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ON THE SCHRÖDINGER EQUATION OUTSIDE STRICTLY CONVEX OBSTACLES

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We prove sharp Strichartz estimates for the semiclassical Schrödinger equation on a compact Riemannian manifold with a smooth, strictly geodesically concave boundary. We deduce classical Strichartz estimates for the Schrödinger equation outside a strictly convex obstacle, local existence for the H^1-critical (quintic) Schrödinger equation, and scattering for the subcritical Schrödinger equation in three dimensions.

1. Introduction

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. Strichartz estimates are a family of dispersive estimates on solutions $u(x,t) : M \times [-T, T] \to \mathbb{C}$ to the Schrödinger equation

$$i\partial_t u + \Delta_g u = 0, \quad u(x,0) = u_0(x), \tag{1-1}$$

where Δ_g denotes the Laplace–Beltrami operator on (M, g). In their most general form, local Strichartz estimates state that

$$\|u\|_{L^q([-T,T],L^r(M))} \leq C\|u_0\|_{H^s(M)}, \tag{1-2}$$

where $H^s(M)$ denotes the Sobolev space over M and $2 \leq q, r \leq \infty$ satisfy $(q, r, n) \neq (2, \infty, 2)$ (for the case $q = 2$ see [Keel and Tao 1998]) and are given by the scaling admissibility condition

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}. \tag{1-3}$$

In $\mathbb{R}^n$ and for $g_{ij} = \delta_{ij}$, Strichartz estimates in the context of the wave and Schrödinger equations have a long history, beginning with the pioneering work [Strichartz 1977], where the particular case $q = r$ for the wave and (classical) Schrödinger equations was proved. This was later generalized to mixed $L^q_t L^r_x$ norms by Ginibre and Velo [1985] for Schrödinger equations, where (q, r) is sharp admissible and $q > 2$; the wave estimates were obtained independently by the same authors [1995] and by Lindblad and Sogge [1995], following [Kapitanskiĭ 1989]. The remaining endpoints for both equations were finally settled by Keel and Tao [1998]. In that case $s = 0$ and $T = \infty$; see also [Kato 1987; Cazenave and Weissler 1990]. Estimates for the flat 2-torus were shown by Bourgain [2003] to hold for $q = r = 4$ and any $s > 0$.

In the variable coefficients case, even without boundaries, the situation is much more complicated: we simply recall the pioneering work of Staffilani and Tataru [2002], dealing with compact, nontrapping perturbations of the flat metric, the works by Hassell et al. [2006], Robbiano and Zuily [2005], and Bouclet and Tzvetkov [2008] which considerably weakens the decay of the perturbation (retaining the

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nontrapping character at spatial infinity). On compact manifolds without boundaries, Burq et al. [2004b] established Strichartz estimates with $s = 1/p$, hence with a loss of derivatives when compared to the case of flat geometries. Recently, Blair et al. [2008] improved on the current results for compact (M, g) where either $\partial M \neq \varnothing$, or $\partial M = \varnothing$ and g Lipschitz, by showing that Strichartz estimates hold with a loss of $s = 4/3p$ derivatives. This appears to be the natural analog of the estimates of Burq et al. for the general boundaryless case.

In this paper we prove that Strichartz estimates for the semiclassical Schrödinger equation also hold on Riemannian manifolds with smooth, strictly geodesically concave boundaries. By the last condition we understand that the second fundamental form on the boundary of the manifold is strictly positive definite. moreover the manifold to be flat at infinity; i.e., the metric coincides with the Euclidean one outside a compact set (though presumably one may use [Bouclet and Tzvetkov 2008] result to combine both situations). We have two main examples of such manifolds in mind: first, we consider the case of a compact manifold with strictly concave boundary, which we shall denote S in the rest of the paper. The second example is the exterior of the strictly convex obstacle in $\mathbb{R}^n$, which will be denoted by Ω.

Assumption 1.1. *Let (S, g) be a smooth n-dimensional compact Riemannian manifold with C^∞ boundary. Assume ∂S is strictly geodesically concave. Let Δ_g be the Laplace–Beltrami operator associated to g.*

Let $0 < \alpha_0 \leq \frac{1}{2}$, $2 \leq \beta_0$, $\Psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ be compactly supported in the interval (α_0, β_0). We introduce the operator $\Psi(-h^2\Delta_g)$ using the Dynkin–Helffer–Sjöstrand formula [Davies 1995] and refer to [Nier 1993], [Davies 1995], or [Ivanovici and Planchon 2008] for a complete overview of its properties. See also [Burq et al. 2004b] for compact manifolds without boundaries.

Definition 1.2. Given $\Psi \in C_0^\infty(\mathbb{R})$, we have

$$\Psi(-h^2\Delta_g) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\Psi}(z)(z + h^2\Delta_g)^{-1}\,dL(z),$$

where $dL(z)$ denotes the Lebesque measure on $\mathbb{C}$ and $\tilde{\Psi}$ is an almost analytic extension of Ψ, for example, with $\langle z\rangle = (1 + |z|^2)^{1/2}$, $N \geq 0$,

$$\tilde{\Psi}(z) = \left(\sum_{m=0}^{N} \frac{\partial^m \Psi(\operatorname{Re} z)(i \operatorname{Im} z)^m}{m!}\right) \tau\left(\frac{\operatorname{Im} z}{\langle \operatorname{Re} z\rangle}\right),$$

where τ is a nonnegative C^∞ function such that $\tau(s) = 1$ if $|s| \leq 1$ and $\tau(s) = 0$ if $|s| \geq 2$.

Our main result is this:

Theorem 1.3. *Under Assumption 1.1, given (q, r) satisfying the scaling condition (1-3), $q > 2$, and $T > 0$ sufficiently small, there exists a constant $C = C(T) > 0$ such that the solution $v(x, t)$ of the semiclassical Schrödinger equation on $S \times \mathbb{R}$ with Dirichlet boundary conditions*

$$\begin{cases} ih\partial_t v + h^2\Delta_g v = 0 \quad \text{on } S \times \mathbb{R}, \\ v(x, 0) = \Psi(-h^2\Delta_g)v_0(x), \\ v|_{\partial S} = 0 \end{cases} \tag{1-4}$$

satisfies

$$\|v\|_{L^q((-T,T),L^r(S))} \leq C h^{-1/q} \left\|\Psi(-h^2\Delta_g)v_0\right\|_{L^2(S)}. \tag{1-5}$$

Remark 1.4. An example of a compact manifold with smooth, strictly concave boundary is given by the Sinai billiard (defined as the complementary of a strictly convex obstacle on a cube of $\mathbb{R}^n$ with periodic boundary conditions).

We deduce from Theorem 1.3 and [Ivanovici and Planchon 2008, Theorem 1.1] (see also Lemma 3.7), as in [Burq et al. 2004b], the following Strichartz estimates with derivative loss:

Corollary 1.5. *Under Assumption 1.1, given (q, r) satisfying the scaling condition (1-3), $q > 2$, and I any finite time interval, there exists a constant $C = C(I) > 0$ such that the solution $u(x, t)$ of the (classical) Schrödinger equation on $S \times \mathbb{R}$ with Dirichlet boundary conditions*

$$\begin{cases} i\partial_t u + \Delta_g u = 0 & \text{on } S \times \mathbb{R}, \\ u(x,0) = u_0(x), & u|_{\partial S} = 0 \end{cases} \tag{1-6}$$

satisfies

$$\|u\|_{L^q((I, L^r(S)))} \leq C(I)\|u_0\|_{H^{1/q}(S)}. \tag{1-7}$$

The proof of Theorem 1.3 is based on the finite speed of propagation of the semiclassical flow [Lebeau 1992] and the energy conservation which allow us to use the arguments of Smith and Sogge [1995] for the wave equation: using the Melrose and Taylor parametrix [1985; 1986] for the stationary wave (see also [Zworski 1990]) we obtain, by Fourier transform in time, a parametrix for the Schrödinger operator near a "glancing" point. Since in the elliptic and hyperbolic regions the solution of (1-8) will clearly satisfy the same Strichartz estimates as on a manifold without boundary (in which case we refer to [Burq et al. 2004b]), we need to restrict our attention only on the glancing region.

As an application of Theorem 1.3 we prove classical, global Strichartz estimates for the Schrödinger equation outside a strictly convex domain in $\mathbb{R}^n$.

Assumption 1.6. *Let $\Omega = \mathbb{R}^n \setminus \Theta$, where Θ is a compact with smooth boundary. Assume that $n \geq 2$ and that $\partial\Omega$ is strictly geodesically concave throughout. Let $\Delta_D = \sum_{j=1}^n \partial_j^2$ denote the Dirichlet Laplace operator (with constant coefficients) on Ω.*

Theorem 1.7. *Under Assumption 1.6, given (q, r) satisfying the scaling condition (1-3), $q > 2$ and $u_0 \in L^2(\Omega)$, there exists a constant $C > 0$ such that the solution $u(x, t)$ of the Schrödinger equation on $\Omega \times \mathbb{R}$ with Dirichlet boundary conditions*

$$\begin{cases} i\partial_t u + \Delta_D u = 0 & \text{on } \Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1-8}$$

satisfies

$$\|u\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}. \tag{1-9}$$

The proof of Theorem 1.7 combines several arguments. First, we perform a time rescaling, first used by Lebeau [1992] in the context of control theory, which transforms the equation into a semiclassical problem for which we can use the time-local semiclassical Strichartz estimates proved in Theorem 1.3. Second, we adapt a result of Burq [2002], which provides Strichartz estimates without loss for a nontrapping problem, with a metric that equals the identity outside a compact set. The proof relies on a local smoothing effect for the free evolution $\exp(it\Delta_D)$, first observed independently by Constantin and Saut [1989], Sjölin [1987], and Vega [1988] in the flat case, and then by Doi [1996] on nontrapping manifolds and by Burq et al. [2004a] on exterior domains. Following a strategy suggested by Staffilani and Tataru [2002], we prove that away from the obstacle the free evolution enjoys the Strichartz estimates exactly as for the free space.

We give two applications of Theorem 1.7. The first is a local existence result for the quintic Schrödinger equation in three dimensions, while the second is a scattering result for the subcritical (subquintic) Schrödinger equation in three-dimensional domains.

Theorem 1.8 (local existence for the quintic Schrödinger equation). *Let Ω be a three dimensional Riemannian manifold satisfying Assumption 1.6. Let $T > 0$ and $u_0 \in H_0^1(\Omega)$. Then there exists a unique solution $u \in C\big([0, T], H_0^1(\Omega)\big) \cap L^5\big((0, T], W^{1,30/11}(\Omega)\big)$ of the quintic nonlinear equation*

$$i\partial_t u + \Delta_D u = \pm |u|^4 u \text{ on } \Omega \times \mathbb{R}, \qquad u|_{t=0} = u_0 \text{ on } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{1-10}$$

Moreover, for any $T > 0$, the flow $u_0 \to u$ is Lipschitz continuous from any bounded set of $H_0^1(\Omega)$ to $C\big([-T, T), H_0^1(\Omega)\big)$. If the initial data u_0 has sufficiently small H^1 norm, then the solution is global in time.

Theorem 1.9 (scattering for subcritical Schrödinger equation). *Let Ω be a three dimensional Riemannian manifold satisfying Assumption 1.6. Let $1 + \frac{4}{3} \le p < 5$ and $u_0 \in H_0^1(\Omega)$. Then the time-global solution of the defocusing Schrödinger equation*

$$i\partial_t u + \Delta_D u = |u|^{p-1} u, \qquad u|_{t=0} = u_0 \text{ on } \Omega, \qquad u|_{\partial\Omega} = 0 \tag{1-11}$$

scatters in $H_0^1(\Omega)$. If $p = 5$ and the gradient ∇u_0 of the initial data has sufficiently small L^2 norm, then the global solution of the critical Schrödinger equation scatters in $H_0^1(\Omega)$.

