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A LICHNEROWICZ ESTIMATE FOR THE FIRST EIGENVALUE OF CONVEX DOMAINS IN KÄHLER MANIFOLDS

VINCENT GUEDJ, BORIS KOLEV AND NADER YEGANEFAR

In this article, we prove a Lichnerowicz estimate for a compact *convex* domain of a Kähler manifold whose Ricci curvature satisfies $\mathrm{Ric} \geq k$ for some constant $k > 0$. When equality is achieved, the boundary of the domain is totally geodesic and there exists a nontrivial holomorphic vector field.

We show that a ball of sufficiently large radius in complex projective space provides an example of a *strongly pseudoconvex* domain which is *not convex*, and for which the *Lichnerowicz estimate* fails.

1. Introduction

Let (M^n, g) be a compact n-dimensional Riemannian manifold. Assume first that M has no boundary. A theorem of Lichnerowicz [1958] asserts that if the Ricci curvature Ric of M satisfies $\mathrm{Ric} \geq k$ for some constant $k > 0$, the first nonzero eigenvalue λ of the Laplace operator satisfies

$$\lambda \geq \frac{n}{n-1} k. \tag{1-1}$$

Here, $nk/(n-1)$ should be viewed as the first nonzero eigenvalue of the round n-dimensional sphere $S^n(k/(n-1))$ of constant curvature $k/(n-1)$. Moreover, by a result of Obata [1962], the equality case in (1-1) is obtained if and only if M is isometric to this sphere. Reilly [1977] considered a similar problem, but for compact manifolds with boundary. Namely, he proved that if M is as in the Lichnerowicz theorem, except that it has a boundary such that its mean curvature with respect to the outward normal vector field is nonnegative, then the first eigenvalue λ of the Laplace operator with the Dirichlet boundary condition still satisfies (1-1). He also proved that the equality case characterizes a hemisphere in $S^n(k/(n-1))$.

In another direction, Lichnerowicz showed that for Kähler manifolds, his estimate (1-1) can be improved, by showing that, in this case, we have

$$\lambda \geq 2k.$$

Moreover, if equality is achieved, there is a nontrivial holomorphic vector field on M.

The purpose of this note is to consider the case of compact Kähler manifolds with boundary. As in Reilly's result, we will have to impose some convexity property on the boundary.

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Theorem 1.1. *Let M be a compact convex domain in a Kähler manifold. Assume that the Ricci curvature satisfies* $\mathrm{Ric} \geq k$ *for some constant $k > 0$. Then the first eigenvalue λ of the Laplacian with the Dirichlet boundary condition satisfies*

$$\lambda \geq 2k.$$

Moreover, if equality is achieved, the boundary ∂M is totally geodesic and there is a nontrivial holomorphic vector field on M.

Remark 1.2. As we will see in the proof, the convexity hypothesis may be relaxed into another condition of mean curvature type. More precisely, let $\boldsymbol{n}$ denote the outward unit normal vector field on the boundary ∂M, and let II and H be respectively the second fundamental form and the mean curvature. Denote also by J the complex structure of M. If we assume that on the boundary we have

$$(n-1)H + \mathrm{II}(J\boldsymbol{n}, J\boldsymbol{n}) \geq 0, \tag{1-2}$$

the Lichnerowicz estimate $\lambda \geq 2k$ holds (see inequality (4-3) and the remark just before Section 3.2). Now, convexity means that II is a nonnegative bilinear symmetric form, so that it obviously implies condition (1-2).

Remark 1.3. Jean-François Grosjean [2002, Theorem 1.1] proves that there is a Lichnerowicz type estimate on compact (real) manifolds with convex boundary and positive Ricci curvature, if there exists a nontrivial parallel p-form with $2 \leq p \leq n/2$. In the Kähler case, we can of course consider the Kähler form which is a nontrivial parallel 2-form, so that the result of Grosjean gives a Lichnerowicz estimate. But this estimate is weaker than ours. Note however that our result was known to Grosjean and is stated without proof in [2002, page 504].

Remark 1.4. It is natural to ask whether our result remains true if one assumes pseudoconvexity of the boundary instead of its convexity. It turns out that a ball of sufficiently large radius in complex projective space provides an example of a strongly pseudoconvex domain which is not convex, and for which the Lichnerowicz estimate fails (see Proposition 5.1 for more details on this).

Remark 1.5. In the real setting, one can consider the Laplacian with the Neumann boundary condition, and again with the convexity condition, one can show that the Lichnerowicz estimate (1-1) still holds for the first nonzero eigenvalue [Pak et al. 1986]. In the Kähler setting, by using the method of proof of Theorem 1.1, it should also be possible to prove that the conclusion of this theorem is true for the first nonzero eigenvalue of the Laplacian with the Neumann boundary condition. It should also be possible to get a similar result for the first nonzero eigenvalue of the $\bar{\partial}$-Laplacian with the absolute $\bar{\partial}$-condition on the boundary.

