle n\rangle^{2s}\Big(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle}\,d\lambda\Big)^2\bigg)^{1/2} \le C(\|\phi\|_{H^s}, F)\delta^\theta \|u\|_{Y_s}^4, \tag{7-4}$$

provided $F \in C^5$. Here C_0 is a suitably large constant, and $C(\|\phi\|_{H^s}, F)$ is a constant independent of δ and u, but which may depend on $\|\phi\|_{H^s}$ and F.

The constant $C(\|\phi\|_{H^s}, F)$ will be specified in the proof of Proposition 7.1, which we postpone to Section 8. We now return to the proof of Theorem 1.3. Proposition 7.1 implies that for δ sufficiently small, T maps a ball

$$\{u \in Y_s : \|u\|_{Y_s} \le C_0\|\phi\|_{H^s}\}$$

into itself. Moreover, using Lemma 5.4 and repeating similar argument as in the proof of Proposition 7.1, one obtains, for $s > 1/2$ and $F \in C^5$,

$$\|T_F u - T_F v\|_{Y_s} \le \delta^\theta C(\|\phi\|_{H^s}, F)\|u - v\|_{Y_s} \tag{7-5}$$

for all u, v in the ball $\{u \in Y_s : \|u\|_{Y_s} \le C_0\|\phi\|_{H^s}\}$. Therefore, for $\delta > 0$ small enough, T_F is a contraction on the ball and the local well-posedness again follows from Picard's fixed-point theorem. This completes the proof of Theorem 1.3.

8. Proof of Proposition 7.1

First we introduce a decomposition of $F(u)$ which was used by Bourgain. Let K be a dyadic number, and define a Fourier multiplier operator P_K by setting

$$P_K u(x,t) = \int \psi_K(y)u(x-y,t)\,dy. \tag{8-1}$$

Here the Fourier transform of ψ_K is a standard bump function supported on $[-2K, 2K]$ and $\widehat{\psi_K}(x) = 1$ for $x \in [-K, K]$. Let u_K denote the Littlewood–Paley Fourier multiplier, that is,

$$u_K = P_K u - P_{K/2} u. \tag{8-2}$$

Then we may decompose $F(u)$ by

$$F(u) = \sum_K (F(P_K u) - F(P_{K/2}u)) = \sum_K F_1(P_K u, P_{K/2}u) u_K + R_1,$$

where R_1 is a function independent of the space variable x. Repeating this procedure for F_1, we obtain

$$\begin{aligned} F(u) &= \sum_{K_1 \ge K_2} F_2(P_{2K_2}u, \dots, P_{K_2/4}u) u_{K_1} u_{K_2} + \sum_{K_1} R_2 u_{K_1} + R_1 \\ &= \sum_{K_1 \ge K_2 \ge K_3} F_3(P_{4K_3}u, \dots, P_{K_3/8}u) u_{K_1} u_{K_2} u_{K_3} + \sum_{K_1 \ge K_2} R_3 u_{K_1} u_{K_2} + \sum_{K_1} R_2 u_{K_1} + R_1 \end{aligned}$$

where R_1, R_2, R_3 are functions independent of the space variable. Set

$$G_{K_3}(x, t) = F_3(P_{4K_3}u, \dots, P_{K_3/8}u). \tag{8-3}$$

Hence we represent w, defined in (7-2), as

$$\begin{aligned} w = \sum_{K_0, K_1 \ge K_2 \ge K_3} &\partial_x u_{K_0} \left(u_{K_1} u_{K_2} u_{K_3} G_{K_3} - \int_{\mathbb{T}} u_{K_1} u_{K_2} u_{K_3} G_{K_3} \, dx \right) \\ &+ \sum_{K_0, K_1 \ge K_2} \partial_x u_{K_0} \left(u_{K_1} u_{K_2} - \int_{\mathbb{T}} u_{K_1} u_{K_2} \, dx \right) R_3 + \sum_{K_0, K_1} \partial_x u_{K_0} \left(u_{K_1} - \int_{\mathbb{T}} u_{K_1} \, dx \right) R_2. \end{aligned}$$

The main contribution of w is from the first term. The remaining terms can be handled by the method presented in Section 6, because R_2, R_3 are functions independent of the space variable x (actually they only depend on the conserved quantity $\int_{\mathbb{T}} u \, dx$). Hence in what follows we only focus on estimating the first term — the most difficult one. Denote the first term by w_1:

$$w_1 = \sum_{K_0, K_1 \ge K_2 \ge K_3} \partial_x u_{K_0} \left(u_{K_1} u_{K_2} u_{K_3} G_{K_3} - \int_{\mathbb{T}} u_{K_1} u_{K_2} u_{K_3} G_{K_3} \, dx \right). \tag{8-4}$$

We should prove

$$\|w_1\|_{s,-1/2} + \left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\hat{w}_1(n, \lambda)|}{\langle \lambda - n^3 \rangle} \, d\lambda \right)^2 \right)^{1/2} \le C \delta^\theta C(\|\phi\|_{H^s}, F) \|u\|_{Y_s}^4. \tag{8-5}$$

In order to specify the constant $C(\|\phi\|_{H^s}, F)$, we define $\mathfrak{M}$ by setting

$$\mathfrak{M} = \sup \big\{ |D^\alpha F_3(u_1, \dots, u_6)| : u_j \text{ satisfies } \|u_j\|_{Y_s} \le C_0 \|\phi\|_{H^s} \text{ for all } j = 1, \dots, 6; \alpha \big\}. \tag{8-6}$$

Here $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_6}^{\alpha_6}$ and α is taken over all tuples $(\alpha_1, \dots, \alpha_6) \in (\mathbb{N} \cup \{0\})^6$ with $\sum |\alpha_j| \le 2$. $\mathfrak{M}$ is a real number. This is because, for $s > 1/2$, $\|u\|_{Y_s} \le C_0 \|\phi\|_{H^s}$ yields that u is bounded by $C \|\phi\|_{H^s}$, and the previous claim follows from $F_3 \in C^2$.

In order to bound $\|w_1\|_{s,-1/2}$, by duality, it suffices to bound

$$\sum_{\substack{K_0,K_1\geq K_2\geq K_3\\ n_0+n_1+n_2+n_3+m=n\\ n_1+n_2+n_3+m\neq 0}} \int \frac{A_{n,\lambda}\langle n\rangle^s n_0}{\langle\lambda-n^3\rangle^{1/2}}\hat{u}_{K_0}(n_0,\lambda-\lambda_1-\lambda_2-\lambda_3-\mu) \cdot\prod_{j=1}^{3}\hat{u}_{K_j}(n_j,\lambda_j)\widehat{G_{K_3}}(m,\mu)\,d\lambda_1\cdots d\lambda_4 d\lambda d\mu, \tag{8-7}$$

where $A_{n,\lambda}$ satisfies

$$\sum_n\int|A_{n,\lambda}|^2 d\lambda=1.$$

The trouble maker is G_{K_3} since there is no way to find a suitable upper bound for its $X_{s,b}$ norm. Because of this, the method in Section 6 is no longer valid, and we have to treat m and μ differently from n and λ, respectively. A delicate analysis must be done to overcome the difficulty caused by G_{K_3}. For simplicity, we assume that $\delta=1$. One can modify the argument to gain a decay of δ^θ by using the technical treatment from Section 6.