Results for the Cauchy problem associated to the critical wave equation outside a strictly convex obstacle were obtained by Smith and Sogge [1995]. Their result was a consequence of the fact that the Strichartz estimates for the Euclidean wave equation also hold on Riemannian manifolds with smooth, compact, and strictly concave boundaries.

Burq et al. [2008] proved that the defocusing quintic wave equation with Dirichlet boundary conditions is globally wellposed on $H^1(M) \times L^2(M)$ for any smooth, compact domain $M \subset \mathbb{R}^3$. Their proof relies on L^p estimates for the spectral projector obtained by Smith and Sogge [2007]. A similar result for the defocusing critical wave equation with Neumann boundary conditions was obtained in [Burq and Planchon 2009].

In the case of Schrödinger equation in $\mathbb{R}^3 \times \mathbb{R}_t$, Colliander et al. [2008] established global well-posedness and scattering for energy-class solutions to the quintic defocusing Schrödinger equation (1-10), which is energy-critical. When the domain is the complementary of an obstacle in $\mathbb{R}^3$, nontrapping but not convex, the counterexamples constructed in [Ivanovici 2010] for the wave equation suggest that losses are likely to occur in the Strichartz estimates for the Schrödinger equation too. In this case Burq et al. [2004a] proved global existence for subcubic defocusing nonlinearities while Anton [2008] proved it for the cubic case. Recently, Planchon and Vega [2009] improved the local well-posedness theory to H^1-subcritical (subquintic) nonlinearities for $n = 3$. Theorem 1.9 is proved in [Planchon and Vega 2009] in the case of the exterior of a star-shaped domain for the particular case $p = 3$, using the estimate

$$\|u\|^4_{L^4_{t,x}} \lesssim \|u_0\|^3_{L^2} \|\nabla u_0\|_{L^2}$$

on the solution to the linear problem, but with no control of the $L^4_t L^\infty_x$ norm one has to use local smoothing estimates close to the boundary, and Strichartz estimates for the usual Laplacian on $\mathbb{R}^3$ away from it. Here we give a simpler proof on the exterior of a strictly convex obstacle and for every $1+\frac{4}{3} < p < 5$ using the Strichartz estimates (1-9).

2. Estimates for the semiclassical Schrödinger equation in a compact domain with strictly concave boundary

In this section we prove Theorem 1.3. In what follows Assumption 1.1 are supposed to hold. We may assume that the metric g is extended smoothly across the boundary, so that S is a geodesically concave subset of a complete, compact Riemannian manifold $\tilde{S}$. By the free semiclassical Schrödinger equation we mean the semiclassical Schrödinger equation on $\tilde{S}$, where the data v_0 has been extended to $\tilde{S}$ by an extension operator preserving the Sobolev spaces. By a broken geodesic in S we mean a geodesic that is allowed to reflect off ∂S according to the reflection law for the metric g.

Restriction in a small neighborhood of the boundary: Elliptic and hyperbolic regions. We consider $\delta > 0$ a small positive number and for $T > 0$ small enough we set

$$S(\delta, T) := \{(x,t) \in S \times [-T, T] : \mathrm{dist}(x, \partial S) < \delta\}.$$

On the complement of $S(\delta, T)$ in $S \times [-T, T]$, the solution $v(x,t)$ equals, in the semiclassical regime and modulo $O_{L^2}(h^\infty)$ errors, the solution of the semiclassical Schrödinger equation on a manifold without boundary for which sharp semiclassical Strichartz estimates follow by the work of Burq et al. [2004b], thus it suffices to establish Strichartz estimates for the norm of v over $S(\delta, T)$.

We show that in order to prove Theorem 1.3 it will be sufficient to consider only data v_0 supported outside a small neighborhood of the boundary. Recall that Lebeau [1992] proved that if Ψ is supported in an interval $[\alpha_0, \beta_0]$ and if $\varphi \in C_0^\infty(\mathbb{R})$ is equal to 1 near the interval $[-\beta_0, -\alpha_0]$, then for t in a bounded set (and for $D_t = i^{-1}\partial_t$) one has

$$\forall N \geq 1, \quad \exists C_N > 0 \quad \left|(1-\varphi)(hD_t)\exp(ith\Delta_g)\Psi(-h^2\Delta_g)v_0\right| \leq C_N h^N. \tag{2-1}$$

For δ and T sufficiently small, let $\chi(x,t) \in C_0^\infty$ be compactly supported and be equal to 1 on $S(\delta, T)$. Let $t_0 > 0$ be such that $T = t_0/4$ and let $A \in C^\infty(\mathbb{R}^n)$, $A = 0$ near ∂S, $A = 1$ outside a neighborhood of the boundary be such that every broken bicharacteristic γ starting at $t = 0$ from the support of $\chi(x,t)$ and for $-\tau \in [\alpha_0, \beta_0]$ (where τ denotes the dual time variable), satisfies

$$\mathrm{dist}\big(\gamma(t), \mathrm{supp}(1-A)\big) > 0 \quad \text{for all } t \in [-2t_0, -t_0]. \tag{2-2}$$

Let $\psi \in C^\infty(\mathbb{R})$, $\psi(t) = 0$ for $t \leq -2t_0$, $\psi(t) = 1$ for $t > -t_0$ and set

$$w(x,t) = \psi(t)\exp(ith\Delta_g)\Psi(-h^2\Delta_g)v_0.$$

Then w satisfies

$$\begin{cases} ih\partial_t w + h^2\Delta_g w = ih\psi'(t)e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0, \\ w|_{\partial S\times\mathbb{R}} = 0, \quad w|_{t\leq -2t_0} = 0, \end{cases}$$

and writing Duhamel's formula we have

$$w(x,t)=\int_{-2t_0}^{t} e^{i(t-s)h\Delta_g}\psi'(s)e^{ish\Delta_g}\Psi(-h^2\Delta_g)v_0\,ds.$$

Notice that $w(x,t)=v(x,t)$ if $t\geq -t_0$, hence for $t\in[-t_0,T]$ we can write

$$v(x,t)=\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g}\psi'(s)e^{ish\Delta_g}\Psi(-h^2\Delta_g)v_0\,ds. \tag{2-3}$$

In particular, for $t\in[-T,T]$, $T=t_0/4$, $v(x,t)=w(x,t)$ is given by (2-3). We want to estimate the $L^q_tL^r_x$ norms of $v(x,t)$ for (x,t) on $S(\delta,T)$ where $v=\chi v$. Let

$$v_Q(x,t)=\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g}\psi'(s)Q(x)e^{ish\Delta_g}\Psi(-h^2\Delta_g)v_0\,ds,\quad \text{where } Q\in\{A,1-A\}.$$

Then $v=v_A+v_{1-A}$, where v_{1-A} solves

$$\begin{cases} ih\partial_t v_{1-A}+h^2\Delta_g v_{1-A}=ih\psi'(t)(1-A)e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0,\\ v_{1-A}|_{\partial S\times\mathbb{R}}=0,\quad v_{1-A}|_{t<-2t_0}=0.\end{cases}$$

We apply Proposition A.8 from Appendix A with $Q=1-A$, $\tilde{\psi}=\psi'$ to deduce that if $\rho_0\in WF_b(v_{1-A})$ then the broken bicharacteristic starting from ρ_0 must intersect the wave front set

$$WF_b\big((1-A)v\big)\cap\big\{t\in[-2t_0,-t_0]\big\}.$$

Since we are interested in estimating the norm of v on $S(\delta,T)$ it is enough to consider only $\rho_0\in WF_b(\chi v_{1-A})$. Thus, if γ is a broken bicharacteristic starting at $t=0$ from ρ_0, $-\tau\in[\alpha_0,\beta_0]$, then Proposition A.8 implies that for some $t\in[-2t_0,-t_0]$, $\gamma(t)$ must intersect $WF_b((1-A)v)$. On the other hand from (2-2) this implies (see Definition A.2) that for every $\sigma\geq 0$

$$\forall N\geq 0\quad \exists C_N>0\quad \|\chi v_{1-A}\|_{H^\sigma(S\times\mathbb{R})}\leq C_N h^N. \tag{2-4}$$

We are thus reduced to estimating $v(x,t)$ for initial data supported outside a small neighborhood of the boundary. Indeed, suppose that the estimates (1-5) hold true for any initial data compactly supported where $A\neq 0$. It follows from (2-3) and (2-4) that

$$\begin{aligned}\|\chi v_A\|_{L^q((-T,T),L^r(S))} &\leq \big\|\psi'(s)A(x)e^{ish\Delta_g}\Psi(-h^2\Delta_g)v_0\big\|_{L^1(s\in(-2t_0,-t_0)),L^2(S))}\\ &\lesssim\Big(\int_{-2t_0}^{-t_0}|\Psi'(s)|\,ds\Big)\big\|\Psi(-h^2\Delta_g)v_0\big\|_{L^2(S)}\\ &=\big\|\Psi(-h^2\Delta_g)v_0\big\|_{L^2(S)},\end{aligned}$$

where we used the fact that the semiclassical Schrödinger flow $\exp(ihs\Delta_g)\Psi(-h^2\Delta_g)$, which maps data at time 0 to data at time s, is an isomorphism on $H^\sigma(S)$ for every $\sigma\geq 0$.

Remark 2.1. When dealing with the wave equation, since the speed of propagation is exact, one can take $\psi(t)=1_{\{t\geq -t_0\}}$ for some small $t_0\geq 0$ and reduce the problem to proving Strichartz estimates for the flow $\exp(ih(t_0+.)\Delta_g)\Psi(-h^2\Delta_g)$ and initial data compactly supported outside a small neighborhood of ∂S. This was precisely the strategy followed by Smith and Sogge [1995].

Let Δ_0 denote the Laplacian on $\tilde{S}$ coming from extending the metric g smoothly across the boundary ∂S. We let $\mathcal{M}$ denote the outgoing solution to the Dirichlet problem for the semiclassical Schrödinger operator on $S \times \mathbb{R}$. Thus, if g is a function on $\partial S \times \mathbb{R}$ which vanishes for $t \le -2t_0$, then $\mathcal{M}g$ is the solution on $S \times \mathbb{R}$ to

$$\begin{cases} ih\partial_t \mathcal{M}g + h^2 \Delta_g \mathcal{M}g = 0, \\ \mathcal{M}g|_{\partial S\times\mathbb{R}} = g. \end{cases} \tag{2-5}$$

Then, for $t \in [-t_0, T]$ and data f supported outside a small neighborhood of the boundary and localized at frequency $1/h$ (that is, such that $f = \Psi(-h^2\Delta_g)f$), we have

$$\begin{aligned}\chi v_A(x,t) &= \chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g} \psi'(s) A(x) e^{ish\Delta_g} f\,ds \\ &= \chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0} \psi'(s) A(x) e^{ish\Delta_0} f\,ds - \mathcal{M}\biggl(\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0} \psi'(s) A(x) e^{ish\Delta_0} f\,ds|_{\partial S\times\mathbb{R}}\biggr).\end{aligned}$$

The cotangent bundle of $\partial S \times \mathbb{R}$ is divided into three disjoint sets: the hyperbolic and elliptic regions, where the Dirichlet problem is respectively hyperbolic and elliptic, and the glancing region, which is the boundary between the two.