An immediate consequence of our theorem is the following.

Corollary 1.6. *Assume that M is a strongly convex domain in a complex manifold which can be endowed with a Kähler metric whose Ricci curvature satisfies* $\mathrm{Ric} \geq k$ *for some constant $k > 0$. Then the first eigenvalue λ of the Laplacian with the Dirichlet boundary condition satisfies*

$$\lambda > 2k.$$

Our proof follows the same strategy as the original proofs of Lichnerowicz and Reilly. We will actually give two slightly different proofs. The first proof is more adapted to the complex setting (see Section 3). We use an appropriate Bochner formula for the $\bar\partial$-Laplacian $\square$ acting on (0, 1)-forms and apply it to $\bar\partial f$, where the function f is an eigenfunction of $\square$ for the first eigenvalue. After integrating the result on M and integrating by parts, we get a Reilly-type formula for the $\bar\partial$-Laplacian which may be of independent interest. The desired eigenvalue estimate follows if we can prove that some boundary term is nonpositive, which is the case under the convexity hypothesis.The second proof rests on the well-known Reilly formula for real manifolds; see [Reilly 1977]. This is done in Section 4.

2. Background material

In this section, we recall some well-known facts that will be used in the proof of our main result.

2.1. ***Decomposition of the Hessian.*** Let f be a real valued smooth function on a Kähler manifold (M, J, g). Its Riemannian Hessian ∇df can be decomposed as the sum of a J-symmetric bilinear form and a J-skew-symmetric bilinear form. More specifically, we have

$$\nabla df = \mathrm{H}^1 f + \mathrm{H}^2 f$$

where for tangent vectors A and B,

$$\mathrm{H}^1 f(A, B) = \tfrac{1}{2}\{\nabla df(A, B) + \nabla df(JA, JB)\}$$

and

$$\mathrm{H}^2 f(A, B) = \tfrac{1}{2}\{\nabla df(A, B) - \nabla df(JA, JB)\}.$$

The two following facts may be easily checked.

(1) The (1, 1)-form associated to $\mathrm{H}^1 f$ by the complex structure J is $i\partial\bar\partial f$:

$$\mathrm{H}^1 f(JA, B) = i\partial\bar\partial f(A, B).$$

(2) In local coordinates, $\mathrm{H}^2 f$ has components

$$(\mathrm{H}^2 f)_{pq} = \overline{(\mathrm{H}^2 f)_{\bar p\bar q}} = \frac{\partial^2 f}{\partial z_p \partial z_q} - \Gamma^r_{pq}\frac{\partial f}{\partial z_r},$$

and the other components vanish. $\mathrm{H}^2 f$ is called the *complex Hessian*.

Since $J^* = J^{-1}$, we have $\|\nabla df\| = \|(\nabla df)^J\|$, where

$$(\nabla df)^J(A, B) := \nabla df(JA, JB).$$

Therefore

$$2\|\mathrm{H}^1 f\|^2 = \|\nabla df\|^2 + \langle \nabla df, (\nabla df)^J\rangle \tag{2-1}$$

and

$$2\|\mathrm{H}^2 f\|^2 = \|\nabla df\|^2 - \langle \nabla df, (\nabla df)^J\rangle. \tag{2-2}$$

2.2. ***Reilly formula for the (real) Laplacian.*** Let (M, g) be a Riemannian manifold. Let f be a smooth function on M and ∇df, Δf, and grad f be its Riemannian Hessian, its Laplacian (Laplace Beltrami), and its gradient on M, respectively. Let $\boldsymbol{n}$ denotes the outward unit normal vector field on ∂M and let II and H be the second fundamental form and the mean curvature, respectively. We choose the convention $\mathrm{II}(X, Y) = \langle \nabla_X \boldsymbol{n}, Y \rangle$ for any $X, Y \in T\partial M$. The Laplacian and the gradient on the boundary ∂M with the induced metric are denoted by $\bar{\Delta}$ and $\overline{\mathrm{grad}}$, respectively. The Reilly formula [Reilly 1977] is given by

$$\int_M \|\nabla df\|^2$$
$$= \int_M (\Delta f)^2 - \int_M \mathrm{Ric}(\mathrm{grad}\, f, \mathrm{grad}\, f) + 2\int_{\partial M} \bar{\Delta} f \frac{\partial f}{\partial \boldsymbol{n}} \sigma - (n-1)\int_{\partial M} H \left(\frac{\partial f}{\partial \boldsymbol{n}}\right)^2 \sigma - \int_{\partial M} \mathrm{II}(\overline{\mathrm{grad}} f, \overline{\mathrm{grad}} f)\sigma.$$