For a dyadic number M, define the Littlewood–Paley Fourier multiplier by

$$g_{K_3,M}=P_M G_{K_3}-P_{M/2}G_{K_3}=(G_{K_3})_M. \tag{8-8}$$

Let v be defined by

$$v(x,t)=\sum_n\int\frac{A_{n,\lambda}}{\langle\lambda-n^3\rangle^{1/2}}e^{i\lambda t}e^{inx}d\lambda. \tag{8-9}$$

To estimate (8-7), it suffices to estimate

$$\sum_{\substack{K,K_0,K_1\geq K_2\geq K_3,M\\ n_0+n_1+n_2+n_3+m=n\\ n_1+n_2+n_3+m\neq 0}} \int \widehat{\langle\partial_x\rangle^s v_K}(n,\lambda)\widehat{\partial_x u_{K_0}}(n_0,\lambda-\lambda_1-\lambda_2-\lambda_3-\mu) \cdot\prod_{j=1}^{3}\hat{u}_{K_j}(n_j,\lambda_j)\widehat{g_{K_3,M}}(m,\mu)\,d\lambda_1\cdots d\lambda_4 d\lambda d\mu. \tag{8-10}$$

Here K is a dyadic number.

As we did in Section 6, we consider three cases:

$$K_0<2^{100}K_2; \tag{8-11}$$

$$2^{100}K_2\leq K_0\leq 2^{10}K_1; \tag{8-12}$$

$$K_0>2^{10}K_1. \tag{8-13}$$

The rest of the paper is devoted to a proof of these three cases. In what follows, we will only provide the details for the estimates of $\|w_1\|_{s,-1/2}$ with $1/2<s<1$ (the case $s\geq 1$ is easier). For the desired estimate of

$$\left(\sum_n\langle n\rangle^{2s}\left(\int\frac{|\hat{w}_1(n,\lambda)|}{\langle\lambda-n^3\rangle}\,d\lambda\right)^2\right)^{1/2},$$

simply replace v by

$$v_1(x,t)=\sum_n\int\frac{C_{n,\lambda}A_n}{\langle\lambda-n^3\rangle}e^{i\lambda t}e^{inx}\,d\lambda, \tag{8-14}$$

and then the desired estimate follows similarly. Here $C_{n,\lambda}\in\mathbb{C}$ satisfies $\sup_\lambda|C_n,\lambda|\le 1$ and $\{A_n\}$ satisfies $\sum_n|A_n|^2\le 1$.

9. Proof of case (8-11)

In this case, we should consider further two subcases:

$$M\le 2^{10}K_1, \tag{9-1}$$

$$M>2^{10}K_1. \tag{9-2}$$

For the contribution of (9-1), noticing that $K\le CK_1$ in this subcase, we estimate (8-10) by

$$\sum_{K_1\ge K_2\ge K_3}\int_{\mathbb{T}\times\mathbb{R}}\left|\Bigl(\sum_{K\le CK_1}\partial_x^s v_K\Bigr)\Bigl(\sum_{K_0\le CK_2}\partial_x u_{K_0}\Bigr)u_{K_1}u_{K_2}u_{K_3}(P_{2^{10}K_1}G_{K_3})\right|dx\,dt, \tag{9-3}$$

which is bounded by

$$\sum_{K_3}\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\int_{\mathbb{T}\times\mathbb{R}}\sum_{K_1}\sum_{K\le CK_1}K^s v_K^*|u_{K_1}|\sum_{K_2}\sum_{K_0\le CK_2}K_0u_{K_0}^*|u_{K_2}|\,dx\,dt, \tag{9-4}$$

where f^* stands for the Hardy–Littlewood maximal function of f. By the Schur's test, (9-4) can be estimated by

$$\sum_{K_3}K_3^{-(2s-1)/2}\|u\|_{Y_s}\mathfrak{M}\int\Bigl(\sum_K|v_K^*|^2\Bigr)^{1/2}\Bigl(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Bigr)^{1/2}\cdot\Bigl(\sum_{K_0}K_0|u_{K_0}^*|^2\Bigr)^{1/2}\Bigl(\sum_{K_2}K_2|u_{K_2}|^2\Bigr)^{1/2}dx\,dt. \tag{9-5}$$

Since $s>1/2$, we obtain, by a use of Hölder's inequality, that (9-4) is majorized by

$$C\mathfrak{M}\|u\|_{Y_s}\left\|\Bigl(\sum_K|v_K^*|^2\Bigr)^{1/2}\right\|_4\left\|\Bigl(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Bigr)^{1/2}\right\|_4\left\|\Bigl(\sum_{K_0}K_0|u_{K_0}^*|^2\Bigr)^{1/2}\right\|_4\left\|\Bigl(\sum_{K_2}K_2|u_{K_2}|^2\Bigr)^{1/2}\right\|_4. \tag{9-6}$$

Observe that

$$\left\|\Bigl(\sum_K|v_K^*|^2\Bigr)^{1/2}\right\|_4\le\left\|\Bigl(\sum_K|v_K|^2\Bigr)^{1/2}\right\|_4\le C\|v\|_4\le C\|v\|_{0,1/3}\le C. \tag{9-7}$$

Here the first inequality is obtained by using Fefferman and Stein's vector-valued inequality on the maximal function, and the second is a consequence of the classical Littlewood–Paley theorem. Similarly,

$$\left\|\Bigl(\sum_{K_0}K_0|u_{K_0}^*|^2\Bigr)^{1/2}\right\|_4\le\left\|\Bigl(\sum_{K_0}K_0|u_{K_0}|^2\Bigr)^{1/2}\right\|_4\le C\|\partial_x^{1/2}u\|_4\le C\|u\|_{1/2,1/3}\le C\|u\|_{Y_s} \tag{9-8}$$

and

$$\Big\| \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2 \Big)^{1/2} \Big\|_4 \le C \|\partial_x^s u\|_4 \le C \|u\|_{s,1/3} \le C \|u\|_{Y_s}. \tag{9-9}$$

Hence, from (9-7), (9-8) and (9-9), we have

$$\text{(8-10)} \le C \mathfrak{M} \|u\|_{Y_s}^4. \tag{9-10}$$

For the contribution of (9-2), since in this subcase $K \le CM$, we estimate (8-10) by

$$\sum_{K_1} \|u_{K_1}\|_\infty \int_{\mathbb{T}\times\mathbb{R}} \sum_{K_3 \le K_1} |u_{K_3}| \sum_M \sum_{K \le CM} K^s v_K^* |g_{K_3,M}| \sum_{K_2} \sum_{K_0 \le CK_2} K_0 u_{K_0}^* |u_{K_2}| \, dx \, dt, \tag{9-11}$$

which is bounded by

$$\begin{aligned} \sum_{K_1} K_1^{-(2s-1)/2} \|u\|_{Y_s} \int_{\mathbb{T}\times\mathbb{R}} \sum_{K_3 \le K_1} |u_{K_3}| \Big(\sum_K |v_K^*|^2 \Big)^{1/2} \Big(\sum_M M^{2s} |g_{K_3,M}|^2 \Big)^{1/2} \\ \cdot \Big(\sum_{K_0} K_0 |u_{K_0}^*|^2 \Big)^{1/2} \Big(\sum_{K_2} K_2 |u_{K_2}|^2 \Big)^{1/2} dx \, dt. \end{aligned} \tag{9-12}$$

By a use of the Cauchy–Schwarz inequality, (9-12) is estimated by

$$\begin{aligned} \sum_{K_1} K_1^{-(2s-1)/2} \|u\|_{Y_s} \int_{\mathbb{T}\times\mathbb{R}} \Big(\sum_K |v_K^*|^2 \Big)^{1/2} \Big(\sum_{K_0} K_0 |u_{K_0}^*|^2 \Big)^{1/2} \Big(\sum_{K_2} K_2 |u_{K_2}|^2 \Big)^{1/2} \\ \cdot \Big(\sum_{K_3} K_3^{2s} |u_{K_3}|^2 \Big)^{1/2} \Big(\sum_{K_3 \le K_1} \sum_M \frac{M^{2s}}{K_3^{2s}} |g_{K_3,M}|^2 \Big)^{1/2} dx \, dt. \end{aligned} \tag{9-13}$$