Let local coordinates be chosen such that $S = \{(x', x_n) : x_n > 0\}$ and $\Delta_g = \partial_{x_n}^2 - r(x, D_{x'})$. A point $(x', t, \eta', \tau) \in T^*(\partial S \times \mathbb{R})$ is classified as one of three distinct types. It is said to be *hyperbolic* if $-\tau + r(x', 0, \eta') > 0$, so that there are two distinct nonzero real solutions η_n to $\tau - r(x', 0, \eta') = \eta_n^2$. These two solutions yield two distinct bicharacteristics, one of which enters S as t increases (the *incoming ray*) and one which exits S as t increases (the *outgoing ray*). The point is *elliptic* if $-\tau + r(x', 0, \eta') < 0$, so there are no real solutions η_n to $\tau - r(x', 0, \eta') = \eta_n^2$. In the remaining case $-\tau + r(x', 0, \eta') = 0$, there is a unique solution which yields a glancing ray, and the point is said to be a *glancing point*. We decompose the identity operator into

$$\mathrm{Id}(x,t) = \frac{1}{(2\pi h)^n} \int e^{(i/h)((x'-y')\eta' + (t-s)\tau)} (\chi_h + \chi_e + \chi_{gl})(y', \eta', \tau)\, d\eta' d\tau,$$

where at (y', η', τ) we have

$$\chi_h := 1_{\{-\tau + r(y',0,\eta') \ge c\}}, \quad \chi_e := 1_{\{-\tau + r(y',0,\eta') \le -c\}}, \quad \chi_{gl} := 1_{\{-\tau + r(y',0,\eta') \in [-c,c]\}},$$

for some $c > 0$ sufficiently small. The corresponding operators with symbols χ_h, χ_e, denoted Π_h, Π_e, respectively, are pseudodifferential cutoffs essentially supported inside the hyperbolic and elliptic regions, while the operator with symbol χ_{gl}, denoted Π_{gl}, is essentially supported in a small set around the glancing region. Thus, on $S(\delta, T)$ we can write χv_A as the sum of four terms:

$$\begin{aligned}\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g} \psi'(s) A(x) e^{ish\Delta_g} f\,ds &= \chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0} \psi'(s) A(x) e^{ish\Delta_0} f\,ds \\ &\quad - \sum_{\Pi \in \{\Pi_e, \Pi_h, \Pi_{gl}\}} \mathcal{M}\Pi\biggl(\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0} \psi'(s) A(x) e^{ish\Delta_0} f\,ds\,\Big|_{\partial S\times\mathbb{R}}\biggr).\end{aligned} \tag{2-6}$$

Remark 2.2. For the first term in the right, $\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0} \psi'(s) A(x) e^{ish\Delta_0} f\,ds$, the desired estimates follow as in the boundaryless case by the results of Staffilani and Tataru [2002] (since we considered the extension of the metric g across the boundary to be smooth).

Elliptic region. From Proposition A.3 in Appendix A there follows the inclusion

$$WF_b\Big(\chi\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}f\,ds\,|_{\partial S\times\mathbb{R}}\Big)\subset\mathcal{H}\cup\mathcal{G},$$

where $\mathcal{H}$ and $\mathcal{G}$ denote the hyperbolic and the glancing regions, respectively. Together with the compactness argument from the proof of Proposition A.7, this implies that the elliptic part satisfies, for all $\sigma\geq 0$,

$$\mathcal{M}\Pi_e\Big(\chi\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}f\,ds\,|_{\partial S\times\mathbb{R}}\Big)=O(h^\infty)\|f\|_{H^\sigma(S)}.$$

For the definition and properties of the b-wave front set see Appendix A.

Hyperbolic region. If local coordinates are chosen such that $S=\{(x',x_n):x_n>0\}$, on the essential support of Π_h the forward Dirichlet problem can be solved locally, modulo smoothing kernels, on an open set in $\tilde{S}\times\mathbb{R}$ around ∂S. Precisely, microlocally near a hyperbolic point, the solution v to (1-4) can be decomposed modulo smoothing operators into an incoming part v_- and an outgoing part v_+ where

$$v_\pm(x,t)=\frac{1}{(2\pi h)^d}\int e^{(i/h)\varphi_\pm(x,t,\xi)}\sigma_\pm(x,t,\xi,h)\,d\xi,$$

where the phases $\varphi_\pm$ satisfy the eikonal equations

$$\begin{cases}\partial_s\varphi_\pm+\langle d\varphi_\pm,d\varphi_\pm\rangle_g=0,\\ \varphi_+|_{\partial S}=\varphi_-|_{\partial S},\quad \partial_{x_n}\varphi_+|_{\partial S}=-\partial_{x_n}\varphi_-|_{\partial S},\end{cases}$$

where $\langle\cdot,\cdot\rangle_g$ denotes the inner product induced by the metric g. The symbols are asymptotic expansions in h and write $\sigma_\pm(\cdot,h)=\sum_{k\geq 0}h^k\sigma_{\pm,k}$, where σ_0 solves the linear transport equation

$$\partial_s\sigma_{\pm,0}+(\Delta_g\varphi_\pm)\sigma_{\pm,0}+\langle d\varphi_\pm,d\sigma_{\pm,0}\rangle_g=0,$$

while for $k\geq 1$, $\sigma_{\pm,k}$ satisfies the nonhomogeneous transport equations

$$\partial_s\sigma_{\pm,k}+(\Delta_g\varphi_\pm)\sigma_{\pm,k}+\langle d\varphi_\pm,d\sigma_{\pm,k}\rangle_g=i\Delta_g\sigma_{\pm,k-1}.$$

A direct computation shows that

$$\Big\|\sum_\pm v_\pm\Big\|^2_{H^\sigma(S\times\mathbb{R})}\simeq\sum_\pm\|v_\pm\|^2_{H^\sigma(S\times\mathbb{R})}\simeq\|v\|^2_{H^\sigma(S\times\mathbb{R})}\simeq\|v\|^2_{L^\infty(\mathbb{R})H^\sigma(S)}.$$

Each component $v_\pm$ is a solution of linear Schrödinger equation (without boundary) and consequently satisfies the usual Strichartz estimates [Burq et al. 2004b].

Note that $\sum_\pm v_\pm$ contains the contribution from

$$\mathcal{M}\Pi_h\Big(\chi\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}\Psi(-h^2\Delta_g)v_0\,ds\,|_{\partial S\times\mathbb{R}}\Big)$$

and a contribution from $\chi\int_{-2t_0}^{-t_0}e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}\Psi(-h^2\Delta_g)v_0\,ds$.

Glancing region. Near a diffractive point we use the Melrose and Taylor construction for the wave equation in order to write, following Zworski [1990], the solution to the wave equation as a finite sum of pseudodifferential cutoffs, each essentially supported in a suitably small neighborhood of a glancing ray. Using the Fourier transform in time we obtain a parametrix for the semiclassical Schrödinger equation (1-4) microlocally near a glancing direction and modulo smoothing operators.

Preliminaries: Parametrix for the wave equation near the glancing region. We start by recalling the results by Melrose and Taylor [1985; 1986] and Zworski [1990, Proposition 4.1] for the wave equation near the glancing region. Let w solve the (semiclassical) wave equation on S with Dirichlet boundary conditions

$$\begin{cases} h^2 D_t^2 w + h^2 \Delta_g w = 0, \quad S\times\mathbb{R}, \quad w|_{\partial S\times\mathbb{R}} = 0, \\ w(x,0) = f(x), \quad D_t w(x,0) = g(x), \end{cases} \tag{2-7}$$

where f, g are compactly supported in S and localized at spatial frequency $1/h$, and where $D_t = i^{-1}\partial_t$.

Proposition 2.3. *Microlocally near a glancing direction the solution to (2-7) can be written, modulo smoothing operators, as*

$$w(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{(i/h)(\theta(x,\xi)+it\xi_1)}\biggl[a(x,\xi/h)\biggl(A_-(\zeta(x,\xi/h)) - A_+(\zeta(x,\xi/h))\frac{A_-(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\biggr) \\ + b(x,\xi/h)\biggl(A'_-(\zeta(x,\xi/h)) - A'_+(\zeta(x,\xi/h))\frac{A_-(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\biggr)\biggr]\times\widehat{K(f,g)}\Bigl(\frac{\xi}{h}\Bigr)\,d\xi, \tag{2-8}$$

where the symbols a, b, and the phases θ, ζ have the following properties: a and b are symbols of type $(1,0)$ and order $\frac{1}{6}$ and $-\frac{1}{6}$, respectively, both of which are supported in a small conic neighborhood of the ξ_1 axis, and the phases θ and ζ are real, smooth and homogeneous of degree 1 and $\frac{2}{3}$, respectively. Further, K is a classical Fourier integral operator of order 0 in f and order -1 in g, compactly supported on both sides. The $A_\pm$ are defined by $A_\pm(z) = \mathrm{Ai}(e^{\mp 2\pi i/3}z)$, where Ai denotes the Airy function.

Remark 2.4. If local coordinates are chosen so that Ω is given by $x_n > 0$, the phase functions θ, ζ satisfy the eikonal equations

$$\begin{cases} \xi_1^2 - \langle d\theta, d\theta\rangle_g + \zeta\langle d\zeta, d\zeta\rangle_g = 0, \\ \langle d\theta, d\zeta\rangle_g = 0, \\ \zeta(x',0,\xi) = \zeta_0(\xi) = -\xi_1^{-1/3}\xi_n, \end{cases} \tag{2-9}$$

in the region $\zeta \le 0$. Here $x' = (x_1,\dots,x_{n-1})$ and $\langle\cdot,\cdot\rangle_g$ denotes the inner product given by the metric g. The phases also satisfy the eikonal equations (2-9) to infinite order at $x_n = 0$ in the region $\zeta > 0$.

Remark 2.5. One can think of $A_-(\zeta)$ (at least away from the boundary $x_n = 0$) as the incoming contribution and of $A_+(\zeta)A_-(\zeta_0)/A_+(\zeta_0)$ as the outgoing one. From [Zworski 1990, Section 2] we have

$$\frac{A_-}{A_+}(z) \simeq \begin{cases} -e^{i\pi/3} + O(z^{-\infty}), & z\to\infty, \\ e^{i(4/3)(-z)^{3/2}}\sum_{j\ge 0}\beta_j z^{-3j/2}, & z\to -\infty, \end{cases}$$

where the part $z \to \infty$ corresponds to the free wave, while the oscillatory one to the billiard ball map shift corresponding to reflection. Using $\mathrm{Ai}(\zeta) = e^{i\pi/3}A_+(\zeta) + e^{-i\pi/3}A_-(\zeta)$, we write

$$A_-(\zeta) - A_+(\zeta)\frac{A_-(\zeta_0)}{A_+(\zeta_0)} = e^{i\pi/3}\left(\mathrm{Ai}(\zeta) - A_+(\zeta)\frac{\mathrm{Ai}(\zeta_0)}{A_+(\zeta_0)}\right).$$

Parametrix for the solution to the semiclassical Schrödinger equation near the glancing region. Let $v(x,t)$ be the solution of the semiclassical Schrödinger equation (1-4) where the initial data $v_0 \in L^2(S)$ is spectrally localized at spatial frequency $1/h$; that is, $v_0(x) = \Psi(-h^2\Delta_g)v_0(x)$. From the discussion at the beginning of this section we see that it will be enough to consider v_0 compactly supported outside some small neighborhood of ∂S. Under this assumption $\Psi(-h^2\Delta_g)v_0$ is a well-defined pseudodifferential operator for which the results of [Burq et al. 2004b, Section 2] apply.