Moreover if we assume that f is vanishing on ∂M, then $\bar{\Delta} f = 0$, $\overline{\mathrm{grad}} f = 0$ and

$$\int_M \|\nabla df\|^2 = \int_M (\Delta f)^2 - \int_M \mathrm{Ric}(\mathrm{grad}\, f, \mathrm{grad}\, f) - (n-1)\int_{\partial M} H \left(\frac{\partial f}{\partial \boldsymbol{n}}\right)^2 \sigma. \tag{2-3}$$

2.3. ***Bochner formula for the (complex) Laplacian.*** Let (M, g) be a Kähler manifold, and denote by ∇ its Levi-Civita connection. If α is a $(0, 1)$-form, we denote by $D''\alpha$ the $(0, 2)$-part of $\nabla\alpha$. More precisely, $\nabla\alpha$ is a section of the bundle $T^*M \otimes (T^*)^{0,1}M$; this bundle decomposes as a direct sum

$$((T^*)^{1,0}M \otimes (T^*)^{0,1}M) \oplus ((T^*)^{0,1}M \otimes (T^*)^{0,1}M),$$

and $D''\alpha$ is the projection of $\nabla\alpha$ on the second factor of this decomposition. In local complex coordinates, we have

$$(D''\alpha)_{\bar{p}\bar{q}} = \frac{\partial \alpha_{\bar{q}}}{\partial \bar{z}_p} - \Gamma^{\bar{r}}_{\bar{p}\bar{q}} \alpha_{\bar{r}}.$$

Now let $(D'')^*$ be the formal adjoint of D''. For a section β of $(T^*)^{0,1}M \otimes (T^*)^{0,1}M$ one can see that locally

$$((D'')^*\beta)_{\bar{p}} = -g^{q\bar{r}} \frac{\partial \beta_{\bar{r}\bar{p}}}{\partial z_q}.$$

Then we have the following Bochner formula for the $\bar{\partial}$-Laplacian $\Box$ acting on $(0, 1)$-forms:

$$\Box = (D'')^* D'' + \mathrm{Ric}\,. \tag{2-4}$$

For future reference, we also give the integration by parts formula for D'' in the presence of a boundary; see, for example, [Taylor 2011, Proposition 9.1]. Here, we assume that M is compact, and we let $\boldsymbol{n}$ denote the outward unit normal vector field on ∂M. The $(0, 1)$ part of the dual 1-form ν corresponding to $\boldsymbol{n}$ by the metric will be denoted by $\nu^{0,1}$. Finally, we let σ denote the measure induced on the boundary by the metric. For smooth α and β, we then have

$$\langle D''\alpha, \beta \rangle_{L^2(M)} = \langle \alpha, (D'')^*\beta \rangle_{L^2(M)} + \int_{\partial M} \langle \nu^{0,1} \otimes \alpha, \beta \rangle \sigma. \tag{2-5}$$

3. Bochner formula and the first eigenvalue

In this section, we will give the first proof of Theorem 1.1. Let $\Box$ denote the $\bar\partial$-Laplacian on M, which is given on forms by

$$\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

Recall that on a Kähler manifold, we have $\Box = \frac{1}{2}\Delta$. We will denote by μ the first eigenvalue of $\Box$ with the Dirichlet boundary condition, so that

$$\mu = \tfrac{1}{2}\lambda.$$

Now let f be a real valued eigenfunction of $\Box$ corresponding to the first eigenvalue μ. Thus $f : \overline{M} \to \mathbb{R}$ is smooth, vanishes on the boundary ∂M, and satisfies $\Box f = \mu f$. (Note that it is possible to choose f to be real valued, because $\Box$ is equal to half the Laplace Beltrami operator Δ.) We write the Bochner formula (2-4) for the (0, 1)-form $\bar\partial f$ and take the L^2-inner product of the resulting equality with $\bar\partial f$ itself:

$$\langle \Box\bar\partial f, \bar\partial f\rangle_{L^2(M)} = \langle (D'')^* D''\bar\partial f, \bar\partial f\rangle_{L^2(M)} + \int_M \operatorname{Ric}(\bar\partial f, \bar\partial f). \tag{3-1}$$

Using the fact that $\Box\bar\partial = \bar\partial\Box$ and $f|_{\partial M} = 0$, we can integrate by parts the left hand side of (3-1) to get

$$\begin{aligned}\langle \Box\bar\partial f, \bar\partial f\rangle_{L^2(M)} &= \langle \bar\partial\Box f, \bar\partial f\rangle_{L^2(M)}\\ &= \langle \bar\partial(\mu f), \bar\partial f\rangle_{L^2(M)}\\ &= \mu\langle \Box f, f\rangle_{L^2(M)}\\ &= \mu^2\|f\|^2_{L^2(M)}.\end{aligned}$$