Using Hölder's inequality, we then bound it further by

$$\begin{aligned} \sum_{K_1} K_1^{-(2s-1)/2} \|u\|_{Y_s} \Big\| \Big(\sum_K |v_K^*|^2 \Big)^{1/2} \Big\|_4 \Big\| \Big(\sum_{K_2} K_0 |u_{K_0}^*|^2 \Big)^{1/2} \Big\|_6 \Big\| \Big(\sum_{K_2} K_2 |u_{K_2}|^2 \Big)^{1/2} \Big\|_6 \\ \cdot \Big\| \Big(\sum_{K_3} K_3^{2s} |u_{K_3}|^2 \Big)^{1/2} \Big\|_4 \Big\| \Big(\sum_{K_3 \le K_1} \sum_M \frac{M^{2s}}{K_3^{2s}} |g_{K_3,M}|^2 \Big)^{1/2} \Big\|_6, \end{aligned} \tag{9-14}$$

which is majorized by

$$\sum_{K_1} K_1^{-(2s-1)/2} \|u\|_{Y_s}^4 \sum_{K_3 \le K_1} K_3^{-s} \Big\| \Big(\sum_M M^{2s} |g_{K_3,M}|^2 \Big)^{1/2} \Big\|_6 \le \sum_{K_1} K_1^{-(2s-1)/2} \|u\|_{Y_s}^4 \sum_{K_3 \le K_1} K_3^{-s} \|\partial_x^s G_{K_3}\|_\infty.$$

From the definition of G_{K_3}, we have

$$\partial_x G_{K_3}(x,t) = \mathrm{O}(\mathfrak{M} K_3) \|u\|_{Y_s} = \mathrm{O}(\mathfrak{M} K_3) \|\phi\|_{H^s}. \tag{9-15}$$

Hence, for $s < 1$,

$$\|\partial_x^s G_{K_3}\|_\infty \le C \mathfrak{M} K_3^s \|\phi\|_{H^s}. \tag{9-16}$$

Since $s > 1/2$, we then have

$$(9\text{-}14) \le C\mathfrak{M}\|\phi\|_{H^s} \sum_{K_1} K_1^{-(2s-1)/2+\varepsilon} \|u\|_{Y_s}^4 \le C\mathfrak{M}\|\phi\|_{H^s}\|u\|_{Y_s}^4. \tag{9-17}$$

This completes our discussion of Case (8-11).

10. Proof of case (8-12)

In this case, it suffices to consider the following subcases:

$$K \le 2^{10} K_2, \tag{10-1}$$

$$K \le 2^{10} M, \tag{10-2}$$

$$K > 2^9(K_2 + M) \quad \text{and} \quad K_3 \ge K_0^{1/2}, \tag{10-3}$$

$$K > 2^9(K_2 + M), \quad K_3 \le K_0^{1/2}, \quad \text{and} \quad M \ge 2^{-10} K_0^{2/3}, \tag{10-4}$$

$$K > 2^9(K_2 + M), \quad K_3 \le K_0^{1/2}, \quad \text{and} \quad M < 2^{-10} K_0^{2/3}. \tag{10-5}$$

The first two cases can be handled in exactly the same way as cases (9-1) and (9-2).

For case (10-3), observe that (8-12) and (10-3) imply

$$K \le C K_1 \tag{10-6}$$

and

$$K_0^{1/2} \le K_2^{1/2} K_3^{1/2}. \tag{10-7}$$

Hence (8-10) is bounded by

$$\int \sum_{K_1} \sum_{K \le C K_1} K^s v_K^* |u_{K_1}| \sum_{\substack{K_0 \ge K_2 \ge K_3 \\ K_0 \le K_3^2}} K_0 u_{K_0}^* |u_{K_2}|\,|u_{K_3}|\,\|G_{K_3}\|_\infty \, dx\, dt. \tag{10-8}$$

Applying Hölder's inequality, we estimate (10-8) by

$$C\mathfrak{M} \int \Big(\sum_K |v_K^*|^2\Big)^{1/2} \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2\Big)^{1/2} \prod_{j=0,2,3} \Big(\sum_{K_j} K_j^{1+\varepsilon} |u_{K_j}|^2\Big)^{1/2} dx\, dt. \tag{10-9}$$

One more use of Hölder's inequality yields that (10-8) is bounded by

$$C\mathfrak{M} \Big\| \Big(\sum_K |v_K|^2\Big)^{1/2} \Big\|_4 \Big\| \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2\Big)^{1/2} \Big\|_4 \prod_{j=0,2,3} \Big\| \Big(\sum_{K_j} K_j^{1+\varepsilon} |u_{K_j}|^2\Big)^{1/2} \Big\|_6.$$

Hence we obtain

$$(10\text{-}8) \le C\mathfrak{M}\|u\|_{Y_s}^4. \tag{10-10}$$

This finishes the proof of (10-3).

For case (10-4), we estimate (8-10) by

$$\sum_{K_2,K_3}\int\sum_{K_1}\sum_{K\le CK_1}K^s v_K^*|u_{K_1}|\sum_{K_0}K_0|u_{K_0}^*||u_{K_2}||u_{K_3}|\sum_{M\ge CK_0^{2/3}}|g_{K_3,M}|\,dx\,dt,\tag{10-11}$$

which is dominated by

$$C\sum_{K_2,K_3}\int\Big(\sum_K|v_K^*|^2\Big)^{1/2}\Big(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Big)^{1/2}|u_{K_2}||u_{K_3}|\cdot\Big(\sum_{K_0}K_0|u_{K_0}^*|^2\Big)^{1/2}\Big(\sum_M M^{3/2}|g_{K_3,M}|^2\Big)^{1/2}dx\,dt.\tag{10-12}$$

By Hölder's inequality with L^4 norms for the first two functions in the integrand, L^{6+} norms for the next three functions, and an L^p norm (very large p) for the last one, (10-12) is dominated by

$$C\|u\|_{Y_s}\sum_{K_2,K_3}\|u_{K_2}\|_{6+}\|u_{K_3}\|_{6+}\Big\|\Big(\sum_{K_0}K_0|u_{K_0}^*|^2\Big)^{1/2}\Big\|_{6+}\|\partial_x^{3/4}G_{K_3}\|_\infty.\tag{10-13}$$

Applying (9-16), we estimate (10-12) by

$$C\mathfrak{M}\|\phi\|_{H_s}\|u\|_{Y_s}^2\prod_{j=2}^3\sum_{K_j}K_j^{3/8}\|u_{K_j}\|_{6+}\le C\mathfrak{M}\|\phi\|_{H_s}\|u\|_{Y_s}^2\prod_{j=2}^3\sum_{K_j}K_j^{3/8}\|u_{K_j}\|_{0+,1/2}\le C\mathfrak{M}\|\phi\|_{H_s}\|u\|_{Y_s}^4,$$

as desired. This completes the discussion of (10-4).