Let $(e_\lambda(x))_{\lambda\geq 0}$ be the eigenbasis of $L^2(S)$ consisting in eigenfunctions of $-\Delta_g$ associated to the eigenvalues (λ^2), so that $-\Delta_g e_\lambda = \lambda^2 e_\lambda$. We write

$$\Psi(-h^2\Delta_g)v_0(x) = \sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)v_\lambda e_\lambda(x), \tag{2-10}$$

and hence

$$e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0(x) = \sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)e^{-ith\lambda^2}v_\lambda e_\lambda(x). \tag{2-11}$$

If δ denotes the Dirac function, the Fourier transform of $v(x,t)$ can be written as

$$\hat{v}\left(x,\frac{\tau}{h}\right) = h\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)\delta_{\{-\tau=h^2\lambda^2\}}v_\lambda e_\lambda(x). \tag{2-12}$$

For $t\in\mathbb{R}$ we can define (since $\hat{v}$ has compact support away from 0)

$$\begin{aligned} w(x,t) &:= \frac{1}{2\pi h}\int_0^\infty e^{it\sigma/h}\hat{v}\left(x,-\frac{\sigma^2}{h}\right)d\sigma = -\frac{1}{4\pi h}\int_{-\infty}^0 e^{it\sqrt{-\tau}/h}\frac{1}{\sqrt{-\tau}}\hat{v}\left(x,\frac{\tau}{h}\right)d\tau \\ &= -\frac{1}{2}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]}\Psi(h^2\lambda^2)\left(\frac{1}{2\pi}\int_{-\infty}^0 e^{it\sqrt{-\tau}/h}\frac{1}{\sqrt{-\tau}}\delta_{\{-\tau=h^2\lambda^2\}}d\tau\right)v_\lambda e_\lambda(x) \\ &= -\frac{1}{2}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]}\frac{1}{h\lambda}\Psi(h^2\lambda^2)e^{it\lambda}v_\lambda e_\lambda(x). \end{aligned} \tag{2-13}$$

Then $w(x,t)$ solves the wave equation

$$\begin{cases} h^2D_t^2w + h^2\Delta_g w = 0 & \text{on } S\times\mathbb{R}, \quad w|_{\partial S\times\mathbb{R}} = 0, \\ w(x,0) = f_h(x), & D_t w(x,0) = g_h(x), \end{cases} \tag{2-14}$$

where the initial data are given by

$$f_h(x) = -\frac{1}{2}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]}\frac{1}{h\lambda}\Psi(h^2\lambda^2)v_\lambda e_\lambda(x), \tag{2-15}$$

$$g_h(x) = -\frac{1}{2h}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]}\Psi(h^2\lambda^2)v_\lambda e_\lambda(x) = -\frac{1}{2h}\Psi(-h^2\Delta_g)v_0(x). \tag{2-16}$$

From (2-15) and (2-16) it follows that

$$h\|g_h\|_{L^2(S)} \simeq \|f_h\|_{L^2(S)} \simeq \left\|\Psi(-h^2\Delta_g)v_0\right\|_{L^2(S)}, \tag{2-17}$$

where by $\alpha \simeq \beta$ we mean that there is $C > 0$ such that $C^{-1}\alpha < \beta < C\alpha$.

Indeed, to prove (2-17) notice that w defined by (2-13) satisfies

$$(hD_t - h\sqrt{-\Delta_g})w = 0$$

and (since Δ_g and D_t commute) we have

$$f_h = w|_{t=0} = \left[(\sqrt{-\Delta_g})^{-1}D_t w\right]\Big|_{t=0} = (\sqrt{-\Delta_g})^{-1}(D_t w|_{t=0}) = (\sqrt{-\Delta_g})^{-1}g_h.$$

Due to the spectral localization and since $g_h = -(1/2h)\Psi(-h^2\Delta_g)v_0$ we deduce (2-17).

By the L^2 continuity of the (classical) Fourier integral operator K introduced in Proposition 2.3 we deduce

$$\left\|K(f_h, g_h)\right\|_{L^2(S)} \le C\left(\|f_h\|_{L^2(S)} + h\|g_h\|_{L^2(S)}\right) \simeq \left\|\Psi(-h^2\Delta_g)v_0\right\|_{L^2(S)}. \tag{2-18}$$

The solution $v(x,t)$ of (1-4) can be written as

$$v(x,t) = \frac{1}{2\pi h}\int_0^\infty e^{-it\sigma^2/h} 2\sigma\hat{v}(x, -\frac{\sigma^2}{h})\,d\sigma = \frac{1}{2\pi h}\int_0^\infty e^{-i\frac{t\sigma^2}{h}} 2\sigma \int_{s\in\mathbb{R}} e^{-i\frac{s\sigma}{h}} w(x,s)\,ds\,d\sigma. \tag{2-19}$$

The next step is to use (2-7) to obtain a representation of $v(x,t)$ near the glancing region: notice that the glancing part of the stationary wave $\hat{w}(x, \sigma/h)$ is given by

$$1_{\{\sigma^2 + r(x',0,\eta') \in [-c,c]\}}\hat{w}\left(x, \frac{\sigma}{h}\right) = 1_{\{\sigma^2 + r(x',0,\eta') \in [-c,c]\}}\hat{v}\left(x, -\frac{\sigma^2}{h}\right) = 1_{\{-\tau + r(x',0,\eta') \in [-c,c]\}}\hat{v}\left(x, \frac{\tau}{h}\right), \tag{2-20}$$

with $\tau = -\sigma^2$ and where $c > 0$ is sufficiently small. The equality in (2-20) follows from (2-13) and from the fact that $\hat{v}$ is essentially supported for the second variable in the interval $[-\beta_0, -\alpha_0]$. Consequently we can apply Equation (2-7) and determine a representation for v near the glancing region (for the Schrödinger equation) as

$$\begin{aligned} v(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{(i/h)(\theta(x,\xi) - t\xi_1^2)} 2\xi_1 \Bigg[a(x,\xi/h)\left(\mathrm{Ai}(\zeta(x,\xi/h)) - A_+(\zeta(x,\xi/h))\frac{\mathrm{Ai}(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\right) \\ + b(x,\xi/h)\left(\mathrm{Ai}'(\zeta(x,\xi/h)) - A'_+(\zeta(x,\xi/h))\frac{\mathrm{Ai}(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\right)\Bigg] \widehat{K(f_h, g_h)}\left(\frac{\xi}{h}\right) d\xi, \end{aligned} \tag{2-21}$$

where a, b and K are those defined in Proposition 2.3 and f_h, g_h are given by (2-15) and (2-16). The initial data f_h, g_h are both supported, like v_0, away from ∂S, so their $\dot{H}^\sigma(S)$ norms for $\alpha < n/2$ will be comparable to the norms of the nonhomogeneous Sobolev space $H^\sigma(\mathbb{R}^n)$. For this reason we shall henceforth work with the latter norms on the data f_h, g_h.

Remark 2.6. It is enough to prove semiclassical Strichartz estimates only for the “outgoing” piece corresponding to the oscillatory term $A_+(\zeta)\,\mathrm{Ai}(\zeta_0)/A_+(\zeta_0)$, since the direct term, corresponding to $\mathrm{Ai}(\zeta)$, has already been dealt with (see Remark 2.2).

We deduce from (2-18) and (2-21) that, to finish the proof of Theorem 1.3, we need only show that the operator A_h defined, for f supported away from ∂S and spectrally localized at the frequency $1/h$ (that is, such that $f = \Psi(-h^2\Delta_g)f$), by

$$A_h f(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} 2\xi_1\Big(a(x,\xi/h)A_+\big(\zeta(x,\xi/h)\big) + b(x,\xi/h)A'_+\big(\zeta(x,\xi/h)\big)\Big) \times e^{(i/h)(\theta(x,\xi)-t\xi_1^2)}\frac{\mathrm{Ai}(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi, \tag{2-22}$$

satisfies

$$\|A_h f\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le Ch^{-1/q}\|f\|_{L^2(\mathbb{R}^n)}. \tag{2-23}$$

Remark 2.7. We introduce a cutoff function $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on the support of f and to 0 near ∂S. Since χ_1 is supported away from the boundary it follows from [Burq et al. 2004b, Proposition 2.1] (which applies here in its adjoint form) that $\Psi(-h^2\Delta_g)\chi_1 f$ is a pseudodifferential operator and can be written in local coordinates as

$$\Psi(-h^2\Delta_g)\chi_1 f = d(x,hD_x)\chi_2 f + O_{L^2(S)}(h^\infty), \tag{2-24}$$

where $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on the support of χ_1 and where $d(x,D_x)$ is defined for x in the suitable coordinate patch using the usual pseudodifferential quantization rule,

$$d(x,D_x)f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}d(x,\xi)\hat{f}(\xi)\,d\xi, \quad d \in C_0^\infty,$$

with symbol d compactly supported for $|\xi|_g^2 := \langle\xi,\xi\rangle_g \in [\alpha_0,\beta_0]$, which follows by the condition of the support of Ψ. Since the principal part of the Laplace operator Δ_g is uniformly elliptic, we can introduce a smooth radial function $\psi \in C_0^\infty\big([\frac{1}{\delta}\alpha_0^{1/2},\delta\beta_0^{1/2}]\big)$ for some $\delta \ge 1$ such that $\psi(|\xi|)d = d$ everywhere. In what follows we shall prove (2-23) where, instead of f we shall write $\psi(|\xi|)f$, keeping in mind that f is supported away from the boundary and localized at spatial frequency $1/h$.

The proof of Theorem 1.3 will be completed once we prove (2-23). To do that, we split the operator A_h into two parts, namely a main term and a diffractive term. To this end, let $\chi(s)$ be a smooth function satisfying

$$\operatorname{supp}\chi \subset (-\infty,-1], \qquad \operatorname{supp}(1-\chi) \subset [-2,\infty).$$

We write this operator as a sum $A_h = M_h + D_h$, by decomposing

$$A_+\big(\zeta(x,\xi)\big) = (\chi A_+)\big(\zeta(x,\xi)\big) + \big((1-\chi)A_+\big)\big(\zeta(x,\xi)\big),$$

and by letting the "main term" be defined for f, as in Remark 2.7, by

$$M_h f(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} 2\xi_1\Big(a(x,\xi/h)(\chi A_+)\big(\zeta(x,\xi/h)\big) + b(x,\xi/h)(\chi A'_+)\big(\zeta(x,\xi/h)\big)\Big) \times e^{(i/h)(\theta(x,\xi)-t\xi_1^2)}\frac{\mathrm{Ai}(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi. \tag{2-25}$$

The diffractive term is then defined for f as before by

$$D_h f(x,t)=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} 2\xi_1\Big(a(x,\xi/h)\big((1-\chi)A_+\big)\big(\zeta(x,\xi/h)\big)+b(x,\xi/h)\big((1-\chi)A'_+\big)\big(\zeta(x,\xi/h)\big)\Big)\\ \times e^{(i/h)(\theta(x,\xi)-t\xi_1^2)}\frac{\mathrm{Ai}(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi. \tag{2-26}$$

We analyze these operators separately following the ideas of [Smith and Sogge 1995].

