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EFFECTIVE INTEGRABLE DYNAMICS FOR A CERTAIN NONLINEAR WAVE EQUATION

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We consider the following degenerate half-wave equation on the one-dimensional torus:

$$i\partial_t u - |D|u = |u|^2 u, \quad u(0,\cdot) = u_0.$$

We show that, on a large time interval, the solution may be approximated by the solution of a completely integrable system — the cubic Szegő equation. As a consequence, we prove an instability result for large H^s norms of solutions of this wave equation.

1. Introduction

Let us consider, on the one-dimensional torus $\mathbb{T}$, the "half-wave" equation

$$i\partial_t u - |D|u = |u|^2 u, \quad u(0,\cdot) = u_0. \tag{1}$$

Here $|D|$ denotes the pseudodifferential operator defined by

$$|D|u = \sum |k| u_k e^{ikx}, \quad u = \sum_k u_k e^{ikx}.$$

This equation can be seen as a toy model for nonlinear Schrödinger equations on degenerate geometries leading to lack of dispersion. For instance, it has the same structure as the cubic nonlinear Schrödinger equation on the Heisenberg group, or associated with the Grušin operator. We refer to [Gérard and Grellier 2010a; 2010b] for more detail.

We endow $L^2(\mathbb{T})$ with the symplectic form

$$\omega(u,v) = \operatorname{Im}(u,v),$$

where (u,v) denotes the inner product on $L^2(\mathbb{T})$. Equation (1) may be seen as the Hamiltonian system related to the energy function $H(u) := \frac{1}{2}(|D|u,u) + \frac{1}{4}\|u\|_{L^4}^4$. In particular, H is invariant by the flow, which also admits the conservation laws

$$Q(u) := \|u\|_{L^2}^2, \quad M(u) := (Du,u).$$

However, Equation (1) is a nondispersive equation. Indeed, it is equivalent to the system

$$i(\partial_t \pm \partial_x)u_\pm = \Pi_\pm(|u|^2 u), \quad u_\pm(0,\cdot) = \Pi_\pm(u_0), \tag{2}$$

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where $u_\pm = \Pi_\pm(u)$. Here, Π_+ denotes the orthogonal projector from $L^2(\mathbb{T})$ onto

$$L^2_+(\mathbb{T}) := \left\{u = \sum_{k\geq 0} u_k e^{ikx},\ (u_k)_{k\geq 0} \in \ell^2\right\}$$

and $\Pi_- := I - \Pi_+$.

Though the scaling is L^2-critical, the first iteration map of the Duhamel formula

$$u(t) = e^{-it|D|}u_0 - i\int_0^t e^{-i(t-\tau)|D|}(|u(\tau)|^2 u(\tau))d\tau$$

is not bounded on H^s for $s < \frac{1}{2}$. Indeed, such boundedness would require the inequality

$$\int_0^1 \|\mathrm{e}^{-it|D|} f\|^4_{L^4(\mathbb{T})}\, dt \lesssim \|f\|^4_{H^{s/2}}.$$

However, testing this inequality on functions localized on positive modes, for instance, shows that this fails if $s < \frac{1}{2}$ (see the Appendix for more detail).

Proceeding as in the case of the cubic Szegő equation [Gérard and Grellier 2010a, Theorem 2.1],

$$i\partial_t w = \Pi_+(|w|^2 w), \tag{3}$$

one can prove the global existence and uniqueness of solutions of (1) in H^s for any $s \geq \frac{1}{2}$. The proof uses in particular the a priori bound of the $H^{1/2}$-norm provided by the energy conservation law.

Proposition 1. *Given $u_0 \in H^{\frac{1}{2}}(\mathbb{T})$, there exists $u \in C(\mathbb{R}, H^{\frac{1}{2}}(\mathbb{T}))$ unique such that*

$$i\partial_t u - |D|u = |u|^2 u, \quad u(0,x) = u_0(x).$$

Moreover if $u_0 \in H^s(\mathbb{T})$ for some $s > \frac{1}{2}$, then $u \in C(\mathbb{R}, H^s(\mathbb{T}))$.

Similarly to the cubic Szegő equation, the proof of Proposition 1 provides only bad large time estimates:

$$\|u(t)\|_{H^s} \lesssim \mathrm{e}^{\mathrm{e}^{C_s t}}.$$

This naturally leads to the question of the large time behavior of solutions of (1). In order to answer this question, a fundamental issue is the decoupling of nonnegative and negative modes in system (2). Assuming that initial data are small and spectrally localized on nonnegative modes, a first step in that direction is given by the next simple proposition, which shows that $u_-(t)$ remains smaller in $H^{1/2}$ uniformly in time.

Proposition 2. *Assume*

$$\Pi_+ u_0 = u_0 = \mathcal{O}(\varepsilon) \quad \text{in } H^{\frac{1}{2}}(\mathbb{T}).$$

Then, the solution u of (1) satisfies

$$\sup_{t\in\mathbb{R}} \|\Pi_- u(t)\|_{H^{\frac{1}{2}}} = \mathcal{O}(\varepsilon^2).$$

Proof. By the energy and momentum conservation laws, we have

$$(|D|u, u) + \tfrac{1}{2}\|u\|_{L^4}^4 = (|D|u_0, u_0) + \tfrac{1}{2}\|u_0\|_{L^4}^4, \; (Du, u) = (Du_0, u_0).$$

Subtracting these equalities, we get

$$2(|D|u_-, u_-) + \tfrac{1}{2}\|u\|_{L^4}^4 = \tfrac{1}{2}\|u_0\|_{L^4}^4 = \mathcal{O}(\varepsilon^4);$$

hence

$$\|u_-\|_{H^{\frac{1}{2}}}^2 = \mathcal{O}(\varepsilon^4). \qquad \square$$

This decoupling result suggests neglecting u_- in the system (2) and hence comparing the solutions of (1) to those of

$$i\partial_t v - Dv = \Pi_+(|v|^2 v),$$

which can be reduced to (3) by the transformation $v(t,x) = w(t, x-t)$.

Our main result is the following.