We now turn to case (10-5). In this case, we have

$$|n_0+n_1|+2K_2+M\ge|n|\ge K/2\ge 2^8(K_2+M),\tag{10-14}$$

which implies

$$|n_0+n_1|\ge 2^5(K_2+M).\tag{10-15}$$

Notice that

$$\begin{aligned}(n_0+n_1+n_2+n_3+m)^3-n_0^3-n_1^3-n_2^3-n_3^3-m^3\\ =3(n_0+n_1)(n_0+n_2+n_3+m)(n_1+n_2+n_3+m)+(n_2+n_3+m)^3-n_2^3-n_3^3-m^3.\end{aligned}\tag{10-16}$$

From (10-15), (10-16), and (10-5), we obtain

$$|n^3-n_0^3-n_1^3-n_2^3-n_3^3-m^3|\ge C(K_2+M)K_0K_1\ge CK_0K_1\ge CK_0^2.\tag{10-17}$$

Hence one of the following four statements must be true:

$$|\lambda-n^3|\ge K_0^2,\tag{10-18}$$

$$|(\lambda-\lambda_1-\lambda_2-\lambda_3-\mu)-n_0^3|\ge K_0^2,\tag{10-19}$$

$$\text{there exists an } i\in\{1,2,3\} \text{ such that } |\lambda_i-n_i^3|\ge K_0^2,\tag{10-20}$$

$$|\mu|\ge K_0^2.\tag{10-21}$$

For case (10-18), we set

$$\tilde v(x,t) = (\hat v \mathbf{1}_{|\lambda-n^3|\ge K_0^2})^\vee(x,t). \tag{10-22}$$

We then estimate (8-10) by

$$\sum_{K_2,K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int |\partial_x u_{K_0}| \sum_{K_1} \sum_{K\le CK_1} K^s \tilde v_K^* |u_{K_1}|\, dx\, dt. \tag{10-23}$$

This is clearly bounded by

$$C\mathfrak{M}\|u\|_{Y_s}^2 \sum_{K_0} \int K_0 |u_{K_0}^*| \Big(\sum_K |\tilde v_K^*|^2\Big)^{1/2} \Big(\sum_{K_1} K_1^{2s}|u_{K_1}|^2\Big)^{1/2} dx\, dt. \tag{10-24}$$

Using the Cauchy–Schwarz inequality, we bound (10-24) by

$$C\mathfrak{M}\|u\|_{Y_s}^2 \int \Big(\sum_{K_0} K_0^\varepsilon |u_{K_0}^*|^2\Big)^{1/2} \Big(\sum_{K_0} K_0^{2-\varepsilon} \sum_K |\tilde v_K^*|^2\Big)^{1/2} \Big(\sum_{K_1} K_1^{2s}|u_{K_1}|^2\Big)^{1/2} dx\, dt. \tag{10-25}$$

By Hölder's inequality, (10-25) is majorized by

$$C\mathfrak{M}\|u\|_{Y_s}^2 \bigg\| \Big(\sum_{K_0} K_0^\varepsilon |u_{K_0}^*|^2\Big)^{1/2} \bigg\|_4 \bigg\| \Big(\sum_{K_0} K_0^{2-\varepsilon} \sum_K |\tilde v_K^*|^2\Big)^{1/2} \bigg\|_2 \bigg\| \Big(\sum_{K_1} K_1^{2s}|u_{K_1}|^2\Big)^{1/2} \bigg\|_4,$$

which is controlled by

$$C\mathfrak{M}\|u\|_{Y_s}^3 \|\partial_x^\varepsilon u\|_4 \Big(\sum_{K_0} K_0^{2-\varepsilon} \|\tilde v\|_2^2\Big)^{1/2} \le C\mathfrak{M}\|u\|_{Y_s}^3 \|\partial_x^\varepsilon u\|_4 \sum_{K_0} K_0^{-\varepsilon/2} \le C\mathfrak{M}\|u\|_{Y_s}^4. \tag{10-26}$$

This finishes the proof of case (10-18).

For case (10-19), let $\tilde u$ be defined by

$$\tilde u = (\hat u \mathbf{1}_{|\lambda-n^3|\ge K_0^2})^\vee. \tag{10-27}$$

Then (8-10) can be estimated by

$$\sum_{K_2,K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int |\partial_x \tilde u_{K_0}| \sum_{K_1} \sum_{K\le CK_1} K^s v_K^* |u_{K_1}|\, dx\, dt. \tag{10-28}$$

By Schur's test and Hölder's inequality, we control (10-28) by

$$\sum_{K_2,K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \|\partial_x \tilde u_{K_0}\|_2 \bigg\| \Big(\sum_K |v_K|^2\Big)^{1/2} \bigg\|_4 \bigg\| \Big(\sum_{K_1} K_1^{2s}|u_{K_1}|^2\Big)^{1/2} \bigg\|_4, \tag{10-29}$$

which is bounded by

$$C\mathfrak{M}\|u\|_{Y_s}^3 \sum_{K_0} \|u_{K_0}\|_{0,1/2} \le C\mathfrak{M}\|u\|_{Y_s}^4. \tag{10-30}$$

This completes the proof of case (10-19).

For case (10-20), if $j = 1$, we dominate (8-10) by

$$\sum_{K_2,K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int |\partial_x u_{K_0}| \sum_{K_1} \sum_{K \le CK_1} K^s v_K^* |\tilde{u}_{K_1}|\, dx\, dt. \tag{10-31}$$

As we did in case (10-19), we bound (10-31) by

$$C\mathfrak{M}\|u\|_{Y_s}^2 \sum_{K_0} \|\partial_x u_{K_0}\|_4 \,\|v\|_4 \left\| \left(\sum_{K_1} K_1^{2s} |\tilde{u}_{K_1}|^2 \right)^{1/2} \right\|_2. \tag{10-32}$$

This can be further controlled by

$$C\mathfrak{M}\|u\|_{Y_s}^3 \sum_{K_0} \frac{1}{K_0} \|\partial_x u_{K_0}\|_4 \,\|v\|_4 \le \mathfrak{M}\|u\|_{Y_s}^3 \sum_{K_0} \frac{1}{K_0} \|u_{K_0}\|_{1,1/3} \le C\mathfrak{M}\|u\|_{Y_s}^4, \tag{10-33}$$

as desired.

We now consider $j = 2$ or $j = 3$. Without loss of generality, assume $j = 2$. In this case, we estimate (8-10) by

$$\sum_{K_3} \|u_{K_3}\| \,\|G_{K_3}\|_\infty \sum_{K_0} \int |\partial_x u_{K_0}| \sum_{K_1} \sum_{K \le CK_1} K^s v_K^* |u_{K_1}| \sum_{K_2 \le CK_0} |\tilde{u}_{K_2}|\, dx\, dt, \tag{10-34}$$

which is bounded by

$$C\mathfrak{M}\|u\|_{Y_s} \sum_{K_0} \|\partial_x u_{K_0}\|_\infty \sum_{K_2 \le K_0} \|\tilde{u}_{K_2}\|_2 \,\|v\|_4 \left\| \left(\sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \right\|_4.$$

Notice that

$$\begin{aligned} \sum_{K_0} \|\partial_x u_{K_0}\|_\infty \sum_{K_2 \le K_0} \|\tilde{u}_{K_2}\|_2 &\le C \sum_{K_0} \frac{1}{K_0} \|\partial_x u_{K_0}\|_\infty \|u\|_{Y_s} \\ &\le C \sum_n \int |\hat{u}(n,\lambda)|\, d\lambda \|u\|_{Y_s} \le C\|u\|_{Y_s}^2. \end{aligned}$$

Hence (10-34) is dominated by

$$C\mathfrak{M}\|u\|_{Y_s}^4. \tag{10-35}$$

This completes case (10-20).