We can deal with the Ricci term in the right hand side of (3-1) in a similar way:

$$\begin{aligned}\int_M \operatorname{Ric}(\bar\partial f, \bar\partial f) &\geq k\langle \bar\partial f, \bar\partial f\rangle_{L^2(M)}\\ &= k\langle \Box f, f\rangle_{L^2(M)}\\ &= k\mu\|f\|^2_{L^2(M)}.\end{aligned}$$

Finally, we can integrate by parts the first term in the right hand side of (3-1) (see formula (2-5)) to get

$$\langle (D'')^* D''\bar\partial f, \bar\partial f\rangle_{L^2(M)} = \|D''\bar\partial f\|^2_{L^2(M)} - \int_{\partial M} \langle D''\bar\partial f, \nu^{0,1}\otimes\bar\partial f\rangle\sigma, \tag{3-2}$$

and, combining this with our previous estimates, we obtain

$$\mu(\mu - k)\|f\|^2_{L^2(M)} \geq \|D''\bar\partial f\|^2_{L^2(M)} - \int_{\partial M} \langle D''\bar\partial f, \nu^{0,1}\otimes\bar\partial f\rangle\sigma. \tag{3-3}$$

As a consequence, if we set

$$I = -\int_{\partial M} \langle D''\bar\partial f, \nu^{0,1}\otimes\bar\partial f\rangle\sigma,$$

we get $\mu \geq k$, provided we can prove that $I \geq 0$. In the next subsection, we see that this is indeed the case under suitable assumptions on the boundary.

3.1. ***Boundary term.*** To estimate the boundary term I, we first notice that as f is real valued, we have

$$(D''\bar{\partial} f)_{\bar{p}\bar{q}} = (\mathrm{H}^2 f)_{\bar{p}\bar{q}}$$

so that

$$I = -\int_{\partial M} \langle \mathrm{H}^2 f, \nu^{0,1} \otimes \bar{\partial} f\rangle\sigma = -\int_{\partial M} \mathrm{H}^2 f(\boldsymbol{n}^{0,1}, (\partial f)^\sharp)\sigma.$$

We then choose a boundary defining function ρ for ∂M. This means that ρ is a smooth real valued function such that $M = \{\rho \le 0\}$, $\partial M = \{\rho = 0\}$, and $d\rho$ does not vanish on ∂M. By multiplying ρ by a suitable smooth positive function if necessary, we may assume that

$$\boldsymbol{n} = \operatorname{grad} \rho.$$

Moreover, near a fixed (but arbitrary) point of the boundary ∂M, we fix a local orthonormal frame adapted to the complex structure J which has the form

$$v_1, Jv_1, \dots, v_m, Jv_m = \boldsymbol{n} = \operatorname{grad} \rho.$$

We also set

$$e_p = \frac{1}{\sqrt{2}}(v_p - iJv_p), \qquad p = 1, \dots, m.$$

Note that as f vanishes on ∂M, its derivatives along tangent vectors to ∂M also vanish and, consequently,

$$(\partial f)^\sharp = \frac{-i}{\sqrt{2}}(\boldsymbol{n} \cdot f)\bar{e}_m, \qquad \boldsymbol{n}^{0,1} = \frac{-i}{\sqrt{2}}\bar{e}_m,$$

where $\boldsymbol{n} \cdot f$ means $df(\boldsymbol{n})$. Therefore,

$$I = \frac{1}{2}\int_{\partial M} (\boldsymbol{n} \cdot f)\nabla df(\bar{e}_m, \bar{e}_m)\sigma,$$

which can be decomposed as $I = I_1 + iI_2$ with

$$I_1 = \frac{1}{4}\int_{\partial M} (\boldsymbol{n} \cdot f)[\nabla df(J\boldsymbol{n}, J\boldsymbol{n}) - \nabla df(\boldsymbol{n}, \boldsymbol{n})]\sigma$$

and

$$I_2 = -\frac{1}{2}\int_{\partial M} (\boldsymbol{n} \cdot f)\nabla df(J\boldsymbol{n}, \boldsymbol{n})\sigma.$$