Theorem 1.1. *Let $s > 1$ and $u_0 = \Pi_+(u_0) \in L^2_+(\mathbb{T}) \cap H^s(\mathbb{T})$ with $\|u_0\|_{H^s} = \varepsilon$, for $\varepsilon > 0$ small enough. Denote by v the solution of the cubic Szegő equation*

$$i\partial_t v - Dv = \Pi_+(|v|^2 v), \quad v(0,\cdot) = u_0. \tag{4}$$

Then, for any $\alpha > 0$, there exists a constant $c = c_\alpha < 1$ such that

$$\|u(t) - v(t)\|_{H^s} = \mathcal{O}(\varepsilon^{3-\alpha}) \quad \textit{for } t \le \frac{c_\alpha}{\varepsilon^2}\log\frac{1}{\varepsilon}. \tag{5}$$

Furthermore, there exists $c > 0$ such that

$$\|u(t)\|_{L^\infty} = \mathcal{O}(\varepsilon) \quad \textit{for all } t \le \frac{c}{\varepsilon^3}. \tag{6}$$

Remarks. 1. If we rescale u as εu, Equation (1) becomes

$$i\partial_t u - |D|u = \varepsilon^2 |u|^2 u, \quad u(0,\cdot) = u_0,$$

with $\|u_0\|_{H^s} = 1$. On the latter equation, it is easy to prove that $u(t) = e^{-it|D|}u_0 + o(1)$ for $t \ll 1/\varepsilon^2$, so that nonlinear effects only start for $1/\varepsilon^2 \lesssim t$. Rescaling v as εv in (4), Theorem 1.1 states that the cubic Szegő dynamics appear as the effective dynamics of (1) on a time interval where nonlinear effects are taken into account.

2. As pointed out before, (4) reduces to (3) by a simple Galilean transformation. Equation (3) has been studied in [Gérard and Grellier 2010a; 2010b; 2012], where its complete integrability is established together with an explicit formula for its generic solutions. Consequently, the first part of Theorem 1.1 provides an accurate description of solutions of (1) for a reasonably large time. Moreover, the second part of Theorem 1.1 claims an L^∞ bound for the solution of (1) on an even larger time. This latter bound is closely related to a special conservation law of (3), namely, some Besov norm of v — see Section 2 below.

3. In the case of small Cauchy data localized on nonnegatives modes, system (2) can be reformulated as a — singular — perturbation of the cubic Szegő equation (3). Indeed, write $u_0 = \varepsilon w_0$ and $u(t,x) = \varepsilon w(\varepsilon^2 t, x-t)$; then $w = w_+ + w_-$ solves the system

$$\begin{aligned} i\partial_t w_+ &= \Pi_+(|w|^2 w),\\ i(\varepsilon^2\partial_t - 2\partial_x)w_- &= \varepsilon^2 \Pi_-(|w|^2 w). \end{aligned} \tag{7}$$

Notice that, for $\varepsilon = 0$ and $\Pi_+ w_0 = w_0$, the solution of this system is exactly the solution of (3). It is therefore natural to ask how much, for $\varepsilon > 0$ small, the solution of system (7) stays close to the solution of Equation (3). Since Equation (3) turns out to be completely integrable, this problem appears as a perturbation of a completely integrable infinite-dimensional system. There is a lot of literature on this subject (see the books [Kuksin 1993; Craig 2000; Kappeler and Pöschel 2003] for KAM theory). In the case of the 1D cubic NLS equation and the modified KdV equation, with special initial data such as solitons or 2-solitons, we refer to [Holmer and Zworski 2007; 2008; Holmer et al. 2007; 2011] and references therein. Here we emphasize that our perturbation is more singular and that we deal with general Cauchy data.

4. The proof of Theorem 1.1 is based on a Poincaré–Birkhoff normal form approach, similarly to [Bambusi 2003; Grébert 2007] for instance. More specifically, we prove that (4) turns out to be a Poincaré–Birkhoff normal form of (1), for small initial data with only nonnegative modes.

As a corollary of Theorem 1.1, we get the following instability result.

Corollary 1. *Let $s > 1$. There exist a sequence of data u_0^n and a sequence of times $t^{(n)}$ such that, for any r,*

$$\|u_0^n\|_{H^r} \to 0,$$

while the corresponding solution of (1) satisfies

$$\|u^n(t^{(n)})\|_{H^s} \simeq \|u_0^n\|_{H^s} \left(\log \frac{1}{\|u_0^n\|_{H^s}}\right)^{2s-1}.$$

It is interesting to compare this result to what is known about the cubic NLS. In the one-dimensional case, the cubic NLS is integrable [Zakharov and Shabat 1972] and admits an infinite number of conservation laws which control the regularity of the solution in Sobolev spaces. As a consequence, no such norm inflation occurs. This is in contrast with the 2D cubic NLS case for which Colliander, Keel, Staffilani, Takaoka, and Tao [2010] exhibited small initial data in H^s which give rise to large H^s solutions after a large time.

In our case, the situation is different. Although the cubic Szegő equation is completely integrable, its conservation laws do not control the regularity of the solutions, which allows a large time behavior similar to the one proved in [Colliander et al. 2010] for 2D cubic NLS [Gérard and Grellier 2010a, Section 6, Corollary 5]. Unfortunately, the time interval on which the approximation (5) holds does not allow to infer large solutions for (1), but only solutions with large relative size with respect to their Cauchy data —

see Section 3 below. A time interval of the form $[0, 1/\varepsilon^{2+\beta}]$ for some $\beta > 0$ would be enough to construct large solutions for (1) for some H^s-norms.

We close this introduction by mentioning that O. Pocovnicu solved a similar problem for Equation (1) on the line by using the renormalization group method instead of the Poincaré–Birkhoff normal form method. Moreover, she improved the approximation in Theorem 1.1 by introducing a quintic correction to the Szegő cubic equation [Pocovnicu 2011].

The paper is organized as follows. In Section 2 we recall some basic facts about the Lax pair structure for the cubic Szegő equation (3). In Section 3, we deduce Corollary 1 from Theorem 1.1. Finally, the proof of Theorem 1.1 is given in Section 4.

2. The Lax pair for the cubic Szegő equation and some of its consequences

In this section, we recall some basic facts about Equation (3) (see [Gérard and Grellier 2010a] for more detail). Given $w \in H^{1/2}(\mathbb{T})$, we define (see, e.g., [Peller 1982; Nikolski 2002]) the Hankel operator of symbol w by

$$H_w(h) = \Pi_+(w\bar{h}), \quad h \in L^2_+.$$

It is easy to check that H_w is a $\mathbb{C}$-antilinear Hilbert–Schmidt operator. In [Gérard and Grellier 2010a], we proved that the cubic Szegő flow admits a Lax pair in the following sense. For simplicity let us restrict ourselves to the case of H^s solutions of (3) for $s > \frac{1}{2}$. By [ibid., Theorem 3.1], there exists a mapping $w \in H^s \mapsto B_w$, valued into $\mathbb{C}$-linear bounded skew-symmetric operators on L^2_+, such that

$$H_{-i\Pi_+(|w|^2w)} = [B_w, H_w]. \tag{8}$$

Moreover,

$$B_w = \frac{i}{2}H_w^2 - iT_{|w|^2},$$

where T_b denotes the Toeplitz operator of symbol b, given by $T_b(h) = \Pi_+(bh)$. Consequently, w is a solution of (3) if and only if