We now turn to the most difficult case, (10-21) in case (8-12). We should decompose G_{K_3}, with respect to the t-variable, into Littlewood–Paley multipliers in the same spirit as before. More precisely, for any dyadic number L, let Q_L be

$$Q_L u(x,t) = \int \psi_L(\tau) u(x, t-\tau)\, d\tau. \tag{10-36}$$

Here the Fourier transform of ψ_L is a bump function supported on $[-2L, 2L]$ and $\widehat{\psi_L}(x) = 1$ if $x \in [-L, L]$. Let

$$\Pi_L u = Q_L u - Q_{L/2} u. \tag{10-37}$$

Then $\Pi_L u$ gives a Littlewood–Paley multiplier with respect to the time variable t. Using this multiplier, we represent

$$u_K = \sum_L u_{K,L}. \tag{10-38}$$

Here $u_{K,L} = \Pi_L(u_K)$. We decompose G_{K_3} as

$$\begin{aligned} G_{K_3} &= C + \sum_L (F_3(Q_L P_{4K_3}u, \dots, Q_L P_{K_3/8}u) - F_3(Q_{L/2} P_{4K_3}u, \dots, Q_{L/2} P_{K_3/8}u)) \\ &= C + \sum_{\substack{j=4,2,1,1/2,1/4,1/8 \\ L}} H_{K_3,L} u_{jK_3,L}, \end{aligned} \tag{10-39}$$

where $H_{K_3,L}$ is given by

$$H_{K_3,L} = F_4\big(Q_{\ell L} P_{4K_3}u, \dots, Q_{\ell L} P_{K_3/8}u;\ \ell = 1, \tfrac{1}{2}\big). \tag{10-40}$$

Let $\mathfrak{M}_1$ be defined by

$$\mathfrak{M}_1 = \sup\{|D^\alpha F_4(u_1, \dots, u_{12})| : u_j \text{ satisfies } \|u_j\|_{Y_s} \le C_0 \|\phi\|_{H^s} \text{ for all } j = 1, \dots, 12; \alpha\}. \tag{10-41}$$

Here $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_{12}}^{\alpha_{12}}$ and α is taken over all tuples $(\alpha_1, \dots, \alpha_{12}) \in (\mathbb{N} \cup \{0\})^{12}$ with $\sum |\alpha_j| \le 1$. $\mathfrak{M}_1$ is a real number because $F_4 \in C^1$.

In order to finish the proof, we need to consider a further three subcases:

$$L \le 2^{10} K_3^3, \tag{10-42}$$

$$2^{10} K_3^3 < L \le 2^{-5} K_0^2, \tag{10-43}$$

$$L > 2^{-5} K_0^2. \tag{10-44}$$

For the contribution of (10-42), we set

$$h_{K_0,jK_3,L} = (\widehat{H_{K_3,L} u_{jK_3,L}} \mathbf{1}_{|\mu| \ge K_0^2})^\vee. \tag{10-45}$$

Here $j = 4, 2, 1, 1/2, 1/4, 1/8$. From the definition of $H_{K_3,L}$, we get

$$\|h_{K_0,jK_3,L}\|_4 \le C \mathfrak{M}_1 \|\phi\|_{H^s} \frac{L}{K_0^2} \|u_{jK_3,L}\|_4. \tag{10-46}$$

Then (8-10) is bounded by

$$\sum_{K_2} \|u_{K_2}\|_\infty \sum_{K_0} \int K_0 u_{K_0}^* \sum_{K_3 \le C K_0^{1/2}} \|u_{K_3}\|_\infty \sum_{L \le C K_3^3} |h_{K_0,jK_3,L}| \sum_{K_1} \sum_{K \le C K_1} K^s v_K^* |u_{K_1}| \, dx\, dt, \tag{10-47}$$

which is majorized by

$$\sum_{K_2} \|u_{K_2}\|_\infty \sum_{K_0} K_0 \sum_{K_3 \le C K_0^{1/2}} \|u_{K_3}\|_\infty \int u_{K_0}^* \cdot \sum_{L \le C K_3^3} |h_{K_0,jK_3,L}| \Big(\sum_K |v_K^*|^2\Big)^{1/2} \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2\Big)^{1/2} dx\, dt. \tag{10-48}$$

Using Hölder's inequality with L^4 norms for the four functions in the integrand, we estimate (10-48) as follows:

$$C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s}^2\sum_{K_0}K_0\|u_{K_0}\|_4\sum_{K_3\le K_0^{1/2}}\|u_{K_3}\|_\infty\sum_{L\le CK_3^3}\frac{L}{K_0^2}\|u_{jK_3,L}\|_4$$
$$\le C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^3\sum_{K_0}K_0^{1/2}\|u_{K_0}\|_{0,1/3}$$
$$\le C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^4. \tag{10-49}$$

This finishes case (10-42).

For the contribution of (10-43), we bound (8-10) by

$$\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\sum_{K_0}|\partial_x u_{K_0}|\sum_{2^{10}K_3^3<L\le 2^{-10}K_0^2}|h_{K_0,jK_3,L}|\sum_{K_1}\sum_{K\le CK_1}K^s v_K^*|u_{K_1}|\,dx\,dt, \tag{10-50}$$

which is dominated by

$$C\|u\|_{Y_s}\sum_{K_3}\|u_{K_3}\|_\infty\sum_{\substack{\Delta\le 2^{-10}\\ \Delta\text{dyadic}}}\int\sum_{K_0}|\partial_x u_{K_0}|\sum_{\substack{2^{10}K_3^3<L\\ (\Delta/2)K_0^2<L\le\Delta K_0^2}}|h_{K_0,jK_3,L}|$$
$$\cdot\Big(\sum_K|v_K^*|^2\Big)^{1/2}\Big(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Big)^{1/2}dx\,dt, \tag{10-51}$$

By the Cauchy–Schwarz inequality, we further estimate (10-51) by

$$C\|u\|_{Y_s}\sum_{K_3}\|u_{K_3}\|_\infty\sum_{\substack{\Delta\le 2^{-10}\\ \Delta\text{dyadic}}}\Delta^{-1/2}\int\sum_{K_0}\frac{|\partial_x u_{K_0}|}{K_0}$$
$$\cdot\Big(\sum_{\substack{2^{10}K_3^3<L\\ (\Delta/2)K_0^2<L\le\Delta K_0^2}}L|h_{K_0,jK_3,L}|^2\Big)^{1/2}\Big(\sum_K|v_K^*|^2\Big)^{1/2}\Big(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Big)^{1/2}dx\,dt. \tag{10-52}$$

Applying Hölder's inequality with an L^∞ norm for the first function in the integrand, an L^2 norm for the second, and L^4 norms for the last two functions, we then majorize (10-52) by

$$C\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{\substack{\Delta\le 2^{-10}\\ \Delta\text{dyadic}}}\Delta^{-1/2}\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0}\Big\|\Big(\sum_{\substack{2^{10}K_3^3<L\\ (\Delta/2)K_0^2<L\le\Delta K_0^2}}L|h_{K_0,jK_3,L}|^2\Big)^{1/2}\Big\|_2. \tag{10-53}$$

Notice that if $L\sim\Delta K_0^2$,

$$\|h_{K_0,jK_3,L}\|_2\le C\mathfrak{M}_1\|\phi\|_{H^s}\Delta\|u_{jK_3,L}\|_2. \tag{10-54}$$

Thus we have

$$\Big\|\Big(\sum_{\substack{2^{10}K_3^3<L\\(\Delta/2)K_0^2<L\le\Delta K_0^2}} L|h_{K_0,jK_3,L}|^2\Big)^{1/2}\Big\|_2 \le C\mathfrak{M}_1\|\phi\|_{H^s}\Delta\Big(\sum_{\substack{2^{10}K_3^3<L\\(\Delta/2)K_0^2<L\le\Delta K_0^2}} L\|u_{jK_3,L}\|_2^2\Big)^{1/2}$$
$$\le C\mathfrak{M}_1\|\phi\|_{H^s}\Delta\|u_{jK_3}\|_{0,1/2}$$
$$\le C\mathfrak{M}_1\|\phi\|_{H^s}^2\Delta. \tag{10-55}$$

From (10-55), (10-53) is bounded by

$$C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{\substack{\Delta\le 2^{-10}\\ \Delta\text{dyadic}}}\Delta^{1/2}\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0}, \tag{10-56}$$

which is clearly majorized by

$$C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^4. \tag{10-57}$$

This finishes case (10-43).

