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ON THE SPECTRUM OF DEFORMATIONS OF COMPACT DOUBLE-SIDED FLAT HYPERSURFACES

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We study the asymptotic behavior of the eigenvalues of the Laplace–Beltrami operator on a compact hypersurface in $\mathbb{R}^{n+1}$ as it is *flattened* into a singular double-sided flat hypersurface. We show that the limit spectral problem corresponds to the Dirichlet and Neumann problems on one side of this flat (Euclidean) limit, and derive an explicit three-term asymptotic expansion for the eigenvalues where the remaining two terms are of orders $\varepsilon^2 \log \varepsilon$ and ε^2.

1. Introduction

In recent years there have been several papers studying the effect that flattening a domain has on the eigenvalues of the Laplace operator [Borisov and Cardone 2011; Borisov and Freitas 2009; 2010; Friedlander and Solomyak 2009]; see also [Nazarov 2001; Panasenko 2005] and the references therein for similar problems with boundary conditions other than Dirichlet. In these papers the main objective has been the derivation of the asymptotics of these eigenvalues in terms of a scalar parameter measuring how thin the domain becomes in one direction, as this parameter approaches zero. As far as we are aware, almost if not all such existing examples in the literature are concerned with domains in Euclidean space where the limiting problem degenerates to a domain of zero measure and therefore eigenvalues approach infinity.

A slightly different set of problems which has been considered consists of domains which are perturbations of singular sets such as thin tubular neighborhoods of graphs, i.e., domains which locally are like thin tubes — see [Exner and Post 2005; 2009], for instance, and also [Grieser 2008] for a review. As in the papers cited above, again the limiting domains have zero measure and the spectrum behaves in quite a different way from the model considered here.

In this paper we study a situation which, although different from that described in the first paragraph, has in common with it the process by which the limiting domain is approached. More precisely, consider the case of a given domain Ω in $\mathbb{R}^{n+1}$ satisfying certain restrictions which for the purpose here may be stated roughly as being bounded from above and below by the graphs of two functions — see Section 2 for a precise formulation. The domain Ω is then flattened towards a domain ω in $\mathbb{R}^n$ via a (continuous) one-parameter family of domains Ω_ε. These domains are obtained as the functions mentioned above are

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multiplied by the parameter ε. The problem that shall concern us here is the study of the evolution of the eigenvalues of the Laplace–Beltrami operator on the one-parameter family of compact hypersurfaces $\mathcal{S}_\varepsilon$ which are the boundaries of the domains Ω_ε described above, as ε approaches zero. One of the differences in this instance is that while the domain Ω_0 has zero $(n+1)$-measure as stated above, $\mathcal{S}_0$ retains positive n-measure, developing instead a singularity on the boundary of the domain ω (when considered as a domain in $\mathbb{R}^n$). We thus expect these eigenvalues to remain finite as the parameter ε approaches zero, and to converge to a limiting spectral problem on the double-sided flat hypersurface. This is indeed the case, and the relevant spectral problems turn out to be the Dirichlet and Neumann problems on the domain ω, with the two next asymptotic terms after that being of orders $\varepsilon^2 \log \varepsilon$ and ε^2. These results have been announced in [Borisov and Freitas 2012].

In order to understand the origin of the $\varepsilon^2 \log \varepsilon$ term in the expansion, it turns out that it is sufficient to consider the case where n equals one, that is when the boundary is basically S^1. Because of this, it is not necessary to take into consideration the geometric intricacies of the problem which appear in higher dimensions and it is possible to obtain the full description of eigenvalues in terms of elliptic integrals.

More precisely, for an ellipse of radii 1 and ε we have that the eigenvalues are given by

$$\lambda_k(\varepsilon) = \frac{k^2\pi^2}{4E^2(1-\varepsilon^2)} \quad \text{for } k \in \mathbb{Z}, \quad \text{where } E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2(\theta)}\, d\theta$$

is the complete elliptic integral of the second type yielding one quarter of the perimeter of the ellipse for $m = 1 - \varepsilon^2$.

Combining the above with the asymptotic expansion for E yields

$$\lambda_k(\varepsilon) = \frac{k^2\pi^2}{4} + \frac{k^2\pi^2}{4}\varepsilon^2 \log\varepsilon + \frac{k^2\pi^2}{2}\left(\frac{1}{4} - \log 2\right)\varepsilon^2 + \mathcal{O}(\varepsilon^{2+\rho}), \quad \rho \in (0,1).$$

In some sense, the purpose of the analysis that we shall carry out in what follows is to show that the above result may actually be extended to higher dimensions. It should be noted here that this expansion depends on the relation between the different variables at the endpoints of the segment, which in this case is of the form $x_1^2 + \varepsilon^2 x_2^2 = 1$. Clearly different relations between the leading powers will lead to different expansions.

More generally, the issue is that the points of the boundary of Ω where there is a tangent in the direction along which the domain is being flattened will play a special role. Throughout the paper we assume this set of points to be contained in a hyperplane orthogonal to the scaling direction, and that this tangency is simple. In the vicinity of these points we take the cross-section of our surface as indicated in Figure 1 which, with the assumptions made, will be similar to the one-dimensional ellipse described above. Our results then state that in the higher-dimensional case the asymptotics for the eigenvalues still behave in a similar fashion and thus the logarithmic terms appearing above persist in this more general setting.

Apart from the intrinsic interest of the behavior of the spectrum close to double-sided flat domains, we point out that such manifolds have appeared in the literature in connection with eigenvalues as maximizers of the invariant eigenvalues among all surfaces isometric to surfaces of revolution in $\mathbb{R}^3$ [Abreu and Freitas 2002] and for hypersurfaces of revolution diffeomorphic to a sphere and isometrically embedded

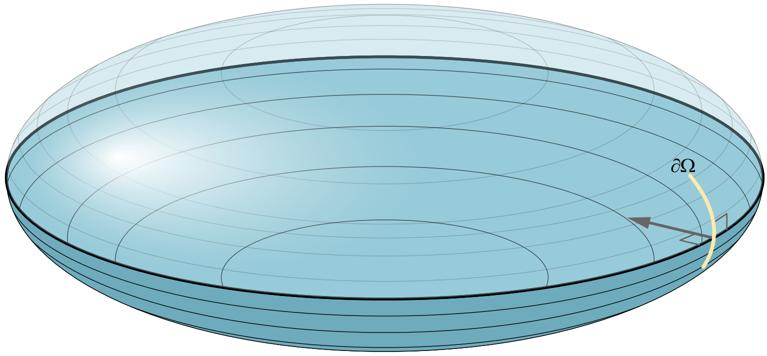

Figure 1. Surface $\mathcal{S}_\varepsilon$ with a cross-section at the edge.

in $\mathbb{R}^{n+1}$ [Colbois et al. 2008]. In fact, it is shown in those papers that these optimal singular *double flat disks* maximize the whole invariant spectrum and not just a specific eigenvalue. Another source of interest for such asymptotic expansions lies with the fact that, in some cases, they turn out to be fairly good approximations for low eigenvalues also for values of the parameter ε away from zero — see [Borisov and Freitas 2009; 2010; Freitas 2007].

We remark in passing that another problem for which it is conjectured that the optimal shape is given by a double-sided flat disk is Alexandrov's conjecture relating the area and diameter of surfaces of nonnegative curvature.

The structure of the paper is as follows. In the next section we give a precise formulation of the problem under consideration and state our main results, namely, the nature of the limiting problem and the relation of the limit and approximating operators. This includes the form of the asymptotic expansion and the expressions for the first three coefficients and an application to the case of the surface of an ellipsoid. Section 3 is then devoted to several preliminaries and auxiliary material used in Sections 4 and 5, where the proofs of the main results are presented.

2. Problem formulation and main results

Let $x' = (x_1, \dots, x_n)$, $x = (x', x_{n+1})$ be Cartesian coordinates in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively, $n \geqslant 2$, and let ω be a bounded domain in $\mathbb{R}^n$ with infinitely smooth boundary. Let also $h_\pm = h_\pm(x') \in C^\infty(\omega) \cap C(\overline{\omega})$ denote two arbitrary functions and define the manifold

$$\mathcal{S}_\varepsilon := \{x : x' \in \overline{\omega}, x_{n+1} = \varepsilon h_+(x')\} \cup \{x : x' \in \overline{\omega}, x_{n+1} = -\varepsilon h_-(x')\}, \tag{2-1}$$

where ε is a small positive parameter. We assume $\mathcal{S}_\varepsilon$ to be infinitely differentiable and to have no self-intersections. To ensure this, we make the following assumptions on $h_\pm$, the first of which ensures the absence of self-intersections:

(A1) The following relations hold true:

$$h_+(x') + h_-(x') > 0, \quad x' \in \omega, \qquad h_+(x') = h_-(x') = 0, \quad x' \in \partial\omega.$$

To state the second assumption we need to introduce some additional notation. Let $\nu = \nu(P)$, $P \in \partial\omega$, be the inward normal to $\partial\omega$, and denote by τ the distance to a point measured in the direction of ν. Consider the equations

$$t = h_+(P + \tau\nu(P)), \quad t > 0, \qquad t = -h_-(P + \tau\nu(P)), \quad t < 0. \tag{2-2}$$

Our second assumption concerns the solvability of these equations with respect to τ and implies the smoothness of $\mathcal{S}_\varepsilon$ in a neighborhood of $\partial\omega$:

(A2) There exists $t_0 > 0$ such that for all $t \in [-t_0, t_0]$, $P \in \partial\omega$, the equations (2-2) have a unique solution given by

$$\tau = a(t, P) \in C^\infty([-t_0, t_0] \times \partial\omega),$$

such that

$$\frac{\partial^2 a}{\partial t^2} > 0 \quad \text{for all } P \in \partial\omega. \tag{2-3}$$

We observe that assumptions (A1) and (A2) imply that

$$h_+(x') \geqslant 0, \quad h_-(x') \leqslant 0 \quad \text{in a small neighborhood of } \partial\omega.$$

The main object of our study is the Laplace–Beltrami operator $\mathcal{H}_\varepsilon$ on $\mathcal{S}_\varepsilon$. We introduce it rigorously as the self-adjoint operator associated with a symmetric lower-semibounded sesquilinear form

$$\mathfrak{h}_\varepsilon[u, v] := (\nabla u, \nabla v)_{L_2(\mathcal{S}_\varepsilon)} \quad \text{on } W_2^1(\mathcal{S}_\varepsilon).$$

We recall that on an arbitrary manifold with metric tensor g this may be written in local coordinates $y = (y_1, \dots, y_n)$ as

$$-\det{}^{-\frac{1}{2}} g \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i} g^{ij} \det{}^{\frac{1}{2}} g \frac{\partial}{\partial y_j},$$

where g^{ij} are the entries of the inverse to the metric tensor. If in our case we take x' as local coordinates on $\mathcal{S}_\varepsilon$, then on each side $\mathcal{S}_\varepsilon^\pm$ the operator $\mathcal{H}_\varepsilon$ may be written in the form

$$\mathcal{H}_\varepsilon = -(1 + \varepsilon^2|\nabla_{x'} h_\pm|^2)^{-\frac{1}{2}} \operatorname{div}_{x'}(1 + \varepsilon^2|\nabla_{x'} h_\pm|^2)^{\frac{1}{2}} (\mathrm{E} + \varepsilon^2 \mathrm{Q}_\pm)^{-1} \nabla_{x'}, \tag{2-4}$$

where E is the $n \times n$ identity matrix and $\mathrm{Q}_\pm$ is the matrix with entries $\frac{\partial h_\pm}{\partial x_i} \frac{\partial h_\pm}{\partial x_j}$. On the boundary $\partial\omega$ the coefficients of such operator have singularities, and this is why in a neighborhood of $\partial\omega$ it is more convenient to employ the coordinates (τ, s), where s are some local coordinates on $\partial\omega$. We do not give here the expression of the operator $\mathcal{H}_\varepsilon$ in such coordinates, as it requires the introduction of additional (cumbersome) notation. These two parametrizations are discussed in detail in Section 3.

The purpose of the present paper is to describe the asymptotic behavior of the resolvent and the spectrum of $\mathcal{H}_\varepsilon$ as $\varepsilon \to +0$. In this limit, the hypersurface $\mathcal{S}_\varepsilon$ collapses to a flat two-sided domain $\boldsymbol{\omega} = (\omega_+, \omega_-)$, where $\omega_\pm$ are two copies of ω understood as the *upper* and *lower* sides of $\boldsymbol{\omega}$. Because of this, it is natural to expect that the limiting operator for $\mathcal{H}_\varepsilon$ as $\varepsilon \to +0$ is the Laplacian on $\boldsymbol{\omega}$, i.e., that on $\omega_\pm$ subject to certain boundary conditions. Indeed, this is true, and it is our first main result. Namely, we introduce the space $L_2(\boldsymbol{\omega})$ as consisting of the vectors $\boldsymbol{u} = (u_+, u_-)$, where the functions $u_\pm$ are defined on $\omega_\pm$ and $u_\pm \in L_2(\omega_\pm)$. We can naturally identify $L_2(\boldsymbol{\omega})$ with $L_2(\omega) \oplus L_2(\omega)$. In the same way we introduce the Sobolev spaces $W_2^j(\boldsymbol{\omega})$ assuming that for each $\boldsymbol{u} \in W_2^j(\boldsymbol{\omega})$ the functions $u_\pm \in W_2^j(\omega_\pm)$ satisfy the boundary conditions

$$\frac{\partial^i u_+}{\partial \tau^i}\bigg|_{\partial\omega} = (-1)^i \frac{\partial^i u_-}{\partial \tau^i}\bigg|_{\partial\omega}, \quad i = 0, 1, \ldots, j-1. \tag{2-5}$$

The meaning of these boundary conditions is that the functions $u_\pm$ should be "glued smoothly" while moving from ω_+ to ω_- via $\partial\omega = \partial\omega_\pm$. We observe that $W_2^j(\boldsymbol{\omega})$ is embedded into $W_2^j(\omega) \oplus W_2^j(\omega)$, but does not coincide. It is also clear that for any $u \in W_2^1(\omega)$ the function $\boldsymbol{u} := (u, u)$ belongs to $W_2^1(\boldsymbol{\omega})$. Similarly, if $u \in W_2^2(\omega)$, $u|_{\partial\omega} = 0$, or, respectively, $u \in W_2^2(\omega)$, $\frac{\partial u}{\partial \tau}|_{\partial\omega} = 0$, then $\boldsymbol{u} = (u, -u) \in W_2^2(\boldsymbol{\omega})$, or, respectively, $\boldsymbol{u} = (u, u) \in W_2^2(\boldsymbol{\omega})$.

Let $\mathcal{H}_0$ be the self-adjoint operator in $L_2(\boldsymbol{\omega})$ associated with the closed symmetric lower-semibounded sesquilinear form

$$\mathfrak{h}_0[\boldsymbol{u}, \boldsymbol{v}] := (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{L_2(\boldsymbol{\omega})} \quad \text{on } W_2^1(\boldsymbol{\omega}).$$

By $\mathfrak{D}(\cdot)$ we denote the domain of an operator, and the symbol $\|\cdot\|_{X\to Y}$ indicates the norm of an operator acting from the Hilbert space X to a Hilbert space Y.

Given any vector $\boldsymbol{u} = (u_+, u_-)$ defined on $\boldsymbol{\omega}$, by $\mathcal{I}_\varepsilon \boldsymbol{u}$ we denote the function on $\mathcal{S}_\varepsilon$ being $u_+(x')$ on $\{x : x' \in \overline{\omega}, x_{n+1} = \varepsilon h_+(x')\}$ and $u_-(x')$ on $\{x : x' \in \overline{\omega}, x_{n+1} = -\varepsilon h_-(x')\}$. And vice versa, given any function u defined on $\mathcal{S}_\varepsilon$, by $\mathcal{I}_\varepsilon^{-1} u$ we denote the vector $\boldsymbol{u} = (u_+, u_-)$, where $u_\pm = u_\pm(x') := u(x')$, $x' \in \omega$, $x_{n+1} = \varepsilon h_\pm(x')$.

Theorem 2.1. *For each $z \in \mathbb{C} \setminus \mathbb{R}$ there exists $C(z) > 0$ such that the following estimate holds true*:

$$\left\|(\mathcal{H}_\varepsilon - z)^{-1} - \mathcal{I}_\varepsilon (\mathcal{H}_0 - z)^{-1} \mathcal{I}_\varepsilon^{-1}\right\|_{L_2(\mathcal{S}_\varepsilon)\to W_2^1(\mathcal{S}_\varepsilon)} \leqslant C(z)\varepsilon^{2/3}. \tag{2-6}$$

Remark 2.2. The statement of this theorem includes the fact that the operator $\mathcal{I}_\varepsilon(\mathcal{H}_0 - z)^{-1}\mathcal{I}_\varepsilon^{-1}$ is well-defined as a bounded one from $L_2(\mathcal{S}_\varepsilon)$ into $W_2^1(\mathcal{S}_\varepsilon)$.

In view of the embedding of $W_2^1(\boldsymbol{\omega})$ into $W_2^1(\omega) \oplus W_2^1(\omega)$, and the compact embedding of the latter into $L_2(\omega) \oplus L_2(\omega) = L_2(\boldsymbol{\omega})$, the operator $\mathcal{H}_\varepsilon$ has a compact resolvent. Hence, it has a pure discrete spectrum accumulating only at infinity. The same is true for the Dirichlet and Neumann Laplacians $-\Delta_\omega^{(D)}$ and $-\Delta_\omega^{(N)}$ on ω. Recall that $-\Delta_\omega^{(D)}$ is the Friedrichs extension in $L_2(\omega)$ of $-\Delta$ from $C_0^\infty(\Omega)$, and $-\Delta_\omega^{(N)}$ is the self-adjoint operator in $L_2(\omega)$ associated with the sesquilinear form $(\nabla u, \nabla v)_{L_2(\Omega)}$ on $W_2^1(\omega)$. In what follows $\sigma_d(\cdot)$ denotes the discrete spectrum of an operator.

Our next result follows from Theorem 2.1 and [Reed and Simon 1980, Theorems VIII.23, VIII.24].

Theorem 2.3. *The eigenvalues of $\mathcal{H}_\varepsilon$ converge to those of $\mathcal{H}_0$ as ε goes to zero. In particular, if $\lambda \notin \sigma_d(\mathcal{H}_0)$, then $\lambda \notin \sigma_d(\mathcal{H}_\varepsilon)$ for ε small enough. For each m-multiple eigenvalue $\lambda \in \sigma_d(\mathcal{H}_0)$ there exist exactly m eigenvalues (counting multiplicities) of $\mathcal{H}_\varepsilon$ converging to λ as $\varepsilon \to +0$. Let $\mathcal{P}_0$ be the projector on the eigenspace associated with λ, $\mathcal{P}_\varepsilon$ be the total projector associated with the eigenvalues* *of $\mathscr{H}_\varepsilon$ converging to λ. Then the following convergence holds true*:

$$\|\mathscr{P}_\varepsilon - \mathscr{I}_\varepsilon \mathscr{P}_0 \mathscr{I}_\varepsilon^{-1}\|_{L_2(\mathscr{S}_\varepsilon)\to W_2^1(\mathscr{S}_\varepsilon)} \to 0, \quad \varepsilon \to +0.$$

Let now λ be an eigenvalue of $\mathscr{H}_0$ with multiplicity m and $\boldsymbol{\psi}_i = (\psi_+^{(i)}, \psi_-^{(i)})$ be associated eigenfunctions orthonormalized in $L_2(\boldsymbol{\omega})$. It will be shown in the next section in Lemma 4.2 that the asymptotics

$$\psi_\pm^{(i)}(x') = \Psi_i^{(0)}(P) \pm \Psi_i^{(1)}(P)\tau + \mathscr{O}(\tau^2), \quad P \in \partial\omega, \quad \tau \to +0, \tag{2-7}$$

hold true, where

$$\Psi_i^{(0)} = \psi_+^{(i)}\big|_{\partial\omega} = \psi_-^{(i)}\big|_{\partial\omega} \in C^\infty(\partial\omega), \quad \Psi_i^{(1)} = \frac{\partial \psi_+^{(i)}}{\partial\tau}\bigg|_{\partial\omega} = -\frac{\partial \psi_-^{(i)}}{\partial\tau}\bigg|_{\partial\omega} \in C^\infty(\partial\omega).$$

By $-\Delta_{\partial\omega}$ we denote the Laplace–Beltrami operator on $\partial\omega$, where the metric $\mathrm{G}_{\partial\omega}$ on $\partial\omega$ is induced by the Euclidean one in $\mathbb{R}^n$. For any smooth functions u, v on $\partial\omega$, we shall denote the pointwise scalar product of their gradients by $\nabla u \cdot \nabla v$.