The main term M_h**.** To estimate the main term M_h we first use the fact that

$$\left|\frac{\mathrm{Ai}(s)}{A_+(s)}\right|\le 2,\qquad \text{for } s\in\mathbb{R}. \tag{2-27}$$

Consequently, since the term $\mathrm{Ai}(\zeta_0)/A_+(\zeta_0)$ acts like a multiplier, as does ξ_1, which by virtue of (2-1) is localized in the interval $[\alpha_0,\beta_0]$, the estimates for M_h will follow from showing that the operator

$$f\to\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\Big(a(x,\xi/h)(\chi A_+)\big(\zeta(x,\xi/h)\big)+b(x,\xi/h)(\chi A'_+)\big(\zeta(x,\xi/h)\big)\Big)\\ \times e^{(i/h)(\theta(x,\xi)-t\xi_1^2)}\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi \tag{2-28}$$

satisfies the same bounds as in (2-23) for f spectrally localized at frequency $1/h$. Following [Zworski 1990, Lemma 4.1], we write χA_+ and $(\chi A_+)'$ in terms of their Fourier transform to express the phase function of this operator

$$\phi(t,x,\xi)=-t\xi_1^2+\theta(x,\xi)-\tfrac{2}{3}(-\zeta)^{3/2}(x,\xi), \tag{2-29}$$

which satisfies the eikonal equation (2-9). Let its symbol be $c_m(x,\xi/h)$, with $c_m(x,\xi)\in\mathcal{S}^0_{2/3,1/3}(\mathbb{R}^n\times\mathbb{R}^n)$ and we also denote the operator defined in (2-28) by W_h^m, thus

$$W_h^m f(x,t)=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{(i/h)\phi(t,x,\xi)}c_m(x,\xi/h)\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi.$$

Proposition 2.8. *Let (q,r) be an admissible pair with $q>2$, let $T>0$ be sufficiently small and for $f=d(x,D_x)\chi_2 f+O_{L^2(\Omega)}(h^\infty)$ as in Remark 2.7 let*

$$W_h f(x,t):=W_h^m f(x,t)=\frac{1}{(2\pi h)^n}\int e^{(i/h)\phi(t,x,\xi)}c_m(x,\xi/h)\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi.$$

Then the following estimates hold:

$$\|W_h f\|_{L^q((0,T],L^r(\mathbb{R}^n))}\le Ch^{-1/q}\|f\|_{L^2(\mathbb{R}^n)}. \tag{2-30}$$

The proof occupies the rest of this section. The first step is a TT* argument. Explicitly,

$$\widehat{W_h^*(F)}\Big(\frac{\xi}{h}\Big)=\int e^{-(i/h)\phi(s,y,\xi)}F(y,s)\overline{c_m(y,\xi/h)}\,dy\,ds,$$

and if we set

$$(T_h F)(x,t) = (W_h W_h^* F)(x,t) = \frac{1}{(2\pi h)^n} \int e^{(i/h)(\phi(t,x,\xi)-\phi(s,y,\xi))} c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\psi^2(|\xi|)F(y,s)\,d\xi\,ds\,dy, \quad (2\text{-}31)$$

then inequality (2-30) is equivalent to

$$\|T_h F\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le Ch^{-2/q}\|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}, \quad (2\text{-}32)$$

where q' and r' satisfy $1/q+1/q'=1$ and $1/r+1/r'=1$. To see, for instance, that (2-32) implies (2-30), notice that the dual version of (2-30) is

$$\|W_h^* F\|_{L^2(\mathbb{R}^n)} \le Ch^{-1/q}\|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))},$$

and we have

$$\|W_h^* F\|^2_{L^2(\mathbb{R}^n)} = \int W_h W_h^* F \bar{F}\,dt\,dx \le \|T_h F\|_{L^q((0,T],L^r(\mathbb{R}^n))}\|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}. \quad (2\text{-}33)$$

Therefore we only need to prove (2-32). Since the symbols are of type $(\frac{2}{3},\frac{1}{3})$ and not of type $(1,0)$, before starting the proof of (2-32) for the operator T_h we need to make a further decomposition: Let $\rho \in C_0^\infty(\mathbb{R})$ satisfy $\rho(s)=1$ near 0 and $\rho(s)=0$ if $|s|\ge 1$. Let

$$T_h F = T_h^f F + T_h^s F,$$

where

$$T_h^s F(x,t) = \int K_h^s(t,x,s,y)F(y,s)\,ds\,dy \quad (2\text{-}34)$$

and

$$K_h^s(t,x,s,y) = \frac{1}{(2\pi h)^n}\int e^{(i/h)(\phi(t,x,\xi)-\phi(s,y,\xi))}\big(1-\rho(h^{-1/3}|t-s|)\big) \times c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\psi^2(|\xi|)\,d\xi, \quad (2\text{-}35)$$

while

$$T_h^f F(x,t) = \int K_h^f(t,x,s,y)F(y,s)\,ds\,dy, \quad (2\text{-}36)$$

and

$$K_h^f(t,x,s,y) = \frac{1}{(2\pi h)^n}\int e^{(i/h)(\phi(t,x,\xi)-\phi(s,y,\xi))}\rho\big(h^{-1/3}|t-s|\big) \times c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\psi^2(|\xi|)\,d\xi. \quad (2\text{-}37)$$

Remark 2.9. The two pieces will be handled differently. The kernel of T_h^f is supported in a suitable small set and it will be estimated by "freezing" the coefficients. To estimate T_h^s we shall use the stationary phase method for type $(1,0)$ symbols. For type $(\frac{2}{3},\frac{1}{3})$ symbols, these stationary phase arguments break down if $|t-s|$ is smaller than $h^{1/3}$, which motivates the decomposition. We use here the same arguments found in [Smith and Sogge 1995].

- The "stationary phase admissible" term T_h^s:

Proposition 2.10. *There is a constant* $1 < C_0 < \infty$ *such that the kernel* K_h^s *of* T_h^s *satisfies*

$$|K_h^s(t,x,s,y)| \le C_N h^N \quad \textit{for all } N \quad \textit{if } \frac{|t-s|}{|x-y|} \notin [C_0^{-1}, C_0]. \tag{2-38}$$

Moreover, there is a function $\xi_c(t,x,s,y)$ *which is smooth in the variables* (t,s), *uniformly over* (x,y), *so that if* $C_0^{-1} \le |t-s|/|x-y| \le C_0$, *then*

$$|K_h^s(t,x,s,y)| \lesssim h^{-n}\Big(1+\frac{|t-s|}{h}\Big)^{-n/2} \quad \textit{for } |t-s| \ge h^{1/3}. \tag{2-39}$$

Proof. We shall use the stationary phase lemma to evaluate the kernel K_h^s of T_h^s. The critical points occur when $|t-s| \simeq |x-y|$. For some constant C_0 and for $|\xi| \in \operatorname{supp}\psi$, ξ_1 in a small neighborhood of 1, we have

$$|\nabla_\xi(\phi(t,x,\xi)-\phi(s,y,\xi))| \simeq |t-s|+|x-y| \ge h^{1/3} \quad \text{if } \frac{|t-s|}{|x-y|} \notin [C_0^{-1}, C_0].$$

Since $c \in S^0_{2/3,1/3}$, an integration by parts leads to (2-38). If $|t-s| \simeq |x-y|$ we introduce a cutoff function $\kappa\big(|x-y|/|t-s|\big)$, with $\kappa \in C_0^\infty(\mathbb{R}\setminus\{0\})$. The phase function can be written as

$$\phi(t,x,\xi)-\phi(s,y,\xi) = (t-s)\Theta(t,x,s,y,\xi) \quad \text{for} \quad |t-s| \simeq |x-y| \ge h^{1/3}.$$

We want to apply the stationary phase method with parameter $|t-s|/h \ge h^{-2/3} \gg 1$ to estimate K_h^s. For x, y, t, s fixed we must show that the critical points of Θ are nondegenerate.

Lemma 2.11. *If* T *is sufficiently small, the phase function* $\Theta(t,x,s,y,\xi)$ *admits a unique, nondegenerate critical point* ξ_c. *Moreover, for* $0 \le t, s \le T$, *the function* $\xi_c(t,x,s,y)$ *solving* $\nabla_\xi\Theta(t,x,s,y,\xi_c) = 0$ *is smooth in* t *and* s, *with uniform bounds on derivatives as* x *and* y *vary, and we have*

$$|\partial_{t,s}^\alpha \partial_{x,y}^\gamma \xi_c(t,x,s,y)| \le C_{\alpha,\gamma} h^{-|\alpha|/3} \quad \textit{if } |x-y| \ge h^{1/3}. \tag{2-40}$$

Proof. The phase $\Theta(t,x,s,y,\xi)$ has the form

$$\begin{aligned}\Theta(t,x,s,y,\xi) &= \xi_1^2 + \frac{1}{t-s}\big(\phi(0,x,\xi)-\phi(0,y,\xi)\big)\\ &= \xi_1^2 + \frac{1}{t-s}\sum_{j=1}^n (x_j-y_j)\partial_{x_j}\phi(0,z_{x,y},\xi),\end{aligned} \tag{2-41}$$

for some $z_{x,y}$ close to x, y (if T is sufficiently small then $|t-s| \simeq |x-y|$ is small), and using the eikonal equations (2-9) we can write

$$\Theta(t,x,s,y,\xi) = \langle \nabla_x\phi, \nabla_x\phi\rangle_g(0,z_{x,y},\xi) - \frac{1}{t-s}\sum_{j=1}^n (x_j-y_j)\partial_{x_j}\phi(0,z_{x,y},\xi).$$

Write $\langle \nabla_x\phi, \nabla_x\phi\rangle_g = \sum_{j,k} g^{j,k}\partial_{x_j}\phi\partial_{x_k}\phi$. We compute $\nabla_\xi\Theta$ explicitly: for each $l \in \{1,\dots,n\}$ we have

$$\partial_{\xi_l}\Theta(t,x,s,y,\xi) = \sum_{j=1}^n \partial^2_{\xi_l,x_j}\phi(0,z_{x,y},\xi)\Big(2\sum_{k=1}^n g^{j,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi) - \frac{x_j-y_j}{t-s}\Big). \tag{2-42}$$

Thus

$$\nabla_\xi \Theta(t,x,s,y,\xi) = \nabla^2_{\xi,x}\phi(0,z_{x,y},\xi)\begin{pmatrix} 2\sum_k g^{1,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi) - \frac{x_1-y_1}{(t-s)} \\ \vdots \\ 2\sum_k g^{n,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi) - \frac{x_n-y_n}{(t-s)} \end{pmatrix}, \tag{2-43}$$

where $\nabla^2_{\xi,x}\phi = (\partial^2_{\xi_l,x_j}\phi)_{l,j\in\{1,\dots,n\}}$ is the matrix $n\times n$ whose elements are the second derivatives of ϕ with respect to ξ and x. We need the following lemma:

Lemma 2.12 [Smith and Sogge 1994, Lemma 3.9]. *For ξ in a conic neighborhood of the ξ_1 axis the mapping*

$$x \to \nabla_\xi\big(\theta(x,\xi) - \tfrac{2}{3}(-\zeta)^{3/2}(x,\xi)\big)$$

is a diffeomorphism on the complement of the hypersurface $\zeta = 0$, with uniform bounds on the Jacobian of the inverse mapping.

Corollary 2.13. *If T is small enough and $|x-y| \simeq |t-s| \le 2T$ then*

$$\det(\nabla^2_{\xi,x}\phi)(0,z_{x,y},\xi) \neq 0. \tag{2-44}$$

We now complete the proof of Lemma 2.11. A critical point for Θ satisfies $\nabla_\xi\Theta(t,x,s,y,\xi) = 0$ and from (2-43) and (2-44) this translates into

$$\big((g^{j,k}(z_{x,y}))_{j,k}\big)(\nabla_x\phi)^t(0,z_{x,y},\xi) = \frac{x-y}{t-s}. \tag{2-45}$$

Since $(g^{j,k})_{j,k}$ is invertible and using again (2-44) we can apply the implicit function's theorem to obtain (for T small enough) a critical point $\xi_c = \xi_c(t,x,s,y)$ for Θ. To show that ξ_c is nondegenerate we compute

$$\begin{aligned}\partial_{\xi_q}\partial_{\xi_l}\Theta(t,x,s,y,\xi) &= \sum_{j=1}^n \partial^3_{\xi_q,\xi_l,x_j}\phi(0,z_{x,y},\xi)\Big(2\sum_{k=1}^n g^{j,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi) - \frac{(x_j-y_j)}{(t-s)}\Big) \\ &\quad + 2\sum_{j=1}^n \partial^2_{\xi_l,x_j}\phi(0,z_{x,y},\xi)\Big(\sum_{k=1}^n g^{j,k}(z_{x,y})\partial^2_{\xi_q,x_k}\phi(0,z_{x,y},\xi)\Big).\end{aligned} \tag{2-46}$$

Consequently at the critical point $\xi = \xi_c$ the hessian matrix $\nabla^2_{\xi,\xi}\Theta$ is given by

$$\nabla^2_{\xi,\xi}\Theta(t,x,s,y,\xi_c) = 2(\nabla^2_{\xi,x}\phi)(g^{ij}(z_{x,y}))_{i,j}(\nabla^2_{\xi,x}\phi)\,\big|_{(0,z_{x,y},\xi_c)},$$

and therefore for T small enough, the critical point ξ_c is nondegenerate by (2-44). □