$$\frac{d}{dt}H_w = [B_w, H_w]. \tag{9}$$

An important consequence of this structure is that the cubic Szegő equation admits an infinite number of conservation laws. Indeed, denoting by $W(t)$ the solution of the operator equation

$$\frac{d}{dt}W = B_w W, \quad W(0) = I,$$

the operator $W(t)$ is unitary for every t, and

$$W(t)^* H_{w(t)} W(t) = H_{w(0)}.$$

Hence, if w is a solution of (3), then $H_{w(t)}$ is unitarily equivalent to $H_{w(0)}$. Consequently, the spectrum of the $\mathbb{C}$-linear positive self-adjoint trace class operator H_w^2 is conserved by the evolution. In particular, the trace norm of H_w is conserved by the flow. A theorem by Peller [1982, Theorem 2, p. 454] states that the trace norm of a Hankel operator H_w is equivalent to the norm of w in the Besov space $B^1_{1,1}(\mathbb{T})$. Recall that the Besov space $B^1 = B^1_{1,1}(\mathbb{T})$ is defined as the set of functions w such that $\|w\|_{B^1_{1,1}}$ is finite, where

$$\|w\|_{B^1_{1,1}} = \|S_0(w)\|_{L^1} + \sum_{j=0}^{\infty} 2^j \|\Delta_j w\|_{L^1};$$

here $w = S_0(w) + \sum_{j=0}^{\infty} \Delta_j w$ stands for the Littlewood–Paley decomposition of w. It is standard that B^1 is an algebra included into L^∞ (in fact into the Wiener algebra). The conservation of the trace norm of H_w therefore provides an L^∞ estimate for solutions of (3) with initial data in B^1.

The space B^1 and formula (8) will play an important role in the proof of Theorem 1.1. In particular, the last part will follow from the fact that $\|u(t)\|_{B^1}$ remains bounded by ε for $t \ll 1/\varepsilon^3$. The fact that $H^s(\mathbb{T}) \subset B^1$ for $s > 1$, explains why we assume $s > 1$ in the statement.

3. Proof of Corollary 1

As observed in [Gérard and Grellier 2010a, Section 6.1, Proposition 7, and Section 6.2, Corollary 5], the equation

$$i\partial_t w = \Pi_+(|w|^2 w) , \quad w(0, x) = \frac{a_0 \, e^{ix} + b_0}{1 - p_0 e^{ix}}$$

with $a_0, b_0, p_0 \in \mathbb{C}$, $|p_0| < 1$ can be solved as

$$w(t, x) = \frac{a(t) \, e^{ix} + b(t)}{1 - p(t) e^{ix}}$$

where a, b, p satisfy an explicitly solvable ODE system.

In the particular case when

$$a_0 = \varepsilon, \quad b_0 = \varepsilon\delta, \quad p_0 = 0, \quad w_\varepsilon(0, x) = \varepsilon(e^{ix} + \delta) ,$$

one finds

$$1 - \left| p\left(\frac{\pi}{2\varepsilon^2\delta}\right) \right|^2 \simeq \delta^2,$$

so that, for $s > \frac{1}{2}$,

$$\left\| w_\varepsilon\left(\frac{\pi}{2\varepsilon^2\delta}\right) \right\|_{H^s} \simeq \frac{\varepsilon}{\delta^{2s-1}}.$$

Let v_ε be the solution of

$$i(\partial_t + \partial_x) v_\varepsilon = \Pi_+(|v_\varepsilon|^2 v_\varepsilon) , \quad v_\varepsilon(0, x) = \varepsilon(e^{ix} + \delta).$$

Then $v_\varepsilon(t, x) = w_\varepsilon(t, x-t)$, so that

$$\left\| v_\varepsilon\left(\frac{\pi}{2\varepsilon^2\delta}\right) \right\|_{H^s} \simeq \frac{\varepsilon}{\delta^{2s-1}}.$$

Choose

$$\varepsilon = \frac{1}{n}, \quad \delta = \frac{C}{\log n},$$

with C large enough that if $t^{(n)} := \pi/(2\varepsilon^2\delta)$ then $t^{(n)} < c\log(1/\varepsilon)/\varepsilon^2$, where $c = c_\alpha$ in Theorem 1.1 for $\alpha = 1$, say. Set $u_0^n := v_\varepsilon(0,\cdot)$. As $\|u_0^n\|_{H^s} \simeq \varepsilon$, the previous estimate reads

$$\left\| v_\varepsilon\left(\frac{\pi}{2\varepsilon^2\delta}\right)\right\|_{H^s} \simeq \|u_0^n\|_{H^s}\left(\log\frac{1}{\|u_0^n\|_{H^s}}\right)^{2s-1}.$$

Applying Theorem 1.1, we get the same information about $\|u_n(t^{(n)})\|_{H^s}$.

4. Proof of Theorem 1.1

First of all, we rescale u as εu so that Equation (1) becomes

$$i\partial_t u - |D|u = \varepsilon^2|u|^2u, \quad u(0,\cdot) = u_0 \tag{10}$$

with $\|u_0\|_{H^s} = 1$.

4.1. *Study of the resonances.* We write the Duhamel formula as

$$u(t) = \mathrm{e}^{-it|D|}\underline{u}(t),$$

with

$$\hat{\underline{u}}(t,k) = \hat{u}_0(k) - i\varepsilon^2 \sum_{k_1-k_2+k_3-k=0} I(k_1,k_2,k_3,k),$$

where

$$I(k_1,k_2,k_3,k) = \int_0^t \mathrm{e}^{-i\tau\Phi(k_1,k_2,k_3,k)}\hat{\underline{u}}(\tau,k_1)\overline{\hat{\underline{u}}(\tau,k_2)}\hat{\underline{u}}(\tau,k_3)\,d\tau,$$

and

$$\Phi(k_1,k_2,k_3,k_4) := |k_1| - |k_2| + |k_3| - |k_4|.$$

If $\Phi(k_1,k_2,k_3,k_4) \neq 0$, an integration by parts in $I(k_1,k_2,k_3,k_4)$ provides an extra factor ε^2; hence the set of (k_1,k_2,k_3,k_4) such that $\Phi(k_1,k_2,k_3,k_4) = 0$ is expected to play a crucial role in the analysis. This set is described in the following lemma.

Lemma 1. *Given* $(k_1,k_2,k_3,k_4) \in \mathbb{Z}^4$,

$$k_1 - k_2 + k_3 - k_4 = 0 \quad \text{and} \quad |k_1| - |k_2| + |k_3| - |k_4| = 0$$

if and only if at least one of the following properties holds:

(a) $k_j \geq 0$ *for all* j.