Let

$$\omega^\delta := \omega \setminus \{x' : 0 < \tau < \delta\}. \tag{2-8}$$

Employing the coefficients of the asymptotics (2-7), we introduce two real symmetric matrices $\Lambda^{(0)}$, $\Lambda^{(1)}$ with entries

$$\Lambda_{ij}^{(0)} := \int_{\partial\omega} \frac{1}{a_2}\big(\lambda \Psi_i^{(0)}\Psi_j^{(0)} - \nabla\Psi_i^{(0)}\cdot\nabla\Psi_j^{(0)} + \Psi_i^{(1)}\Psi_j^{(1)}\big)\, d\omega, \tag{2-9}$$

$$\begin{aligned}
\Lambda_{ij}^{(1)} := -\lim_{\delta\to+0}\bigg[&\frac{1}{2}\int_{\omega^\delta} |\nabla_{x'}h_+|^2\big(\lambda\psi_+^{(i)}\psi_+^{(j)} - (\nabla_{x'}\psi_+^{(i)}, \nabla_{x'}\psi_+^{(j)})_{\mathbb{R}^d}\big)\, dx' \\
&+ \frac{1}{2}\int_{\omega^\delta} |\nabla_{x'}h_-|^2\big(\lambda\psi_-^{(i)}\psi_-^{(j)} - (\nabla_{x'}\psi_-^{(i)}, \nabla_{x'}\psi_-^{(j)})_{\mathbb{R}^d}\big)\, dx' \\
&+ \int_{\omega^\delta} (\nabla_{x'}h_+, \nabla_{x'}\psi_+^{(i)})_{\mathbb{R}^d}(\nabla_{x'}h_+, \nabla_{x'}\psi_+^{(j)})_{\mathbb{R}^d}\, dx' \\
&+ \int_{\omega^\delta} (\nabla_{x'}h_-, \nabla_{x'}\psi_-^{(i)})_{\mathbb{R}^d}(\nabla_{x'}h_-, \nabla_{x'}\psi_-^{(j)})_{\mathbb{R}^d}\, dx' \\
&+ \ln\delta \int_{\partial\omega} \frac{1}{4a_2}\big(\Psi_i^{(1)}\Psi_j^{(1)} + \lambda\Psi_i^{(0)}\Psi_j^{(0)} - \nabla\Psi_i^{(0)}\cdot\nabla\Psi_j^{(0)}\big)\, ds\bigg] \\
- \int_{\partial\omega} \frac{1 + 4\ln 2 + \ln a_2}{4a_2}&\big(\Psi_i^{(1)}\Psi_j^{(1)} + \lambda\Psi_i^{(0)}\Psi_j^{(0)} - \nabla\Psi_i^{(0)}\cdot\nabla\Psi_j^{(0)}\big)\, ds,
\end{aligned} \tag{2-10}$$

where

$$a_2(P) := \frac{1}{2}\frac{\partial^2 a}{\partial t^2}(0, P).$$

It will be shown in Section 4 that the matrix $\Lambda^{(1)}$ is well-defined. By the theorem on simultaneous diagonalization of two quadratic forms, in what follows the eigenfunctions $\boldsymbol{\psi}_i$ are supposed to be orthonormalized in $L_2(\boldsymbol{\omega})$ and the matrix $\Lambda^{(0)} + \frac{1}{\ln\varepsilon}\Lambda^{(1)}$ to be diagonal. The eigenfunctions $\boldsymbol{\psi}_i$ chosen in this way depend on ε, but it is clear that the norms $\|\psi_{\pm}^{(i)}\|_{C^k(\overline{\omega})}$ are bounded uniformly in ε for all $k \geqslant 0$, $i = 1, \ldots, m$.

Theorem 2.4. *Let λ be an m-multiple eigenvalue of $\mathcal{H}_0$ and $\boldsymbol{\psi}_i$, $i = 1, \ldots, m$, be the associated eigenfunctions of $\mathcal{H}_0$ chosen as described above. Then there exist exactly m eigenvalues $\lambda_k(\varepsilon)$, $k = 1, \ldots, m$ (counting multiplicity) of $\mathcal{H}_\varepsilon$ converging to λ. These eigenvalues satisfy the asymptotic expansions*

$$\lambda_k(\varepsilon) = \lambda + \varepsilon^2 \ln \varepsilon \, \mu_k\left(\frac{1}{\ln \varepsilon}\right) + \mathcal{O}(\varepsilon^{2+\rho}), \tag{2-11}$$

where μ_k are the eigenvalues of the matrix $\Lambda^{(0)} + \frac{1}{\ln \varepsilon}\Lambda^{(1)}$, and ρ is any constant in $(0, 1/2)$. The eigenvalues $\mu_k\left(\frac{1}{\ln \varepsilon}\right)$ are holomorphic in $\frac{1}{\ln \varepsilon}$ and converge to the eigenvalues of $\Lambda^{(0)}$ as $\varepsilon \to 0$.

In addition to the asymptotic expansions for the eigenvalues $\lambda_i(\varepsilon)$ given in this theorem, we also obtain the asymptotics for the total projector associated with these eigenvalues. However, to formulate this result we have to introduce additional notation and it is thus more convenient to postpone its statement which will then be made at the end of Section 5 — see Theorem 5.3.

Let us describe briefly the main ideas employed in the proofs of the main results. The proof of the uniform resolvent convergence in Theorem 2.1 is based on the analysis of the quadratic forms associated with the perturbed and the limiting operators and on the accurate estimates of the functions in certain weighted Sobolev spaces. The proof of the first theorem uses essentially the method of matching asymptotic expansions [Il′in 1992] for formal construction of the asymptotics for the eigenfunctions associated with $\lambda_k(\varepsilon)$. These asymptotics are constructed as a combination of outer and inner expansions. The former depends on x' and its coefficients have singularities at $\partial\omega$. In the vicinity of $\partial\omega$ we introduce a special rescaled variable $\xi := a^{1/2}(x_{n+1}\varepsilon^{-1}, P)\varepsilon^{-1}$ as $x_{n+1} > 0$ and $\xi := -a^{1/2}(x_{n+1}\varepsilon^{-1}, P)\varepsilon^{-1}$ as $x_{n+1} < 0$. This variable then describes the slope of $\mathcal{S}_\varepsilon$ in the vicinity of ε — see also the equations (3-11) giving the parametrization of $\mathcal{S}_\varepsilon$ in the vicinity of $\partial\omega$. After rewriting the eigenvalue equation in the variables (ξ, s), where s are local coordinates on $\partial\omega$, its leading term is in fact the Laplace–Beltrami operator on the ellipse giving rise to the logarithmic terms in the asymptotics for both the eigenvalues and the eigenfunctions.

Despite the fact that we are only presenting the leading terms of the asymptotics for $\lambda_k(\varepsilon)$ and for the associated total projector in Theorems 2.4 and 5.3, respectively, our approach also allows us to construct the complete asymptotic expansions if required. Although this would need to be checked in a way similar to what was done here for the first few terms, the ansatzes (5-1) and (5-39) suggest that the complete asymptotic expansion for the eigenvalues should be

$$\lambda_k(\varepsilon) = \lambda + \varepsilon^2 \ln \varepsilon \mu_k(\varepsilon) + \sum_{i=2}^{\infty} \varepsilon^{2i} \ln^i \varepsilon \mu_k^{(i)}\left(\frac{1}{\ln \varepsilon}\right),$$

where $\mu_k^{(i)}$ are functions holomorphic in $\frac{1}{\ln \varepsilon}$. These higher-order terms would then still reflect the behavior observed in the ellipse example given in the Introduction.

Although the above formulas for $\Lambda_{ij}^{(0)}$ and (especially) $\Lambda_{ij}^{(1)}$ may look quite cumbersome at a first glance, they will actually simplify when computed for particular cases as some of the terms involved will vanish depending on whether we are considering Dirichlet or Neumann boundary conditions on $\partial\omega$. We note that a similar effect was already present when computing the coefficients in the expansions obtained in [Borisov and Freitas 2009; 2010]. This is particularly clear in the second of these papers dealing with dimensions higher than two, where the general expression is quite complicated and needs to be computed specifically in each case. When this is done for general ellipsoids in any dimension, for instance, it yields a much simpler one-line expression.

We shall illustrate this by considering a thin ellipsoidal surface. To this end take ω to be the unit disk centered at the origin with

$$h_\pm(x') := \sqrt{1-r^2}, \quad r = |x'|, \quad \tau = 1-r, \quad a_2 = \tfrac{1}{2}. \tag{2-12}$$

Under such definition this surface converges to the unit disk ω regarded as a double-sided surface. In this instance the limiting eigenvalues may be found via separation of variables and they will be of the form κ^2, where κ are the zeroes of the Bessel function J_κ and its derivative J_κ', corresponding to eigenfunctions satisfying Dirichlet and Neumann boundary conditions on $\partial\omega$, respectively. The following examples illustrating both cases are taken from [Borisov and Freitas 2012], where the details may be found.

We consider the case of Dirichlet boundary conditions first; i.e.,

$$J_0(\kappa) = 0, \quad \lambda = \kappa^2, \quad \psi(x) = -\frac{J_0(\kappa r)}{\sqrt{2\pi}\,J_1(\kappa)}, \quad \boldsymbol{\psi} = (\psi, -\psi), \quad \Psi^{(0)} = 0, \quad \Psi^{(1)} = -\frac{\kappa}{\sqrt{2\pi}}.$$

Substituting these formulas and (2-12) into (2-9) and (2-10), we then obtain

$$\Lambda_{11}^{(0)} = 2\lambda \quad \text{and} \quad \Lambda_{11}^{(1)} = -\frac{\lambda}{J_1^2(\kappa)} \int_0^1 \frac{r^3}{1-r^2}\big(J_0^2(\kappa r) + J_1^2(\kappa r) - J_1^2(\kappa)\big)\,dr - \lambda \ln 2.$$

The asymptotics (2-11) thus become

$$\lambda_\kappa(\varepsilon) = \lambda + \varepsilon^2(2\lambda \ln\varepsilon + \Lambda_{11}^{(1)}) + \mathcal{O}(\varepsilon^{2+\rho})$$

and, for a particular eigenvalue, the remaining integral may be computed numerically. We illustrate this by considering the case corresponding to the first Dirichlet eigenvalue on the disk which yields

$$\lambda_1(\varepsilon) = j_{0,1}^2 + \varepsilon^2(2j_{0,1}^2 \ln\varepsilon + \Lambda_{11}^{(1)}) + \mathcal{O}(\varepsilon^{2+\rho}) \approx 5.7831 + 11.5664\,\varepsilon^2 \ln\varepsilon - 6.0871\,\varepsilon^2 + \mathcal{O}(\varepsilon^{2+\rho}).$$

As an example of a limiting multiple eigenvalue we consider the first nontrivial Neumann eigenvalue of the disk. In two dimensions this is a double eigenvalue with associated (normalized) eigenfunctions

$$\psi_1(x) = \frac{J_1(\kappa' r)\cos\theta}{J_0(\kappa')\sqrt{\pi(\kappa'^2-1)}}, \quad \psi_2(x) = \frac{J_1(\kappa' r)\sin\theta}{J_0(\kappa')\sqrt{\pi(\kappa'^2-1)}},$$

where θ is the polar angle corresponding to x and κ' is the first nontrivial zero of J_1'.

The eigenfunctions in $L_2(\boldsymbol{\omega})$ are then given by $\boldsymbol{\psi}_i = (\psi_i, \psi_i)$, $i = 1, 2$, from which we have

$$\Psi_1^{(0)} = \frac{J_1(\kappa')\cos\theta}{J_0(\kappa')\sqrt{\pi(\kappa'^2-1)}}, \quad \Psi_2^{(0)} = \frac{J_1(\kappa')\sin\theta}{J_0(\kappa')\sqrt{\pi(\kappa'^2-1)}}$$

and $\Psi_i^{(1)}=0$, $i=1,2$. Proceeding as before, we have

$$\Lambda_{11}^0=\Lambda_{22}^0=\frac{2J_1^2(\kappa')}{J_0^2(\kappa')}=2\kappa'^2=2\lambda \quad \text{and} \quad \Lambda_{ij}^0=0 \quad (i\neq j).$$

For the next term we now obtain

$$\Lambda_{ii}^{(1)}=-\frac{\kappa'^2}{J_0^2(\kappa')(\kappa'^2-1)}\int_0^1\frac{r^3}{1-r^2}\Big[J_1^2(\kappa'r)-J_1^2(\kappa')+J_0^2(\kappa'r)+J_0^2(\kappa')-\frac{2}{\kappa'r}J_0(\kappa'r)J_1(\kappa'r)\Big]\,dr-\lambda\ln 2$$

for $i=1,2$ and $\Lambda_{ij}=0$ for $i\neq j$.

From this, and again computing the relevant integrals numerically, we obtain

$$\lambda_i(\varepsilon)=(j'_{1,1})^2+\varepsilon^2(2\lambda\ln\varepsilon+\Lambda_{ii}^{(1)})+\mathcal{O}(\varepsilon^{2+\rho})\approx 3.3900+6.7799\,\varepsilon^2\ln\varepsilon-1.8555\,\varepsilon^2+\mathcal{O}(\varepsilon^{2+\rho}),\quad i=1,2.$$

Due to the radial symmetry of ω, it is clear that these two eigenvalues should coincide, and the associated eigenfunctions converge to $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$.

3. Preliminaries

In this section we discuss two parametrizations of the surface $\mathcal{S}_\varepsilon$ and prove three auxiliary lemmas which will be used in the next sections for proving Theorems 2.1, 2.4.

First parametrization of $\mathcal{S}_\varepsilon$. The first parametrization is that used in the definition of $\mathcal{S}_\varepsilon$ in (2-1); i.e., each point on $\mathcal{S}_\varepsilon$ is described as $x_{n+1}=\pm\varepsilon h_\pm(x')$, $x'\in\overline{\omega}$, where the sign corresponds to the upper or lower part of $\mathcal{S}_\varepsilon$. Let us first calculate the metrics on $\mathcal{S}_\varepsilon$ in terms of the variables x'.

The tangential vectors to $\mathcal{S}_\varepsilon$ at the point $x'\in\omega$, $x_{n+1}=\varepsilon h_\pm(x')$ are

$$\left(0,\dots,0,1,0,\dots,0,\varepsilon\frac{\partial h_\pm}{\partial x_i}\right),\quad i=1,\dots,n,$$

where "1" stands on i-th position. Thus, the metric tensor has the form

$$\mathrm{G}_\pm(x',\varepsilon):=\begin{pmatrix} 1+\varepsilon^2\big(\frac{\partial h_\pm}{\partial x_1}\big)^2 & \varepsilon^2\frac{\partial h_\pm}{\partial x_1}\frac{\partial h_\pm}{\partial x_2} & \varepsilon^2\frac{\partial h_\pm}{\partial x_1}\frac{\partial h_\pm}{\partial x_3} & \dots & \varepsilon^2\frac{\partial h_\pm}{\partial x_1}\frac{\partial h_\pm}{\partial x_n}\\ \varepsilon^2\frac{\partial h_\pm}{\partial x_2}\frac{\partial h_\pm}{\partial x_1} & 1+\varepsilon^2\big(\frac{\partial h_\pm}{\partial x_2}\big)^2 & \varepsilon^2\frac{\partial h_\pm}{\partial x_2}\frac{\partial h_\pm}{\partial x_3} & \dots & \varepsilon^2\frac{\partial h_\pm}{\partial x_2}\frac{\partial h_\pm}{\partial x_n}\\ \varepsilon^2\frac{\partial h_\pm}{\partial x_3}\frac{\partial h_\pm}{\partial x_1} & \varepsilon^2\frac{\partial h_\pm}{\partial x_3}\frac{\partial h_\pm}{\partial x_2} & 1+\varepsilon^2\big(\frac{\partial h_\pm}{\partial x_3}\big)^2 & \dots & \varepsilon^2\frac{\partial h_\pm}{\partial x_3}\frac{\partial h_\pm}{\partial x_n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \varepsilon^2\frac{\partial h_\pm}{\partial x_n}\frac{\partial h_\pm}{\partial x_1} & \varepsilon^2\frac{\partial h_\pm}{\partial x_n}\frac{\partial h_\pm}{\partial x_2} & \varepsilon^2\frac{\partial h_\pm}{\partial x_{n-1}}\frac{\partial h_\pm}{\partial x_3} & \dots & 1+\varepsilon^2\big(\frac{\partial h_\pm}{\partial x_n}\big)^2 \end{pmatrix}.$$

It easy to see that

$$\mathrm{G}_\pm(x',\varepsilon)=\mathrm{E}+\varepsilon^2\mathrm{Q}_\pm,\quad \mathrm{Q}_\pm:=(\nabla_{x'}h_\pm)(\nabla_{x'}h_\pm)^*, \tag{3-1}$$

where $\nabla_{x'}h_\pm$ is treated as a column vector, and "$*$" denotes transposition.

Lemma 3.1. *The matrix* $G_\pm$ *has two eigenvalues, the* $(n-1)$*-multiple eigenvalue* 1, *and the simple eigenvalue* $(1+\varepsilon^2|\nabla_{x'}h_\pm|^2)$. *The following identity holds true*:

$$d\,\mathcal{S}_\varepsilon = J_\varepsilon^\pm\,dx', \quad J_\varepsilon^\pm := \sqrt{1+\varepsilon^2|\nabla_{x'}h_\pm|^2}, \quad dx' = dx_1\,dx_2\cdots dx_n. \tag{3-2}$$

Proof. From (3-1) we may write the eigenvalue problem for the matrix $G_\pm$ as

$$(\mathrm{E}+\varepsilon^2 vv^*)u = zu \quad \text{and} \quad (z-1)u = \varepsilon^2 vv^*u,$$

where $v=\nabla_{x'}h_\pm$. We thus see that any vector orthogonal to v is an eigenvector for the above equation with eigenvalue z equal to one. This yields an eigenvalue of multiplicity $n-1$ if v is not zero, and n in case v vanishes. In the former case, we easily see that v is also an eigenvector, now with eigenvalue $1+\varepsilon^2|v|^2$, which will have multiplicity one. The determinant of $G_\pm$ is thus $g^\pm = 1+\varepsilon^2|v|^2$, yielding the volume element to be $\sqrt{1+\varepsilon^2|v|^2}$ as desired. □

In what follows we shall make use of the differential expression for the operator $\mathcal{H}_\varepsilon$, namely, its expansion with respect to ε. The expression itself is given by (2-4), while using (3-1) allows us to expand some of the terms in this expression in powers of ε:

$$(\mathrm{E}+\varepsilon^2\mathrm{Q}_\pm)^{-1} = \mathrm{E}-\varepsilon^2\mathrm{Q}_\pm+\mathcal{O}(\varepsilon^4), \quad (1+\varepsilon^2|\nabla_{x'}h_\pm|^2)^{\pm\frac{1}{2}} = 1\pm\varepsilon^2\frac{|\nabla_{x'}h_\pm|^2}{2}+\mathcal{O}(\varepsilon^4),$$

where the plus and minus signs correspond to the upper and lower parts of S_ε, respectively. We substitute these formulas into (2-4) and get

$$\mathcal{H}_\varepsilon = -\Delta_{x'} - \varepsilon^2\left(\frac{|\nabla_{x'}h_\pm|^2}{2}\Delta_{x'} + \operatorname{div}_{x'}\left(\frac{|\nabla_{x'}h_\pm|^2}{2}-\mathrm{Q}_\pm\right)\nabla_{x'}\right)+\mathcal{O}(\varepsilon^4). \tag{3-3}$$

The disadvantage of the parametrization by the variables x' is that the functions $h_\pm$ are not smooth in a vicinity of $\partial\omega$ and their derivatives blow up at the boundary $\partial\omega$. We shall show this below while introducing the second parametrization. The main idea of the second parametrization is to use special coordinates in a vicinity of $\partial\omega$ so that they involve smooth functions only; this parametrization is purely local and will be used only in a vicinity of $\partial\omega$. It is natural to expect the existence of such coordinates since the surface $\mathcal{S}_\varepsilon$ is infinitely differentiable.

Second parametrization of $\mathcal{S}_\varepsilon$**.** In a neighborhood of $\partial\omega$ we introduce new coordinates (τ,s), where $s=(s_1,\dots,s_{n-1})$ are local coordinates on $\partial\omega$ corresponding to a C^∞-atlas, and τ, we remind, is the distance to a point measured in the direction of the inward normal $\nu=\nu(s)$ to $\partial\omega$. Let $r=r(s)$ be the vector-function describing $\partial\omega$. We have

$$x' = r(s)+\tau\nu(s), \quad \nabla_{(\tau,s)} = \mathrm{M}(\tau,s)\nabla_{x'}, \quad \mathrm{M}=\mathrm{M}(\tau,s) = \begin{pmatrix} \nu \\ \frac{\partial r}{\partial s_1}+\tau\frac{\partial\nu}{\partial s_1} \\ \vdots \\ \frac{\partial r}{\partial s_{n-1}}+\tau\frac{\partial\nu}{\partial s_{n-1}} \end{pmatrix}, \tag{3-4}$$

where $\nu(s)$ and the other vectors in the definition of M are treated as rows. The vectors $\frac{\partial r}{\partial s_i}$ are tangential to M and linearly independent, while $\nu(s)$ is orthogonal to $\partial\omega$. Thus, the matrix M is invertible for all sufficiently small τ and all $s \in \partial\omega$. The inequalities

$$C_1 \leqslant \mathrm{M}(\tau, s) \leqslant C_2, \quad C_2^{-1} \leqslant \mathrm{M}^{-1}(\tau, s) \leqslant C_1^{-1}, \quad s \in \partial\omega, \quad \tau \in [-\tau_0, \tau_0] \tag{3-5}$$

are valid, where C_1, C_2 are positive constants independent of (τ, s). It follows from these estimates and (3-4) that the matrix $M^{-1}(\tau, s)$ is infinitely differentiable in the neighborhood $\{x : |\tau| < \tau_0\}$ of $\partial\omega$.

Consider now the equations (2-2). By assumption (A2) they have the smooth solution $\tau = a(x_{n+1}, P)$ and, for small x_{n+1}, the function a behaves as

$$a(x_{n+1}, P) = a_2(P)x_{n+1}^2 + \mathcal{O}(x_{n+1}^3).$$

Hence,

$$h_\pm(P + \tau\nu(P)) = x_{n+1} = \pm a_2^{-\frac{1}{2}}(P)\tau^{\frac{1}{2}} + \mathcal{O}(\tau), \quad \tau \to +0, \qquad \nabla_{x'} h_\pm = \mathrm{M}^{-1}\nabla_{(\tau,s)} h_\pm,$$
$$C_3\tau^{-1} \leqslant |\nabla_{x'} h_\pm|^2 \leqslant C_4\tau^{-1}, \quad \tau \in (0, \tau_0], \tag{3-6}$$

where C_3, C_4 are positive constants independent of (τ, s). As we see from the last estimates, the functions $h_\pm$ are not smooth at the point $\tau = 0$, i.e., at $\partial\omega$.

We employ once again assumption (A2) and pass from the equations $x_{n+1} = \pm\varepsilon h_\pm(x')$ to

$$\tau = a(t, P), \quad x_{n+1} = \varepsilon t, \quad x' = r(s) + \tau\nu(s). \tag{3-7}$$

It follows from (2-3) that the function $a(t, P)$ can be represented as $t^2\widetilde{a}(t, P)$, where $\widetilde{a} \in C^\infty([-t_0, t_0] \times \partial\omega)$ and $\widetilde{a} > 0$ for sufficiently small t_0.