On the support of κ it follows that the kernel K^s_h has the form

$$K^s_h(t,x,s,y) = \frac{1}{(2\pi h)^n}\int e^{(i/h)|t-s|\Theta(t,x,s,y,\xi)}\psi^2(|\xi|)\big(1-\rho(h^{-1/3}|t-s|)\big)\times c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\,d\xi, \tag{2-47}$$

where, if $\omega = |t-s|/h$ and $\xi_1 \simeq 1$, the symbol satisfies

$$\left|\partial^\alpha_{t,s}\partial^k_\omega \sigma_h\big(t,x,s,y,\omega\xi/|t-s|\big)\right| \le C_{\alpha,k} h^{-|\alpha|/3}\big(|t-s|^{3/2}/h\big)^{-2k/3},$$

where we have set

$$\sigma_h\big(t,x,s,y,\omega\xi/|t-s|\big) = \big(1-\rho(h^{-1/3}|t-s|)\big)c_m\big(x,\omega\xi/|t-s|\big)\overline{c_m\big(y,\omega\xi/|t-s|\big)}.$$

Indeed, since $c_m \in S^0_{2/3,1/3}$, for $\alpha = 0$ one has

$$|\partial^k_\omega \sigma_h| \le |\xi||t-s|^{-k}\left|(\partial^k_\xi c_m)\big(t,x,\omega\xi/|t-s|\big)\right| \le C_{0,k}|t-s|^{-k}(\omega/|t-s|)^{-2k/3} = C_{0,k}|t-s|^{-k}h^{2k/3}.$$

We conclude using the next lemma with $\omega = |t-s|/h$ and $\delta = |t-s|^{3/2} \ge h^{1/2} \gg h$.

Lemma 2.14. *Suppose that* $\Theta(z,\xi) \in C^\infty(\mathbb{R}^{2(n+1)} \times \mathbb{R}^n)$ *is real,* $\nabla_\xi\Theta(z,\xi_c(z)) = 0$, $\nabla_\xi\Theta(z,\xi) \ne 0$ *if* $\xi \ne \xi_c(z)$, *and*

$$|\det \nabla^2_{\xi\xi}\Theta| \ge c_0 > 0 \quad \text{if } |\xi| \le 1.$$

Suppose also that

$$|\partial^\alpha_z \partial^\beta_\xi \Theta(z,\xi)| \le C_{\alpha,\beta} h^{-|\alpha|/3} \quad \text{for all } \alpha, \beta.$$

In addition, suppose that the symbol $\sigma_h(z,\xi,\omega)$ *vanishes when* $|\xi| \ge 1$ *and satisfies*

$$|\partial^\alpha_z \partial^\gamma_\xi \partial^k_\omega \sigma_h(z,\xi,\omega)| \le C_{k,\alpha,\gamma}\, h^{-(|\alpha|+|\gamma|)/3}(\delta/h)^{-2k/3} \quad \text{for all } k, \alpha, \gamma,$$

where on the support of σ_h *we have* $\omega \ge h^{-2/3}$ *and* $\delta > 0$. *Then we can write*

$$\int_{\mathbb{R}^n} e^{i\omega\Theta(z,\xi)}\sigma_h(z,\xi,\omega)\,d\xi = \omega^{-n/2}e^{i\omega\Theta(z,\xi_c(z))}b_h(z,\omega),$$

where b_h *satisfies*

$$|\partial^k_\omega \partial^\alpha_z b_h(z,\omega)| \le C_{k,\alpha} h^{-|\alpha|/3}(\delta/h)^{-2k/3}$$

and where each of the constants depend only on c_0 *and the size of finitely many of the constants* $C_{\alpha,\beta}$ *and* $C_{k,\alpha,\gamma}$ *above. In particular, the constants are uniform in* δ *if* $1 \ge \delta \ge h$.

This lemma, used in [Smith and Sogge 1995, Lemma 2.6] and also in Grieser's thesis [1992], follows easily from the proof of the standard stationary phase lemma [Sogge 1993, page 45]. Its application concludes the proof of Proposition 2.10. □

For each t, s, let $T^s_h(t,s)$ be the "frozen" operator defined by

$$T^s_h(t,s)g(x) = \int K^s_h(t,x,s,y)g(y)\,dy.$$

From Proposition 2.10 we deduce

$$\|T^s_h(t,s)g\|_{L^\infty(\mathbb{R}^n)} \le C\max\big(h^{-n},(h|t-s|)^{-n/2}\big)\|g\|_{L^1(\mathbb{R}^n)}. \tag{2-48}$$

Lemma 2.15. *If* T *is small enough then for* t, s *fixed the frozen operators* $T^s_h(t,s)$, $T^f_h(t,s)$ *are bounded on* $L^2(\mathbb{R}^n)$; *that is, for all* $g \in L^2(\mathbb{R}^n)$ *we have*

$$\|T^s_h(t,s)g\|_{L^2(\mathbb{R}^n)} \le C\|g\|_{L^2(\mathbb{R}^n)}. \tag{2-49}$$

Proof. If $f \in L^2(\mathbb{R}^n)$ then

$$\|W_h f(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi h)^{2n}} \int_{\xi,\eta} \int_x e^{(i/h)(\phi(t,x,\xi)-\phi(t,x,\eta))} c_m(x,\xi/h)\overline{c_m(x,\eta/h)} \times \psi(|\xi|)\psi(|\eta|)\hat{f}\Big(\frac{\xi}{h}\Big)\bar{\hat{f}}\Big(\frac{\eta}{h}\Big)\,dx\,d\xi\,d\eta. \tag{2-50}$$

From Lemma 2.12 it follows that the mapping

$$\chi := \Big(x \to -t(\xi_1+\eta_1,0,\dots,0) + \int_0^1 \nabla_\xi \phi(0,x,(1-w)\xi + w\eta)\,dw\Big)$$

is a diffeomorphism away from the hypersurface $\zeta = 0$ with uniform bounds on the Jacobian of χ^{-1}. This change of variables reduces the problem to the L^2-continuity of semiclassical pseudodifferential operators with symbols of type $(\frac{2}{3},\frac{1}{3})$. □

Interpolation between (2-48) and (2-49) with weights $1-2/r$ and $2/r$ respectively yields

$$\|T_h^s(t,s)g\|_{L^r(\mathbb{R}^n)} \le C h^{-n(1-2/r)}\Big(1+\frac{|t-s|}{h}\Big)^{-n(1/2-1/r)} \|g\|_{L^{r'}(\mathbb{R}^n)} \tag{2-51}$$

and hence

$$\|T_h^s F\|_{L^q(0,T],L^r(\mathbb{R}^n)} \le C h^{-n/2(1-2/r)} \Big\| \int_{1 \ll \frac{|t-s|}{h}}^T |t-s|^{-n/2(1-2/r)} \|F(.,s)\|_{L^{r'}(\mathbb{R}^n)}\,ds \Big\|_{L^{q'}((0,T])}.$$

Since $n\big(\frac{1}{2}-\frac{1}{r}\big) = \frac{2}{q} < 1$ the application $|t|^{-2/q} : L^{q'} \to L^q$ is bounded and by Hardy–Littlewood–Sobolev inequality we deduce

$$\|T_h^s F\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le C h^{-2/q} \|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}. \tag{2-52}$$

- The "frozen" term T_h^f:

To estimate T_h^f it suffices to obtain bounds for its kernel K_h^f with both the variables (t,x) and (s,y) restricted to lie in a cube of $\mathbb{R}^{n+1}$ of side length comparable to $h^{1/3}$. Let us decompose S_T into disjoint cubes $Q = Q_x \times Q_t$ of side length $h^{1/3}$. We then have

$$\|T_h^f F\|^q_{L^q([0,T],L^r(\mathbb{R}^n))} = \int_0^T \Big(\sum_{Q=Q_x\times Q_t} \|\chi_Q T_h^f F\|^r_{L^r(Q_x)}\Big)^{q/r} dt = \sum_Q \|\chi_Q T_h^f F\|^q_{L^q([0,T],L^r(\mathbb{R}^n))},$$

where by χ_Q we denoted the characteristic function of the cube Q. In fact, by the definition, the integral kernel $K_h^f(t,x,s,y)$ of T_h^f vanishes if $|t-s| \ge h^{1/3}$. If $|t-s| \le h^{1/3}$ and $|x-y| \ge C_0 h^{1/3}$, then the phase

$$\phi(t,x,\xi) - \phi(s,y,\xi)$$

has no critical points with respect to ξ_1 (on the support of ψ), so that

$$|K_h^f(t,x,s,y)| \le C_N h^N \quad \text{for all } N \qquad \text{if } |x-y| \ge C_0 h^{1/3}.$$

It therefore suffices to estimate $\|\chi_Q T_h^f \chi_{Q^*} F\|_{L^q([0,T],L^r(\mathbb{R}^n))}$, where Q^* is the dilate of Q by some fixed factor independent of h. Since $q > 2 > q'$, $r \ge 2 \ge r'$, where q', r' are such that $1/q + 1/q' = 1$,

$1/r + 1/r' = 1$, we shall obtain

$$\sum_Q \|\chi_Q T_h^f \chi_{Q^*} F\|^q_{L^q([0,T],L^r(\mathbb{R}^n))} \le C_1 \sum_Q \|\chi_{Q^*} F\|^q_{L^{q'}([0,T],L^{r'}(\mathbb{R}^n))} \le C_2 \|F\|^q_{L^{q'}([0,T],L^{r'}(\mathbb{R}^n))}. \tag{2-53}$$

To prove (2-53) we shall use the following proposition:

Proposition 2.16. *Let $b(\xi) \in L^\infty(\mathbb{R}^n)$ be elliptic near $\xi_1 \simeq 1$, $b_h(\xi) := b(\xi/h)$, then for $h \ll |t-s| \le h^{1/3}$, $h \ll |x-y| \le h^{1/3}$ the operator defined by*

$$B_h f(x,t) = \frac{1}{(2\pi h)^n} \int e^{(i/h)\phi(t,x,\xi)} \psi(|\xi|) b_h(\xi) \hat{f}\Big(\frac{\xi}{h}\Big)\, d\xi \tag{2-54}$$

satisfies

$$\|B_h f\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le C h^{-1/q} \|f\|_{L^2(\mathbb{R}^n)}. \tag{2-55}$$

Proof. We use again the TT* argument. Since $b(\xi)$ acts as an L^2 multiplier we can apply the stationary phase theorem in the integral

$$\int e^{(i/h)(\phi(t,x,\xi)-\phi(s,y,\xi))} \psi(|\xi|)\, d\xi$$

to obtain

$$\|B_h B_h^* F\|_{L^q((0,T],L^r((\mathbb{R}^n)))} \lesssim h^{-2/q} \|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}.$$

Notice that we haven't used the special properties of the phase function at $t = 0$. □

Let now Q be a fixed cube in $\mathbb{R}^{n+1}$ of side length $h^{1/3}$. Let

$$b_h(t,x,s,y,\xi) = \rho(h^{-1/3}|t-s|) c_m(x,\xi/h) \overline{c_m(y,\xi/h)},$$

and write

$$b_h(t,x,s,y,\xi) = b_h(0,0,s,y,\xi) + \int_0^t \partial_t b_h(r,0,s,y,\xi)\, dr \\ + \cdots + \int_0^t \cdots \int_0^{x_n} \partial_t \cdots \partial_{x_n} b_h(r,z_1,\ldots,z_n,s,y,\xi)\, dr\, dz. \tag{2-56}$$

If the symbol c is independent of t and x, the estimates (2-30) follow from Proposition 2.16. We use this, for instance, to deduce

$$\|\chi_Q T_h^f \chi_{Q*} F\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le C h^{-n/2(1/2-1/r)} \tag{2-57}$$
$$\times \Bigg(\Big\| \iint e^{(i/h)(x\xi-\phi(s,y,\xi))} \psi(|\xi|) b_h(0,0,s,y,\xi) F(y,s)\, d\xi\, ds\, dy \Big\|_{L^2(\mathbb{R}^n)} \\ + \cdots + \int_0^{h^{1/3}} \int_0^{h^{1/3}} \Big\| \iint e^{(i/h)(x\xi-\phi(s,y,\xi))} \partial_t \ldots \partial_{x_n} \psi(|\xi|) b_h(r,z,s,y,\xi) F(y,s)\, d\xi\, ds\, dy \Big\|_{L^2(\mathbb{R}^n)} dr\, dz \Bigg).$$

Each derivative of $b_h(t,x,s,y,\xi)$ loses a factor of $h^{-1/3}$, but this is compensated by the integral over (r,z), so that it suffices to establish uniform estimates for fixed (r,z). By duality, we have to establish the estimate

$$\Big\| \iint e^{(i/h)\phi(s,y,\xi)} \psi(|\xi|) b_h(0,0,s,y,\xi) \hat{f}\Big(\frac{\xi}{h}\Big)\, d\xi \Big\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le C \|f\|_{L^2(\mathbb{R}^n)},$$

which follows by using the same argument of freezing the variables (s, y) together with Proposition 2.16.