(b) $k_j \leq 0$ *for all* j.

(c) $k_1 = k_2$ *and* $k_3 = k_4$.

(d) $k_1 = k_4$ *and* $k_3 = k_2$.

Proof. Consider $(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4$ such that $k_1 - k_2 + k_3 - k_4 = 0$, $|k_1| - |k_2| + |k_3| - |k_4| = 0$, and the k_j are not all nonnegative or all nonpositive. Let us prove in that case that either $k_1 = k_2$ and $k_3 = k_4$, or $k_1 = k_4$ and $k_3 = k_2$. Without loss of generality, we can assume that at least one of the k_j is positive, for instance k_1. Then, subtracting both equations, we get that $|k_3| - k_3 = |k_2| - k_2 + |k_4| - k_4$. If k_3 is nonnegative, both k_2 and k_4 must be nonnegative; hence all the k_j are nonnegative. Assume now that k_3 is negative. At least one among k_2, k_4 is negative. If both of them are negative, then $k_3 = k_2 + k_4$ but this would imply $k_1 = 0$ which is impossible by assumption. So we get either that $k_3 = k_2$ (and so $k_1 = k_4$) or $k_3 = k_4$ (and so $k_1 = k_2$). This completes the proof of the lemma. □

4.2. ***First reduction.*** We get rid of the resonances corresponding to cases (c) and (d) by applying the transformation

$$u(t) \mapsto \mathrm{e}^{2it\varepsilon^2\|u_0\|^2_{L^2}} u(t) \tag{11}$$

which, since the L^2 norm of u is conserved, leads to the equation

$$i\partial_t u - |D|u = \varepsilon^2(|u|^2 - 2\|u\|^2_{L^2})u, \quad u(0,\cdot) = u_0. \tag{12}$$

Notice that this transformation does not change the H^s norm. The Hamiltonian function associated to (12) is given by

$$H(u) = \tfrac{1}{2}(|D|u, u) + \tfrac{1}{4}\varepsilon^2(\|u\|^4_{L^4} - 2\|u\|^4_{L^2}) = H_0(u) + \varepsilon^2 R(u),$$

where

$$H_0(u) := \tfrac{1}{2}(|D|u, u),$$
$$R(u) := \tfrac{1}{4}(\|u\|^4_{L^4} - 2\|u\|^4_{L^2}) = \frac{1}{4}\Bigg(\sum_{\substack{k_1-k_2+k_3-k_4=0\\ k_1\neq k_2,k_4}} u_{k_1}\overline{u_{k_2}}u_{k_3}\overline{u_{k_4}} - \sum_{k\in\mathbb{Z}} |u_k|^4\Bigg).$$

4.3. ***The Poincaré–Birkhoff normal form.*** We claim that under a suitable canonical transformation on u, H can be reduced to the Hamiltonian

$$\tilde{H}(u) = H_0(u) + \varepsilon^2\tilde{R}(u) + O(\varepsilon^4),$$

where

$$\tilde{R}(u) = \frac{1}{4}\sum_{(k_1,k_2,k_3,k_4)\in\mathcal{R}} u_{k_1}\overline{u_{k_2}}u_{k_3}\overline{u_{k_4}},$$

with

$$\mathcal{R} = \big\{(k_1,k_2,k_3,k_4) : k_1 - k_2 + k_3 - k_4 = 0;\ k_1 \neq k_2;\ k_1 \neq k_4;\ k_j \geq 0 \text{ for all } j \text{ or } k_j \leq 0 \text{ for all } j\big\}.$$

We look for a canonical transformation as the value at time 1 of some Hamiltonian flow. In other words, we look for a function F such that its Hamiltonian vector field is smooth on H^s and on B^1, so that our canonical transformation is φ_1, where φ_σ is the solution of

$$\frac{d}{d\sigma}\varphi_\sigma(u) = \varepsilon^2 X_F(\varphi_\sigma(u)), \quad \varphi_0(u) = u. \tag{13}$$

Recall that, given a smooth real valued function F, its Hamiltonian vector field X_F is defined by

$$dF(u).h =: \omega(h, X_F(u)),$$

and, given functions F, G admitting Hamiltonian vector fields, their Poisson bracket $\{F, G\}$ is defined by

$$\{F, G\}(u) = \omega(X_F(u), X_G(u)).$$

Let us make some preliminary remarks about the Poisson brackets.

In view of the expression of ω, we have

$$\{F, G\} := dG.X_F = \frac{2}{i}\sum_k (\partial_{\bar{k}} F \partial_k G - \partial_{\bar{k}} G \partial_k F)$$

where $\partial_k F$ stands for $\partial F/\partial u_k$ and $\partial_{\bar{k}} F$ for $\partial F/\partial \overline{u_k}$. In particular, if F and G are respectively homogeneous of order p and q, then their Poisson bracket is homogeneous of order $p+q-2$.

Lemma 2. *Set*

$$F(u) := \sum_{k_1-k_2+k_3-k_4=0} f_{k,k_2,k_3,k_4} u_{k_1}\overline{u_{k_2}}u_{k_3}\overline{u_{k_4}},$$

where

$$f_{k_1,k_2,k_3,k_4} = \begin{cases} \dfrac{i}{4(|k_1|-|k_2|+|k_3|-|k_4|)} & \textit{if } |k_1|-|k_2|+|k_3|-|k_4| \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Then X_F is smooth on H^s, $s > \frac{1}{2}$, as well as on B^1, and

$$\{F, H_0\} + R = \tilde{R},$$

$$\|DX_F(u)h\| \lesssim \|u\|^2\|h\|,$$

where the norm is taken either in H^s, $s > \frac{1}{2}$, or in B^1.

Proof. First we make a formal calculation with F given by

$$F(u) := \sum_{k_1-k_2+k_3-k_4=0} f_{k_1,k_2,k_3,k_4} u_{k_1}\overline{u_{k_2}}u_{k_3}\overline{u_{k_4}}$$

for some coefficients f_{k_1,k_2,k_3,k_4} to be determined later. We compute

$$\{F, H_0\} = \frac{1}{i}\sum_{k_1-k_2+k_3-k_4=0} (-|k_1|+|k_2|-|k_3|+|k_4|) f_{k_1,k_2,k_3,k_4} u_{k_1}\overline{u_{k_2}}u_{k_3}\overline{u_{k_4}},$$

so that equality $\{F, H_0\} + R = \tilde{R}$ requires

$$f_{k_1,k_2,k_3,k_4} = \begin{cases} \dfrac{i}{4(|k_1|-|k_2|+|k_3|-|k_4|)} & \text{if } |k_1|-|k_2|+|k_3|-|k_4| \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

One can easily check that the function F is explicitly given by

$$F(u) = \frac{1}{2}\mathrm{Im}\big((D_0^{-1}u_-, |u_+|^2u_+) - (D_0^{-1}u_+, |u_-|^2u_-) - (D_0^{-1}|u_+|^2, |u_-|^2)\big),$$

where D_0^{-1} is the operator defined by

$$D_0^{-1}u(x) = \sum_{k\neq 0} \frac{u_k}{k}\, \mathrm{e}^{ikx}.$$

Notice that, since functions $|u_+|^2$ and $|u_-|^2$ are real valued, the quantity $(D_0^{-1}|u_+|^2, |u_-|^2)$ is purely imaginary, and therefore is equal to i times its imaginary part.