We introduce a new variable $\zeta = t\widetilde{a}^{\frac{1}{2}}(t, P)$. From assumption (A2) we conclude that

$$t = b(\zeta, P) \in C^\infty([-\zeta_0, \zeta_0] \times \partial\omega) \tag{3-8}$$

for a fixed small constant ζ_0, and the Taylor series for a and b read

$$a(t, P) = \sum_{i=2}^{\infty} a_i(P)t^i, \quad t \to +0, \tag{3-9}$$

$$b(\zeta, P) = \sum_{i=1}^{\infty} b_i(P)\zeta^i, \quad \zeta \to 0, \qquad b_1 := a_2^{-\frac{1}{2}}, \tag{3-10}$$

where a_i, $b_i \in C^\infty(\partial\omega)$. We define a rescaled variable $\xi := \zeta\varepsilon^{-1}$. The final form of the second parametrization for $\mathcal{S}_\varepsilon$ is

$$x' = r(s) + \varepsilon^2\xi^2\nu(s), \quad x_{n+1} = \varepsilon^2 b_\varepsilon(\xi, r(s)), \quad \xi \in [-\zeta_0\varepsilon^{-1}, \zeta_0\varepsilon^{-1}], \tag{3-11}$$

where $b_\varepsilon(\xi, P) := \varepsilon^{-1}b(\varepsilon\xi, P)$ and ζ_0 is a fixed sufficiently small number. We observe that, by the definition of ζ,

$$\tau = a(t, P) = \zeta^2 = \varepsilon^2\xi^2. \tag{3-12}$$

As in (3-3), we shall also employ the expansion in ε of the differential expression for $\mathcal{H}_\varepsilon$ corresponding to the second parametrization. We find first the tangential vectors to S_ε corresponding to the parametrization (3-11):

$$T_{s_i} = \left(\frac{\partial r}{\partial s_i} + \varepsilon^2\xi^2\frac{\partial \nu}{\partial s_i}, \varepsilon^2\frac{\partial b_\varepsilon}{\partial s_i}\right), \quad T_\xi = \varepsilon^2\left(2\xi\nu, \frac{\partial b_\varepsilon}{\partial \xi}\right). \tag{3-13}$$

It is clear that the vectors $\frac{\partial r}{\partial s_i}, \frac{\partial \nu}{\partial s_i}$ belong to the tangential plane and are orthogonal to ν. Employing this fact and (3-13), we calculate the metric tensor:

$$(T_\xi, T_\xi)_{\mathbb{R}^{n+1}} = \varepsilon^4\left(4\xi^2 + \left(\frac{\partial b_\varepsilon}{\partial \xi}\right)^2\right), \quad (T_\xi, T_{s_i})_{\mathbb{R}^{n+1}} = \varepsilon^4\frac{\partial b_\varepsilon}{\partial \xi}\frac{\partial b_\varepsilon}{\partial s_i},$$
$$(T_{s_i}, T_{s_j})_{\mathbb{R}^{n+1}} = \left(\frac{\partial r}{\partial s_i} + \varepsilon^2\xi^2\frac{\partial \nu}{\partial s_i}, \frac{\partial r}{\partial s_j} + \varepsilon^2\xi^2\frac{\partial \nu}{\partial s_j}\right)_{\mathbb{R}^{n+1}} + \varepsilon^4\frac{\partial b_\varepsilon}{\partial s_i}\frac{\partial b_\varepsilon}{\partial s_j}.$$

By the Weingarten equations we see that

$$((T_{s_i}, T_{s_j})_{\mathbb{R}^{n+1}})_{i,j=\overline{1,n}} = \mathrm{A},$$

where

$$\begin{aligned}\mathrm{A} &:= \mathrm{G}_{\partial\omega} - 2\varepsilon^2\xi^2\mathrm{B} + \varepsilon^4\xi^4\mathrm{B}\mathrm{G}_{\partial\omega}^{-1}\mathrm{B} + \varepsilon^4(\nabla_s b_\varepsilon)(\nabla_s b_\varepsilon)^* \\ &= \mathrm{G}_{\partial\omega}(\mathrm{E} - \varepsilon^2\xi^2\mathrm{G}_{\partial\omega}^{-1}\mathrm{B})^2 + \varepsilon^4(\nabla_s b_\varepsilon)(\nabla_s b_\varepsilon)^*,\end{aligned} \tag{3-14}$$

$\mathrm{G}_{\partial\omega}$ is the metric tensor of $\partial\omega$ associated with the coordinates s, B is the second fundamental form of $\partial\omega$ corresponding to the orientation defined by ν. Hence, the metric tensor G_ε of S_ε associated with the parametrization (3-11) reads

$$\mathrm{G}_\varepsilon = \begin{pmatrix} \varepsilon^4\left(4\xi^2 + \left(\frac{\partial b_\varepsilon}{\partial \xi}\right)^2\right) & \varepsilon^4\mathrm{p}^* \\ \varepsilon^4\mathrm{p} & \mathrm{A}\end{pmatrix}, \quad \mathrm{p} := \frac{\partial b_\varepsilon}{\partial \xi}\nabla_s b_\varepsilon.$$

By direct calculations we check that

$$\mathrm{G}_\varepsilon^{-1} = \begin{pmatrix} \varepsilon^{-4}\beta & -\beta\mathrm{p}^*\mathrm{A}^{-1} \\ -\beta\mathrm{A}^{-1}\mathrm{p} & \mathrm{A}^{-1} + \varepsilon^4\beta\mathrm{A}^{-1}\mathrm{p}\mathrm{p}^*\mathrm{A}^{-1}\end{pmatrix}, \quad \beta := \left(4\xi^2 + \left(\frac{\partial b_\varepsilon}{\partial \xi}\right)^2 - \varepsilon^4\mathrm{p}^*\mathrm{A}^{-1}\mathrm{p}\right)^{-1}. \tag{3-15}$$

The quantities in (3-15) are well-defined provided ζ_0 is sufficiently small. Indeed, by (3-9),

$$\mathrm{A} = \mathrm{G}_{\partial\omega} + \mathcal{O}(\zeta^2), \quad \mathrm{p} = \mathcal{O}(1), \quad \frac{\partial b}{\partial \zeta}(\zeta, P) = \mathcal{O}(1), \quad \zeta \to 0,$$

which implies the existence of A^{-1} and β. In what follows we assume that ζ_0 is chosen in such a way.

By $K_i = K_i(s)$, $i = 1, \dots, n-1$, we denote the principal curvatures of $\partial\omega$, and $K := \sum_{i=1}^{n-1} K_i$. We note that $(n-1)^{-1}K$ is the mean curvature of $\partial\omega$ and let

$$\mathrm{a} := \det\bigl((\mathrm{E} - \varepsilon^2\xi^2\mathrm{G}_{\partial\omega}^{-1}\mathrm{B})^2 + \varepsilon^4\mathrm{G}_{\partial\omega}^{-1}(\nabla_s b_\varepsilon)(\nabla_s b_\varepsilon)^*\bigr).$$

Lemma 3.2. *The following identities hold true*:

$$b_\varepsilon=\sum_{i=1}^{\infty} b_i(P)\varepsilon^{i-1}\xi^i,\quad \mathrm{A}^{-1}=\mathrm{G}_{\partial\omega}^{-1}+\mathcal{O}(\varepsilon^2\xi^2),\quad \mathrm{p}=\xi b_1\nabla_s b_1+\mathcal{O}(\varepsilon\xi^2), \tag{3-16}$$

$$\det \mathrm{G}_\varepsilon=\varepsilon^4\beta^{-1}\det\mathrm{A}, \tag{3-17}$$

$$\det\mathrm{A}=\mathrm{a}\det\mathrm{G}_{\partial\omega},\quad \mathrm{a}=\sum_{i=0}^{2}\varepsilon^{2i}\alpha_{2i}+\mathcal{O}(\varepsilon^4\xi^4), \tag{3-18}$$

$$\alpha_0:=1,\quad \alpha_2:=-2\xi^2K. \tag{3-19}$$

Proof. The identities (3-16) follow directly from the definitions of b_ε, A, and p.

We make linear transformations in (3-15) to calculate the determinant of G_ε:

$$(\det\mathrm{G}_\varepsilon)^{-1}=\det{}^{-1}\mathrm{G}_\varepsilon=\begin{vmatrix}\varepsilon^{-4}\beta & -\beta\mathrm{p}^*\mathrm{A}^{-1}\\ 0 & \mathrm{A}^{-1}\end{vmatrix}=\varepsilon^{-4}\beta\det{}^{-1}\mathrm{A},$$

which proves (3-17).

It is easy to see that

$$\det\mathrm{A}=\mathrm{a}\det\mathrm{G}_{\partial\omega}. \tag{3-20}$$

In view of (3-14) we get

$$\begin{aligned}\mathrm{a}&=\det\big(\mathrm{E}+\varepsilon^4(\mathrm{E}-\varepsilon^2\xi^2\mathrm{G}_{\partial\omega}^{-1}\mathrm{B})^{-2}\mathrm{G}_{\partial\omega}^{-1}(\nabla_s b_\varepsilon)(\nabla_s b_\varepsilon)^*\big)\det(\mathrm{E}-\varepsilon^2\xi^2\mathrm{G}_{\partial\omega}^{-1}\mathrm{B})^2\\
&=\big(1+\varepsilon^4\operatorname{Tr}(\mathrm{E}-\varepsilon^2\xi^2\mathrm{G}_{\partial\omega}^{-1}\mathrm{B})^{-2}\mathrm{G}_{\partial\omega}^{-1}(\nabla_s b_\varepsilon)(\nabla_s b_\varepsilon)^*+\mathcal{O}(\varepsilon^8\xi^2)\big)\prod_{i=1}^{n-1}(1-\varepsilon^2\xi^2K_i)^2\\
&=\big(1+\varepsilon^4\operatorname{Tr}\mathrm{G}_{\partial\omega}^{-1}(\nabla_s b_\varepsilon)(\nabla_s b_\varepsilon)^*+\mathcal{O}(\varepsilon^6\xi^4)\big)\big(1-2\varepsilon^2\xi^2K+\mathcal{O}(\varepsilon^4\xi^4)\big)\\
&=\big(1+\varepsilon^4|\nabla b_\varepsilon|^2+\mathcal{O}(\varepsilon^6\xi^4)\big)\big(1-2\varepsilon^2\xi^2K+\mathcal{O}(\varepsilon^4\xi^4)\big).\end{aligned}$$

We substitute the obtained formula and (3-10) into (3-20) and arrive at (3-18). □

Employing (3-14), (3-16), by direct calculations we check

$$\mathrm{p}^*\mathrm{A}^{-1}\mathrm{p}=\left(\frac{\partial b_\varepsilon}{\partial\xi}\right)^2(\nabla_s b_\varepsilon)^*\mathrm{G}_{\partial\omega}^{-1}(\nabla_s b_\varepsilon)+\mathcal{O}(\varepsilon^2\xi^2)=\left(\frac{\partial b_\varepsilon}{\partial\xi}\right)^2|\nabla b_\varepsilon|^2+\mathcal{O}(\varepsilon^2\xi^2)=b_1^2\xi^2|\nabla b_1|^2+\mathcal{O}(\varepsilon\xi^2).$$

Hence, by (3-17), (3-18) and the definition of β,

$$\varepsilon^{-2}\det{}^{\frac12}\mathrm{G}_\varepsilon=\beta^{-\frac12}\det{}^{\frac12}\mathrm{A}=\beta^{-1}\beta_{\mathrm{A}}\det{}^{\frac12}\mathrm{G}_{\partial\omega},\quad \beta_{\mathrm{A}}:=\beta^{\frac12}\mathrm{a}^{\frac12}=\sum_{i=0}^{4}\varepsilon^i\beta_{i-4}+\mathcal{O}\big(\varepsilon^5(|\xi|^2+\xi^4)\big),$$

where $\beta_i=\beta_i(\xi,P)\in C^\infty(\mathbb{R}\times\partial\omega)$ are some functions. In particular,

$$\beta_{-4}:=\frac{1}{(4\xi^2+b_1^2)^{\frac12}},\quad \beta_{-3}:=-\frac{2b_1b_2\xi}{(4\xi^2+b_1^2)^{\frac32}},$$

$$\beta_{-2}:=-\frac{3b_1b_3\xi^2}{(4\xi^2+b_1^2)^{\frac32}}-\frac{4\xi^2(2\xi^2-b_1^2)b_2^2}{(4\xi^2+b_1^2)^{\frac52}}-\frac{\xi^2K}{(4\xi^2+b_1^2)^{\frac12}}, \tag{3-21}$$

while the functions β_{-1}, β_0 satisfy the uniform in ξ and P estimates

$$|\beta_{-1}| \leqslant \frac{C|\xi|^3}{1+|\xi|^3}, \quad |\beta_0| \leqslant C\xi^2(1+|\xi|).$$

The obtained formulas, Lemma 3.2, and (3-15) allow us to write the expansion for $\mathrm{G}_\varepsilon^{-1}$:

$$\varepsilon^{-2}(\det^{\frac{1}{2}} \mathrm{G}_\varepsilon)\mathrm{G}_\varepsilon^{-1} = \det^{\frac{1}{2}} \mathrm{G}_{\partial\omega} \sum_{i=-4}^{0} \varepsilon^i \mathrm{G}_i + \mathcal{O}(\varepsilon), \tag{3-22}$$

$$\mathrm{G}_i := \begin{pmatrix} \beta_i & 0 \\ 0 & 0 \end{pmatrix}, \quad i=-4,\dots,-1, \quad \mathrm{G}_0 := \begin{pmatrix} \beta_0 & -b_1\xi\beta_{-4}(\nabla_s b_1)^* \mathrm{G}_{\partial\omega}^{-1} \\ -b_1\xi\beta_{-4}\mathrm{G}_{\partial\omega}^{-1}\nabla_s b_1 & \beta_{-4}^{-1}\mathrm{G}_{\partial\omega}^{-1} \end{pmatrix}. \tag{3-23}$$

Taking into account (3-17), (3-18), we write the operator $\mathcal{H}_\varepsilon$ in terms of the variables (s_0, s), where $s_0 := \xi$:

$$\mathcal{H}_\varepsilon = -\frac{1}{\det^{\frac{1}{2}} \mathrm{G}_\varepsilon} \sum_{i,j=0}^{n-1} \frac{\partial}{\partial s_i} G_\varepsilon^{ij} \det^{\frac{1}{2}} \mathrm{G}_\varepsilon \frac{\partial}{\partial s_j} = -\frac{\varepsilon^{-2}\beta_{\mathrm{A}}}{\mathrm{a}\det^{\frac{1}{2}} \mathrm{G}_{\partial\omega}} \sum_{i,j=0}^{n-1} \frac{\partial}{\partial s_i} G_\varepsilon^{ij} \det^{\frac{1}{2}} \mathrm{G}_\varepsilon \frac{\partial}{\partial s_j}, \tag{3-24}$$

and G_ε^{ij} are the entries of the inverse matrix in (3-15). It follows from the last formula and (3-15) that

$$\mathcal{H}_\varepsilon = \varepsilon^{-4}\mathrm{a}^{-1}\beta_{\mathrm{A}}\frac{\partial}{\partial\xi}\beta_{\mathrm{A}}\frac{\partial}{\partial\xi} + \mathcal{O}(1).$$

We employ the obtained equation, (3-24), (3-22) and (3-23), and expand the coefficients of $\mathcal{H}_\varepsilon$ in powers of ε leading us to the identities

$$\mathcal{H}_\varepsilon = \sum_{i=-4}^{0} \varepsilon^i \mathcal{L}_i + \mathcal{O}(\varepsilon), \tag{3-25}$$

$$\begin{gathered}\mathcal{L}_{-4} := \mathcal{L}^{(-4)}, \quad \mathcal{L}_{-3} := \mathcal{L}^{(-3)}, \quad \mathcal{L}_{-2} := \mathcal{L}^{(-2)} + \alpha^{(2)}\mathcal{L}^{(-4)}, \mathcal{L}_{-1} := \mathcal{L}^{(-1)} + \alpha^{(2)}\mathcal{L}^{(-3)}, \\ \mathcal{L}_0 := \mathcal{L}^{(0)} + \alpha^{(2)}\mathcal{L}^{(-2)} + \alpha^{(4)}\mathcal{L}^{(-4)}, \quad \alpha^{(2)} := 2\xi^2 K, \quad \alpha^{(4)} = \alpha^{(4)}(\xi, s),\end{gathered} \tag{3-26}$$

$$\mathcal{L}^{(i)} := -\sum_{j=0}^{i+4} \beta_{j-4}\frac{\partial}{\partial\xi}\beta_{i-j}\frac{\partial}{\partial\xi}, \quad i=-4,\dots,-1, \tag{3-27}$$

$$\begin{aligned}\mathcal{L}^{(0)} := &-\sum_{l=0}^{4} \beta_{l-4}\frac{\partial}{\partial\xi}\beta_{-l}\frac{\partial}{\partial\xi} + b_1\beta_{-4}\frac{\partial}{\partial\xi}\xi\beta_{-4}(\nabla_s b_1)^*\mathrm{G}_{\partial\omega}^{-1}\nabla_s \\ &+ \beta_{-4}\det^{-\frac{1}{2}} \mathrm{G}_{\partial\omega} \operatorname{div}_s b_1\beta_{-4}\xi \det^{\frac{1}{2}} \mathrm{G}_{\partial\omega}(\nabla_s b_1)^*\mathrm{G}_{\partial\omega}^{-1}\frac{\partial}{\partial\xi} \\ &- \beta_{-4}\det^{-\frac{1}{2}} \mathrm{G}_{\partial\omega} \operatorname{div}_s \beta_{-4}^{-1}(\det^{\frac{1}{2}} \mathrm{G}_{\partial\omega})\mathrm{G}_{\partial\omega}^{-1}\nabla_s.\end{aligned} \tag{3-28}$$

Auxiliary lemmas. We proceed to the auxiliary lemmas which will be used for proving Theorem 2.4.

Lemma 3.3. *In a vicinity of $\partial\omega$ the identities*

$$\det \mathrm{M} = (\det^{\frac{1}{2}} \mathrm{G}_{\partial\omega}) \prod_{i=1}^{n-1} (1-\tau K_i), \quad -\Delta_{x'} = -\frac{1}{\det \mathrm{M}} \operatorname{div}_{(\tau,s)} (\det \mathrm{M}) \widehat{\mathrm{M}} \nabla_{(\tau,s)} \tag{3-29}$$

hold true, *where*

$$\widehat{\mathrm{M}} := (\mathrm{M}^{-1})^* \mathrm{M}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & (\mathrm{E} - \tau \mathrm{G}_{\partial\omega}^{-1} \mathrm{B})^{-2} \mathrm{G}_{\partial\omega}^{-1} \end{pmatrix}. \tag{3-30}$$

Proof. It follows from (3-4) and the Weingarten formulas that

$$\mathrm{M} = \begin{pmatrix} \nu \\ \frac{\partial r}{\partial s_i} - \tau \sum_{k=1}^{n-1} B_i^k \frac{\partial r}{\partial s_k} \end{pmatrix},$$

where B_i^k are the entries of the matrix $\mathrm{G}_{\partial\omega}^{-1}\mathrm{B}$, and all vectors are treated as rows.

A straightforward direct calculation allows us to check that the inverse matrix M^{-1} reads

$$\mathrm{M}^{-1} = \begin{pmatrix} \nu \\ \sum_{k=1}^{n-1} c_i^k \frac{\partial r}{\partial s_k} \end{pmatrix}^*, \tag{3-31}$$

where * indicates matrix transposition, and c_i^k are the entries of the matrix $\mathrm{C} = (\mathrm{E} - \tau \mathrm{G}_{\partial\omega}^{-1}\mathrm{B})^{-1}\mathrm{G}_{\partial\omega}^{-1}$.

Let $u_1, u_2 \in C_0^\infty(\omega)$ be any two functions with the corresponding supports located in a neighborhood of $\partial\omega$, where the coordinates (τ, s) are well-defined. We integrate by parts:

$$\begin{aligned}(-\Delta_{x'} u, v)_{L_2(\omega)} &= (\nabla_{x'} u, \nabla_{x'} v)_{L_2(\omega)} = (\mathrm{M}^{-1} \nabla_{(\tau,s)} u, (\det \mathrm{M}) \mathrm{M}^{-1} \nabla_{(\tau,s)} v)_{L_2((0,\tau_0)\times\partial\omega)} \\ &= \big(-\operatorname{div}_{(\tau,s)} (\det \mathrm{M}) (\mathrm{M}^{-1})^* (\mathrm{M}^{-1}) \nabla_{(\tau,s)} u, v\big)_{L_2((0,\tau_0)\times\partial\omega)} \\ &= \big(-(\det^{-1} \mathrm{M}) \operatorname{div}_{(\tau,s)} (\det \mathrm{M}) (\mathrm{M}^{-1})^* \mathrm{M}^{-1} \nabla_{(\tau,s)} u, v\big)_{L_2(\omega)}.\end{aligned}$$

Hence,

$$-\Delta_{x'} = -(\det^{-1} \mathrm{M}) \operatorname{div}_{(\tau,s)} (\det \mathrm{M}) (\mathrm{M}^{-1})^* \mathrm{M}^{-1} \nabla_{(\tau,s)}. \tag{3-32}$$

In view of (3-31) we have

$$(\mathrm{M}^{-1})^* \mathrm{M}^{-1} = \begin{pmatrix} \nu \\ \sum_{k=1}^{n-1} c_i^k \frac{\partial r}{\partial s_k} \end{pmatrix} \begin{pmatrix} \nu \\ \sum_{k=1}^{n-1} c_i^k \frac{\partial r}{\partial s_k} \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{C}\mathrm{G}_{\partial\omega}\mathrm{C} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (\mathrm{E} - \tau \mathrm{G}_{\partial\omega}^{-1} \mathrm{B})^{-2} \mathrm{G}_{\partial\omega}^{-1} \end{pmatrix},$$

$$\det^{-2} \mathrm{M} = \det (\mathrm{M}^{-1})^* \mathrm{M}^{-1} = \det(\mathrm{E} - \tau \mathrm{G}_{\partial\omega}^{-1} \mathrm{B})^{-2} \det \mathrm{G}_{\partial\omega}^{-1},$$

$$\det \mathrm{M} = \det^{\frac{1}{2}} \mathrm{G}_{\partial\omega} \det(\mathrm{E} - \tau \mathrm{G}_{\partial\omega}^{-1} \mathrm{B}) = \det^{\frac{1}{2}} \mathrm{G}_{\partial\omega} \prod_{i=1}^{n-1} (1 - \tau K_i).$$

The obtained formulas and (3-32) imply the statement of the lemma. □

We recall that the set ω^δ was introduced in (2-8).