Actually I_2 vanishes because I is a real number. (This follows from the fact that in Equation (3-1), the left hand side and the Ricci term are real numbers, so that the term involving D'' is also a real number. This implies, by Equation (3-2), that the boundary term I is a real number as well. There is also a more conceptual reason for the vanishing of I_2; see Section 3.2.) We now turn our attention to I_1. As $\triangle f = \mu f = 0$ on ∂M, the trace of ∇df is also zero on ∂M:

$$\nabla df(J\boldsymbol{n}, J\boldsymbol{n}) - \nabla df(\boldsymbol{n}, \boldsymbol{n}) = \sum_{k=1}^{m-1}[\nabla df(v_k, v_k) + \nabla df(Jv_k, Jv_k)] + 2\nabla df(J\boldsymbol{n}, J\boldsymbol{n}).$$

We notice that all vectors appearing in the right hand side are tangent to the boundary. For such a vector u, we have on ∂M

$$\begin{aligned}\nabla df(u,u) &= -\langle \nabla_u u, \boldsymbol{n}\rangle(\boldsymbol{n}\cdot f)\\ &= \langle \nabla_u \boldsymbol{n}, u\rangle(\boldsymbol{n}\cdot f)\\ &= (\boldsymbol{n}\cdot f)\nabla d\rho(u,u).\end{aligned}$$

This implies

$$I_1 = \frac{1}{4}\int_{\partial M}(\boldsymbol{n}\cdot f)^2\left(\sum_{k=1}^{m-1}[\nabla d\rho(v_k,v_k)+\nabla d\rho(Jv_k,Jv_k)]+2\nabla d\rho(J\boldsymbol{n},J\boldsymbol{n})\right)\sigma. \tag{3-4}$$

If we assume that ∂M is convex, all terms in the integrand of the right hand side are nonnegative, so that $I = I_1 \geq 0$ as desired. This proves that $\mu \geq k$ in the convex case.

It remains to deal with the equality case. If we assume that $\mu = k$, then, by (3-3), we must have $D''\bar{\partial}f = 0$ and $I = 0$. On the one hand, $D''\bar{\partial}f = 0$ means that the $(1,0)$-vector field associated to $\bar{\partial}f$ by the metric is a (nonzero) holomorphic vector field. On the other hand, from $I = 0$, we infer that the integrand in Equation (3-4) has to vanish identically on the boundary:

$$(\boldsymbol{n}\cdot f)^2\left(\sum_{k=1}^{m-1}\{\nabla d\rho(v_k,v_k)+\nabla d\rho(Jv_k,Jv_k)\}+2\nabla d\rho(J\boldsymbol{n},J\boldsymbol{n})\right)=0.$$

Assume by contradiction that ∂M is not totally geodesic (but is still convex of course). Then the term between the brackets is positive at some point and we will get the vanishing of $\boldsymbol{n}.f$ on an open subset of ∂M. But f is in the kernel of the elliptic operator $\square - \mu$ and vanishes on ∂M. By the unique continuation principle for elliptic operators (see, for example, [Booß-Bavnbek and Wojciechowski 1993]), f has to vanish on M as well, which is absurd. Therefore, ∂M is totally geodesic. This completes the proof of Theorem 1.1.

Remark. With our conventions, $\nabla d\rho$ is nothing but the second fundamental form of ∂M. Thus, we recover condition (1-2) of Remark 1.2.

3.2. ***A direct proof that the boundary term is real.*** The fact that

$$I_2 = -\frac{1}{2}\int_{\partial M}(\boldsymbol{n}\cdot f)\nabla df(J\boldsymbol{n},\boldsymbol{n})\sigma$$

vanishes is also a consequence of the fact that the expression

$$(\boldsymbol{n}\cdot f)\nabla df(J\boldsymbol{n},\boldsymbol{n})\sigma = (\boldsymbol{n}\cdot f)(J\boldsymbol{n}\cdot\boldsymbol{n}\cdot f)\sigma$$

is an exact differential form on the closed manifold ∂M. Indeed, the vector field $J\boldsymbol{n} = J\,\mathrm{grad}\,\rho$ is the Hamiltonian vector field associated to ρ. This means that if ω is the Kähler form,

$$i_{J\boldsymbol{n}}\omega = -d\rho.$$

Hence

$$di_{J\boldsymbol{n}}i_{\boldsymbol{n}}\omega^m = -md(\boldsymbol{n}\cdot\rho)\wedge\omega^{m-1} - m(m-1)d\rho\wedge di_{\boldsymbol{n}}\omega\wedge\omega^{m-2}.$$