In view of the formula above, the Hamiltonian vector field $X_F(u)$ is a sum of products of terms involving the maps $f \mapsto \bar{f}$, $f \mapsto D_0^{-1}f$, $f \mapsto \Pi_\pm f$, $(f, g) \mapsto fg$. These maps are continuous on H^s and on B^1. Hence, X_F is smooth and its differential satisfies the claimed estimate on H^s, $s > \frac{1}{2}$, and B^1. □

The proof of the following technical lemma is based on straightforward calculations.

Lemma 3. *The function $\tilde{R}$ and its Hamiltonian vector field are given by*

$$\tilde{R}(u) = \frac{1}{4}(\|u_+\|_{L^4}^4 + \|u_-\|_{L^4}^4) + \mathrm{Re}((u, 1)\,(u_-, u_-^2)) - \frac{1}{2}(\|u_+\|_{L^2}^4 + \|u_-\|_{L^2}^4),$$

$$iX_{\tilde{R}}(u) = \Pi_+(|u_+|^2u_+) + \Pi_-(|u_-|^2u_-) - 2\|u_+\|_{L^2}^2u_+ - 2\|u_-\|_{L^2}^2u_- + (u_-^2, u_-) + 2(1, u)\Pi_-(|u_-|^2) + (1, u)u_-^2,$$

where we have set $u_\pm := \Pi_\pm(u)$.

The maps $X_{\{F,R\}}$ and $X_{\{F,\tilde{R}\}}$ are smooth homogeneous polynomials of degree five on B^1 and on H^s for every $s > \frac{1}{2}$.

We now perform the canonical transformation

$$\chi_\varepsilon := \exp(\varepsilon^2 X_F).$$

Lemma 4. *Set $\varphi_\sigma := \exp(\varepsilon^2\sigma X_F)$ for $-1 \le \sigma \le 1$. There exist $m_0 > 0$ and $C_0 > 0$ so that, for any $u \in B^1$ so that $\varepsilon\|u\|_{B^1} \le m_0$, $\varphi_\sigma(u)$ is well defined for $\sigma \in [-1, 1]$ and*

$$\|\varphi_\sigma(u)\|_{B^1} \le \frac{3}{2}\|u\|_{B^1},$$

$$\|\varphi_\sigma(u) - u\|_{B^1} \le C_0\varepsilon^2\|u\|_{B^1}^3,$$

$$\|D\varphi_\sigma(u)\|_{B^1\to B^1} \le e^{C_0\varepsilon^2\|u\|_{B^1}^2}.$$

Moreover, the same estimates hold in H^s, $s > \frac{1}{2}$, with some constants $m(s)$ and $C(s)$.

Proof. Write φ_σ as the integral of its derivative and use Lemma 2 to get

$$\sup_{|\sigma|\le\tau} \|\varphi_\sigma(u)\|_{B^1} \le \|u\|_{B^1} + C\varepsilon^2 \sup_{|\sigma|\le\tau} \|\varphi_\sigma(u)\|_{B^1}^3, \quad 0 \le \tau \le 1. \tag{14}$$

We now use the following standard bootstrap lemma.

Lemma 5. *Let $a, b, T > 0$ and $\tau \in [0, T] \mapsto M(\tau) \in \mathbb{R}_+$ be a continuous function satisfying*

$$M(\tau) \le a + bM(\tau)^3 \quad \text{for all } \tau \in [0, T].$$

Assume that $\sqrt{3b}\, M(0) < 1$ and $\sqrt{3b}\, a < \frac{2}{3}$. Then $M(\tau) \le \frac{3}{2}a$ for all $\tau \in [0, T]$.

Proof. For the convenience of the reader, we give the proof of Lemma 5. The function $f : z \ge 0 \mapsto z - bz^3$ attains its maximum at $z_c = 1/\sqrt{3b}$, equal to $f_m = 2/(3\sqrt{3b})$. Consequently, since a is smaller than f_m by the second inequality,

$$\{z \ge 0 : f(z) \le a\} = [0, z_-] \cup [z_+, +\infty)$$

with $z_- < z_c < z_+$ and $f(z_-) = a$. Since $M(\tau)$ belongs to this set for every τ and since $M(0)$ belongs to the first interval by the first inequality, we conclude by continuity that $M(\tau) \le z_-$ for every τ. By the concavity of f, $f(z) \ge \frac{2}{3}z$ for $z \in [0, z_c]$, hence $z_- \le \frac{3}{2}a$. □

Let us come back to the proof of Lemma 4. If $\varepsilon \|u\|_{B^1} < \dfrac{2}{3\sqrt{3C}}$, Equation (14) and Lemma 5 imply that

$$\sup_{|\sigma| \le 1} \|\varphi_\sigma(u)\|_{B^1} \le \tfrac{3}{2}\|u\|_{B^1}, \tag{15}$$

which is the first estimate. For the second one, we write for $|\sigma| \le 1$,

$$\|\varphi_\sigma(u) - u\|_{B^1} = \|\varphi_\sigma(u) - \varphi_0(u)\|_{B^1} \le |\sigma| \sup_{|s| \le |\sigma|} \left\| \frac{d}{ds}\varphi_s(u) \right\|_{B^1} \le C_0 \varepsilon^2 \|u\|^3_{B^1},$$

where the last inequality comes from Lemma 2 and estimate (15).

It remains to prove the last estimate. We differentiate the equation satisfied by φ_σ and use again Lemma 2 to obtain

$$\begin{aligned} \|D\varphi_\sigma(u)\|_{B^1 \to B^1} &\le 1 + \varepsilon^2 \left| \int_0^\sigma \|DX_F(\varphi_\tau(u))\|_{B^1 \to B^1} \|D\varphi_\tau(u)\|_{B^1 \to B^1}\, d\tau \right| \\ &\le 1 + C_0 \varepsilon^2 \|u\|^2_{B^1} \left| \int_0^\sigma \|D\varphi_\tau(u)\|_{B^1 \to B^1} d\tau \right|, \end{aligned}$$

and Gronwall's lemma yields the result. Analogous proofs give the estimates in H^s. □

Let u satisfy the assumption of Lemma 4 in B^1 or in H^s for some $s > \frac{1}{2}$.