For the contribution of (10-44), we estimate (8-10) by

$$\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\sum_{K_0}|\partial_x u_{K_0}|\sum_{L>2^{-5}K_0^2}|h_{K_0,jK_3,L}|\sum_{K_1}\sum_{K\le CK_1}K^s v_K^*|u_{K_1}|\,dx\,dt, \tag{10-58}$$

which is bounded by

$$\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\Big(\sum_{K_0}\frac{|\partial_x u_{K_0}|^2}{K_0^2}\Big)^{1/2}$$
$$\cdot\Big(\sum_{L>2^{-5}K_0^2}L|h_{K_0,jK_3,L}|^2\Big)^{1/2}\Big(\sum_K|v_K^*|^2\Big)^{1/2}\Big(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Big)^{1/2}dx\,dt. \tag{10-59}$$

Applying Hölder's inequality, we further have

$$\text{(10-59)}\le C\mathfrak{M}_1\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0}\Big(\sum_{L>2^{-5}K_0^2}L\|u_{jK_3,L}\|_2^2\Big)^{1/2}$$
$$\le C\mathfrak{M}_1\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0}\|u_{jK_3}\|_{0,1/2}. \tag{10-60}$$

This is clearly majorized by

$$C\mathfrak{M}_1\|\phi\|_{H_s}\|u\|_{Y_s}^4. \tag{10-61}$$

Hence we complete case (10-44).

11. Proof of case (8-13)

In this case, it suffices to consider the following subcases:

$$M \geq 2^{-10} K_0^{2/3}, \tag{11-1}$$

$$M < 2^{-10} K_0^{2/3} \quad \text{and} \quad K_2^2 K_3 \geq 2^{-10} K_0^2, \tag{11-2}$$

$$M < 2^{-10} K_0^{2/3} \quad \text{and} \quad K_2^2 M \geq 2^{-10} K_0^2, \tag{11-3}$$

$$M < 2^{-10} K_0^{2/3}, \quad K_2^2 K_3 < 2^{-10} K_0^2 \quad \text{and} \quad K_2^2 M < 2^{-10} K_0^2. \tag{11-4}$$

For case (11-1), notice that we have

$$K \leq C M^{3/2}. \tag{11-5}$$

Hence we estimate (8-10) by

$$\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_M \sum_{K \leq CM^{3/2}} K^s v_K^* \sum_{K_0 \leq CM^{3/2}} K_0 u_{K_0}^* |g_{K_3,M}| \, dx \, dt, \tag{11-6}$$

which is bounded by

$$\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_M M^{(3/2)(1-s)} |g_{K_3,M}| \sum_{K \leq CM^{3/2}} K^s v_K^* \biggl(\sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \biggr)^{1/2} dx \, dt, \tag{11-7}$$

since $1/2 < s < 1$. Applying Schur's test, we estimate (11-7) by

$$\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \biggl(\sum_M M^3 |g_{K_3,M}|^2 \biggr)^{1/2} \biggl(\sum_K |v_K^*|^2 \biggr)^{1/2} \biggl(\sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \biggr)^{1/2} dx \, dt. \tag{11-8}$$

By Hölder's inequality and $s > 1/2$, (11-8) is majorized by

$$\begin{aligned} C \sum_{K_1 \geq K_2 \geq K_3} \|\partial_x^{3/2} G_{K_3}\|_\infty \biggl(\prod_{j=1}^{3} \|u_{K_j}\|_{6+} \biggr) \biggl\| \biggl(\sum_K |v_K|^2 \biggr)^{1/2} \biggr\|_4 \biggl\| \biggl(\sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \biggr)^{1/2} \biggr\|_4 \\ \leq C\mathfrak{M}(\|\phi\|_{H^s} + \|\phi\|_{H^s}^2) \|u\|_{Y_s} \sum_{K_1 \geq K_3 \geq K_3} K_3^{3/2} \prod_{j=1}^{3} \|u_{K_j}\|_{6+} \\ \leq C\mathfrak{M}(\|\phi\|_{H^s} + \|\phi\|_{H^s}^2) \|u\|_{Y_s} \prod_{j=1}^{3} \sum_{K_j} K_j^{1/2} \|u_{K_j}\|_{0+,1/2} \\ \leq C\mathfrak{M}(\|\phi\|_{H^s} + \|\phi\|_{H^s}^2) \|u\|_{Y_s}^4. \end{aligned} \tag{11-9}$$

This finishes case (11-1).

For case (11-2), observe that, in this case,

$$K_0 \leq C K_1^{1/2} K_2^{1/2} K_3^{1/2}. \tag{11-10}$$

We estimate (8-10) by

$$\int \sum_{K_1\geq K_2\geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_{K\leq CK_0} K^s v_K^* \sum_{K_0\leq C(K_1K_2K_3)^{1/2}} K_0 u_{K_0}^* \|G_{K_3}\|_\infty dx\,dt, \tag{11-11}$$

which is bounded by

$$C\mathfrak{M}\int \Big(\sum_K |v_K^*|^2\Big)^{1/2}\Big(\sum_{K_0} K_0^{2s}|u_{K_0}^*|^2\Big)^{1/2} \prod_{j=1}^3 \sum_{K_j} K_j^{1/2}|u_{K_j}|\,dx\,dt. \tag{11-12}$$

Using Hölder's inequality with L^4 norms for the first two functions and L^6 norms for the last three functions in the integrand, we obtain

$$C\mathfrak{M}\|u\|_{Y_s}\prod_{j=1}^3 \Big\|\sum_{K_j} K_j^{1/2}|u_{K_j}|\Big\|_6 \leq C\mathfrak{M}\|u\|_{Y_s}^4. \tag{11-13}$$

This completes case (11-2).

For case (11-3) we have

$$K_0 \leq CK_1^{1/2}K_2^{1/2}M^{1/2}. \tag{11-14}$$

Hence we dominate (8-10) by

$$\int \sum_{K_1\geq K_2\geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_M |g_{K_3,M}| \sum_{K\leq CK_0} K^s v_K^* \sum_{K_0\leq C(K_1K_2M)^{1/2}} K_0 u_{K_0}^* dx\,dt, \tag{11-15}$$

which is bounded by

$$C\sum_{K_3}\int \Big(\sum_K |v_K^*|^2\Big)^{1/2}\Big(\sum_{K_0} K_0^{2s}|u_{K_0}^*|^2\Big)^{1/2} |u_{K_3}| \cdot \Big(\sum_M M|g_{K_3,M}|^2\Big)^{1/2}\prod_{j=1}^2\sum_{K_j} K_j^{1/2}|u_{K_j}|\,dx\,dt. \tag{11-16}$$

Using Hölder's inequality with L^4 norms for the first two functions, L^6 norms for the third, an L^p norm with p very large for the fourth, and L^{6+} for the last two functions in the integrand, we obtain

$$C\|u\|_{Y_s}\prod_{j=1}^2\Big\|\sum_{K_j}K_j^{1/2}|u_{K_j}|\Big\|_{6+}\sum_{K_3}\|u_{K_3}\|_6\,\|\partial_x^{1/2}G_{K_3}\|_\infty. \tag{11-17}$$

Clearly (11-17) is dominated by

$$C\mathfrak{M}\|\phi\|_{H^s}\|u\|_{Y_s}^3\sum_{K_3}K_3^{1/2}\|u_{K_3}\|_6 \leq C\mathfrak{M}\|\phi\|_{H^s}\|u\|_{Y_s}^4. \tag{11-18}$$

Hence case (11-3) is done.