Let $j : \partial M \to M$ be the inclusion map. Since the functions $\boldsymbol{n} \cdot \rho$ and ρ are constant on ∂M, we have

$$j^*(d i_{J\boldsymbol{n}} i_{\boldsymbol{n}} \omega^m) = 0.$$

Now, $J\boldsymbol{n}$ is a vector field defined on a neighborhood of ∂M whose restriction to ∂M is tangent to ∂M, so that

$$j^*(i_{J\boldsymbol{n}}\beta) = i_{J\boldsymbol{n}} j^*(\beta)$$

for any differential form β. As a consequence, we get

$$d i_{J\boldsymbol{n}} j^*(i_{\boldsymbol{n}}\omega^m) = 0.$$

Finally, we have

$$j^*(i_{\boldsymbol{n}}\omega^m) = \sigma$$

and

$$d i_{J\boldsymbol{n}}\sigma = 0.$$

Defining a vector field X by

$$X = \tfrac{1}{2}(\boldsymbol{n} \cdot f)^2 J\boldsymbol{n},$$

it follows that, on ∂M, we have

$$d i_X \sigma = (\boldsymbol{n} \cdot f)(J\boldsymbol{n} \cdot \boldsymbol{n} \cdot f)\sigma.$$

4. Reilly formula and the first eigenvalue

In this section, we present an alternative proof of our main result which was indicated by the referee. It is based on Reilly's formula, a well-known result in *real* Riemannian geometry, which is probably the tool used in [Grosjean 2002, page 504].

This complements nicely the arguments given in Section 3, which have a complex geometry flavor. The complex proof is a bit longer, as we first need to establish a Reilly-type formula for the $\bar\partial$-Laplacian. Given the importance of the $\bar\partial$-Laplacian in complex geometry, it is likely that this (*complex*) Reilly formula will have other applications.

Let M be a compact smooth domain in a Kähler manifold of complex dimension m and real dimension $n = 2m$, with metric g and Ricci curvature bounded from below by some positive constant k. The outward unit normal vector field on the boundary ∂M is denoted by $\boldsymbol{n}$. Our aim is to prove a Lichnerowicz estimate for the first eigenvalue by using the Reilly formula. We begin with some general facts.

Let G be a symmetric, covariant 2-tensor field and X a vector field. We have

$$\operatorname{div}(G(X, \cdot)) = (\operatorname{div} G)(X) + \langle G, DX^\flat\rangle,$$

where $DX^\flat$ is the symmetric part of the covariant 2-tensor field $\nabla X^\flat$. Specializing this formula for $G = (\nabla df)^J$ and $X = \operatorname{grad} f$, for some smooth real function f, we get

$$\operatorname{div}\alpha = \operatorname{Tr}[\nabla^2 df(\cdot, J\cdot, J \operatorname{grad} f)] + \langle(\nabla df)^J, \nabla df\rangle,$$

where

$$\alpha(X) := \nabla df(JX, J\,\mathrm{grad}\, f).$$

Given an orthonormal basis $(e_i)_{1\le i\le n}$ at a point x in M, we have

$$\begin{aligned}
\mathrm{Tr}[\nabla^2 df(\cdot, J\cdot, J\,\mathrm{grad}\, f)] &= \tfrac{1}{2}\{\nabla^2 df(e_i, Je_i, J\,\mathrm{grad}\, f) - \nabla^2 df(Je_i, e_i, J\,\mathrm{grad}\, f)\}\\
&= -\tfrac{1}{2}[R(e_i, Je_i)\, df](J\,\mathrm{grad}\, f)\\
&= \tfrac{1}{2} R(e_i, Je_i, J\,\mathrm{grad}\, f, \mathrm{grad}\, f)\\
&= -\mathrm{Ric}(\mathrm{grad}\, f, \mathrm{grad}\, f).
\end{aligned}$$

Hence we get

$$\mathrm{div}\,\alpha = -\mathrm{Ric}(\mathrm{grad}\, f, \mathrm{grad}\, f) + \langle (\nabla df)^J, \nabla df\rangle.$$

Integrating by parts we find

$$\int_M \langle (\nabla df)^J, \nabla df\rangle = \int_M \mathrm{Ric}(\mathrm{grad}\, f, \mathrm{grad}\, f) + \int_{\partial M} \alpha(\boldsymbol{n})\sigma,$$

but, for a point $m \in \partial M$, we have

$$\begin{aligned}
\alpha(\boldsymbol{n})_m &= (\nabla df)_m(J\boldsymbol{n}, J\,\mathrm{grad}\, f)\\
&= (\nabla df)_m\left(J\boldsymbol{n}, J\left(\overline{\mathrm{grad}} f + \frac{\partial f}{\partial \boldsymbol{n}}\boldsymbol{n}\right)\right)\\
&= (\nabla df)_m(J\boldsymbol{n}, J\overline{\mathrm{grad}} f) + \frac{\partial f}{\partial \boldsymbol{n}}(\nabla df)_m(J\boldsymbol{n}, J\boldsymbol{n}).
\end{aligned}$$