Let us compute $H \circ \chi_\varepsilon = H \circ \varphi_1$ as the Taylor expansion of $H \circ \varphi_\sigma$ at time 1 around 0. One gets

$$\begin{aligned} H \circ \chi_\varepsilon &= H \circ \varphi_1 = H_0 \circ \varphi_1 + \varepsilon^2 R \circ \varphi_1 \\ &= H_0 + \frac{d}{d\sigma}[H_0 \circ \varphi_\sigma]_{\sigma=0} + \varepsilon^2 R + \int_0^1 \left((1-\sigma)\frac{d^2}{d\sigma^2}[H_0 \circ \varphi_\sigma] + \varepsilon^2 \frac{d}{d\sigma}[R \circ \varphi_\sigma] \right) d\sigma \\ &= H_0 + \varepsilon^2(\{F, H_0\} + R) + \varepsilon^4 \int_0^1 ((1-\sigma)\{F, \{F, H_0\}\} + \{F, R\}) \circ \varphi_\sigma \, d\sigma \\ &= H_0 + \varepsilon^2 \tilde{R} + \varepsilon^4 \int_0^1 \Big((1-\sigma)\{F, \tilde{R}\} + \sigma\{F, R\} \Big) \circ \varphi_\sigma \, d\sigma \\ &=: H_0 + \varepsilon^2 \tilde{R} + \varepsilon^4 \int_0^1 G(\sigma) \circ \varphi_\sigma \, d\sigma. \end{aligned}$$

By Lemma 3, one gets

$$\sup_{0\le\sigma\le 1} \|X_{G(\sigma)}(w)\| \le C\|w\|^5$$

where the norm stands for the B^1 norm or the H^s norm. Since

$$X_{G(\sigma)\circ\varphi_\sigma}(u) = D\varphi_{-\sigma}(\varphi_\sigma(u)).X_{G(\sigma)}(\varphi_\sigma(u)),$$

we conclude from Lemma 4 that, if $\varepsilon\|u\|_{B^1} \le m_0$,

$$\|X_{G(\sigma)\circ\varphi_\sigma}(u)\|_{B^1} \le C\|u\|_{B^1}^5 .$$

As a consequence, one can write

$$X_{H\circ\chi_\varepsilon} = X_{H_0} + \varepsilon^2 X_{\tilde R} + \varepsilon^4 Y,$$

where, if $\varepsilon\|u\|_{B^1} \le m_0$, then

$$\|Y(u)\|_{B^1} \lesssim \|u\|_{B^1}^5 .$$

An analogous estimate holds in H^s, $s > \frac{1}{2}$.

4.4. ***End of the proof.*** We first deal with the B^1-norm of a solution u of (12). We are going to prove that $\|u(t)\|_{B^1} = \mathcal{O}(1)$ for $t \ll 1/\varepsilon^3$ by the following bootstrap argument. We assume that for some K large enough with respect to $\|u_0\|_{B^1}$, for some $T > 0$, for all $t \in [0, T]$, we have $\|u(t)\|_{B^1} \le 10K$, and we prove that if $T \ll 1/\varepsilon^3$, $\|u(t)\|_{B^1} \le K$ for $t \in [0, T]$. This will prove the result by continuity.

Set, for $t \in [0, T]$,

$$\tilde u(t) := \chi_\varepsilon^{-1}(u(t)),$$

so that $\tilde u$ is a solution of

$$i\partial_t \tilde u - |D|\tilde u = \varepsilon^2 i X_{\tilde R}(\tilde u) + \varepsilon^4 i Y(\tilde u).$$

Moreover, by Lemma 4,

$$\|\tilde u(t) - u(t)\|_{B^1} \lesssim \varepsilon^2 \|u\|_{B^1}^3$$

and so by the hypothesis, $\|\tilde u(t)\|_{B^1} \le 11K$ if ε is small enough. In view of the expression of the Hamiltonian vector field of $\tilde R$ in Lemma 3, the equation for $\tilde u$ reads

$$\begin{cases} i\partial_t \tilde u_+ - D\tilde u_+ = \varepsilon^2\left(\Pi_+(|\tilde u_+|^2\tilde u_+) - 2\|\tilde u_+\|_{L^2}^2 \tilde u_+ + \int_{\mathbb{T}} |\tilde u_-|^2 \tilde u_-\right) + \varepsilon^4 i Y_+(\tilde u),\\ i\partial_t \tilde u_- + D\tilde u_- = \varepsilon^2\left(\Pi_-(|\tilde u_-|^2\tilde u_-) - 2\|\tilde u_-\|_{L^2}^2 \tilde u_- + 2(1,\tilde u)\Pi_-(|\tilde u_-|^2) + (1,\tilde u)\tilde u_-^2\right) + \varepsilon^4 i Y_-(\tilde u). \end{cases}$$

Notice that all the Hamiltonian functions we have dealt with so far are invariant by multiplication by complex numbers of modulus 1, hence their Hamiltonian vector fields satisfy

$$X(\mathrm{e}^{i\theta} z) = \mathrm{e}^{i\theta} z,$$

so that the corresponding Hamiltonian flows conserve the L^2 norm. Hence $\tilde u$ has the same L^2 norm as u, which is the L^2 norm of u_0. In particular, $|(1,\tilde u)| \le \|u_0\|_{L^2}$.

Moreover, as $\|u_0\|_{B^1} \lesssim \|u_0\|_{H^s} = \mathcal{O}(1)$ since $s > 1$, $\tilde{u}_0$ satisfies

$$\|\tilde{u}_0 - u_0\|_{B^1} \lesssim \varepsilon^2$$

by Lemma 4, so that, as $u_{0-} = 0$, we get $\|\tilde{u_0}_-\|_{B^1} = \mathcal{O}(\varepsilon^2)$. Then we obtain from the second equation

$$\sup_{0\le\tau\le t} \|\tilde{u}_-(\tau)\|_{B^1} \lesssim \varepsilon^2 + \varepsilon^2 t\Big(\sup_{0\le\tau\le t} \|\tilde{u}_-(\tau)\|_{B^1}^3 + \sup_{0\le\tau\le t} \|\tilde{u}_-(\tau)\|_{B^1}^2\Big) + \varepsilon^4 t K^5.$$