The diffractive term D_h**.** To estimate the diffractive term we shall proceed again as in [Smith and Sogge 1995, Section 2].

Lemma 2.17. *For* $x_n \geq 0$ *and for* ξ *in a small conic neighborhood of the positive* ξ_1 *axis, the symbol* q *of* S_h *can be written in the form*

$$\begin{aligned} q(x,\xi) &:= \big(a(x,\xi)((1-\chi)A_+)(\zeta(x,\xi)) + b(x,\xi)((1-\chi)A_+)'(\zeta(x,\xi))\big)\frac{\mathrm{Ai}(\zeta_0(\xi))}{A_+(\zeta_0(\xi))} \\ &= p(x,\xi,\zeta(x,\xi)), \end{aligned}$$

where, for some $c > 0$

$$\big|\partial_\xi^\alpha \partial_\zeta^j \partial_{x'}^\beta \partial_{x_n}^k p\big(x,\xi,\zeta(x,\xi)\big)\big| \leq C_{\alpha,j,\beta,k}\xi_1^{1/6-|\alpha|+2k/3} e^{-cx_n^{3/2}\xi_1 - |\zeta|^{3/2}/2}.$$

Proof. Since

$$\big|\partial_\zeta^k((1-\chi)A_+)(\zeta)\big| \leq C_{k,\epsilon} e^{(2/3+\epsilon)|\zeta|^{3/2}} \quad \text{for all } \epsilon > 0$$

and a and b belong to $S^{1/6}_{1,0}$, the result will follow by showing that $\dfrac{\mathrm{Ai}}{A_+}(\zeta_0(\xi)) = \tilde{p}(x,\xi',\zeta(x,\xi))$ in the region $\zeta(x,\xi) \geq -2$, where, if $\xi' = (\xi_1,\dots,\xi_{n-1})$,

$$\big|\partial_{\xi'}^\alpha \partial_\zeta^j \partial_{x'}^\beta \partial_{x_n}^k \tilde{p}(x,\xi',\zeta)\big| \leq C_{\alpha,j,\beta,k,\epsilon}\xi_1^{-|\alpha|+2k/3} e^{-cx_n^{3/2}\xi_1-(4/3-\epsilon)|\zeta|^{3/2}}. \tag{2-58}$$

At $x_n = 0$, one has $\zeta = \zeta_0$, $\partial_{x_n}\zeta < 0$. It follows that for some $c > 0$

$$\zeta_0(x,\xi) \geq \zeta(x,\xi) + cx_n\xi_1^{2/3}.$$

By the asymptotic behavior of the Airy function we have, in the region $\zeta(x,\xi) \geq -2$

$$\Big|\Big(\frac{\mathrm{Ai}}{A_+}\Big)^{(k)}(\zeta_0)\Big| \leq C_{k,\epsilon} e^{-cx_n^{3/2}\xi_1-(4/3-\epsilon)|\zeta(x,\xi)|^{3/2}}. \tag{2-59}$$

We introduce a new variable $\tau(x,\xi) = \xi_1^{1/3}\zeta(x,\xi)$. At $x_n = 0$ one has $\tau = -\xi_n$, so that we can write $\xi_n = \sigma(x,\xi',\tau)$, where σ is homogeneous of degree 1 in (ξ',τ). We set

$$\tilde{p}(x,\xi',\zeta) = \frac{\mathrm{Ai}}{A_+}(-\xi_1^{-1/3}\sigma(x,\xi',\xi^{1/3}\zeta)).$$

The estimates (2-58) will follow by showing that

$$\Big|\partial_{\xi'}^\alpha \partial_\tau^j \partial_{x'}^\beta \partial_{x_n}^k \frac{\mathrm{Ai}}{A_+}(-\xi_1^{-1/3}\sigma(x,\xi',\tau))\Big| \leq C_{\alpha,j,\beta,k,\epsilon}\xi_1^{-|\alpha|-j+2k/3} e^{-cx_n^{3/2}\xi_1-(4/3-\epsilon)|\tau|^{3/2}\xi_1^{-1/2}}. \tag{2-60}$$

For $k = 0$, the estimates (2-60) follow from (2-59), together with the fact that

$$\Big|\partial_{\xi'}^\alpha \partial_\tau^j \partial_{x'}^\beta \frac{\mathrm{Ai}}{A_+}(-\xi_1^{-1/3}\sigma(x,\xi',\tau))\Big| \leq C_{\alpha,\beta,j}\big(x_n\xi_1^{2/3} + \xi_1^{-1/3}|\tau|\big)\xi_1^{-|\alpha|-j},$$

which, in turn, holds by homogeneity, together with the fact that $\sigma(x,\xi',\tau) = 0$ if $x_n = \tau = 0$. If $k > 0$, the estimate (2-60) follows by observing that the effect of differentiating in x_n is similar to multiplying by a symbol of order $2/3$. This concludes the proof of Lemma 2.17. □

Lemma 2.18. *The Schwartz kernel of the diffractive term D_h can be written in the form*

$$\int e^{i(\theta(x,\xi)-ht\xi_1^2)}\psi(h|\xi|)q(x,\xi)\,d\xi = \int e^{i(\theta(x,\xi)-ht\xi_1^2+\sigma\xi_1^{-2/3}\zeta(x,\xi)+\sigma^3/3\xi_1^2-\langle y,\xi\rangle)}\psi(h|\xi|)c_d(x,\xi,\sigma)\,d\sigma\,d\xi, \quad (2\text{-}61)$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product and where

$$|\partial_\xi^\alpha\partial_\sigma^j\partial_{x'}^\beta\partial_{x_n}^k c_d(x,\xi,\sigma)| \le C_{\alpha,j,\beta,k,N}\xi_1^{-1/2-|\alpha|-2j/3+2k/3}e^{-cx_n^{3/2}\xi_1}(1+\xi_1^{-4/3}\sigma^2)^{-N/2} \quad \textit{for all } N.$$

Proof. The symbol c_d of the Schwartz kernel of D_h can be expressed as a product of two symbols

$$c_d(x,\xi,\sigma) = c_1(x,\xi,\sigma\xi_1^{-2/3})c_2\big(x,\xi,\zeta(x,\xi)\big),$$

where

$$c_1(x,\xi,\sigma\xi_1^{-2/3}) = \xi_1^{-2/3}\Psi_+(\xi_1^{-2/3}\sigma)\big(a(x,\xi)+\sigma\xi_1^{-2/3}b(x,\xi)\big) \in S^{-1/2}_{2/3,1/3}(\mathbb{R}^n_x,\mathbb{R}^{n+1}_{\xi,\sigma})$$

comes from the Fourier transform of A_+ (here Ψ_+ is a symbol of order 0) and where c_2 satisfies for all $N \ge 0$ (for $\sigma^2\xi_1^{-4/3}+\zeta(x,\xi)=0$)

$$\big|\partial_{\xi'}^\alpha\partial_\sigma^j\partial_{x'}^\beta\partial_{x_n}^k c_2\big(x,\xi',-(\sigma^2\xi_1^{-4/3})\big)\big| \le C_{\alpha,j,\beta,k,N}\xi_1^{-2j/3}|\sigma\xi_1^{-2/3}|^j\xi_1^{-|\alpha|+2k/3}e^{-cx_n^{3/2}\xi_1}(1+\xi_1^{-4/3}\sigma^2)^{-N/2}, \quad (2\text{-}62)$$

which follows from (2-58). We use the exponential factor $e^{-cx_n^{3/2}\xi_1}$ to deduce from (2-62) that

$$|x_n^j\partial_{x_n}^k c_2(x,\xi',-(\sigma^2\xi_1^{-4/3}))| \le C_{j,k,N}(x_n\xi_1^{2/3})^j e^{-c(x_n\xi_1^{2/3})^{3/2}}\xi_1^{2/3(k-j)}(1+\xi_1^{-4/3}\sigma^2)^{-N/2} \quad \text{for all } N. \qquad \square$$

From now on we proceed as for the main term and we reduce the problem to considering the operator

$$W_h^d f(x,t) = \frac{1}{(2\pi h)^n}\int e^{(i/h)\tilde\phi(t,x,\xi,\sigma)}c_d(x,\xi/h,\sigma)\psi(|\xi|)\hat f\Big(\frac{\xi}{h}\Big)\,d\xi,$$

where $x_n^j\partial_{x_n}^k c_d \in S^{2(k-j)/3}_{2/3,1/3}(\mathbb{R}^{n-1}_{x'}\times\mathbb{R}^n_\xi)$ uniformly over x_n and where we have set

$$\tilde\phi(t,x,\xi,\sigma) := -t\xi_1^2+\theta(x,\xi)+\sigma\xi_1^{1/3}\zeta(x,\xi)+\tfrac{1}{3}\xi_1\sigma^3, \quad (2\text{-}63)$$

obtained after the changes of variables $\sigma\to\sigma\xi_1$, $\xi\to\xi/h$ in (2-61). Using the freezing arguments behind the proof of the estimates for T_h^f and Minkowski inequality we have

$$\begin{aligned}\|W_h^d f\|_{L^q((0,T],L^r(\mathbb{R}^n))} &\le \Big\|\frac{1}{(2\pi h)^n}\int e^{(i/h)\tilde\phi(t,x,\xi,\sigma)}c_d(x',0,\xi/h,\sigma)\psi(|\xi|)\hat f\Big(\frac{\xi}{h}\Big)\,d\sigma\,d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^n))}\\ &+h^{-2/3}\int_0^{h^{2/3}}\Big\|\frac{1}{(2\pi h)^n}\int e^{(i/h)\tilde\phi(t,x,\xi,\sigma)}h^{2/3}\partial_{x_n}c_d(x',r,\xi/h,\sigma)\psi(|\xi|)\hat f\Big(\frac{\xi}{h}\Big)\,d\sigma\,d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^{n-1}))}dr\\ &+h^{2/3}\int_{r>h^{2/3}}\frac{dr}{r^2}\Big\|\frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\tilde\phi(t,x,\xi,\sigma)}h^{-2/3}r^2\partial_{x_n}c_d(x',r,\xi/h,\sigma)\psi(|\xi|)\hat f\Big(\frac{\xi}{h}\Big)\,d\sigma\,d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^{n-1}))}.\end{aligned}$$