Now, recall that the second fundamental form II of ∂M is defined as follows (see [Gallot et al. 2004, Chapter 5] for details). Let U, V be local vector fields in M which extend some vector fields u, v on ∂M, in a neighborhood of $m \in \partial M$. We have

$$(\nabla_U V)_m = (\overline{\nabla}_u v)_m - \mathrm{II}_m(u, v)\boldsymbol{n},$$

from which we deduce that

$$(\nabla df)_m(u, v) = (\overline{\nabla} df)_m(u, v) + \frac{\partial f}{\partial \boldsymbol{n}} \mathrm{II}_m(u, v).$$

Therefore

$$\alpha(\boldsymbol{n})_m = (\nabla df)_m(J\boldsymbol{n}, J\overline{\mathrm{grad}} f) + \frac{\partial f}{\partial \boldsymbol{n}}\overline{\nabla} df(J\boldsymbol{n}, J\boldsymbol{n}) + \left(\frac{\partial f}{\partial \boldsymbol{n}}\right)^2 \mathrm{II}(J\boldsymbol{n}, J\boldsymbol{n}).$$

If we assume furthermore that f vanishes on the boundary, the first two terms of the right hand side of the equation above vanish as well, so we finally obtain

$$\int_M \langle (\nabla df)^J, \nabla df\rangle = \int_M \mathrm{Ric}(\mathrm{grad}\, f, \mathrm{grad}\, f) + \int_{\partial M} \left(\frac{\partial f}{\partial \boldsymbol{n}}\right)^2 \mathrm{II}(J\boldsymbol{n}, J\boldsymbol{n})\sigma. \tag{4-1}$$

On the left side of the Reilly formula (2-3), we can first use (2-2) to replace $\|\nabla df\|^2$ by

$$\|\nabla df\|^2 = 2\|\mathrm{H}^2 f\|^2 + \langle \nabla df, (\nabla df)^J\rangle,$$

and then use (4-1) to get

$$2\int_M \|\mathrm{H}^2 f\|^2 = \int_M (\Delta f)^2 - 2\int_M \mathrm{Ric}(\operatorname{grad} f, \operatorname{grad} f) - \int_{\partial M} [(n-1)H + \mathrm{II}(J\boldsymbol{n}, J\boldsymbol{n})]\left(\frac{\partial f}{\partial \boldsymbol{n}}\right)^2 \sigma. \quad (4\text{-}2)$$

Suppose now that f is a real valued eigenfunction of Δ corresponding to the first eigenvalue λ of Δ, so that $f : \overline{M} \to \mathbb{R}$ is smooth, vanishes on the boundary ∂M, and satisfies $\Delta f = \lambda f$. The hypothesis on the Ricci curvature implies that

$$\int_M \mathrm{Ric}(\operatorname{grad} f, \operatorname{grad} f) \geq k\|df\|^2_{L^2} = k\langle \Delta f, f\rangle_{L^2} = k\lambda \|f\|^2_{L^2}.$$

From (4-2), we then infer

$$\lambda(\lambda - 2k)\|f\|^2_{L^2} \geq \int_{\partial M} [(n-1)H + \mathrm{II}(J\boldsymbol{n}, J\boldsymbol{n})]\left(\frac{\partial f}{\partial \boldsymbol{n}}\right)^2 \sigma. \quad (4\text{-}3)$$

Finally, if we assume that the boundary is convex, II is by definition a symmetric bilinear form which is nonnegative, so that its trace H is also nonnegative. Therefore, the left hand side of the previous equation is nonnegative, and we get $\lambda \geq 2k$, as desired. For the equality case, we can argue as in the end of Section 3.1.

5. Counterexample in the pseudoconvex case

We use the notation introduced in Section 3. It is clear from the proof of Theorem 1.1 that in order to get the estimate $\mu \geq k$, it is enough to assume that on the boundary we have

$$\sum_{k=1}^{m-1} \{\nabla d\rho(v_k, v_k) + \nabla d\rho(Jv_k, Jv_k)\} + 2\nabla d\rho(J\boldsymbol{n}, J\boldsymbol{n}) \geq 0, \quad (5\text{-}1)$$

and not necessarily the convexity of ∂M. We may rewrite this condition as

$$\sum_{k=1}^{m-1} \mathrm{H}^1\rho(v_k, v_k) + \nabla d\rho(J\boldsymbol{n}, J\boldsymbol{n}) \geq 0.$$

Here, $\sum_{k=1}^{m-1} \mathrm{H}^1\rho(v_k, v_k)$ is the trace of the Levi form of the boundary, which would be nonnegative if ∂M were assumed to be only pseudoconvex. The extra term $\nabla d\rho(J\boldsymbol{n}, J\boldsymbol{n})$, however, can usually not be controlled in the pseudoconvex case. This suggests that the conclusion of Theorem 1.1 does not generally hold in this case, as we now explain.