Lemma 3.4. *Let the functions $f_\pm \in C^\infty(\omega_\pm)$ satisfy the differentiable asymptotics*

$$f_\pm(x') = \sum_{j=-4}^{\infty} f^\pm_{j/2}(P)\tau^{\frac{j}{2}}, \quad \tau \to +0 \tag{3-33}$$

uniformly in $P \in \partial\omega_\pm$, where $f^\pm_{j/2} \in C^\infty(\partial\omega_\pm)$, and $V^{(0)}, V^{(1)} \in C^\infty(\partial\omega)$ are some functions. Suppose the condition

$$\begin{aligned}\lim_{\delta\to+0}\Big[&(f_+,\psi_+^{(i)})_{L_2(\omega^\delta)} + (f_-,\psi_-^{(i)})_{L_2(\omega^\delta)} - \delta^{-1}\int_{\partial\omega}(f^+_{-2}+f^-_{-2})\Psi_i^{(0)}\,ds\\
&-2\delta^{-1/2}\int_{\partial\omega}(f^+_{-3/2}+f^-_{-3/2})\Psi_i^{(0)}\,ds\\
&-\ln\delta\int_{\partial\omega}\big((K(f^+_{-2}+f^-_{-2})-f^+_{-1}-f^-_{-1})\Psi_i^{(0)}-(f^+_{-2}-f^-_{-2})\Psi_i^{(1)}\big)\,ds\Big]\\
&-\int_{\partial\omega}(f^+_{-2}-f^-_{-2})\Psi_i^{(1)}\,ds+\int_{\partial\omega}(f^+_{-2}+f^-_{-2})\Psi_i^{(0)}K\,ds\\
&+2\int_{\partial\omega}\big(V^{(0)}\Psi_i^{(1)}-V^{(1)}\Psi_i^{(0)}\big)\,ds=0, \quad i=1,\dots,m,\end{aligned} \tag{3-34}$$

holds true. Then there exist the unique solutions $u_\pm \in C^\infty(\omega_\pm)$ to the equations

$$(-\Delta_{x'}-\lambda)u_\pm = f_\pm, \quad x \in \omega_\pm, \tag{3-35}$$

these solutions satisfy differentiable asymptotics

$$\begin{aligned}u_\pm(x') = f^\pm_{-2}(P)\ln\tau + U^{(0)}(P) \pm V^{(0)}(P) + 4f^\pm_{-3/2}(P)\tau^{1/2} + \tau(V^{(1)}(P) \pm U^{(1)}(P))\\
+\tau(1-\ln\tau)\big(f^\pm_{-1}(P)-K(P)f^\pm_{-2}(P)\big)+\mathcal{O}(\tau^{3/2}), \quad \tau\to 0,\end{aligned} \tag{3-36}$$

uniformly in $P \in \partial\omega_\pm$, where $U^{(0)}, U^{(1)} \in C^\infty(\partial\omega_\pm)$ are some functions, and the condition

$$(U_0, \Psi_i^{(0)})_{L_2(\partial\omega)} + (U_1, \Psi_i^{(1)})_{L_2(\partial\omega)} = 0, \quad i=1,\dots,m, \tag{3-37}$$

holds true.

Proof. Let $\chi(\tau)$ be the cut-off function introduced in the proof of Lemma 4.4. We introduce the functions

$$\begin{aligned}\hat{u}_\pm(x') := \big(f^\pm_{-2}(P)\ln\tau \pm V^{(0)}(P) + 4f^\pm_{-3/2}(P)\tau^{1/2} + \tau(1-\ln\tau)\big(f^\pm_{-1}(P)-K(P)f^\pm_{-2}(P)\big)\\
+\tau V^{(1)}(P) - \tfrac{4}{3}\tau^{3/2}\big(f^\pm_{-1/2}(P)-2K(P)f^\pm_{-3/2}(P)\big)\big)\chi(\tau).\end{aligned}$$

Employing Lemma 3.3, one can check that

$$(-\Delta_{x'}-\lambda)\hat{u}_\pm(x') = \chi(\tau)\sum_{j=-4}^{-1} f^\pm_{j/2}(P)\tau^j + \hat{f}_\pm(x'), \tag{3-38}$$

where $\hat{f}_\pm \in C^\infty(\omega_\pm) \cap L_2(\omega_\pm)$.

We construct the solutions to (3-35) as

$$u_\pm = \widehat{u}_\pm + \widetilde{u}_\pm.$$

Substituting this identity and (3-38) into (3-35), we obtain the equations for $\widetilde{u}_\pm$:

$$(-\Delta_{x'} - \lambda)\widetilde{u}_\pm = \widetilde{f}_\pm, \quad \widetilde{f}_\pm := f_\pm - \chi \sum_{j=-4}^{-1} f^\pm_{j/2}\tau^j - \widehat{f}_\pm, \tag{3-39}$$

and by (3-33) we have $\widetilde{f}_\pm \in L_2(\omega_\pm)$. Hence, we can rewrite these equations as

$$(\mathcal{H}_0 - \lambda)\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{f}}, \quad \widetilde{\boldsymbol{u}} := (\widetilde{u}_+, \widetilde{u}_-), \quad \widetilde{\boldsymbol{f}} := (\widetilde{f}_+, \widetilde{f}_-). \tag{3-40}$$

Since λ is a discrete eigenvalue of $\mathcal{H}_0$, the solvability condition of the last equation is

$$(\widetilde{\boldsymbol{f}}, \boldsymbol{\psi}_i)_{L_2(\boldsymbol{\omega})} = 0, \quad k = 1, \dots, m,$$

which can be rewritten as

$$(\widetilde{f}_+, \psi^{(i)}_+)_{L_2(\omega)} + (\widetilde{f}_+, \psi^{(i)}_+)_{L_2(\omega)} = 0, \quad k = 1, \dots, m,$$

or, equivalently,

$$\lim_{\delta\to 0}\big((\widetilde{f}_+, \psi^{(i)}_+)_{L_2(\omega^\delta)} + (\widetilde{f}_-, \psi^{(i)}_-)_{L_2(\omega^\delta)}\big) = 0, \quad k = 1, \dots, m. \tag{3-41}$$

Integrating by parts and taking into account (3-38), (3-39), we get

$$(\widetilde{f}_\pm, \psi^{(i)}_\pm)_{L_2(\omega^\delta)} = \big(f_\pm + (\Delta_{x'} + \lambda)\widetilde{u}_\pm, \psi^{(i)}_\pm\big)_{L_2(\omega^\delta)} = (f_\pm, \psi^{(i)}_\pm)_{L_2(\omega^\delta)} - \int_{\partial\omega^\delta}\left(\psi^{(i)}_\pm \frac{\partial \widetilde{u}_\pm}{\partial\tau} - \widetilde{u}_\pm \frac{\partial \psi^{(i)}_\pm}{\partial\tau}\right) ds.$$

Here we have used that the normal derivative on $\partial\omega^\delta$ is that with respect to τ up to the sign. We parametrize the points of $\partial\omega^\delta$ by those on $\partial\omega$ via the relation $x' = r(s) + \delta\nu(s)$. In view of (3-4) and (3-29) we have

$$\int_{\partial\omega^\delta} \cdot\, ds = \int_{\partial\omega} \cdot \prod_{j=1}^{n-1}(1 - \tau K_j)\, ds. \tag{3-42}$$

Taking this formula into account, we continue the calculations:

$$\begin{aligned}
(\widetilde{f}_\pm, \psi^{(i)}_\pm)_{L_2(\omega^\delta)} &= (f_\pm, \psi^{(i)}_\pm)_{L_2(\omega^\delta)} - \int_{\partial\omega}\left(\psi^{(i)}_\pm \frac{\partial \widetilde{u}_\pm}{\partial\tau} - \widetilde{u}_\pm \frac{\partial \psi^{(i)}_\pm}{\partial\tau}\right)\bigg|_{x'=r(s)+\delta\nu(s)} \prod_{j=1}^{n-1}(1 - \tau K_j)\, ds \\
&= (f_\pm, \psi^{(i)}_\pm)_{L_2(\omega^\delta)} - \delta^{-1}\int_{\partial\omega} f^\pm_{-2}\Psi^{(0)}_k\, ds - 2\delta^{-1/2}\int_{\partial\omega} f^\pm_{-3/2}\Psi^{(0)}_k\, ds \\
&\quad - \ln\delta \int_{\partial\omega}\big((K f^\pm_{-2} - f^\pm_{-1})\Psi^{(0)}_i \mp f^\pm_{-2}\Psi^{(1)}_i\big)\, ds \\
&\quad + \int_{\partial\omega} f^\pm_{-2}\big(\Psi^{(0)}_i K \mp \Psi^{(1)}_i\big)\, ds + \int_{\partial\omega}\big(V^{(0)}\Psi^{(1)}_i - V^{(1)}\Psi^{(0)}_i\big)\, ds + \mathcal{O}(\delta^{1/2}).
\end{aligned}$$

We substitute the last identities into (3-41) and arrive at (3-34). Thus, the condition (3-34) implies the existence of solutions to (3-35).

The functions $\widetilde{u}_\pm \in W_2^2(\omega_\pm)$ satisfy (2-5) in the sense of traces. Define

$$U^{(0)} := \widetilde{u}_\pm|_{\partial\omega}, \quad U^{(1)} := \frac{\partial \widetilde{u}_\pm}{\partial \tau}\bigg|_{\partial\omega}, \quad U^{(0)}, U^{(1)} \in L_2(\partial\omega).$$

The solution to (3-40) is defined up to a linear combination of the eigenfunctions. In view of the belongings $U^{(0)}$, $U^{(1)} \in L_2(\partial\omega)$ we can choose the mentioned linear combination of the eigenfunctions so that the condition (3-37) is satisfied. Then the solution to (3-40) is unique and the same is obviously true for (3-35). To prove the asymptotics (3-36) it is sufficient to study the smoothness of $\widetilde{u}_\pm$ at $\partial\omega$.

By standard smoothness improving theorems we conclude that $\widetilde{u}_\pm \in C^\infty(\omega)$. Moreover, given any $N > 0$, it is easy to construct the function $\widehat{u}_\pm^{(N)}$ similar to $\widehat{u}_\pm$ such that

$$\widehat{u}_\pm^{(N)}(x') = \widehat{u}_\pm(x') + \mathcal{O}(\tau^2), \quad \tau \to 0, \quad (-\Delta_{x'} - \lambda)\widehat{u}_\pm^{(N)}(x') = \chi(\tau) \sum_{j=-4}^{N} f_{j/2}^\pm(P)\tau^j + \widehat{f}_\pm^{(N)}(x'),$$

where $\widehat{f}_\pm^{(N)} \in C^\infty(\omega_\pm) \cap C^{N_1}(\overline{\omega}_\pm)$, and $N_1 = N_1(N) \to +\infty$, $N \to +\infty$. Then, proceeding as above, we can construct the solutions to (3-35) as $u_\pm = \widetilde{u}_\pm + \widehat{u}_\pm$, where $\widetilde{\boldsymbol{u}}^{(N)} := (\widetilde{u}_+^{(N)}, \widetilde{u}_-^{(N)})$ solves the equation

$$(\mathcal{H}_0 - \lambda)\widetilde{\boldsymbol{u}}^{(N)} = \widetilde{\boldsymbol{f}}^{(N)}, \quad \widetilde{\boldsymbol{f}}^{(N)} := (\widetilde{f}_+^{(N)}, \widetilde{f}_-^{(N)}), \quad \widetilde{f}_\pm^{(N)}(x') := f_\pm(x') - \chi(\tau) \sum_{j=-4}^{N} f_{j/2}^\pm(P)\tau^j - \widehat{f}_\pm^{(N)}.$$

It is clear that $\widetilde{f}_\pm^{(N)}$ belongs to $C^{N_2}(\overline{\omega}_\pm)$, where $N_2 = N_2(N) \to +\infty$ as $N \to +\infty$. Hence, by the smoothness improving theorems, $\widetilde{u}_\pm^{(N)} \in C^{N_3}(\overline{\omega}_\pm)$, $N_3 = N_3(N) \to +\infty$, $N \to +\infty$. Choosing N large enough, we arrive at the asymptotics (3-36). □

Lemma 3.5. *For all $u, v \in C^\infty(\overline{\omega})$ in a small vicinity of $\partial\omega$ the identities*

$$\operatorname{div}_{x'} \mathrm{Q}_\pm \nabla_{x'} u = \frac{1}{\det \mathrm{M}} \operatorname{div}_{(\tau,s)} (\det \mathrm{M}) \widehat{\mathrm{M}} \nabla_{(\tau,s)} h_\pm (\nabla_{(\tau,s)} h_\pm)^* \widehat{\mathrm{M}} \nabla_{(\tau,s)} u, \tag{3-43}$$

$$(\nabla_{x'} u, \nabla_{x'} v)_{\mathbb{R}^d} = \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \tau} + \nabla u \cdot (\mathrm{E} - \tau \mathrm{B}\mathrm{G}_{\partial\omega}^{-1})^{-2} \nabla v \tag{3-44}$$

hold true.

Proof. Let $u, v \in C^\infty(\overline{\omega})$ be two arbitrary functions with supports in a small vicinity $\{x' : 0 \leqslant \tau < \tau_0\}$, where τ_0 is a small fixed number. We choose τ_0 so that in this vicinity the coordinates (τ, s) are well-defined.

Taking (3-1) and (3-4) into account, we pass to the variables (τ, s) and integrate by parts to obtain

$$\begin{aligned}\int_\omega v \operatorname{div}_{x'} \mathrm{Q}_\pm \nabla_{x'} u \, dx' &= -\int_\omega (\nabla_{x'} v, \nabla_{x'} h_\pm (\nabla_{x'} h_\pm)^* \nabla_{x'} u)_{\mathbb{R}^n} \, dx' \\ &= -\int_{[0,\tau_0)\times\partial\omega} \big(\mathrm{M}^{-1}\nabla_{(\tau,s)} v, \mathrm{M}^{-1}\nabla_{(\tau,s)} h_\pm (\nabla_{(\tau,s)} h_\pm)^* \widehat{\mathrm{M}} \nabla_{(\tau,s)} u\big)_{\mathbb{R}^n} (\det \mathrm{M}) \, d\tau \, ds \\ &= \int_{[0,\tau_0)\times\partial\omega} v \operatorname{div}_{(\tau,s)} (\det \mathrm{M}) \widehat{\mathrm{M}} \nabla_{(\tau,s)} h_\pm (\nabla_{(\tau,s)} h_\pm)^* \widehat{\mathrm{M}} \nabla_{(\tau,s)} u \, d\tau \, ds \\ &= \int_\omega v (\det{}^{-1} \mathrm{M}) \operatorname{div}_{(\tau,s)} (\det \mathrm{M}) \widehat{\mathrm{M}} \nabla_{(\tau,s)} h_\pm (\nabla_{(\tau,s)} h_\pm)^* \widehat{\mathrm{M}} \nabla_{(\tau,s)} u \, dx',\end{aligned}$$

which proves (3-43).

The identity (3-44) follows from (3-4) and (3-30):

$$\begin{aligned}(\nabla_{x'} u, \nabla_{x'} v)_{\mathbb{R}^n} &= (\mathrm{M}^{-1}\nabla_{x'} u, \mathrm{M}^{-1}\nabla_{x'} v)_{\mathbb{R}^n} = (\nabla_{x'} u, \widehat{\mathrm{M}} \nabla_{x'} v)_{\mathbb{R}^n} \\ &= \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \tau} + \big(\nabla_s u, (\mathrm{E} - \tau \mathrm{G}_{\partial\omega}^{-1} \mathrm{B})^{-2} \mathrm{G}_{\partial\omega}^{-1} \nabla_s u\big)_{\mathbb{R}^n} = \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \tau} + \nabla u \cdot (\mathrm{E} - \tau \mathrm{B} \mathrm{G}_{\partial\omega}^{-1})^{-2} \nabla v. \qquad \square\end{aligned}$$

4. Uniform resolvent convergence

In this section we prove Theorem 2.1. We begin with two auxiliary lemmas.

Lemma 4.1. *The identity $\mathfrak{D}(\mathcal{H}_0) = W_2^2(\boldsymbol{\omega})$ holds true and for each $\boldsymbol{u} \in \mathfrak{D}(\mathcal{H}_0)$ the operator $\mathcal{H}_0$ acts as $\mathcal{H}_0(\boldsymbol{u}) = (-\Delta_{x'} u_+, -\Delta_{x'} u_-)$. For each $z \in \mathbb{C} \setminus \mathbb{R}$ the estimate*

$$\|(\mathcal{H}_0 - z)^{-1}\|_{L_2(\boldsymbol{\omega}) \to W_2^2(\boldsymbol{\omega})} \leqslant \frac{C}{|\mathrm{Im}(z)|} \tag{4-1}$$

holds for some constant C, where $\mathrm{Im}(z)$ denotes the imaginary part of z.

Proof. The first part follows from the definitions and the considerations above for the space $W_2^2(\boldsymbol{\omega})$. The second part of the statement follows from the fact that the operator $\mathcal{H}_0$ is self-adjoint with compact resolvent. $\square$

The description of the spectrum of $\mathcal{H}_0$ as being made up of the union of the Dirichlet and Neumann spectra is given in the following lemma, together with some properties which will be useful in the sequel.

Lemma 4.2. *The spectrum of $\mathcal{H}_0$ coincides with the union of spectra of $-\Delta_\omega^{(D)}$ and $-\Delta_\omega^{(N)}$ counting multiplicities. Namely, if λ is an $m^{(D)}$-multiple eigenvalue of $-\Delta_\omega^{(D)}$ with the associated eigenfunctions $\psi_i^{(D)}$, $i = 1, \dots, m^{(D)}$, and is an $m^{(N)}$-multiple eigenvalue of $-\Delta_\omega^{(N)}$ with the associated eigenfunctions $\psi_i^{(N)}$, $i = 1, \dots, m^{(N)}$, then λ is an $(m^{(D)} + m^{(N)})$-multiple eigenvalue of $\mathcal{H}_0$ with the associated eigenfunctions $\boldsymbol{\psi}_i = (\psi_i^{(D)}, -\psi_i^{(D)})$ and $\boldsymbol{\psi}_i = (\psi_i^{(N)}, \psi_i^{(N)})$. For any eigenfunction $\boldsymbol{\psi} = (\psi_+, \psi_-)$ of $\mathcal{H}_0$ we have $\psi_\pm \in C^\infty(\overline{\omega})$ and the asymptotics*

$$\psi_\pm(x') = \Psi^{(0)}(P) \pm \tau \Psi^{(1)}(P) + \mathcal{O}(\tau^2), \quad P \in \partial\omega,$$

where

$$\Psi^{(0)}=\psi_+|_{\partial\omega}=\psi_-|_{\partial\omega}\in C^\infty(\partial\omega),\quad \Psi^{(1)}=\frac{\partial\psi_+}{\partial\tau}\bigg|_{\partial\omega}=-\frac{\partial\psi_-}{\partial\tau}\bigg|_{\partial\omega}\in C^\infty(\partial\omega)$$

and

$$x'=P+\tau\nu(P)\quad \textit{for small positive } \tau.$$

Proof. Clearly if λ is an eigenvalue of $-\Delta_\omega^{(D)}$ with eigenfunction u, then λ is an eigenvalue of $\mathcal{H}_0$ with eigenfunction $(u,-u)$. Similarly, an eigenvalue of $-\Delta_\omega^{(N)}$ with eigenfunction v will also be an eigenvalue of $\mathcal{H}_0$ with eigenfunction (v,v).

Assume now that (u,v) is an eigenfunction of $\mathcal{H}_0$ and consider the functions $w_1=u-v$ and $w_2=u+v$. Then, provided they do not vanish identically, w_1 and w_2 will be eigenfunctions of $-\Delta_\omega^{(D)}$ and $-\Delta_\omega^{(N)}$, respectively. In case w_1 vanishes identically, then $u=v$ and u will be an eigenfunction of $-\Delta_\omega^{(N)}$, while if w_2 vanishes $u=-v$ and this will be an eigenfunction of $-\Delta_\omega^{(D)}$.

The remaining part of the lemma follows from standard arguments. □

By $L_2(\boldsymbol{\omega},J_\varepsilon\,dx')$ we indicate the subspace of $L_2(\boldsymbol{\omega})$ consisting of the functions $\boldsymbol{u}$ with the finite norm

$$\|\boldsymbol{u}\|^2_{L_2(\boldsymbol{\omega},J_\varepsilon dx')}=\|u_+\|^2_{L_2(\omega_+,J_\varepsilon^+dx')}+\|u_-\|^2_{L_2(\omega_-,J_\varepsilon^-dx')},\quad \|u_\pm\|^2_{L_2(\omega,J_\varepsilon^\pm dx')}=\int_{\omega_\pm}|u_\pm(x')|^2J_\varepsilon^\pm(x')\,dx'.$$

In the same way we introduce the space $W_2^1(\boldsymbol{\omega},J_\varepsilon\,dx')$ as consisting of $\boldsymbol{u}\in W_2^1(\boldsymbol{\omega})$ with the finite norm

$$\|\boldsymbol{u}\|^2_{W_2^1(\boldsymbol{\omega},J_\varepsilon\,dx')}=\|\nabla_{x'}\boldsymbol{u}\|^2_{L_2(\boldsymbol{\omega},J_\varepsilon\,dx')}+\|\boldsymbol{u}\|^2_{L_2(\boldsymbol{\omega},J_\varepsilon\,dx')},$$

where $\nabla_{x'}\boldsymbol{u}=(\nabla_{x'}u_+,\nabla_{x'}u_-)$.

Lemma 4.3. *The spaces $L_2(\mathcal{S}_\varepsilon)$ and $L_2(\boldsymbol{\omega},J_\varepsilon\,dx')$ are isomorphic and the isomorphism is the operator $\mathcal{I}_\varepsilon:L_2(\boldsymbol{\omega},J_\varepsilon\,dx')\to L_2(\mathcal{S}_\varepsilon)$. If $\boldsymbol{u}\in W_2^1(\boldsymbol{\omega},J_\varepsilon\,dx')$, then $\mathcal{I}_\varepsilon\boldsymbol{u}\in W_2^1(\mathcal{S}_\varepsilon)$, and if $u\in W_2^1(\mathcal{S}_\varepsilon)$, then $\mathcal{I}_\varepsilon^{-1}u\in W_2^1(\boldsymbol{\omega},J_\varepsilon\,dx')$. The inequality*

$$\|J_\varepsilon^{-\frac12}\nabla_{x'}\boldsymbol{u}\|_{L_2(\boldsymbol{\omega})}\leqslant\|\nabla\mathcal{I}_\varepsilon\boldsymbol{u}\|_{L_2(\mathcal{S}_\varepsilon)}\leqslant\|\nabla_{x'}\boldsymbol{u}\|_{L_2(\boldsymbol{\omega},J_\varepsilon\,dx')}\tag{4-2}$$

holds true, where $J_\varepsilon^{-\frac12}\nabla_{x'}\boldsymbol{u}:=((J_\varepsilon^+)^{-\frac12}\nabla_{x'}u_+,(J_\varepsilon^-)^{-\frac12}\nabla_{x'}u_-)$, $\boldsymbol{u}=(u_+,u_-)$.