Let $M(t) = \dfrac{1}{\varepsilon} \sup_{0\le\tau\le t} \|\tilde{u}_-(\tau)\|_{B^1}$, so that, if $t \le T$,

$$M(t) \lesssim \varepsilon + \varepsilon^3 T M(t)^2 (1 + \varepsilon M(t)) + \varepsilon^3 T.$$

As $3m^2 \le 1 + 2m^3$ for any $m \ge 0$, we get

$$M(t) \lesssim \varepsilon + \varepsilon^3 T M(t)^3 + \varepsilon^3 T.$$

Using Lemma 5, we conclude that, if $T \ll 1/\varepsilon^3$,

$$\sup_{0\le\tau\le T} \|\tilde{u}_-(\tau)\|_{B^1} \ll \varepsilon.$$

For further reference, notice that, if $T \lesssim \dfrac{1}{\varepsilon^2} \log \dfrac{1}{\varepsilon}$, this estimate can be improved to

$$\sup_{0\le\tau\le T} \|\tilde{u}_-(\tau)\|_{B^1} \lesssim \varepsilon^{2-\alpha} \quad \text{for all } \alpha > 0.$$

We come back to the case $T \ll 1/\varepsilon^3$. From the estimate on $\tilde{u}_-$, we infer

$$\|\tilde{u}_+\|_{L^2}^2 = \|\tilde{u}\|_{L^2}^2 + \mathcal{O}(\varepsilon^2) = \|u_0\|_{L^2}^2 + \mathcal{O}(\varepsilon^2),$$

and the equation for $\tilde{u}_+$ reads

$$i\partial_t \tilde{u}_+ - D\tilde{u}_+ = \varepsilon^2 \left(\Pi_+(|\tilde{u}_+|^2 \tilde{u}_+) - 2\|u_0\|_{L^2}^2 \tilde{u}_+\right) + \varepsilon^4 i Y_+(\tilde{u}) + \mathcal{O}(\varepsilon^5) + \mathcal{O}(\varepsilon^4)\tilde{u}_+.$$

Since $\tilde{u_0}_+$ is not small in B^1, we have to use a different strategy to estimate $\tilde{u}_+$. We use the complete integrability of the cubic Szegő equation, especially its Lax pair and the conservation of the B^1-norm.

At this stage it is of course convenient to cancel the linear term $\|u_0\|_{L^2}^2 \tilde{u}_+$ by multiplying $\tilde{u}_+(t)$ by $\mathrm{e}^{2i\varepsilon^2 t \|u_0\|_{L^2}^2}$. As pointed out before, this change of unknown is completely transparent to the above system. This leads to

$$i\partial_t \tilde{u}_+ - D\tilde{u}_+ = \varepsilon^2 \Pi_+(|\tilde{u}_+|^2 \tilde{u}_+) + \varepsilon^4 Y_+(\tilde{u}) + \mathcal{O}(\varepsilon^5) + \mathcal{O}(\varepsilon^4)\tilde{u}_+.$$

Notice that all the $\mathcal{O}$ terms above are measured in B^1 norm. We now appeal to the results recalled in Section 2. We introduce the unitary family $U(t)$ defined by

$$i\partial_t U - DU = \varepsilon^2 (T_{|\tilde{u}_+|^2} - \tfrac{1}{2} H_{\tilde{u}_+}^2) U, \quad U(0) = I,$$

so that, using formula (8),

$$i\partial_t (U(t)^* H_{\tilde{u}_+(t)} U(t)) = \varepsilon^4 U(t)^* H_{Y_+(\tilde{u}) + \mathcal{O}(\varepsilon) + \mathcal{O}(1)\tilde{u}_+} U(t).$$

Then, we use the theorem from [Peller 1982] that states, as recalled in Section 2, that the trace norm of a Hankel operator of symbol b is equivalent to the B^1-norm of b to obtain

$$\begin{aligned}\|\tilde u_+(t)\|_{B^1} &\simeq \mathrm{Tr}|H_{\tilde u_+(t)}|\\ &\lesssim \mathrm{Tr}|H_{\tilde u_{0+}}| + \varepsilon^4\int_0^t (\mathrm{Tr}|H_{Y_+(\tilde u)}(\tau)| + \mathrm{Tr}|H_{\tilde u_+}(\tau)| + \varepsilon)\,d\tau\\ &\lesssim \|\tilde u_{0+}\|_{B^1} + \varepsilon^4\int_0^t (\|\tilde u(\tau)\|_{B^1}^5 + \|\tilde u_+(\tau)\|_{B^1} + \varepsilon)\,d\tau\end{aligned}$$

so that as $\|\tilde u(t)\|_{B^1} \le 11K$,

$$\|\tilde u_+(t)\|_{B^1} \lesssim \|\tilde{u_0}_+\|_{B^1} + \varepsilon^4 t(11K)^5,$$

and, if $t \ll 1/\varepsilon^3$ and ε is small enough,

$$\|\tilde u(t)\|_{B^1} \le \frac{K}{10}.$$

Using again the second estimate in Lemma 4, we infer

$$\|u(t)\|_{B^1} \le K.$$

Finally, using the inverse of transformation (11) and multiplying u by ε, we obtain estimate (6) of Theorem 1.1.

We now estimate the difference between the solution of the wave equation and the solution of the cubic Szegő equation. Since we have applied transformation (11), we have to compare in B^1 the solution u of (12) to the solution v of equation

$$i\partial_t v - Dv = \varepsilon^2(\Pi_+(|v|^2 v) - 2\|u_0\|_{L^2}^2 v)\ , \quad v(0) = u_0.$$

Notice that, as u_0 is bounded in H^s, $s > 1$, and as the B^1 norm is conserved by the cubic Szegő flow,

$$\|v(t)\|_{B^1} \simeq \|u_0\|_{B^1} \lesssim \|u_0\|_{H^s} = \mathcal{O}(1).$$

We shall prove that, for every $\alpha > 0$, there exists $c_\alpha > 0$ such that,

$$\|u(t) - v(t)\|_{B^1} \le \varepsilon^{2-\alpha} \quad \text{for all } t \le \frac{c_\alpha}{\varepsilon^2}\log\frac{1}{\varepsilon}.$$

In view of the previous estimates, it is enough to prove that, on the same time interval,

$$\|\tilde u_+(t) - v(t)\|_{B^1} \le \varepsilon^{2-\alpha},$$

where $\tilde u_+$ satisfies

$$\begin{cases} i\partial_t \tilde u_+ - D\tilde u_+ = \varepsilon^2\left(\Pi_+(|\tilde u_+|^2\tilde u_+) - 2\|u_0\|_{L^2}^2\tilde u_+\right) + \mathcal{O}(\varepsilon^4),\\ \qquad \tilde u_+(0) = \tilde u_{0,+}.\end{cases} \tag{16}$$