We consider here the complex m-dimensional projective space $\mathbb{P}^m(\mathbb{C})$ equipped with the Fubini–Study metric normalized so that the holomorphic sectional curvature is 4 (the Einstein constant is thus $2(m+1)$ and the diameter is $\pi/2$).

Proposition 5.1. *Fix some point $x \in \mathbb{P}^m(\mathbb{C})$, some $r_0 \in]0, \pi/2[$, and let M be the geodesic ball centered at x, of radius r_0.*

(i) *If $r_0 \in]\pi/4, \pi/2[$, M is strongly pseudoconvex, not convex.*

(ii) *The first eigenvalue of M with Dirichlet boundary conditions goes to 0 as r_0 approaches $\pi/2$.*

Proof. The first point is a well-known result. For completeness, we outline the proof here. Denote by r the distance function from x, and set $\rho = r^2 - r_0^2$, so that ρ is a smooth defining function for M. We want to compute the eigenvalues of the Hessian of ρ. As

$$\nabla d\rho = 2r\nabla dr + 2dr \otimes dr,$$

we only have to compute the eigenvalues of ∇dr. To do this, we proceed as in the proof of [Greene and Wu 1979, Theorem A, page 19]. Recall that for a tangent vector u, the curvature $R(u, .)u$ of $\mathbb{P}^m(\mathbb{C})$ is given by [Berger et al. 1971, Proposition F.34]

$$R(u, .)u = \begin{cases} 0 & \text{on } \mathbb{R}u, \\ 4\mathrm{Id} & \text{on } \mathbb{R}Ju, \\ \mathrm{Id} & \text{on the orthogonal complement of } (u, Ju). \end{cases}$$

Let γ be a normal geodesic starting from x. We can choose a parallel frame along γ which has the form $v_1, Jv_1, \dots, v_m, Jv_m = \operatorname{grad} r$. Using the explicit expression of R, it is then easy to check that the space of Jacobi fields V along γ satisfying $V(0) = 0$ and $V \perp \dot{\gamma}$ has as a basis $V_i = \sin(r)v_i$, JV_i, $i = 1, \dots, m-1$ and $V_m = \sin(2r)v_m$. Using the second variation formula, we see that ∇dr is diagonalized in the basis $v_1, Jv_1, \dots, v_m, Jv_m$ with eigenvalues $\cot(r)$ (of order $2m-2$), $2\cot(2r)$, and 0. If $r = r_0 \in\,]\pi/4, \pi/2[$, we infer that the Levi form of ρ is positive definite, being equal to $2r_0\cot(r_0)\mathrm{Id}$ on the Levi distribution. In other words, M is strongly pseudoconvex. However, M is not convex because the principal curvature $2\cot(2r_0)$ is negative.

As for the second point of our proposition, it is, for example, a consequence of [Chavel and Feldman 1978, Theorem 1], which states the following: Let X be a compact Riemannian manifold and let $X' \subset X$ be a submanifold. For small $\varepsilon > 0$, let X'_ε be the ε-neighborhood of X' in X and denote by Ω_ε the set $X \setminus X'_\varepsilon$. Let (λ_j) be the spectrum of X and let $(\lambda_j(\varepsilon))$ be the spectrum of Ω_ε with Dirichlet boundary conditions. If the codimension of X' in X is at least 2, then, for all j, $\lambda_j(\varepsilon) \to \lambda_{j-1}$ as $\varepsilon \to 0$. In our case, we can take $X = \mathbb{P}^m(\mathbb{C})$ and $X' = \mathbb{P}^{m-1}(\mathbb{C})$, which we view as the cut locus of our fixed point x. If $\varepsilon = \pi/2 - r_0$, Ω_ε actually coincides with M and we get (ii). $\square$

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VINCENT GUEDJ: vincent.guedj@math.univ-toulouse.fr
Institut Universitaire de France and Institut Mathematiques de Toulouse, Universite Paul Sabatier, 118 route de Narbonne, F-31062 Toulouse Cedex 9, France

BORIS KOLEV: kolev@cmi.univ-mrs.fr
Laboratoire d'Analyse, Topologie, Probabilités, CNRS & Aix-Marseille University, CMI, 39 rue F. Joliot-Curie, 13453 Marseille Cedex 13, France

NADER YEGANEFAR: nader.yeganefar@cmi.univ-mrs.fr
Laboratoire d'Analyse, Topologie, Probabilités, Aix-Marseille University, CMI, 39 rue F. Joliot-Curie, 13453 Marseille Cedex 13, France

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