Proof. The fact that $\mathcal{I}_\varepsilon$ is a bijection between the two spaces follows directly from its definition.

Regarding the inequalities we have

$$\begin{aligned}\|J_\varepsilon^{-\frac12}\nabla_{x'}\boldsymbol{u}\|^2_{L_2(\boldsymbol{\omega})}&=\int_{\omega_+}(J_\varepsilon^+)^{-1}|\nabla_{x'}u_+|^2\,dx'+\int_{\omega_-}(J_\varepsilon^-)^{-1}|\nabla_{x'}u_-|^2\,dx'\\&=\int_{\omega_+}J_\varepsilon^+(J_\varepsilon^+)^{-2}|\nabla_{x'}u_+|^2\,dx'+\int_{\omega_-}J_\varepsilon^-(J_\varepsilon^-)^{-2}|\nabla_{x'}u_-|^2\,dx'\\&\le\int_{\omega_+}J_\varepsilon^+(\nabla_{x'}u_+)^*G_+^{-1}\nabla_{x'}u_+\,dx'+\int_{\omega_-}J_\varepsilon^-(\nabla_{x'}u_-)^*G_-^{-1}\nabla_{x'}u_-\,dx'\\&=\|\nabla\mathcal{I}_\varepsilon\boldsymbol{u}\|_{L_2(\mathcal{S}_\varepsilon)}\le\int_{\omega_+}J_\varepsilon^+|\nabla_{x'}u_+|^2\,dx'+\int_{\omega_-}J_\varepsilon^-|\nabla_{x'}u_-|^2\,dx'=\|\nabla_{x'}\boldsymbol{u}\|_{L_2(\boldsymbol{\omega},J_\varepsilon\,dx')},\end{aligned}$$

where we have used the knowledge of the eigenvalues of $G_\pm$ and the fact that $1 \le J_\varepsilon^\pm$. □

Define $\omega_\delta := \omega \cap \{x' : 0 < \tau < \delta\}$. We recall that the set ω^δ was introduced in (2-8), and in what follows $\boldsymbol{\omega}^\delta$ is ω^δ considered as a two-sided domain.

Lemma 4.4. *If $\boldsymbol{u} \in W_2^1(\boldsymbol{\omega})$, or, respectively, $\boldsymbol{u} \in W_2^2(\boldsymbol{\omega})$, then $\boldsymbol{u} \in L_2(\boldsymbol{\omega}, J_\varepsilon\, dx')$, or, respectively, $\boldsymbol{u} \in W_2^1(\boldsymbol{\omega}, J_\varepsilon\, dx')$. The inequalities*

$$\|\boldsymbol{u}\|_{L_2(\boldsymbol{\omega}, J_\varepsilon\, dx')} \leqslant C\|\boldsymbol{u}\|_{W_2^1(\boldsymbol{\omega})}, \tag{4-3}$$

$$\|\boldsymbol{u}\|_{L_2(\boldsymbol{\omega}_{\varepsilon^{4/3}}, J_\varepsilon\, dx')} \leqslant C\varepsilon^{2/3}\|\boldsymbol{u}\|_{W_2^1(\boldsymbol{\omega})}, \tag{4-4}$$

$$\|\boldsymbol{u}\|_{L_2(\boldsymbol{\omega}_{\varepsilon^{4/3}})} \leqslant C\varepsilon^{2/3}\|\mathcal{I}_\varepsilon \boldsymbol{u}\|_{W_2^1(\mathcal{S}_\varepsilon)}, \tag{4-5}$$

$$\|\boldsymbol{u}\|_{W_2^1(\boldsymbol{\omega}, J_\varepsilon\, dx')} \leqslant C\|\boldsymbol{u}\|_{W_2^2(\boldsymbol{\omega})},$$

$$\|\boldsymbol{u}\|_{W_2^1(\omega_{\varepsilon^{4/3}}, J_\varepsilon\, dx')} \leqslant C\varepsilon^{2/3}\|\boldsymbol{u}\|_{W_2^2(\boldsymbol{\omega})} \tag{4-6}$$

hold true, where C denotes positive constants independent of ε and $\boldsymbol{u}$.

Proof. Let $\boldsymbol{u} \in W_2^1(\boldsymbol{\omega})$; then $u_\pm \in W_2^1(\boldsymbol{\omega})$, and for almost all $P \in \partial\omega$ the function $u_\pm(P + \cdot\,\nu(P))$ belongs to $W_2^1(0, \tau_0)$. Let $\chi = \chi(\tau)$ be an infinitely differentiable cut-off function vanishing as $\tau \geqslant \tau_0$ and being one as $\tau \leqslant \tau_0/2$. Then $u_\pm = u_\pm\chi$ for $\tau \in [0, \tau_0/2]$, and

$$u_\pm = \int_{\tau_0}^{\tau} \frac{\partial(u_\pm\chi)}{\partial\tau}\, d\tau, \quad |u_\pm(P + \tau\nu(P))|^2 \leqslant C\|u_\pm(P + \cdot\,\nu(P))\|^2_{W_2^1(0,\tau_0)}, \quad \tau \in [0, \tau_0/2],$$

where C is a positive constant independent of P and $u_\pm$. We multiply the last inequality by $J_\varepsilon^\pm$, integrate over $\partial\omega$, and take into account (3-5) to obtain

$$\int_{\partial\omega} \big|u_\pm(P + \tau\nu(P))\big|^2 |\det{}^{-1} \mathrm{M}|\, d\omega \leqslant C\|u_\pm\|^2_{W_2^1(\omega^{\tau_0})},$$

where C is a positive constant independent of $P \in \partial\omega$ and $u_\pm$. The above estimate, inequality (3-6), the definition (3-2) of $J_\varepsilon^\pm$ and the smoothness of $h_\pm$ imply

$$\int_\omega |u_\pm|^2 J_\varepsilon^\pm\, dx' = \int_{\omega_\delta} |u_\pm|^2 J_\varepsilon^\pm\, dx' + \int_{\omega^\delta} |u_\pm|^2 J_\varepsilon^\pm\, dx', \quad \delta \in (0, \tau_0/2],$$

$$\int_{\omega^\delta} |u_\pm|^2 J_\varepsilon^\pm\, dx' \leqslant C(\delta)\|u_\pm\|^2_{L_2(\omega^\delta)},$$

$$\int_{\omega_\delta} |u_\pm|^2 J_\varepsilon^\pm\, dx' = \int_0^\delta d\tau \int_{\partial\omega} |u_\pm|^2 J_\varepsilon^\pm |\det{}^{-1} \mathrm{M}|\, d\omega \leqslant C\|u_\pm\|^2_{W_2^1(\omega)} \int_0^\delta \sqrt{1 + C_4\varepsilon^2\tau^{-1}}\, d\tau, \tag{4-7}$$

where the constants C and $C(\delta)$ are independent of ε and $u_\pm$, and C is independent of δ. Taking $\delta = \tau_0/2$, we see that $\boldsymbol{u} \in L_2(\boldsymbol{\omega}, J_\varepsilon\, dx')$ and thus the estimate (4-3) holds. If we now take $\delta = \varepsilon^{4/3}$ in (4-7) instead and use the identity

$$\int_0^\delta \sqrt{1 + \varepsilon^2 C_4\tau^{-1}}\, d\tau = \mathring{J}_\varepsilon^\pm(\delta) := \sqrt{\delta^2 + C_4\varepsilon^2\delta} + \frac{C_4}{2}\varepsilon^2 \ln \frac{C_4\varepsilon^2 + 2\delta + 2\sqrt{\delta^2 + C_4\varepsilon^2\delta}}{C_4\varepsilon^2},$$

we obtain (4-4).

Let us prove (4-5). We integrate by parts as follows:

$$\int_{\omega_{\varepsilon^{4/3}}} |u_\pm|^2 J_\varepsilon^\pm\, dx' \leqslant C \int_{\partial\omega} d\omega \int_0^{\varepsilon^{4/3}} |u_\pm|^2 J_\varepsilon^\pm\, d\tau,$$

$$\int_0^{\varepsilon^{4/3}} |u_\pm|^2 J_\varepsilon^\pm\, d\tau = |u_\pm|^2 \mathring{J}_\varepsilon^\pm \big|_{\tau=0}^{\tau=\varepsilon^{4/3}} - 2\int_0^{\varepsilon^{4/3}} \mathring{J}_\varepsilon^\pm(\tau)\operatorname{Re} u_\pm \frac{\partial \overline{u}_\pm}{\partial\tau}\, d\tau$$

$$\leqslant \mathring{J}_\varepsilon^\pm(\varepsilon^{4/3})\Big(|u_\pm|^2\big|_{\tau=\varepsilon^{4/3}} + \int_0^{\varepsilon^{4/3}} |u_\pm|^2 J_\varepsilon^\pm\, d\tau + \int_0^{\varepsilon^{4/3}} \frac{1}{J_\varepsilon^\pm}\Big|\frac{\partial u_\pm}{\partial\tau}\Big|^2 d\tau\Big),$$

$$\int_{\omega_{\varepsilon^{4/3}}} |u_\pm|^2 J_\varepsilon^\pm\, dx' \leqslant C\varepsilon^{4/3}\Big(\int_{\partial\omega} |u_\pm|^2\big|_{\tau=\varepsilon^{4/3}}\, d\omega + \int_{\omega_{\varepsilon^{4/3}}} \Big(\frac{1}{J_\varepsilon^\pm}|\nabla_{x'}u_\pm|^2 + J_\varepsilon^\pm |u_\pm|^2\Big)\, dx'\Big).$$

By the embedding of $W_2^1(\omega^{\varepsilon^{4/3}})$ into $L_2(\{x : \tau = \varepsilon^{4/3}\})$ we have the estimate

$$\int_{\partial\omega} |u_\pm|^2\big|_{\tau=\varepsilon^{4/3}}\, d\omega \leqslant C\|u_\pm\|^2_{W_2^1(\omega^{\varepsilon^{4/3}})} \leqslant C\|\mathcal{I}_\varepsilon \boldsymbol{u}\|^2_{W_2^1(\mathcal{S}_\varepsilon)},$$

where the constants C are independent of ε and $\boldsymbol{u}$. These two last estimates together with (4-2) yield (4-5).

To prove the second part of the lemma related to the case $\boldsymbol{u} \in W_2^2(\omega)$ it is sufficient to note that since $u_\pm, \nabla_{x'}u_\pm \in W_2^1(\omega)$, by the first part of the lemma these functions belong to $L_2(\omega, J_\varepsilon^\pm\, dx')$, and the estimates (4-3), (4-4) are valid for $\boldsymbol{u}$ replaced by $\nabla_{x'}\boldsymbol{u}$. This completes the proof. □

Proof of Theorem 2.1. Let $f \in L_2(\mathcal{S}_\varepsilon)$; then $\boldsymbol{f} := \mathcal{I}_\varepsilon f \in L_2(\boldsymbol{\omega}, J_\varepsilon\, dx') \subset L_2(\boldsymbol{\omega})$. Let $u^{(\varepsilon)} := (\mathcal{H}_\varepsilon - z)^{-1} f$, $\boldsymbol{u}^{(0)} := (\mathcal{H}_0 - z)^{-1}\mathcal{I}_\varepsilon^{-1} f$. By the definitions of $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ we have

$$\mathfrak{h}_\varepsilon[u^{(\varepsilon)}, \varphi] - z(u^{(\varepsilon)}, \varphi)_{L_2(\mathcal{S}_\varepsilon)} = (f, \varphi)_{L_2(\mathcal{S}_\varepsilon)} \quad \text{for each } \varphi \in W_2^1(\mathcal{S}_\varepsilon), \tag{4-8}$$

$$\mathfrak{h}_0[\boldsymbol{u}^{(0)}, \boldsymbol{\varphi}] - z(\boldsymbol{u}^{(0)}, \boldsymbol{\varphi})_{L_2(\boldsymbol{\omega})} = (\boldsymbol{f}, \boldsymbol{\varphi})_{L_2(\boldsymbol{\omega})} \quad \text{for each } \varphi \in W_2^1(\boldsymbol{\omega}). \tag{4-9}$$

Since $\boldsymbol{u}^{(0)} \in W_2^2(\boldsymbol{\omega})$, by Lemmas 3.1 and 4.4, $u^{(0)} := \mathcal{I}_\varepsilon \boldsymbol{u}^{(0)} \in W_2^1(\mathcal{S}_\varepsilon)$. Hence, $v^{(\varepsilon)} := u^{(\varepsilon)} - u^{(0)} \in W_2^1(S_\varepsilon)$ and this can be used as a test function in (4-8):

$$\mathfrak{h}_\varepsilon[u^{(\varepsilon)}, v^{(\varepsilon)}] - z(u^{(\varepsilon)}, v^{(\varepsilon)})_{L_2(S_\varepsilon)} = (f, v^{(\varepsilon)})_{L_2(S_\varepsilon)}.$$

The identity $u^{(\varepsilon)} = v^{(\varepsilon)} + u^{(0)}$ yields

$$\|\nabla v^{(\varepsilon)}\|^2_{L_2(S_\varepsilon)} - z\|v^{(\varepsilon)}\|^2_{L_2(S_\varepsilon)} = (f, v_\varepsilon)_{L_2(S_\varepsilon)} - (\nabla u^{(0)}, \nabla v^{(\varepsilon)})_{L_2(S_\varepsilon)} + z(u^{(0)}, v^{(\varepsilon)})_{L_2(S_\varepsilon)}. \tag{4-10}$$

We parametrize S_ε as $x' = x'$, $x_{n+1} = \pm\varepsilon h_\pm(x')$, and use the definition of the scalar product of $\nabla u^{(0)}$ and $\nabla v^{(\varepsilon)}$ in $L_2(S_\varepsilon)$. It implies

$$\begin{aligned}
&(f, v^{(\varepsilon)})_{L_2(S_\varepsilon)} - (\nabla u^{(0)}, \nabla v^{(\varepsilon)})_{L_2(S_\varepsilon)} + z(u^{(0)}, v^{(\varepsilon)})_{L_2(S_\varepsilon)} \\
&\quad = (f_+, J_\varepsilon^+ v_+^{(\varepsilon)})_{L_2(\omega_+)} + (f_-, J_\varepsilon^- v_-^{(\varepsilon)})_{L_2(\omega_-)} - \big((J_\varepsilon^+ G_+^{-1}\nabla_{x'}u_+^{(0)}, \nabla_{x'}v_+^{(\varepsilon)})_{L_2(\omega_+)} \\
&\qquad + (J_\varepsilon^- G_-^{-1}\nabla_{x'}u_-^{(0)}, \nabla_{x'}v_-^{(\varepsilon)})_{L_2(\omega_-)}\big) + z(u_+^{(0)}, J_\varepsilon^+ v_+^{(\varepsilon)})_{L_2(\omega_+)} + z(u_-^{(0)}, J_\varepsilon^- v_-^{(\varepsilon)})_{L_2(\omega_-)},
\end{aligned}$$

where $\boldsymbol{v}^{(\varepsilon)}=(v_+^{(\varepsilon)},v_-^{(\varepsilon)})=\mathcal{I}_\varepsilon^{-1}v^{(\varepsilon)}$ and $G_\pm^{ij}$ are the entries of the inverse matrix $G_\pm^{-1}$. We substitute the last formula into (4-10) and then sum it with (4-9), where we take $\boldsymbol{\varphi}=\boldsymbol{v}^{(\varepsilon)}\in W_2^1(\boldsymbol{\omega},J_\varepsilon\,dx')\subset W_2^1(\boldsymbol{\omega})$:

$$\|\nabla v^{(\varepsilon)}\|_{L_2(S_\varepsilon)}^2-z\|v^{(\varepsilon)}\|_{L_2(S_\varepsilon)}^2=R^++R^-, \tag{4-11}$$

$$R^\pm:=(f_\pm,(J_\varepsilon^\pm-1)v_\pm^{(\varepsilon)})_{L_2(\omega)}-(J_\varepsilon^\pm G_\pm^{-1}\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega_\pm)}$$
$$-(\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega)}+z(u_\pm^{(0)},(J_\varepsilon^\pm-1)v_\pm^{(\varepsilon)})_{L_2(\omega)}.$$

Let us estimate $R^\pm$ which we shall write as

$$R^\pm=R_1^\pm+R_2^\pm, \tag{4-12}$$

where

$$R_1^\pm:=(f_\pm,(J_\varepsilon^\pm-1)v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)}-(J_\varepsilon^\pm G_\pm^{-1}\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)}$$
$$-(\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)}+z(u_\pm^{(0)},(J_\varepsilon^\pm-1)v^{(\varepsilon)})_{L_2(\omega^\delta)},$$
$$R_2^\pm:=(f_\pm,(J_\varepsilon^\pm-1)v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)}-(J_\varepsilon^\pm G_\pm^{-1}\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)}$$
$$-(\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)}+z(u_\pm^{(0)},(J_\varepsilon^\pm-1)v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)},$$

and $\delta:=\varepsilon^{4/3}$. As $x'\in\omega_\delta$, by (3-6) we have

$$\varepsilon^2|\nabla_{x'}h_\pm|^2\leqslant C\varepsilon^{2/3},\quad \|\mathrm{G}_\pm^{-1}-\mathrm{E}\|\leqslant C\varepsilon^{2/3},\quad |J_\varepsilon^\pm-1|\leqslant C\varepsilon^{2/3},\quad |(J_\varepsilon^\pm)^{-1}-1|\leqslant C\varepsilon^{2/3}.$$

Hereinafter by C we indicate nonessential positive constants independent of ε, $u^{(\varepsilon)}$, $\boldsymbol{u}^{(0)}$, and f. Hence, by Lemmas 3.1, 4.4 and Schwarz's inequality,

$$\big|(f_\pm,(J_\varepsilon^\pm-1)v_\pm^{(\varepsilon)})_{L_2(\omega_\delta)}\big|\leqslant C\varepsilon^{2/3}\|f_\pm\|_{L_2(\omega,J_\varepsilon^\pm dx')}\|v_\pm^{(\varepsilon)}\|_{L_2(\omega,J_\varepsilon^\pm dx')}\leqslant C\varepsilon^{2/3}\|f\|_{L_2(S_\varepsilon)}\|v^{(\varepsilon)}\|_{L_2(S_\varepsilon)},$$
$$\big|z(u_\pm^{(0)},(J_\varepsilon^\pm-1)v_\pm^{(\varepsilon)})_{L_2(\omega_\delta)}\big|\leqslant C\varepsilon^{2/3}\|u^{(0)}\|_{L_2(\omega)}\|v^{(\varepsilon)}\|_{L_2(S_\varepsilon)},$$
$$\big|(\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega_\delta)}-(J_\varepsilon^\pm G_\pm^{-1}\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega_\delta)}\big|$$
$$\leqslant C\varepsilon^{2/3}\|\boldsymbol{u}^{(0)}\|_{W_2^1(\boldsymbol{\omega})}\|\nabla_{x'}v_\pm^{(\varepsilon)}\|_{L_2(\omega_\delta)}\leqslant C\varepsilon^{2/3}\|\boldsymbol{u}^{(0)}\|_{W_2^1(\boldsymbol{\omega})}\|J_\varepsilon^{-\frac12}\nabla_{x'}\boldsymbol{v}^{(\varepsilon)}\|_{L_2(\boldsymbol{\omega}_\delta)}$$
$$\leqslant C\varepsilon^{2/3}\|\boldsymbol{u}^{(0)}\|_{W_2^1(\boldsymbol{\omega})}\|J_\varepsilon^{-1}\nabla_{x'}\boldsymbol{v}^{(\varepsilon)}\|_{L_2(\boldsymbol{\omega}_\delta,J_\varepsilon\,dx')}\leqslant C\varepsilon^{2/3}\|\boldsymbol{u}^{(0)}\|_{W_2^1(\boldsymbol{\omega})}\|\nabla v^{(\varepsilon)}\|_{L_2(S_\varepsilon)},$$

and therefore

$$|R_1^++R_1^-|\leqslant C\varepsilon^{2/3}\|\boldsymbol{u}^{(0)}\|_{W_2^1(\boldsymbol{\omega})}\|v^{(\varepsilon)}\|_{W_2^1(S_\varepsilon)}. \tag{4-13}$$

To estimate $R_2^\pm$ we employ (4-3), (4-4), (4-5). We begin with the first term in $R_2^\pm$ applying again Schwarz's inequality and (4-5) to obtain

$$|(f_\pm,(J_\varepsilon^\pm-1)v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)}|\leqslant\|f_\pm\|_{L_2(\omega^\delta,J_\varepsilon^\pm dx')}\|\big(1-(J_\varepsilon^\pm)^{-1}\big)v_\pm^{(\varepsilon)}\|_{L_2(\omega^\delta,J_\varepsilon^\pm dx')}$$
$$\leqslant\|f\|_{L_2(S_\varepsilon)}\|v_\pm^{(\varepsilon)}\|_{L_2(\omega^\delta,J_\varepsilon^\pm dx')}\leqslant C\varepsilon^{2/3}\|f\|_{L_2(S_\varepsilon)}\|v^{(\varepsilon)}\|_{W_2^1(S_\varepsilon)}. \tag{4-14}$$

Employing (4-2), (4-3) and (4-5) in the same way we get two more estimates:

$$\begin{aligned}\big|z\big(u_\pm^{(0)},(J_\varepsilon^\pm-1)v^{(\varepsilon)}\big)_{L_2(\omega^\delta)}\big| &\leqslant C\|u_\pm^{(0)}\|_{L_2(\omega^\delta,J_\varepsilon^\pm\,dx')}\|v_\pm^{(\varepsilon)}\|_{L_2(\omega^\delta,J_\varepsilon^\pm\,dx')}\\ &\leqslant C\varepsilon^{2/3}\|\boldsymbol{u}^{(0)}\|_{W_2^1(\boldsymbol{\omega})}\|v^{(\varepsilon)}\|_{W_2^1(S_\varepsilon)},\\ \big|(\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)}\big| &\leqslant \|(J_\varepsilon^\pm)^{\frac12}\nabla_{x'}u_\pm^{(0)}\|_{L_2(\omega^\delta)}\|(J_\varepsilon^\pm)^{-\frac12}\nabla_{x'}v_\pm^{(\varepsilon)}\|_{L_2(\omega^\delta)}\\ &\leqslant C\varepsilon^{2/3}\|\boldsymbol{u}^{(0)}\|_{W_2^2(\boldsymbol{\omega})}\|\nabla v^{(\varepsilon)}\|_{L_2(S_\varepsilon)}.\end{aligned}\tag{4-15}$$