As $\|\tilde u(t)\|_{B^1} \lesssim 1$, $\|v(t)\|_{B^1} \lesssim 1$, $\|\tilde u_{0,+} - u_0\|_{B^1} \le \varepsilon^2\|u_0\|_{B^1} \lesssim \varepsilon^2$ and

$$(i\partial_t - D)(\tilde u_+ - v) = \varepsilon^2\Pi_+(|\tilde u_+|^2\tilde u_+ - |v|^2 v - 2\|u_0\|_{L^2}^2(\tilde u_+ - v)) + \mathcal{O}(\varepsilon^4),$$

we get, using that B^1 is an algebra on which Π_+ acts,

$$\|\tilde{u}_+(t) - v(t)\|_{B^1} \lesssim \varepsilon^2 + \varepsilon^4 t + \varepsilon^2 \int_0^t \|\tilde{u}_+(\tau) - v(\tau)\|_{B^1} \, d\tau.$$

This yields

$$\|\tilde{u}_+(t) - v(t)\|_{B^1} \lesssim (\varepsilon^2 + \varepsilon^4 t) e^{\varepsilon^2 t};$$

hence, for $t \leq \dfrac{c_\alpha}{\varepsilon^2} \log \dfrac{1}{\varepsilon}$,

$$\|\tilde{u}_+(t) - v(t)\|_{B^1} \leq \varepsilon^{2-\alpha}.$$

We now turn to the estimates in H^s for $s > 1$.

From the equation on v and the a priori estimate in B^1, it follows that $\|v(t)\|_{H^s} \leq A e^{A\varepsilon^2 t}$, $t > 0$, so that $\|v(t)\|_{H^s} \leq N(\varepsilon)$ for $t \leq (c/\varepsilon^2) \log(1/\varepsilon)$, $0 < c \ll 1$, where $N(\varepsilon) := A\varepsilon^{-cA}$.

Let us assume that for some $T > 0$,

$$\|u(t)\|_{H^s} \leq 10N(\varepsilon) \quad \text{for all } t \in [0, T].$$

We are going to prove that, for every $\alpha > 0$, there exists $c_\alpha > 0$ such that, if

$$T \leq \frac{c_\alpha}{\varepsilon^2} \log \frac{1}{\varepsilon},$$

then

$$\|u(t) - v(t)\|_{H^s} \leq \varepsilon^{2-\alpha} \quad \text{for all } t \in [0, T]$$

Since $\|v(t)\|_{H^s} \leq N(\varepsilon)$ for $t \leq (c/\varepsilon^2) \log(1/\varepsilon)$, this will prove the result by a bootstrap argument.

As before, we perform the same canonical transformation

$$\tilde{u}(t) := \chi_\varepsilon^{-1}(u(t)),$$

to get the solution of

$$i\partial_t \tilde{u} - |D| \, \tilde{u} = \varepsilon^2 i X_{\tilde{R}}(\tilde{u}) + \varepsilon^4 i Y(\tilde{u}).$$

By Lemma 4,

$$\|\tilde{u}(t) - u(t)\|_{H^s} \lesssim \varepsilon^2 N(\varepsilon)^3$$

and so $\|\tilde{u}(t)\|_{H^s} \lesssim N(\varepsilon)$. Therefore it suffices to prove that

$$\|\tilde{u}(t) - v(t)\|_{H^s} \leq \varepsilon^{2-\alpha} \quad \text{for all } t \in [0, T].$$

We first deal with $\tilde{u}_-$. A similar argument to the one developed in B^1 gives that for, for $0 \leq t \lesssim \dfrac{1}{\varepsilon^2} \log \dfrac{1}{\varepsilon}$,

$$\sup_{0 \leq \tau \leq t} \|\tilde{u}_-(\tau)\|_{H^s} \leq C_\alpha \varepsilon^{2-\alpha}$$

for every $\alpha > 0$.

It remains to estimate the H^s norm of $\tilde{u}_+ - v$. Notice that

$$\|\tilde{u}_{0,+} - u_0\|_{H^s} \leq \varepsilon^2$$

by Lemma 4. We use the following inequality — recall that $B^1 \subset L^\infty$:

$$\| \Pi_+(|u|^2u - |v|^2v) \|_{H^s} \lesssim \big(\|u\|_{B^1}^2 + \|v\|_{B^1}^2 \big) \|u-v\|_{H^s} + \big(\|v\|_{H^s} + \|u-v\|_{H^s} \big) \big(\|u\|_{B^1} + \|v\|_{B^1} \big) \|u-v\|_{B^1}.$$

Plugging this into a Gronwall inequality, in view of the previous estimates, we finally get

$$\|\tilde{u}_+(t) - v(t)\|_{H^s} \le \varepsilon^{2-\alpha}$$

for $t \le \frac{c_\alpha}{\varepsilon^2} \log \frac{1}{\varepsilon}$. This completes the proof.

Appendix: A necessary condition for wellposedness

In this section, we justify that the boundedness in H^s of the first iteration map of the Duhamel formula

$$F(t) = e^{-it|D|} f - i \int_0^t e^{-i(t-\tau)|D|} (|F(\tau)|^2 F(\tau)) \, d\tau$$

implies

$$\int_0^1 \|e^{-it|D|} f\|_{L^4(\mathbb{T})}^4 dt \lesssim \|f\|_{H^{s/2}}^4.$$

Indeed, assume the inequality

$$\left\| \int_0^1 e^{-i(1-\tau)|D|} (|e^{-i\tau|D|} f|^2 e^{-i\tau|D|} f) d\tau \right\|_{H^s} \lesssim \|f\|_{H^s}^3.$$

We compute the scalar product of the expression in the left hand side with $e^{-i|D|} f$ and we get

$$\int_0^1 \|e^{-i\tau|D|} f\|_{L^4}^4 d\tau \lesssim \|f\|_{H^s}^3 \|f\|_{H^{-s}}.$$

If we assume first that f is spectrally supported, that is if $f = \Delta_N f$ for some N, then $\|f\|_{H^{\pm s}} \simeq N^{\pm s} \|f\|_{L^2}$ and the preceding inequality becomes

$$\int_0^1 \|e^{-i\tau|D|} f\|_{L^4}^4 d\tau \lesssim N^{2s} \|f\|_{L^2}^4.$$

Finally, for general $f = \sum_N \Delta_N(f)$, we used the Littlewood–Paley estimate

$$\|g\|_{L^4}^4 \lesssim \sum_N \|\Delta_N g\|_{L^4}^4$$

to get

$$\int_0^1 \|e^{-i\tau|D|} f\|_{L^4}^4 d\tau \lesssim \|f\|_{H^{s/2}}^4.$$

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