For case (11-4) we observe that

$$M^2K_2 \leq 2^{-10}K_0^2. \tag{11-19}$$

In fact, if (11-19) does not hold, then, from (11-4),

$$M^2K_2 > 2^{-10}K_0^2 > K_2^2M.$$

Thus $M > K_2$, which immediately yields

$$M^3 > M^2K_2 > 2^{-10}K_0^2,$$

contradicting $M < 2^{-10}K_0^{2/3}$. Hence (11-19) must be true. From (11-19), $K_2^2K_3 + K_2^2M < 2^{-9}K_0^2$, we get

$$|(n_2+n_3+m)^3 - n_2^3 - n_3^3 - m^3| \le 2^{-5}K_0^2. \tag{11-20}$$

Since $n_1+n_2+n_3+m \neq 0$, from (8-13), (11-4), and (11-20), the crucial arithmetic observation (10-16) yields

$$|n^3 - n_0^3 - n_1^3 - n_2^3 - n_3^3 - m^3| \ge 2K_0^2. \tag{11-21}$$

Hence one of the following statements must be true:

$$|\lambda - n^3| \ge K_0^2, \tag{11-22}$$

$$|(\lambda - \lambda_1 - \lambda_2 - \lambda_3 - \mu) - n_0^3| \ge K_0^2, \tag{11-23}$$

$$\text{there exists an } i \in \{1,2,3\} \text{ such that } |\lambda_i - n_i^3| \ge K_0^2, \tag{11-24}$$

$$|\mu| \ge K_0^2. \tag{11-25}$$

For case (11-22), we estimate (8-10) by

$$\sum_{K_1,K_2,K_3} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int K_0 |u_{K_0}^*| \Big| \sum_{K \le CK_0} \partial_x^s \tilde{v}_K \Big| \, dx\, dt. \tag{11-26}$$

Then the Cauchy–Schwarz inequality yields

$$\begin{aligned} C\mathfrak{M}\|u\|_{Y_s}^3 \Bigg\| \bigg(\sum_{K_0} K_0^{2-2s} \Big| \sum_{K \le CK_0} \partial_x^s \tilde{v}_K \Big|^2 \bigg)^{1/2} \Bigg\|_2 \Bigg\| \bigg(\sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \bigg)^{1/2} \Bigg\|_2 \\ \le C\mathfrak{M}\|u\|_{Y_s}^4 \bigg(\sum_{K_0} K_0^{2-2s} \sum_{K \le CK_0} \big\| \partial_x^s \tilde{v}_K \big\|_2^2 \bigg)^{1/2} \le C\mathfrak{M}\|u\|_{Y_s}^4. \end{aligned} \tag{11-27}$$

This finishes the proof of case (11-22).

For case (11-23), (8-10) can be estimated by

$$\sum_{K_1,K_2,K_3} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int K_0 |\tilde{u}_{K_0}^*| \sum_{K \le CK_0} K^s v_K^* dx\, dt. \tag{11-28}$$

By Schur's test and Hölder's inequality, we control (11-28) by

$$C\mathfrak{M}\|u\|_{Y_s}^3 \Bigg\| \bigg(\sum_{K} |v_K^*|^2 \bigg)^{1/2} \Bigg\|_2 \Bigg\| \bigg(\sum_{K_0} K_0^{2s+2} |\tilde{u}_{K_0}|^2 \bigg)^{1/2} \Bigg\|_2, \tag{11-29}$$

which is clearly bounded by

$$C\mathfrak{M}\|u\|_{Y_s}^3\Big(\sum_{K_0} K_0^{2s}\|u_{K_0}\|_{0,1/2}^2\Big)^{1/2} \le C\mathfrak{M}\|u\|_{Y_s}^4. \tag{11-30}$$

This completes the proof of case (11-23).

For case (11-24), without loss of generality, assume $j=1$. We then dominate (8-10) by

$$\sum_{K_2,K_3}\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_1}\sum_{K_0}\int K_0|u_{K_0}^*||\tilde{u}_{K_1}|\sum_{K\le CK_0}K^s v_K^*\,dx\,dt. \tag{11-31}$$

By Hölder's inequality, we bound (11-31) by

$$\begin{aligned}\sum_{K_2,K_3}&\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_1}\sum_{K_0}\sum_{K\le CK_0}K^sK_0\|u_{K_0}\|_4\,\|\tilde{u}_{K_1}\|_2\,\|v_K\|_4\\ &\le\sum_{K_2,K_3}\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_1}\|u_{K_1}\|_{0,1/2}\sum_{K_0}\sum_{K\le CK_0}K^s\|u_{K_0}\|_4\|v_K\|_4.\end{aligned} \tag{11-32}$$

By Schur's test, we dominate (11-32) by

$$\begin{aligned}C\mathfrak{M}\|u\|_{Y_s}^2\sum_{K_1}\|u_{K_1}\|_{0,1/2}&\Big(\sum_{K_0}K_0^{2s}\|u_{K_0}\|_4^2\Big)^{1/2}\Big(\sum_K\|v_K\|_4^2\Big)^{1/2}\\ &\le C\mathfrak{M}\|u\|_{Y_s}^3\Big(\sum_{K_0}K_0^{2s}\|u_{K_0}\|_{0,1/3}^2\Big)^{1/2}\Big(\sum_K\|v_K\|_{0,1/3}^2\Big)^{1/2}\\ &\le C\mathfrak{M}\|u\|_{Y_s}^4.\end{aligned} \tag{11-33}$$

Hence case (11-24) is done.

In order to finish the proof, as is done in (10-36), we need to consider three further subcases:

$$L\le 2^{10}K_3^3, \tag{11-34}$$

$$2^{10}K_3^3<L\le 2^{-5}K_0^2, \tag{11-35}$$

$$L>2^{-5}K_0^2. \tag{11-36}$$

For the contribution of (11-34), notice that

$$\|h_{K_0,jK_3,L}\|_6\le C\mathfrak{M}_1\|\phi\|_{H^s}\frac{L}{K_0^2}\|u_{jK_3,L}\|_6. \tag{11-37}$$

Here $h_{K_0,jK_3,L}$ is defined as in (10-45). In this case we also have $K_3\le K_0^{2/3}$, from

$$K_2^2K_3\le 2^{-10}K_0^2.$$

Then (8-10) is bounded by

$$\int\sum_{K_0}K_0u_{K_0}^*\sum_{K\le CK_0}K^sv_K^*\sum_{\substack{K_1\ge K_2\ge K_3\\ K_3\le K_0^{2/3}}}|u_{K_1}||u_{K_2}||u_{K_3}|\sum_{L\le CK_3^3}|h_{K_0,jK_3,L}|\,dx\,dt. \tag{11-38}$$

Write (11-38) as

$$\sum_{\substack{\Delta\text{dyadic}\\ \Delta\le 1}}\int\sum_{K_0}K_0u^*_{K_0}\sum_{K\le CK_0}K^s v^*_K\sum_{\substack{K_1\ge K_2\ge K_3\\ \Delta K_0^{2/3}/2<K_3\le\Delta K_0^{2/3}}}|u_{K_1}||u_{K_2}||u_{K_3}|\sum_{L\le CK_3^3}|h_{K_0,jK_3,L}|\,dx\,dt. \tag{11-39}$$

Observe that if $\Delta K_0^{2/3}/2<K_3\le\Delta K_0^{2/3}$, we have

$$K_0\le\Delta^{-3/2}K_1^{1/2}K_2^{1/2}K_3^{1/2}. \tag{11-40}$$

Hence

$$C\|u\|_{Y_s}\sum_{K_0}\sum_{K\le K_0}K^s\sum_{K_1,K_2}K_1^{1/2}K_2^{1/2}\sum_{\Delta\le 1}\Delta^{-3/2}\sum_{K_3\sim\Delta K_0^{2/3}}K_3^{1/2}\cdot\int u^*_{K_0}v^*_K|u_{K_1}||u_{K_2}|\sum_{L\le CK_3^3}|h_{K_0,jK_3,L}|\,dx\,dt. \tag{11-41}$$