Since

$$(G_\pm^{-1}\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{\mathbb{R}^n}=\nabla\mathcal{I}_\varepsilon\boldsymbol{u}^{(0)}\cdot\nabla v^{(\varepsilon)},$$

by Schwarz's inequality we have

$$\begin{aligned}\big|(G_\pm^{-1}\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)}\big| &\leqslant \|\nabla v^{(\varepsilon)}\|_{L_2(S_\varepsilon)}(G_\pm^{-1}\nabla_{x'}u_\pm^{(0)},\nabla_{x'}u_\pm^{(0)})^{\frac12}_{L_2(\omega^\delta)}\\ &\leqslant \|\nabla v^{(\varepsilon)}\|_{L_2(S_\varepsilon)}\|(J_\varepsilon^\pm)^{\frac12}\nabla_{x'}u_\pm^{(0)}\|_{L_2(\omega^\delta)}.\end{aligned}$$

Here we have used the inequality

$$\sum_{i,j=1}^n G_\pm^{ij}\xi_i\xi_j\leqslant\sum_{i=1}^n|\xi_i|^2,$$

which follows from Lemma 3.1. Using (4-6) we get

$$\big|(G_\pm^{-1}\nabla_{x'}u_\pm^{(0)},\nabla_{x'}v_\pm^{(\varepsilon)})_{L_2(\omega^\delta)}\big|\leqslant\|\nabla v^{(\varepsilon)}\|_{L_2(S_\varepsilon)}\|\boldsymbol{u}^{(0)}\|_{W_2^1(\boldsymbol{\omega}^\delta)}\leqslant C\varepsilon^{2/3}\|\nabla v^{(\varepsilon)}\|_{L_2(S_\varepsilon)}\|\boldsymbol{u}^{(0)}\|_{W_2^2(\boldsymbol{\omega})},$$

which with (4-14) and (4-15) yields

$$|R_2^++R_2^-|\leqslant C\varepsilon^{2/3}\|\boldsymbol{u}^{(0)}\|_{W_2^2(\boldsymbol{\omega})}\|v^{(\varepsilon)}\|_{W_2^1(S_\varepsilon)}.$$

Together with (4-1), (4-11), (4-12), (4-13) it follows that

$$\big|\|\nabla v^{(\varepsilon)}\|^2_{L_2(S_\varepsilon)}-z\|v^{(\varepsilon)}\|^2_{L_2(S_\varepsilon)}\big|\leqslant C\varepsilon^{2/3}\|\boldsymbol{u}^{(0)}\|_{W_2^2(\boldsymbol{\omega})}\|v^{(\varepsilon)}\|_{W_2^1(S_\varepsilon)}\leqslant C\varepsilon^{2/3}\|\boldsymbol{f}\|_{L_2(\boldsymbol{\omega})}\|v^{(\varepsilon)}\|_{W_2^1(S_\varepsilon)}.$$

Since

$$\big|\|\nabla v^{(\varepsilon)}\|^2_{L_2(S_\varepsilon)}-z\|v^{(\varepsilon)}\|^2_{L_2(S_\varepsilon)}\big|\geqslant C\|v^{(\varepsilon)}\|^2_{W_2^1(S_\varepsilon)},$$

we arrive at (2-6), completing the proof. □

Remark 4.5. The proof above uses the estimates from Lemma 4.4 which include a measure of the boundary behavior by means of the weight function J_ε. A different approach which may also be used to prove convergence of the resolvent in similar situations is based on inequalities of Hardy type instead, possibly allowing for a better control of the behavior near the boundary — see [Krejčiřík and Zuazua 2010] for an illustration of this principle.

In the proof of Theorem 2.4 in the next section we shall use the following auxiliary lemma which is convenient to prove in this section.

Lemma 4.6. *Let λ be a m-multiple eigenvalue of $\mathcal{H}_0$, and $\lambda_i(\varepsilon)$, $i = 1, \dots, m$, be the eigenvalues of $\mathcal{H}_\varepsilon$ taken counting multiplicity and converging to λ, and $\psi_\varepsilon^{(i)}$ be the associated eigenfunctions orthonormalized in $L_2(S_\varepsilon)$. For z close to λ the representation*

$$(\mathcal{H}_\varepsilon - z)^{-1} = \sum_{i=1}^{m} \frac{\psi_\varepsilon^{(i)}}{\lambda_i(\varepsilon) - z}(\,\cdot\,, \psi_\varepsilon^{(i)})_{L_2(S_\varepsilon)} + \mathcal{R}_\varepsilon(z)$$

holds true, where the operator $\mathcal{R}_\varepsilon(z) : L_2(S_\varepsilon) \to W_2^1(S_\varepsilon)$ is bounded uniformly in ε and z. The range of $\mathcal{R}_\varepsilon(z)$ is orthogonal to all $\psi_\varepsilon^{(i)}$, $i = 1, \dots, m$.

Proof. We choose a fixed δ so that the disk $B_\delta(\lambda) := \{z : |z - \lambda| < \delta\}$ contains no eigenvalues of $\mathcal{H}_0$ except λ and

$$\operatorname{dist}\{\partial B_\delta(\lambda), \sigma_d(\mathcal{H}_0)\} \geqslant \delta.$$

Then, by Theorem 2.3, for sufficiently small ε this disk contains the eigenvalues $\lambda_i(\varepsilon)$, $i = 1, \dots, m$, and no other eigenvalues of $\mathcal{H}_\varepsilon$, and

$$\operatorname{dist}\bigl\{B_\delta(\lambda), \sigma_d(\mathcal{H}_\varepsilon) \setminus \{\lambda_i(\varepsilon), i = 1, \dots, m\}\bigr\} \geqslant \frac{\delta}{2}. \tag{4-16}$$

Denote by V_ε the orthogonal complement to $\psi_\varepsilon^{(i)}$, $i = 1, \dots, m$, in $L_2(S_\varepsilon)$. By [Kato 1966, Chapter V, Section 3.5, Equations (3.21)] the representation (3-29) holds true, where $\mathcal{R}_\varepsilon(z)$ is the part of the resolvent $(\mathcal{H}_\varepsilon - z)^{-1}$ acting in V_ε and

$$\|\mathcal{R}_\varepsilon(z)\|_{V_\varepsilon \to V_\varepsilon} \leqslant \frac{1}{\operatorname{dist}\bigl\{B_\delta(\lambda), \sigma_d(\mathcal{H}_\varepsilon) \setminus \{\lambda_i(\varepsilon), i = 1, \dots, m\}\bigr\}} \leqslant \frac{2}{\delta} \tag{4-17}$$

for $z \in B_\delta(\lambda)$, where we have used (4-16). Hence, the range of $\mathcal{R}_\varepsilon(z)$ is orthogonal to $\psi_\varepsilon^{(i)}$, $i = 1, \dots, m$. It is easy to check that the function $u_\varepsilon := \mathcal{R}_\varepsilon(z) f$, $f \in L_2(S_\varepsilon)$, solves the equation

$$(\mathcal{H}_\varepsilon - z)u_\varepsilon = f_\varepsilon, \quad f_\varepsilon := f - \sum_{i=1}^{m} \psi_\varepsilon^{(i)}(f, \psi_\varepsilon^{(i)})_{L_2(S_\varepsilon)}, \quad \|f_\varepsilon\|_{L_2(S_\varepsilon)} \leqslant \|f\|_{L_2(S_\varepsilon)}.$$

Hence, by the definition of $\mathcal{H}_\varepsilon$ and (4-17),

$$\|\nabla u_\varepsilon\|^2_{L_2(S_\varepsilon)} = z\|u_\varepsilon\|^2_{L_2(S_\varepsilon)} + (f_\varepsilon, u_\varepsilon)_{L_2(S_\varepsilon)} \leqslant |z|\|u_\varepsilon\|^2_{L_2(S_\varepsilon)} + \|f_\varepsilon\|_{L_2(S_\varepsilon)}\|u_\varepsilon\|_{L_2(S_\varepsilon)} \leqslant C(\delta)\|f\|^2_{L_2(S_\varepsilon)},$$

where the constant $C(\delta)$ is independent of ε and f. The last estimate and (4-17) complete the proof. □

5. Asymptotic expansions

In this section we give the proof of Theorem 2.4 which will be divided into two parts. We first build the asymptotic expansions formally, where the core of the formal construction is the method of matching asymptotic expansions [Il′in 1992]. The second part is devoted to the justification of the asymptotics, i.e., obtaining estimates for the error terms.

The formal construction consists of determining the outer and inner expansions on the base of the perturbed eigenvalue problem and the matching of these expansions. The outer expansion is used to approximate the perturbed eigenfunctions outside a small neighborhood of $\partial\omega$. It is constructed in terms of the variables x' using the first parametrization of $\mathcal{S}_\varepsilon$ given in the previous sections. In a vicinity of $\partial\omega$ the perturbed eigenfunctions are approximated by the inner expansion which is based on the second parametrization of $\mathcal{S}_\varepsilon$ and is constructed in terms of the variables (ξ, s).

Outer expansion: First term. By Theorem 2.3 there exist exactly m eigenvalues of $\mathcal{H}_\varepsilon$ converging to λ counting multiplicities. We denote these eigenvalues by $\lambda_k(\varepsilon)$, $k = 1, \dots, m$, while the symbols $\psi_\varepsilon^{(k)}$ will denote the associated eigenfunctions. We construct the asymptotics for $\lambda_k(\varepsilon)$ as

$$\lambda_k(\varepsilon) = \lambda + \varepsilon^2 \ln\varepsilon\, \mu_k\left(\frac{1}{\ln\varepsilon}\right) + \cdots. \tag{5-1}$$

Hereinafter terms like $\ln\varepsilon A$ are understood as $(\ln\varepsilon)A$. In accordance with the method of matching asymptotic expansions we form the asymptotics for $\psi_\varepsilon^{(k)}$ as the sum of outer and inner expansions. The outer expansion is built as

$$\psi_{\varepsilon,\text{ex}}^{(k)} = \mathcal{I}_\varepsilon(\boldsymbol{\psi}_k + \varepsilon^2 \ln\varepsilon\, \boldsymbol{\phi}_k + \cdots), \tag{5-2}$$

where $\boldsymbol{\phi}_k = (\phi_+^{(k)}, \phi_-^{(k)})$, $\phi_\pm^{(k)} = \phi_\pm^{(k)}(x', \varepsilon)$, and the eigenfunctions $\boldsymbol{\psi}_k$ are chosen as described before the statement of Theorem 2.4. We also recall that these functions depend on ε in the case where λ is a multiple eigenvalue.

We substitute the identities (5-1), (5-2), and (3-3) into the eigenvalue equation

$$\mathcal{H}_\varepsilon \psi_\varepsilon^{(k)} = \lambda_k(\varepsilon)\psi_\varepsilon^{(k)}, \tag{5-3}$$

and take into account the eigenvalue equations for $\boldsymbol{\psi}_i$. It implies the equations for $\boldsymbol{\phi}_k$, namely,

$$\begin{gathered}(-\Delta_{x'} - \lambda)\phi_\pm^{(k)} = \frac{1}{\ln\varepsilon} f_{2,\pm}^{(k)} + \mu_k \psi_\pm^{(k)}, \quad x' \in \omega_\pm, \qquad f_{2,\pm}^{(k)} := \mathcal{H}_\pm^{(2)} \psi_\pm^{(k)}, \\ \mathcal{H}_\pm^{(2)} := -\operatorname{div}_{x'} \mathrm{Q}_\pm \nabla_{x'} - \frac{|\nabla_{x'} h_\pm|^2}{2}\Delta_{x'} + \tfrac{1}{2}\operatorname{div}_{x'} |\nabla_{x'} h_\pm|^2 \nabla_{x'}.\end{gathered} \tag{5-4}$$

The functions $\psi_\pm^{(i)}$ are infinitely differentiable in $\overline{\omega}_\pm$, and thus

$$\psi_\pm^{(k)}(x', \varepsilon) = \Psi_k^{(0)}(P, \varepsilon) \pm \Psi_k^{(1)}(P, \varepsilon)\tau + \Psi_k^{(2,\pm)}(P, \varepsilon)\tau^2 + \mathcal{O}(\tau^3), \quad P \in \partial\omega, \tag{5-5}$$

as $\tau \to +0$, where, by the definition of the domain of $\mathcal{H}_0$,

$$\Psi_k^{(0)} := \psi_+^{(k)}\big|_{\partial\omega} = \psi_-^{(k)}\big|_{\partial\omega}, \quad \Psi_k^{(1)} := \frac{\partial\psi_+^{(k)}}{\partial\tau}\bigg|_{\partial\omega} = -\frac{\partial\psi_-^{(k)}}{\partial\tau}\bigg|_{\partial\omega}, \quad \Psi_k^{(2,\pm)} := \frac{1}{2}\frac{\partial^2\psi_\pm^{(k)}}{\partial\tau^2}\bigg|_{\partial\omega},$$
$$\Psi_k^{(j)}, \Psi_k^{(2,\pm)} \in C^\infty(\partial\omega).$$

The functions $\Psi_k^{(i)}$ depend on ε only if λ is a multiple eigenvalue, since the same is true for the functions $\boldsymbol{\psi}_k$.

In view of the identity (3-12) we rewrite (5-5) as

$$\begin{aligned}\psi_\pm^{(k)}(x',\varepsilon)&=\Psi_k^{(0)}(P,\varepsilon)\pm\Psi_k^{(1)}(P,\varepsilon)\zeta^2+\Psi_k^{(2,\pm)}(P,\varepsilon)\zeta^4+\mathcal{O}(\zeta^6), && \zeta\to+0.\\ \psi_\pm^{(k)}(x',\varepsilon)&=\Psi_k^{(0)}(P,\varepsilon)\pm\varepsilon^2\Psi_k^{(1)}(P,\varepsilon)\xi^2+\varepsilon^4\Psi_k^{(2,\pm)}(P,\varepsilon)\xi^4+\mathcal{O}(\varepsilon^6\xi^6), && \varepsilon\xi\to 0.\end{aligned}\tag{5-6}$$

Inner expansion. In accordance with the method of matching asymptotic expansions the identities (5-2), (5-6) yield that the inner expansion for the eigenfunctions $\psi_\varepsilon^{(k)}$ should read

$$\psi_{\varepsilon,\mathrm{in}}^{(k)}(\xi,P,\varepsilon)=\sum_{i=0}^{4}\varepsilon^i v_i^{(k)}(\xi,P,\varepsilon)+\cdots,\tag{5-7}$$

where the coefficients must satisfy the following asymptotics as $\xi\to\pm\infty$:

$$v_0^{(k)}(\xi,P,\varepsilon)=\Psi_k^{(0)}(P,\varepsilon)+o(1),\tag{5-8}$$

$$v_1^{(k)}(\xi,P,\varepsilon)=o(|\xi|),\tag{5-9}$$

$$\begin{aligned}v_2^{(k)}(\xi,P,\varepsilon)&=\pm\Psi_k^{(1)}(P,\varepsilon)\xi^2+o(|\xi|^2),\\ v_3^{(k)}(\xi,P,\varepsilon)&=o(|\xi|^3),\\ v_4^{(k)}(\xi,P,\varepsilon)&=\Psi_k^{(2,\pm)}(P,\varepsilon)\xi^4+o(|\xi|^4).\end{aligned}\tag{5-10}$$

These asymptotics mean that the first term of the outer expansion is matched with the inner expansion.

We substitute (5-1), (5-7), (3-25), (3-21) into the eigenvalue equation (5-3) and equate the coefficients of ε^{-4}. This implies the equation for $v_0^{(k)}$:

$$\mathcal{L}_{-4}v_0^{(k)}\equiv-\frac{1}{\sqrt{4\xi^2+b_1^2}}\frac{\partial}{\partial\xi}\frac{1}{\sqrt{4\xi^2+b_1^2}}\frac{\partial v_0^{(k)}}{\partial\xi}=0\quad\text{on }\mathbb{R}\times\partial\omega.$$

The solution to the last equation satisfying (5-8) is obviously

$$v_0^{(k)}(\xi,P,\varepsilon)\equiv\Psi_k^{(0)}(P,\varepsilon).\tag{5-11}$$

We then substitute this identity and (5-1), (5-7), (3-25), (3-26), (3-27), (3-25) into (5-3) and equate the coefficients at ε^i, $i=-3,\dots,0$, leading us to the equations for $v_i^{(k)}$, $i=1,\dots,4$:

$$\mathcal{L}_{-4}v_1^{(k)}=0\qquad\text{on }\mathbb{R}\times\partial\omega,\tag{5-12}$$

$$\mathcal{L}_{-4}v_2^{(k)}=0\qquad\text{on }\mathbb{R}\times\partial\omega,\tag{5-13}$$

$$\mathcal{L}_{-4}v_3^{(k)}+\mathcal{L}_{-3}v_2^{(k)}+\mathcal{L}_{-2}v_1^{(k)}=0\qquad\text{on }\mathbb{R}\times\partial\omega,\tag{5-14}$$

$$\mathcal{L}_{-4}v_4^{(k)}+\mathcal{L}_{-3}v_3^{(k)}+\mathcal{L}_{-2}v_2^{(k)}+\mathcal{L}_{-1}v_1^{(k)}+\mathcal{L}_0v_0^{(k)}=\lambda v_0^{(k)}\qquad\text{on }\mathbb{R}\times\partial\omega,\tag{5-15}$$

where we have used that

$$\mathcal{L}_iv_0^{(k)}\equiv 0,\quad i=-3,\dots,-1,$$

due to (3-26), (3-27), (5-11). The only solution to (5-12) satisfying (5-9) is independent of ξ:

$$v_1^{(k)}(\xi, P, \varepsilon) \equiv C_1^{(k,0)}(P, \varepsilon), \tag{5-16}$$

where $C_1^{(k,0)}$ is an unknown function to be determined.

Equation (5-13) can be solved, and the solution satisfying (5-10) is

$$v_2^{(k)}(\xi, P, \varepsilon) = \Psi_k^{(1)}(P, \varepsilon) X_1(\xi, b_1(P)) + C_2^{(k,0)}(P, \varepsilon), \tag{5-17}$$

$$X_1(\xi, b) := \tfrac{1}{2}\xi(4\xi^2 + b^2)^{\frac{1}{2}} + \frac{b^2}{4}\ln\bigl(2\xi + (4\xi^2 + b^2)^{\frac{1}{2}}\bigr) - \frac{b^2}{4}\ln b, \tag{5-18}$$

where $C_2^{(k,0)}$ is an unknown function to be determined.

In view of (5-16), (5-17), (3-26), (3-27) and (5-13), Equation (5-14) may be written as

$$\beta_{-4}\frac{\partial}{\partial\xi}\beta_{-4}\frac{\partial v_3^{(k)}}{\partial\xi} = -\beta_{-4}\frac{\partial}{\partial\xi}\beta_{-3}\frac{\partial v_2^{(k)}}{\partial\xi} \quad \text{on } \mathbb{R}\times\partial\omega.$$

Employing the formulas (3-21), (5-17) and (5-18), we solve the last equation:

$$\begin{aligned} v_3^{(k)}(\xi, P, \varepsilon) &= \frac{\Psi_0^{(k,1)}(P, \varepsilon) b_1(P) b_2(P)}{2\beta_{-4}(\xi, P)} + C_3^{(k,1)}(P, \varepsilon) X_1(\xi) + C_3^{(k,0)}(P, \varepsilon) \\ &= \tfrac{1}{2}\Psi_k^{(1)}(P, \varepsilon) b_1(P) b_2(P)(4\xi^2 + b_1^2(P))^{\frac{1}{2}} + C_3^{(k,1)}(P, \varepsilon) X_1(\xi) + C_3^{(k,0)}(P, \varepsilon), \end{aligned} \tag{5-19}$$

where $C_3^{(k,1)}$ and $C_3^{(k,0)}$ are unknown functions to be determined.

We substitute (5-16), (5-17), (5-18), (5-19), (3-26), (3-27), (3-28), (3-19) and (3-21) into (5-15) and then solve it to obtain

$$\begin{aligned} v_4^{(k)} &= \tfrac{1}{16}\Psi_0^{(k,1)}\xi\biggl(K(4\xi^2 + b_1^2)^{\frac{3}{2}} + 12 b_1 b_3 (4\xi^2 + b_1^2)^{\frac{1}{2}} + \frac{8b_2^2(8\xi^2 + 3b_1^2)}{(4\xi^2 + b_1^2)^{\frac{1}{2}}}\biggr) \\ &\quad + \tfrac{1}{2}C_3^{(k,1)} b_1 b_2 (4\xi^2 + b_1^2)^{\frac{1}{2}} - \tfrac{1}{2}X_1^2(\Delta_{\partial\omega} + \lambda)\Psi_k^{(0)} + \tfrac{1}{2}X_2 b_1 \nabla b_1 \cdot \nabla\Psi_k^{(0)} + C_4^{(k,1)} X_1 + C_4^{(k,0)}, \end{aligned}$$

where $X_1 = X_1(\xi, b_1(P))$,

$$X_2 = X_2(\xi, b) := \xi^2 - b^2 X_3\biggl(\frac{2\xi + \sqrt{4\xi^2 + b^2}}{b}\biggr), \quad X_3(z) := \tfrac{1}{8}\ln^2 z + \frac{1}{16}\Bigl(z^2 - \frac{1}{z^2}\Bigr)\ln z - \frac{1}{32}\Bigl(z^2 + \frac{1}{z^2}\Bigr),$$

and $C_4^{(k,0)} = C_4^{(k,0)}(P, \varepsilon)$ and $C_4^{(k,1)} = C_4^{(k,1)}(P, \varepsilon)$ are unknown functions to be determined.