Applying Hölder's inequality with L^4 norms for first two functions and L^6 for the last three, and then using (11-37), we get

$$C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s}\sum_{K_0}\sum_{K\le K_0}K^s\sum_{K_1,K_2}K_1^{1/2}K_2^{1/2}\sum_{\Delta\le 1}\Delta^{-3/2}\cdot\sum_{K_3\sim\Delta K_0^{2/3}}K_3^{1/2}\|u_{K_0}\|_4\,\|v^*_K\|_4\,\|u_{K_1}\|_6\,\|u_{K_2}\|_6\sum_{L\le CK_3^3}\frac{L}{K_0^2}\|u_{jK_3,L}\|_6, \tag{11-42}$$

which is bounded by

$$\begin{aligned}
&C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s}\sum_{K_0}\sum_{K\le K_0}K^s\sum_{\Delta\le 1}\Delta^{-3/2}\sum_{L\le C\Delta^3K_0^2}\frac{L}{K_0^2}\|u_{K_0}\|_4\,\|v^*_K\|_4\\
&\qquad\cdot\sum_{K_1}K_1^{1/2}\|u_{K_1}\|_{0+,1/2}\sum_{K_2}K_2^{1/2}\|u_{K_2}\|_{0+,1/2}\sum_{K_3}K_3^{1/2}\|u_{jK_3,L}\|_{0+,1/2}\\
&\le C\mathfrak{M}_1\|\phi\|^2_{H^s}\|u\|^3_{Y_s}\sum_{\Delta\le 1}\Delta^{3/2}\sum_{K_0}\sum_{K\le CK_0}K^s\|u_{K_0}\|_4\,\|v_K\|_4\\
&\le C\mathfrak{M}_1\|\phi\|^2_{H^s}\|u\|^3_{Y_s}\Bigl(\sum_{K_0}K_0^{2s}\|u_{K_0}\|^2_{0,1/3}\Bigr)^{1/2}\Bigl(\sum_{K}\|v_K\|^2_{0,1/3}\Bigr)^{1/2}\\
&\le C\mathfrak{M}_1\|\phi\|^2_{H^s}\|u\|^4_{Y_s}.
\end{aligned} \tag{11-43}$$

This completes case (11-34).

For the contribution of (11-35), we bound (8-10) by

$$\sum_{K_1}\|u_{K_1}\|_\infty\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\sum_{K_0}\sum_{K\le CK_0}K^s v^*_K K_0u^*_{K_0}\sum_{2^{10}K_3^3<L\le 2^{-5}K_0^2}|h_{K_0,jK_3,L}|\,dx\,dt, \tag{11-44}$$

which is dominated by

$$C\|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta \le 2^{-5} \\ \Delta \text{dyadic}}} \sum_{K_0} \sum_{K \le C K_0} K^s \int K_0 u_{K_0}^* v_K^* \sum_{\substack{2^{10} K_3^3 < L \\ (\Delta/2) K_0^2 < L \le \Delta K_0^2}} |h_{K_0, jK_3, L}| \, dx \, dt. \tag{11-45}$$

Using the Cauchy–Schwarz inequality, we further estimate (11-45) by

$$C\|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta \le 2^{-5} \\ \Delta \text{dyadic}}} \Delta^{-\frac{1}{2}} \sum_{K_0} \sum_{K \le C K_0} K^s \int u_{K_0}^* v_K^* \Bigg(\sum_{\substack{2^{10} K_3^3 < L \\ (\Delta/2) K_0^2 < L \le \Delta K_0^2}} L |h_{K_0, jK_3, L}|^2 \Bigg)^{1/2} dx \, dt. \tag{11-46}$$

Employing Hölder's inequality with L^4 norms for the first two functions and an L^2 for the last one, we bound (11-46) by

$$C\|u\|_{Y_s}^2 \cdot \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta \le 2^{-10} \\ \Delta \text{dyadic}}} \Delta^{-1/2} \sum_{K_0} \sum_{K \le C K_0} K^s \|u_{K_0}\|_4 \, \|v_K\|_4 \Bigg\| \Bigg(\sum_{\substack{2^{10} K_3^3 < L \\ (\Delta/2) K_0^2 < L \le \Delta K_0^2}} L |h_{K_0, jK_3, L}|^2 \Bigg)^{1/2} \Bigg\|_2. \tag{11-47}$$

From (10-55), (11-47) is majorized by

$$\begin{aligned} C\mathfrak{M}_1 \|\phi\|_{H^s}^2 \|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta \le 2^{-10} \\ \Delta \text{dyadic}}} \Delta^{\frac{1}{2}} \sum_{K_0} \sum_{K \le C K_0} K^s \|u_{K_0}\|_4 \, \|v_K\|_4 \\ \le C\mathfrak{M}_1 \|\phi\|_{H^s}^2 \|u\|_{Y_s}^3 \Bigg(\sum_{K_0} K_0^{2s} \|u_{K_0}\|_{0,\frac{1}{3}}^2 \Bigg)^{1/2} \Bigg(\sum_{K} \|v_K\|_{0,\frac{1}{3}}^2 \Bigg)^{1/2} \\ \le C\mathfrak{M}_1 \|\phi\|_{H^s}^2 \|u\|_{Y_s}^4. \end{aligned} \tag{11-48}$$

This finishes the proof for case (11-35).

For the contribution of (11-36), we estimate (8-10) by

$$\sum_{K_1, K_2} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \sum_{K_3} \|u_{K_3}\|_\infty \int \sum_{K_0} K_0 u_{K_0}^* \sum_{L > 2^{-5} K_0^2} |h_{K_0, jK_3, L}| \sum_{K \le C K_0} K^s v_K^* \, dx \, dt. \tag{11-49}$$

By the Cauchy–Schwarz inequality, (11-49) is bounded by

$$\sum_{K_1, K_2} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \sum_{K_3} \|u_{K_3}\|_\infty \sum_{K_0} \sum_{K \le C K_0} K^s \int v_K^* u_{K_0}^* \Bigg(\sum_{L > 2^{-10} K_0^2} L |h_{K_0, jK_3, L}|^2 \Bigg)^{1/2} dx \, dt. \tag{11-50}$$

Employing Hölder's inequality with L^4 norms for the first two functions and an L^2 for the last one, we dominate (11-50) by

$$C\mathfrak{M}_1\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{K_0}\sum_{K\le CK_0}K^s\|u_{K_0}\|_4\,\|v_K\|_4\,\biggl\|\biggl(\sum_{L>2^{-5}K_0^2}L|u_{jK_3,L}|^2\biggr)^{1/2}\biggr\|_2$$
$$\le C\mathfrak{M}_1\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{K_0}\sum_{K\le CK_0}K^s\|u_{K_0}\|_{0,1/3}\|v_K\|_{0,1/3}\|u\|_{0,1/2}$$
$$\le C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s}^4. \tag{11-51}$$

Hence we complete case (11-36).

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Received 13 Oct 2011. Revised 16 Jun 2012. Accepted 21 Jul 2012.

YI HU: yihu1@illinois.edu
Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, United States
Current address: *Department of Mathematical Sciences, Georgia Southern University, 65 Georgia Ave. Room 3008, Statesboro, GA 30458, United States*

XIAOCHUN LI: xcli@math.uiuc.edu
Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, United States

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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ANALYSIS & PDE

Volume 6 No. 4 2013