To determine the coefficient $\boldsymbol{\phi}^{(k)}$ in the outer expansion and the functions $C_i^{k,j}$ in the inner one, we should match the constructed functions $v_i^{(k)}$ with the outer expansion. In order to do it, we must find the asymptotics for the functions $v_i^{(k)}$ as $\xi \to \pm\infty$. We observe that the functions $X_1, X_2 \in C^\infty(\mathbb{R}\times(0, +\infty))$ satisfy the identities

$$\begin{aligned} X_1(\xi, b) &= \pm\xi^2 \pm \frac{b^2}{8}(2\ln|\xi| + 1 + 4\ln 2 - 2\ln b) + \mathcal{O}(\xi^{-2}), &\quad \xi \to \pm\infty, \\ X_2(\xi, b) &= \xi^2\bigl(\tfrac{3}{2} - 2\ln 2 + \ln b - \ln|\xi|\bigr) + \mathcal{O}(\ln^2|\xi|), &\quad \xi \to \pm\infty, \end{aligned}$$

uniformly in $b \geqslant b_0 > 0$, with b_0 any fixed constant. Taking these asymptotics into account, we write the asymptotics for $v_i^{(k)}$ as $\xi \to \pm\infty$ and then pass to the variables (τ, P):

$$\sum_{i=0}^{4}\varepsilon^i v_i^{(k)}(\xi,P,\varepsilon)=\Psi_k^{(0)}(P,\varepsilon)\pm\Psi_k^{(1)}(P,\varepsilon)\tau+\tfrac{1}{2}\big(\pm\Psi_k^{(1)}(P,\varepsilon)K(P)-\Delta_{\partial\omega}\Psi_k^{(0)}(P,\varepsilon)-\lambda\Psi_k^{(0)}(P,\varepsilon)\big)\tau^2$$
$$+\varepsilon(\pm C_3^{(k,1)}(P,\varepsilon)\tau+C_1^{(k,0)})+\varepsilon^2\big(\ln\varepsilon\, W_{2,1,\pm}^{(k)}(x',\varepsilon)+W_{2,0,\pm}^{(k)}(x',\varepsilon)\big)+\mathcal{O}(\varepsilon^3+\varepsilon^4\tau^{-1}),$$

where

$$W_{2,1,\pm}^{(k)} := \frac{1}{4}b_1^2\left(\mp\Psi_k^{(1)}+\tau\left(\Delta_{\partial\omega}+\frac{2}{b_1}\nabla b_1\cdot\nabla+\lambda\right)\Psi_k^{(0)}\right), \tag{5-20}$$

$$\begin{aligned}W_{2,0,\pm}^{(k)} :=\ & \pm\tfrac{1}{8}b_1^2\Psi_k^{(1)}\ln\tau\pm\frac{b_1^2}{8}(1+4\ln 2-2\ln b_1)\Psi_k^{(1)}+C_2^{(k,0)}+\Psi_k^{(1)}b_1b_2\tau^{1/2}\\ &-\tfrac{1}{8}b_1^2\tau\ln\tau\left(\Delta_{\partial\omega}+\frac{2}{b_1}\nabla b_1\cdot\nabla+\lambda\right)\Psi_k^{(0)}+\tau\big(-\tfrac{1}{8}b_1^2(1+4\ln 2-2\ln b_1)(\Delta_{\partial\omega}+\lambda)\Psi_k^{(0)}\\ &-\tfrac{1}{2}\big(2\ln 2-\ln b_1-\tfrac{3}{2}\big)b_1\nabla b_1\cdot\nabla\Psi_k^{(0)}\pm\tfrac{1}{16}(3Kb_1^2+32b_2^2+24b_1b_3)\Psi_k^{(1)}\pm C_4^{(k,1)}\big).\end{aligned} \tag{5-21}$$

Taking into account the obtained formulas and (5-2), in accordance with the method of matching asymptotic expansions we conclude that

$$C_3^{(k,1)}(P,\varepsilon)=C_1^{(k,0)}(P,\varepsilon)\equiv 0, \tag{5-22}$$

while the solutions to (5-4) should satisfy the asymptotics

$$\phi_\pm^{(k)}(x',\varepsilon)=W_{2,1,\pm}^{(k)}(x',\varepsilon)+\frac{1}{\ln\varepsilon}W_{2,0,\pm}^{(k)}(x',\varepsilon)+o(\tau),\quad \tau\to 0. \tag{5-23}$$

Moreover, the identity

$$\tfrac{1}{2}\big(\pm\Psi_k^{(1)}K-\Delta_{\partial\omega}\Psi_k^{(0)}-\lambda\Psi_k^{(0)}\big)=\Psi_k^{(2,\pm)} \tag{5-24}$$

should hold.

Outer expansion: Second term. We substitute (3-29) and (5-5) into the eigenvalue equation for $\psi_\pm^{(k)}$ and equate the coefficient of τ^0. This leads us to identity (5-24).

We proceed to the problem (5-4), (5-23). To study its solvability we shall make use of one more auxiliary lemma. Recall that the matrices M and $\widehat{\mathrm{M}}$ are defined in (3-4) and (3-30), respectively.

Lemma 5.1. *The functions $f_{2,\pm}^{(k)}$ introduced in (5-4) satisfy the hypothesis of Lemma 3.4. In particular, the asymptotics (3-33) holds true with*

$$f_{-2}^{\pm}=\pm\frac{b_1^2}{8\ln\varepsilon}\Psi_k^{(1)},\quad f_{-3/2}^{\pm}=\frac{b_1b_2}{4\ln\varepsilon}\Psi_k^{(1)},\quad f_{-1}^{\pm}=-\frac{b_1^2}{4\ln\varepsilon}\left(\Psi_k^{(2,\pm)}-\frac{1}{b_1}\nabla b_1\cdot\nabla\Psi_k^{(0)}\mp K\Psi_k^{(1)}\right). \tag{5-25}$$

Proof. We begin with an obvious identity:

$$f_{2,\pm}^{(k)}=\frac{1}{\ln\varepsilon}\big(-\operatorname{div}_{x'}\mathrm{Q}_\pm\nabla_{x'}\psi_\pm^{(k)}+\tfrac{1}{2}\big(\nabla_{x'}|\nabla_{x'}h_\pm|^2,\nabla_{x'}\psi_\pm^{(k)}\big)_{\mathbb{R}^n}\big), \tag{5-26}$$

which follows from the definition of $f^{(k)}_{2,\pm}$ in (5-4). To prove the lemma, we shall pass to the variables (τ, s) in the obtained identity. It follows from (3-7), (3-12) and the definition of S_ε that

$$h_\pm(x') = t, \quad \pm t > 0.$$

Hence, by (3-8), (3-10),

$$h_\pm(x') = b(\pm\sqrt{\tau}, P) = \sum_{i=1}^{\infty} b_i(P)(\pm\sqrt{\tau})^i, \quad \tau \to +0. \tag{5-27}$$

Thus, employing (3-4) and (5-26), we conclude that the functions $f^{(k)}_{2,0,\pm}$ satisfy the hypothesis of Lemma 3.4 and in particular the asymptotics (3-33) holds true. It remains to prove the identities (5-25).

It follows from (3-44) that

$$|\nabla_{x'} h_\pm|^2 = \left|\frac{\partial h_\pm}{\partial \tau}\right|^2 + \nabla h_\pm \cdot (\mathrm{E} - \tau \mathrm{B} \mathrm{G}^{-1}_{\partial\omega})^{-2} \nabla h_\pm. \tag{5-28}$$

We substitute (5-27) into the obtained identity and arrive at the asymptotics for $|\nabla_{x'} h_\pm|^2$:

$$|\nabla_{x'} h_\pm|^2 = \sum_{j=-2}^{\infty} h^\pm_{j/2}(P)\tau^{j/2}, \quad h^\pm_{-1} = \tfrac{1}{4} b_1^2, \quad h^\pm_{-1/2} = \pm b_1 b_2, \quad \tau \to +0. \tag{5-29}$$

Employing these formulas and (3-4), (3-30), (5-5) and (3-44) we rewrite the second term in the right-hand side of (5-26) as

$$\begin{aligned} \tfrac{1}{2}\big(\nabla_{x'}|\nabla_{x'} h_\pm|^2, \nabla_{x'}\psi^{(k)}_\pm\big)_{\mathbb{R}^n} &= \frac{1}{2}\frac{\partial |\nabla_{x'} h_\pm|^2}{\partial \tau}\frac{\partial \psi^{(k)}_\pm}{\partial \tau} + \tfrac{1}{2}\nabla|\nabla_{x'} h_\pm|^2 \cdot (\mathrm{E} - \tau \mathrm{B} \mathrm{G}^{-1}_{\partial\omega})^{-2} \nabla \psi^{(k)}_\pm \\ &= \sum_{j=-4}^{\infty} f^{\pm,2}_{j/2} \tau^{j/2}, \end{aligned} \tag{5-30}$$

where $f^{\pm,2}_{j/2} \in C^\infty(\partial\omega)$ are some functions, and, in particular,

$$f^{\pm,2}_{-2} = \mp\frac{1}{8\ln\varepsilon} b_1^2 \Psi^{(1)}_k, \quad f^{\pm,2}_{-3/2} = -\frac{1}{4\ln\varepsilon} b_1 b_2 \Psi^{(1)}_k, \quad f^{\pm,2}_{-1} = -\frac{b_1^2}{4\ln\varepsilon}\left(\Psi^{(2,\pm)}_k + \frac{1}{b_1}\nabla b_1 \cdot \nabla \Psi^{(0)}_k\right). \tag{5-31}$$

To obtain the same asymptotics for the first term in the right-hand side of (5-26), we employ first (3-43):

$$-\operatorname{div}_{x'} \mathrm{Q}_\pm \nabla_{x'} \psi^{(k)}_\pm = -\frac{1}{\det \mathrm{M}} \operatorname{div}_{(\tau,s)} (\det \mathrm{M}) \nabla_{(\tau,s)} h_\pm (\nabla_{(\tau,s)} h_\pm)^* \widehat{\mathrm{M}} \nabla_{(\tau,s)} \psi^{(k)}_\pm. \tag{5-32}$$

It follows from the equations (3-29), (3-30), (5-27) that

$$(\nabla_{(\tau,s)} h_\pm)^* \widehat{\mathrm{M}} \nabla_{(\tau,s)} \psi^{(k)}_\pm = \frac{\partial h_\pm}{\partial \tau}\frac{\partial \psi^{(k)}_\pm}{\partial \tau} + \nabla h_\pm \cdot (\mathrm{E} - \tau \mathrm{B} \mathrm{G}^{-1}_{\partial\omega})^{-2} \nabla \psi^{(k)}_\pm = \sum_{j=-1}^{\infty} c^\pm_{j/2} \tau^{j/2}, \quad \tau \to +0,$$

$$(\det \mathrm{M}) \widehat{\mathrm{M}} \nabla_{(\tau,s)} h_\pm = \sum_{j=-1}^{\infty} \mathbf{c}^\pm_{j/2} \tau^{j/2}, \quad \tau \to +0,$$

where $c^{\pm}_{j/2} = c^{\pm}_{j/2}(P) \in C^\infty(\partial\omega)$ are some functions, $\mathbf{c}^{\pm}_{j/2} = \mathbf{c}^{\pm}_{j/2}(P) \in C^\infty(\partial\omega)$ are some n-dimensional vector-functions, and

$$c^{\pm}_{-1/2} = \tfrac{1}{2} b_1, \quad c^{\pm}_0 = \pm b_2 \Psi^{(1)}_k, \quad \mathbf{c}^{\pm}_{-1/2} = \pm\tfrac{1}{2} b_1 \mathbf{e}_1, \quad \mathbf{c}^{\pm}_0 = b_2 \mathbf{e}_1,$$

and $\mathbf{e}_1 = (1, 0, \dots, 0)^*$. We substitute the last identities into (5-32), which yields

$$-\operatorname{div}_{x'} \mathrm{Q}_\pm \nabla_{x'} \psi^{(k)}_\pm = \sum_{j=-4}^{\infty} f^{\pm,1}_{j/2} \tau^{j/2}, \quad \tau \to +0,$$

$$f^{\pm,1}_{-2} = \pm\frac{1}{4\ln\varepsilon} b_1^2 \Psi^{(1)}_k, \quad f^{\pm,1}_{-3/2} = \frac{1}{2\ln\varepsilon} b_1 b_2 \Psi^{(1)}_k, \quad f^{\pm,1}_{-1} = \pm\frac{1}{4\ln\varepsilon} b_1^2 K \Psi^{(1)}_k.$$

The last identity, (5-30), (5-31), (5-26) imply the formulas (5-25). □

Taking into account (5-5), we apply Lemma 5.1 to problem (5-4). It implies that the right-hand side of (5-4) satisfies the hypothesis of Lemma 3.4 with the first four coefficients given by (5-25).

Given some functions $V^{(0)}_k, V^{(1)}_k \in C^\infty(\partial\omega)$, suppose the solvability condition (3-34) holds true. Then by (3-36), (5-24), (5-25) there exists the unique solution to (5-4) with the asymptotics

$$\begin{aligned}
\phi^{(k)}_\pm &= \frac{1}{\ln\varepsilon}\big(\pm\tfrac{1}{8} b_1^2 \Psi^{(1)}_k \ln\tau + b_1 b_2 \Psi^{(1)}_k \tau^{1/2} + \tau(1-\ln\tau)\big(-\tfrac{1}{4} b_1^2 \Psi^{(2,\pm)}_k + \tfrac{1}{4} b_1 \nabla b_1 \cdot \nabla \Psi^{(0)}_k \pm \tfrac{1}{8} K b_1^2 \Psi^{(1)}_k\big)\big) \\
&\quad + U^{(0)}_k \pm V^{(0)}_k + \tau(V^{(1)}_k \pm U^{(1)}_k) + \mathcal{O}(\tau^{3/2}) \\
&= \frac{1}{\ln\varepsilon}\left(\pm\tfrac{1}{8} b_1^2 \Psi^{(1)}_k \ln\tau + b_1 b_2 \Psi^{(1)}_k \tau^{1/2} + \tau(1-\ln\tau)\left(\Delta_{\partial\omega} + \frac{2}{b_1}\nabla b_1 \cdot \nabla + \lambda\right)\Psi^{(0)}_k\right) \\
&\quad + U^{(0)}_k \pm V^{(0)}_k + \tau(V^{(1)}_k \pm U^{(1)}_k), \quad \tau \to +0,
\end{aligned} \tag{5-33}$$

where $U^{(0)}_k, U^{(1)}_k \in C^\infty(\partial\omega)$ are some functions satisfying (3-37). We compare the last asymptotics with (5-20), (5-21), (5-23), take into consideration the identity (5-24) and arrive at the formulas for $V^{(0)}_k$, $V^{(1)}_k$, $C^{(k,0}_2$ and $C^{(k,1)}_4$:

$$\begin{aligned}
V^{(0)}_k &= -\frac{b_1^2}{4}\Psi^{(1)}_k + \frac{b_1^2}{8\ln\varepsilon}(1 + 4\ln 2 - 2\ln b_1)\Psi^{(1)}_k, \quad C^{(k,0)}_2 = \ln\varepsilon\, U^{(0)}_k, \\
V^{(1)}_k &= \frac{b_1^2}{4}\left(\Delta_{\partial\omega} + \frac{2}{b_1}\nabla b_1 \cdot \nabla + \lambda\right)\Psi^{(0)}_k \\
&\qquad - \frac{b_1^2}{4\ln\varepsilon}\left((2\ln 2 - \ln b_1 + 1)(\Delta_{\partial\omega} + \lambda)\Psi^{(0)}_k + \frac{4\ln 2 - 2\ln b_1 - 2}{b_1}\nabla b_1 \cdot \nabla\Psi^{(0)}_k\right), \\
C^{(k,1)}_4 &= \ln\varepsilon\, U^{(1)}_k - \tfrac{1}{16}(3Kb_1^2 + 32b_2^2 + 24b_1 b_3)\Psi^{(1)}_k.
\end{aligned}$$

In what follows the functions $V^{(0)}_k$, $V^{(1)}_k$, $C^{(k,0}_2$ and $C^{(k,1)}_4$ are supposed to be chosen in accordance with the above given formulas. Bearing these formulas, (5-24) and (5-25) in mind, we write the solvability conditions (3-34) for (5-4):

$$\frac{1}{\ln\varepsilon}\lim_{\delta\to+0}\Big[(f^{(k)}_{2,+},\psi^{(i)}_{+})_{L_2(\omega^\delta)}+(f^{(k)}_{2,-},\psi^{(i)}_{-})_{L_2(\omega^\delta)}-\delta^{-1/2}\int_{\partial\omega} b_1b_2\Psi^{(1)}_k\Psi^{(0)}_i\,ds$$
$$+\ln\delta\int_{\partial\omega}\frac{b_1^2}{4}\bigg(\Psi^{(1)}_i\Psi^{(1)}_k+\Psi^{(0)}_i\Big(\Delta_{\partial\omega}+\frac{2}{b_1}\nabla b_1\cdot\nabla+\lambda\Big)\Psi^{(0)}_k\bigg)\,ds\Big]$$
$$+\int_{\partial\omega}\frac{b_1^2}{2\ln\varepsilon}(2\ln2-\ln b_1+1)\Psi^{(0)}_i(\Delta_{\partial\omega}+\lambda)\Psi^{(0)}_k\,ds$$
$$+\int_{\partial\omega}\frac{b_1}{\ln\varepsilon}(2\ln2-\ln b_1-1)\Psi^{(0)}_i\nabla b_1\cdot\nabla\Psi^{(0)}_k\,ds+\int_{\partial\omega}\frac{b_1^2}{2\ln\varepsilon}(2\ln2-\ln b_1)\Psi^{(1)}_k\Psi^{(1)}_i\,ds$$
$$-\int_{\partial\omega}\frac{b_1^2}{2}\bigg(\Psi^{(1)}_k\Psi^{(1)}_i+\Psi^{(0)}_i\Big(\Delta_{\partial\omega}+\frac{2}{b_1}\nabla b_1\cdot\nabla+\lambda\Big)\Psi^{(0)}_k\bigg)\,ds+\mu_k\delta_{ik}=0,\quad i,k=1,\dots,m. \tag{5-34}$$

Let us simplify the obtained identity. We first rewrite the formulas (5-4) of $f^{(k)}_{2,\pm}$ in a more convenient form employing the eigenvalue equation for $\psi^{(k)}_\pm$ and the definition of the matrix $\mathrm{Q}_\pm$:

$$f^{(k)}_{2,\pm}=-\operatorname{div}_{x'}\Phi^{(k)}_\pm\nabla_{x'}h_\pm+\frac{\lambda}{2}|\nabla_{x'}h_\pm|^2\psi^{(k)}_\pm+\tfrac12\operatorname{div}_{x'}|\nabla_{x'}h_\pm|^2\nabla_{x'}\psi^{(k)}_\pm,\quad \Phi^{(k)}_\pm:=(\nabla_{x'}h_\pm,\nabla_{x'}\psi^{(k)}_\pm)_{\mathbb{R}^n}.$$

Employing this representation, we integrate by parts to obtain

$$(f^{(k)}_{2,\pm},\psi^{(i)}_\pm)_{L_2(\omega^\delta)}=\int_{\partial\omega^\delta}\bigg(\Phi^{(k)}_\pm\frac{\partial h_\pm}{\partial\tau}-\frac12|\nabla_{x'}h_\pm|^2\frac{\partial\psi^{(i)}_\pm}{\partial\tau}\bigg)\psi^{(i)}_\pm\,ds+\int_{\omega^\delta}\Phi^{(i)}_\pm\Phi^{(k)}_\pm\,dx'$$
$$+\frac{\lambda}{2}\int_{\omega^\delta}|\nabla_{x'}h_\pm|^2\psi^{(i)}_\pm\psi^{(k)}_\pm\,dx'-\frac12\int_{\omega^d}|\nabla_{x'}h_\pm|^2(\nabla_{x'}\psi^{(i)}_\pm,\nabla_{x'}\psi^{(k)}_\pm)_{\mathbb{R}^d}\,dx'. \tag{5-35}$$

Applying (3-44), we have

$$\Phi^{(k)}_\pm=\frac{\partial h_\pm}{\partial\tau}\frac{\partial\psi^{(k)}_\pm}{\partial\tau}+\nabla h_\pm\cdot(\mathrm{E}-\tau\mathrm{B}\mathrm{G}^{-1}_{\partial\omega})^{-2}\nabla\psi^{(k)}_\pm$$

in a vicinity of $\partial\omega$. Hence, by (5-5), (5-27) and (5-28),

$$\Phi^{(k)}_\pm=\frac{b_1}{2\sqrt{\tau}}\Psi^{(1)}_k+\mathcal{O}(1),\quad \tau\to+0, \tag{5-36}$$

$$\bigg(\Phi^{(k)}_\pm\frac{\partial h_\pm}{\partial\tau}-\frac12|\nabla_{x'}h_\pm|^2\frac{\partial\psi^{(i)}_\pm}{\partial\tau}\bigg)\psi^{(i)}_\pm\prod_{j=1}^{n-1}(1-\tau K_j)\,ds$$
$$=\pm\frac{1}{8\tau}b_1^2\Psi^{(1)}_i\Psi^{(1)}_k+\frac{1}{2\sqrt{\tau}}b_1b_2\Psi^{(0)}_i\Psi^{(1)}_k+\tfrac18 b_1^2\Psi^{(1)}_i\Psi^{(1)}_k\mp\tfrac18 b_1^2K\Psi^{(0)}_i\Psi^{(1)}_k$$
$$+\tfrac14(b_1^2\Psi^{(2,\pm)}_k\pm3b_1b_3\Psi^{(1)}_k\pm2b_2^2\Psi^{(1)}_k+2b_1\nabla b_1\cdot\Psi^{(0)}_k)\Psi^{(0)}_i+\mathcal{O}(\sqrt{\tau}),\quad \tau\to+0.$$

Substituting the last identity into (5-35) and using (3-42) and (5-24), we get

$$
\begin{aligned}
&(f_{2,+}^{(k)},\psi_+^{(i)})_{L_2(\omega^\delta)}+(f_{2,-}^{(k)},\psi_-^{(i)})_{L_2(\omega^\delta)}\\
&\quad=\int_{\omega^\delta}\frac{|\nabla_{x'}h_+|^2}{2}\big(\lambda\psi_+^{(i)}\psi_+^{(k)}-(\nabla_{x'}\psi_+^{(i)},\nabla_{x'}\psi_+^{(k)})_{\mathbb{R}^d}\big)\,dx'\\
&\qquad+\int_{\omega^\delta}\frac{|\nabla_{x'}h_-|^2}{2}\big(\lambda\psi_-^{(i)}\psi_-^{(k)}-(\nabla_{x'}\psi_-^{(i)},\nabla_{x'}\psi_-^{(k)})_{\mathbb{R}^d}\big)\,dx'\\
&\qquad+\int_{\omega^\delta}(\Phi_+^{(i)}\Phi_+^{(k)}+\Phi_-^{(i)}\Phi_-^{(k)})\,dx'+\delta^{-1/2}\int_{\partial\omega}b_1b_2\Psi_i^{(0)}\Psi_k^{(0)}\,ds\\
&\qquad+\int_{\partial\omega}\frac{b_1^2}{4}\Psi_i^{(1)}\Psi_k^{(1)}\,ds-\int_{\partial\omega}\frac{b_1^2}{4}\Psi_i^{(0)}(\Delta_{\partial\omega}+\lambda)\Psi_k^{(0)}\,ds\\
&\qquad+\int_{\partial\omega}b_1\Psi_i^{(0)}\nabla b_1\cdot\nabla\Psi_k^{(0)}\,ds+\mathcal{O}(\delta^{1/2}),\quad \delta\to+0.
\end{aligned}
$$

We integrate by parts once again, this time over $\partial\omega$, and we have

$$
\int_{\partial\omega}b_1^2\Psi_i^{(0)}\bigg(\Delta_{\partial\omega}+\frac{2}{b_1}\nabla b_1\cdot\nabla+\lambda\bigg)\Psi_k^{(0)}\,ds=\int_{\partial\omega}b_1^2\big(\lambda\Psi_i^{(0)}\Psi_k^{(0)}-\nabla\Psi_i^{(0)}\cdot\nabla\Psi_k^{(0)}\big)\,ds. \tag{5-37}
$$

Substituting the last two identities into (5-34) yields

$$
\begin{aligned}
&\frac{1}{\ln\varepsilon}\lim_{\delta\to+0}\bigg[\int_{\omega^\delta}\frac{|\nabla_{x'}h_+|^2}{2}\big(\lambda\psi_+^{(i)}\psi_+^{(k)}-(\nabla_{x'}\psi_+^{(i)},\nabla_{x'}\psi_+^{(k)})_{\mathbb{R}^d}\big)\,dx'\\
&\qquad+\int_{\omega^\delta}\frac{|\nabla_{x'}h_-|^2}{2}\big(\lambda\psi_-^{(i)}\psi_-^{(k)}-(\nabla_{x'}\psi_-^{(i)},\nabla_{x'}\psi_-^{(k)})_{\mathbb{R}^d}\big)\,dx'+\int_{\omega^\delta}(\Phi_+^{(i)}\Phi_+^{(k)}+\Phi_-^{(i)}\Phi_-^{(k)})\,dx'\\
&\qquad+\ln\delta\int_{\partial\omega}\frac{b_1^2}{4}\big(\Psi_i^{(1)}\Psi_k^{(1)}+\lambda\Psi_i^{(0)}\Psi_k^{(0)}-\nabla\Psi_i^{(0)}\cdot\nabla\Psi_k^{(0)}\big)\,ds\bigg]\\
&\quad+\int_{\partial\omega}\frac{b_1^2}{4\ln\varepsilon}(1+4\ln 2-2\ln b_1)\big(\Psi_i^{(1)}\Psi_k^{(1)}+\Psi_i^{(0)}(\Delta_{\partial\omega}+\lambda)\Psi_k^{(0)}\big)\,ds\\
&\quad+\int_{\partial\omega}\frac{b_1}{\ln\varepsilon}(2\ln 2-\ln b_1)\Psi_i^{(0)}\nabla b_1\cdot\nabla\Psi_k^{(0)}\,ds\\
&\quad-\int_{\partial\omega}\frac{b_1^2}{2}\bigg(\Psi_k^{(1)}\Psi_i^{(1)}+\Psi_i^{(0)}\bigg(\Delta_{\partial\omega}+\frac{2}{b_1}\nabla b_1\cdot\nabla+\lambda\bigg)\Psi_k^{(0)}\bigg)\,ds+\mu_k\delta_{ik}=0,
\end{aligned} \tag{5-38}
$$

as $i,k=1,\dots,m$. It follows from (5-36), (5-29) and (5-5) that

$$
\begin{aligned}
|\nabla_{x'}h_+|^2\big(\lambda\psi_+^{(i)}\psi_+^{(k)}-(\nabla_{x'}\psi_+^{(i)},\nabla_{x'}\psi_+^{(k)})_{\mathbb{R}^d}\big)+|\nabla_{x'}h_-|^2\big(\lambda\psi_-^{(i)}\psi_-^{(k)}-(\nabla_{x'}\psi_-^{(i)},\nabla_{x'}\psi_-^{(k)})_{\mathbb{R}^d}\big)&\\
=\frac{b_1^2}{2\tau}(\lambda\Psi_i^{(0)}\Psi_k^{(0)}-\nabla\Psi_i^{(0)}\cdot\nabla\Psi_k^{(0)})+\mathcal{O}(\tau^{-1/2}),\quad \tau\to+0,&
\end{aligned}
$$

$$
\Phi_\pm^{(i)}\Phi_\pm^{(k)}=\frac{b_1^2}{4\tau}\Psi_i^{(1)}\Psi_k^{(1)}+\mathcal{O}(\tau^{-1/2}),\quad \tau\to+0.
$$

Hence, the limit in (5-38) is finite. To calculate the boundary integrals in (5-38) we integrate by parts:

$$\int_{\partial\omega}\frac{b_1^2}{4}(1+4\ln 2-2\ln b_1)\big(\Psi_i^{(1)}\Psi_k^{(1)}+\Psi_i^{(0)}(\Delta_{\partial\omega}+\lambda)\Psi_k^{(0)}\big)\,ds+\int_{\partial\omega}b_1(2\ln 2-\ln b_1)\Psi_i^{(0)}\nabla b_1\cdot\nabla\Psi_k^{(0)}\,ds$$
$$=\int_{\partial\omega}\frac{b_1^2}{4}(1+4\ln 2-2\ln b_1)\big(\Psi_i^{(1)}\Psi_k^{(1)}+\lambda\Psi_i^{(0)}\Psi_k^{(0)}-\nabla\Psi_i^{(0)}\cdot\nabla\Psi_k^{(0)}\big)\,ds.$$

Due to this identity, (5-37), the definition of b_1 in (3-10) and the definitions (2-9) and (2-10) of the matrices $\Lambda^{(0)}$ and $\Lambda^{(1)}$, respectively, we can rewrite (5-38) in the final form

$$\mu_k\delta_{ik}=\Lambda_{ik}^{(0)}+\frac{1}{\ln\varepsilon}\Lambda_{ik}^{(1)}.$$

Since the matrix on the right-hand side of the last identity is diagonal, we conclude that the solvability condition for the problem (5-4), (5-23) is satisfied provided μ_k are the eigenvalues of the matrix $\Lambda^{(0)}+\frac{1}{\ln\varepsilon}\Lambda^{(1)}$. It follows from [Kato 1966, Chapter II, Section 6.1, Theorem 6.1] that the eigenvalues of this matrix are holomorphic in $\frac{1}{\ln\varepsilon}$ and converge to those of $\Lambda^{(0)}$ as $\varepsilon\to 0$.

In view of the choice of μ_i the problems (5-4), (5-33) are solvable. We observe that each of the functions $\phi_\pm^{(k)}$ is defined up to a linear combination of the eigenfunctions $\psi_\pm^{(i)}$. The exact values of the coefficients of these linear combinations can be determined while constructing the next terms in the asymptotic expansions for $\lambda_k(\varepsilon)$ and $\psi_\varepsilon^{(k)}$. The formal constructing of the asymptotic expansions is complete.

Justification of the asymptotics. In order to justify the obtained asymptotics, one has to construct additional terms. This is a general and standard situation for singularly perturbed problems. In our case one should construct the terms of order up to $\mathcal{O}(\varepsilon^4)$ in the outer expansion for the eigenfunctions and for the eigenvalues, and the terms of order up to $\mathcal{O}(\varepsilon^6)$ in the inner expansion for the eigenfunctions. The asymptotics with the additional terms read

$$\lambda_k(\varepsilon)=\lambda+\varepsilon^2\ln\varepsilon\,\mu_k\left(\frac{1}{\ln\varepsilon}\right)+\varepsilon^4\ln^2\varepsilon\,\eta_k(\varepsilon)+\cdots,$$
$$\psi_{\varepsilon,\mathrm{ex}}^{(k)}=\mathcal{I}_\varepsilon(\boldsymbol{\psi}_k+\varepsilon^2\ln\varepsilon\,\boldsymbol{\phi}_k+\varepsilon^4\ln^2\varepsilon\,\boldsymbol{\theta}_k+\cdots),\qquad \psi_{\varepsilon,\mathrm{in}}^{(k)}=v_0^{(k)}+\sum_{i=2}^{6}\varepsilon^i v_i^{(k)}+\cdots,\tag{5-39}$$

where $\boldsymbol{\theta}_k=(\theta_+^{(k)},\theta_-^{(k)})$, $\theta_\pm^{(k)}=\theta_\pm^{(k)}(x',\varepsilon)$, $v_i^{(k)}=v_i^{(k)}(\xi,P,\varepsilon)$, and we used that $v_1^{(k)}=0$ by (5-16), (5-22). The equations for $\theta_\pm^{(k)}$ are

$$(-\Delta_{x'}-\lambda)\theta_\pm^{(k)}=\frac{1}{\ln\varepsilon}\mathcal{H}_\pm^{(2)}\phi_\pm^{(k)}+\frac{1}{\ln^2\varepsilon}\mathcal{H}_\pm^{(4)}\psi_\pm^{(k)}+\mu_k\phi_\pm^{(k)}+\eta_k\psi_\pm^{(k)},\quad x'\in\omega_\pm,$$
$$\mathcal{H}_\pm^{(4)}:=\tfrac{3}{8}|\nabla_{x'}h_\pm|^4\Delta_{x'}-\tfrac{1}{2}|\nabla_{x'}h_\pm|^2\operatorname{div}_{x'}\big(\tfrac{1}{2}|\nabla_{x'}h_\pm|^2\mathrm{E}-\mathrm{Q}_\pm\big)\nabla_{x'}$$
$$-\operatorname{div}_{x'}\big(\tfrac{1}{8}|\nabla_{x'}h_\pm|^4\mathrm{E}+\tfrac{1}{2}\mathrm{Q}_\pm|\nabla_{x'}h_\pm|^2+\mathrm{Q}_\pm^2\big)\nabla_{x'}.$$

The functions $\theta_\pm^{(k)}$ should satisfy the asymptotics

$$\theta_\pm^{(k)}(x',\varepsilon)=W_{4,2,\pm}^{(k)}(x',\varepsilon)+\frac{1}{\ln\varepsilon}W_{4,1,\pm}^{(k)}(x',\varepsilon)+\frac{1}{\ln^2\varepsilon}W_{4,0,\pm}^{(k)}(x',\varepsilon)+o(1),\quad \tau\to+0,$$
$$W_{4,2,\pm}^{(k)}=-\tfrac{1}{32}b_1^3\big(b_1(\Delta_{\partial\omega}+\lambda)\Psi_k^{(0)}+2\nabla b_1\cdot\nabla\Psi_k^{(0)}\big),$$
$$W_{4,1,\pm}^{(k)}=\tfrac{1}{32}b_1^3(\ln\tau+1+4\ln 2-2\ln b_1)\big(b_1(\Delta_{\partial\omega}+\lambda)\Psi_k^{(0)}+2\nabla b_1\cdot\nabla\Psi_k^{(0)}\big),$$
$$\begin{aligned}W_{4,0,\pm}^{(k)}=\pm\frac{1}{128}\frac{\Psi_k^{(1)}b_1^4}{\tau}+\frac{1}{8}\frac{\Psi_k^{(1)}b_1^3b_2}{\sqrt{\tau}}&\\ -\tfrac{1}{128}b_1^3\big(b_1(\Delta_{\partial\omega}+\lambda)\Psi_k^{(0)}+2\nabla b_1\cdot\nabla\Psi_k^{(0)}\big)(\ln\tau+4\ln 2-2\ln b_1+1)^2&\\ -\tfrac{1}{128}b_1^3\big(b_1(\Delta_{\partial\omega}+\lambda)\Psi_k^{(0)}-2\nabla b_1\cdot\nabla\Psi_k^{(0)}\big)\pm\tfrac{1}{256}\Psi_k^{(1)}(3Kb_1^4+48b_1^3b_3+128b_1^2b_2^2).&\end{aligned}$$

The equations for the functions $v_5^{(k)}$, $v_6^{(k)}$ are obtained in the same way as those for $v_i^{(k)}$, $i=0,\dots,4$, from

$$\mathscr{L}_{-4}v_5^{(k)}+\sum_{i=-3}^{-1}\mathscr{L}_i v_{1-i}^{(k)}\mathscr{L}_1 v_0^{(k)}=0 \qquad \text{on } \mathbb{R}\times\partial\omega,$$
$$\mathscr{L}_{-4}v_6^{(k)}+\sum_{i=-3}^{0}\mathscr{L}_i v_{2-i}^{(k)}+\mathscr{L}_2 v_0^{(k)}=\lambda v_2^{(k)}+\ln\varepsilon\,\eta_k v_0^{(k)} \quad \text{on } \mathbb{R}\times\partial\omega,$$

where the operators $\mathscr{L}_1$, $\mathscr{L}_2$ are the next terms in the expansion (3-25). It can be shown that the problem for $\theta_\pm^{(k)}$ is solvable for some $\eta_k(\varepsilon)$. The equations for $v_5^{(k)}$ and $v_6^{(k)}$ can be solved explicitly. The arbitrary coefficients $C_{5,1}^{(k)}$, $C_{5,0}^{(k)}$, $C_{6,1}^{(k)}$, $C_{6,0}^{(k)}$ appearing in $v_5^{(k)}$, $v_6^{(k)}$ can be determined while matching the inner and outer expansions.

We now introduce the partial sums

$$\widehat{\lambda}_\varepsilon^{(k)}=\lambda+\varepsilon^2\ln\varepsilon\,\mu_k\left(\frac{1}{\ln\varepsilon}\right)+\varepsilon^4\ln^2\varepsilon\,\eta_k(\varepsilon),$$

$$\widehat{\psi}_{\varepsilon,\mathrm{ex}}^{(k)}=\mathscr{I}_\varepsilon(\boldsymbol{\psi}_k+\varepsilon^2\ln\varepsilon\,\boldsymbol{\phi}_k+\varepsilon^4\ln^2\varepsilon\,\boldsymbol{\theta}_k),\quad \widehat{\psi}_{\varepsilon,\mathrm{in}}^{(k)}=v_0^{(k)}+\sum_{i=2}^{6}\varepsilon^i v_i^{(k)}$$

and define the final approximation for the eigenfunctions as

$$\widehat{\psi}_\varepsilon^{(k)}(x)=\widehat{\psi}_{\varepsilon,\mathrm{ex}}^{(k)}(x)\chi\left(\frac{\tau}{\varepsilon^\alpha}\right)+\widehat{\psi}_{\varepsilon,\mathrm{in}}^{(k)}(\xi,P)\left(1-\chi\left(\frac{\tau}{\varepsilon^\alpha}\right)\right),$$

where $\alpha\in(0,1)$ is a fixed constant, and χ is the cut-off function introduced in the proof of Lemma 4.4.

Lemma 5.2. *The function $\widehat{\psi}_\varepsilon^{(k)}\in C^\infty(S_\varepsilon)$ satisfies the convergence*

$$\|\widehat{\psi}_\varepsilon^{(k)}-\mathscr{I}_\varepsilon\boldsymbol{\psi}_k\|_{L_2(S_\varepsilon)}\to 0,\quad \varepsilon\to+0, \tag{5-40}$$

and the equation

$$(\mathscr{H}_\varepsilon-\widehat{\lambda}_\varepsilon^{(k)})\widehat{\psi}_\varepsilon^{(k)}=F_\varepsilon^{(k)}, \tag{5-41}$$

where for the right-hand side the uniform in ε estimate

$$\|F_\varepsilon^{(k)}\|_{L_2(S_\varepsilon)} \leqslant C\varepsilon^{5\alpha/2} \tag{5-42}$$

holds true. The relations

$$(\mathcal{I}_\varepsilon \boldsymbol{\psi}_i, \mathcal{I}_\varepsilon \boldsymbol{\psi}_j)_{L_2(S_\varepsilon)} \to \delta_{ij}, \quad \varepsilon \to +0, \tag{5-43}$$

are valid.

The proof of this lemma is not very difficult and is based on lengthy and rather technical, but straightforward, calculations. Because of this, and in order not to overload the text with long technical formulas, we shall skip these here.

It follows from Lemma 4.6 and (5-41) that

$$\widehat{\psi}_\varepsilon^{(k)} = \sum_{i=1}^{m} \frac{\psi_\varepsilon^{(i)}}{\lambda_i(\varepsilon)-\lambda_k(\varepsilon)} (F_\varepsilon^{(k)}, \psi_\varepsilon^{(i)})_{L_2(S_\varepsilon)} + \mathcal{R}_\varepsilon(\lambda_k(\varepsilon)) F_\varepsilon^{(k)}, \tag{5-44}$$

and, by (5-42),

$$\|\mathcal{R}_\varepsilon(\lambda_k(\varepsilon)) F_\varepsilon^{(k)}\|_{W_2^1(S_\varepsilon)} \leqslant C\varepsilon^{5\alpha/2}, \quad k=1,\dots,m, \tag{5-45}$$

where the constant C is independent of ε. We calculate the scalar products of the functions $\widehat{\psi}_\varepsilon^{(k)}$ in $L_2(S_\varepsilon)$ taking into consideration (5-44) and the properties of the operator $\mathcal{R}_\varepsilon$ described in Lemma 4.6:

$$(\widehat{\psi}_\varepsilon^{(k)}, \widehat{\psi}_\varepsilon^{(p)})_{L_2(S_\varepsilon)} = \sum_{i=1}^{m} \gamma_i^{(k)}(\varepsilon)\gamma_i^{(p)}(\varepsilon) + \big(\mathcal{R}_\varepsilon(\lambda_k(\varepsilon)) F_\varepsilon^{(k)}, \mathcal{R}_\varepsilon(\lambda_\varepsilon^{(p)}) F_\varepsilon^{(p)}\big)_{L_2(S_\varepsilon)},$$

$$\gamma_\varepsilon^{(k)}(\varepsilon) := \frac{1}{\lambda_i(\varepsilon)-\widehat{\lambda}_\varepsilon^{(k)}} (F_\varepsilon^{(k)}, \psi_\varepsilon^{(i)})_{L_2(S_\varepsilon)}.$$

The identities obtained and (5-45), (5-40), (5-43) yield

$$\sum_{i=1}^{m} \gamma_i^{(k)}(\varepsilon)\gamma_i^{(p)}(\varepsilon) \to \delta_{kp}, \quad \varepsilon \to +0. \tag{5-46}$$

In particular, as $p=k$ it implies

$$|\gamma_i^{(k)}(\varepsilon)| \leqslant \tfrac{3}{2} \tag{5-47}$$

for sufficiently small ε. We introduce the matrix $\mathrm{R}_\varepsilon := (\gamma_i^{(k)}(\varepsilon))$ and rewrite (5-46) as $\mathrm{R}_\varepsilon \mathrm{R}_\varepsilon^* \to \mathrm{E}$, $\varepsilon \to +0$, where * denotes matrix transposition. Thus, $|\det \mathrm{R}_\varepsilon| \to 1$ as $\varepsilon \to +0$. Therefore, for each sufficiently small ε there exists a permutation $(i_1(\varepsilon), i_2(\varepsilon), \dots, i_m(\varepsilon))$ such that

$$\left|\prod_{i=1}^{m} \gamma_{i_k(\varepsilon)}^{(k)}(\varepsilon)\right| \geqslant \frac{1}{2m!}. \tag{5-48}$$

For a given ε we rearrange the eigenvalues $\lambda_i(\varepsilon)$ so that $i_k(\varepsilon)=k$, which by (5-47), (5-48) yields

$$|\gamma_i^{(i)}(\varepsilon)| \geqslant \frac{2^{m-2}}{3^{m-1} m!}, \quad i=1,\dots,m.$$

In view of the definition of $\gamma_k^{(k)}(\varepsilon)$, (5-42), and the normalization of $\psi_\varepsilon^{(i)}$ it follows that

$$|\lambda_i(\varepsilon)-\widehat{\lambda}_i(\varepsilon)| \leqslant \frac{3^{m-1}m!}{2^{m-2}}\big|(F_\varepsilon^{(i)},\psi_\varepsilon^{(i)})_{L_2(S_\varepsilon)}\big| \leqslant C\varepsilon^{5\alpha/2}.$$

Choosing $\alpha > 4/5$, we arrive at the asymptotics (2-11).

Define now

$$\widetilde{\psi}_\varepsilon^{(k)} = \mathcal{I}_\varepsilon(\boldsymbol{\psi}_k+\varepsilon^2\ln\varepsilon\,\boldsymbol{\phi}_k)\chi\left(\frac{\tau}{\varepsilon^\alpha}\right)+\left(v_0^{(k)}+\sum_{i=2}^{4}\varepsilon^i v_i^{(k)}\right)\left(1-\chi\left(\frac{\tau}{\varepsilon^\alpha}\right)\right).$$

By direct calculations one can check that

$$\|\widehat{\psi}_\varepsilon^{(k)}-\widetilde{\psi}_\varepsilon^{(k)}\|_{W_2^1(S_\varepsilon)} = \mathcal{O}(\varepsilon^{\frac{5\alpha}{2}}).$$

This identity and (5-45) imply

$$\sum_{i=1}^{m}\gamma_i^{(k)}(\varepsilon)\psi_\varepsilon^{(i)} = \psi_\varepsilon^{(k)}+\mathcal{O}(\varepsilon^{\frac{5\alpha}{2}}),\quad k=1,\dots,m.$$

Since the right-hand sides of these identities are linearly independent, the functions $\sum_{i=1}^m \gamma_i^{(k)}(\varepsilon)\psi_\varepsilon^{(i)}$ form a basis spanned over the eigenfunctions $\psi_\varepsilon^{(i)}$, $i=1,\dots,m$. Hence, we arrive at:

Theorem 5.3. *Let $\mathcal{P}_\varepsilon$ be the total projector associated with the eigenvalues $\lambda_i(\varepsilon)$, $i=1,\dots,m$, and $\widetilde{\mathcal{P}}_\varepsilon$ be the projector on the space spanned over $\widetilde{\psi}_\varepsilon^{(i)}$, $i=1,\dots,m$. Then*

$$\mathcal{P}_\varepsilon = \widetilde{\mathcal{P}}_\varepsilon+\mathcal{O}(\varepsilon^{2+\rho}),$$

where ρ is any constant in $(0,1/2)$.

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