Since $c_d(x',0,\xi,\sigma)$ and $h^{2/3}(1+h^{-4/3}r^2)\partial_{x_n}c_d(x',r,\xi,\sigma)$ are symbols of order 0 and type $(\frac{2}{3},\frac{1}{3})$ with uniform estimates over r, the estimates for the diffractive term also follow from Proposition 2.8. Indeed, the term on the second line loses a factor $h^{-2/3}$, but this is compensated by the integral over $r \le h^{2/3}$. The term on the last line can be bounded by above by

$$h^{2/3}\int_{r>h^{2/3}} \frac{dr}{r^2}\left\| \frac{1}{(2\pi h)^n}\int e^{(i/h)\tilde{\phi}(t,x,\xi,\sigma)}(h^{-2/3}r^2\partial_{x_n}c_d(x',r,\xi/h,\sigma))\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\sigma\,d\xi\right\|_{L^q((0,T],L^r(\mathbb{R}^n))}$$
$$\le \left\| \frac{1}{(2\pi h)^n}\int e^{(i/h)\tilde{\phi}(t,x,\xi,\sigma)}(h^{-2/3}r^2\partial_{x_n}c_d(x',r,\xi/h))\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\sigma\,d\xi\right\|_{L^q((0,T],L^r(\mathbb{R}^n))}.$$

We conclude by using the same arguments as in the proof of Proposition 2.8, where now W_h is replaced by operators with symbols $c_d(x', 0, \xi, \sigma)$, However, for this term we can't directly apply Lemma 2.11, since the expansion of the Airy function giving the phase function (2-29) is available only for $\zeta(x, \xi/h) \le -1$. Writing the phase function of (2-61) in the form $\tilde{\phi}(t, x, \xi, \sigma) - \langle y, \xi\rangle$, we notice that at $t = 0$ this phase is homogeneous of degree 1 in ξ and the proof of the nondegeneracy of the critical points in the TT* argument of Lemma 2.11 reduces to checking that the Jacobian J of the mapping

$$(\xi,\sigma) \to (\nabla_x(\theta(x,\xi)+\sigma\zeta(x,\xi)), \zeta(x,\xi)+\sigma^2) \tag{2-64}$$

does not vanish at the critical point of the phase of (2-61). Hence we will obtain a phase function $\breve{\phi}(t, x, \xi)$ which will satisfy $\nabla^2_{x,\xi}\breve{\phi}(0, x, \xi) \neq 0$ and this will hold also for small $|t| \le T$ and we can use the same argument as in Lemma 2.11. To prove that the Jacobian of the application (2-64) doesn't vanish we use [Smith and Sogge 1994, Lemma A.2]. Precisely, at this (critical) point $\sigma = \zeta(x, \xi) = 0$, $y = 0$, and $\nabla_{x'}\zeta(x, \xi) = 0$. Since $\partial_{x_n}\zeta(x, \xi) \neq 0$ and $\partial_{\xi_n}\zeta(x, \xi) \neq 0$ at this point, the result follows by the nonvanishing of $|\nabla_{x'}\nabla_{\xi'}\theta(x, \xi)|$. In fact we have

$$\det\begin{pmatrix} \nabla_{x'}\nabla_{\xi'}\theta & \nabla_{\xi'}\partial_{x_n}\theta & \nabla_{\xi'}\zeta \\ \partial_{\xi_n}\nabla_{x'}\theta & \partial_{\xi_n}\partial_{x_n}\theta & \partial_{\xi_n}\zeta \\ \nabla_{x'}\zeta & \partial_{x_n}\zeta & 2\sigma \end{pmatrix}\Bigg|_{\sigma^2=-\zeta=0} \neq 0.$$

3. Strichartz estimates for the classical Schrödinger equation outside a strictly convex obstacle in $\mathbb{R}^n$

In this section we prove Theorem 1.7 under Assumption 1.6. We shall work with the Laplace operator with constant coefficients $\Delta_D = \sum_{j=1}^n \partial_j^2$ acting on $L^2(\Omega)$ to avoid technicalities, where Ω is the exterior in $\mathbb{R}^n$ of a strictly convex domain Θ.

In the proof of Theorem 1.7 we distinguish two main steps. We start by performing a time rescaling which transforms the Equation (1-8) into a semiclassical problem. Due to the finite speed of propagation (proved by Lebeau [1992]), we can use the (local) semiclassical result of Theorem 1.3 together with the smoothing effect (following Staffilani and Tataru [2002] and Burq [2002]) to obtain classical Strichartz estimates near the boundary. Outside a fixed neighborhood of $\partial\Omega$ we use a method suggested by Staffilani and Tataru [2002] which considers the Schrödinger flow as a solution of a problem in the whole space $\mathbb{R}^n$, for which the Strichartz estimates are known.

We start by proving that using Theorem 1.3 on a compact manifold with strictly concave boundary we can deduce sharp Strichartz estimates for the semiclassical Schrödinger flow on Ω. More precisely, we prove the following result, and then show how it can be used to prove Theorem 1.7.

Proposition 3.1. *Given (q, r) satisfying the scaling condition (1-3) with $q > 2$ there exists a constant $C > 0$ such that the (classical) Schrödinger flow on $\Omega \times \mathbb{R}$ with Dirichlet boundary condition and spectrally localized initial data $\Psi(-h^2\Delta_D)u_0$, where $\Psi \in C_0^\infty(\mathbb{R} \setminus 0)$, satisfies*

$$\left\| e^{it\Delta_D}\Psi(-h^2\Delta_D)u_0 \right\|_{L^q(\mathbb{R})L^r(\Omega)} \le C \left\| \Psi(-h^2\Delta_D)u_0 \right\|_{L^2(\Omega)}. \tag{3-1}$$

Proof. We use a method similar to the one given in our recent paper [Ivanovici and Planchon 2009] in collaboration with F. Planchon. Let $\tilde{\Psi} \in C_0^\infty(\mathbb{R} \setminus \{0\})$ be such that $\tilde{\Psi} = 1$ on the support of Ψ, hence

$$\tilde{\Psi}(-h^2\Delta_D)\Psi(-h^2\Delta_D) = \Psi(-h^2\Delta_D).$$

Following [Burq 2002; Ivanovici and Planchon 2009], we split $e^{it\Delta_D}\Psi(-h^2\Delta_D)u_0(x)$ as a sum of two terms,

$$\tilde{\Psi}(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0 + \tilde{\Psi}(-h^2\Delta_D)(1-\chi)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0,$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ equals 1 in a neighborhood of $\partial\Omega$.

- Study of $\tilde{\Psi}(-h^2\Delta_D)(1-\chi)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0$:
  Set $w_h(x,t) = (1-\chi)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0(x)$. Then w_h satisfies

$$\begin{cases} i\partial_t w_h + \Delta_D w_h = -[\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0, \\ w_h|_{t=0} = (1-\chi)\Psi(-h^2\Delta_D)u_0. \end{cases} \tag{3-2}$$

Since χ is equal to 1 near the boundary $\partial\Omega$, the solution to (3-2) also solves a problem in the whole space $\mathbb{R}^n$. Consequently, the Duhamel formula gives

$$w_h(t,x) = e^{it\Delta}(1-\chi)\Psi(-h^2\Delta_D)u_0 - \int_0^t e^{i(t-s)\Delta}[\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0(s)\,ds, \tag{3-3}$$

where Δ denotes the free Laplacian on $\mathbb{R}^n$ and therefore the contribution of $e^{it\Delta}(1-\chi)\Psi(-h^2\Delta_D)u_0$ satisfies the usual Strichartz estimates. For the second term on the right in (3-3) we use the next lemma:

Lemma 3.2 [Christ and Kiselev 2001]. *Consider a bounded operator*

$$T : L^{q'}(\mathbb{R}, B_1) \to L^q(\mathbb{R}, B_2)$$

given by a locally integrable kernel $K(t,s)$ with values in bounded operators from B_1 to B_2, where B_1 and B_2 are Banach spaces. Suppose that $q' < q$. Then the operator

$$\tilde{T} f(t) = \int_{s<t} K(t,s) f(s)\,ds$$

is bounded from $L^{q'}(\mathbb{R}, B_1)$ to $L^q(\mathbb{R}, B_2)$ and

$$\|\tilde{T}\|_{L^{q'}(\mathbb{R},B_1)\to L^q(\mathbb{R},B_2)} \le C(1-2^{-(1/q-1/q')})^{-1}\|T\|_{L^{q'}(\mathbb{R},B_1)\to L^q(\mathbb{R},B_2)}.$$

Since $q > 2$, this lemma allows us to replace the study of the second term in the right-hand side of (3-3) by that of

$$\int_0^\infty e^{i(t-s)\Delta}[\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0(s)(s)\,ds =: U_0U_0^* f(x,t),$$

where $U_0 = e^{it\Delta}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^q(\mathbb{R}, L^r(\mathbb{R}^n))$ and U_0^* is bounded from $L^2(\mathbb{R}, H^{-1/2}_{\text{comp}})$ to $L^2(\mathbb{R}^n)$ and where we set $f := [\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0$ which belongs to $L^2 H^{-1/2}_{\text{comp}}(\Omega)$ by Burq et al. [2004a, Proposition 2.7]. The estimates for w_h follow as in [Burq et al. 2004a] and we find

$$\|w_h\|_{L^q(\mathbb{R},L^r(\Omega))} \le C\|(1-\chi)\Psi(-h^2\Delta_D)u_0\|_{L^2(\mathbb{R}^n)} + \|[\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R}, H^{-1/2}_{\text{comp}}(\Omega))}. \quad (3\text{-}4)$$

The last term in (3-4) can be estimated using [Burq et al. 2004a, Proposition 2.7] by

$$C\|\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R}, H^{1/2}_{\text{comp}}(\Omega))} \le C\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}. \quad (3\text{-}5)$$

Finally, we conclude this part using [Ivanovici and Planchon 2008, Theorem 1.1] which gives

$$\|\Psi(-h^2\Delta_D)w_h\|_{L^r(\Omega)} \le \|w_h\|_{L^r(\Omega)}. \quad (3\text{-}6)$$

- Study of $\tilde{\Psi}(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0$:
  Let $\varphi \in C_0^\infty((-1, 2))$ equal to 1 on $[0, 1]$. For $l \in \mathbb{Z}$ set

$$v_{h,l} = \varphi(t/h - l)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0, \quad (3\text{-}7)$$

$$V_{h,l} = \Big(\varphi(t/h - l)[\Delta_D, \chi] + i\frac{\varphi'(t/h - l)}{h}\chi\Big)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0. \quad (3\text{-}8)$$

The quantity in (3-7) is a solution to

$$\begin{cases} i\partial_t v_{h,l} + \Delta_D v_{h,l} = V_{h,l}, \\ v_{h,l}|_{t<hl-h} = 0, \quad v_{h,l}|_{t>hl+2h} = 0. \end{cases} \quad (3\text{-}9)$$

Let $Q \subset \mathbb{R}^n$ be an open cube sufficiently large such that $\partial\Omega$ is contained in the interior of Q. We denote by S the punctured torus obtained from removing the obstacle Θ (recall that $\Omega = \mathbb{R}^n \setminus \Theta$) in the compact manifold obtained from Q with periodic boundary conditions on ∂Q. Notice that S, when defined in this way, coincides with the Sinai billiard. Let $\Delta_S := \sum_{j=1}^n \partial_j^2$ denote the Laplace operator on the compact domain S.

On S, we may define a spectral localization operator using eigenvalues λ_k and eigenvectors e_k of Δ_S: if $f = \sum_k c_k e_k$, then

$$\Psi(-h^2\Delta_S)f = \sum_k \Psi(-h^2\lambda_k^2)c_k e_k. \quad (3\text{-}10)$$

Remark 3.3. In a neighborhood of the boundary, the domains of Δ_S and Δ_D coincide, thus if $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ is supported near $\partial\Omega$ then $\Delta_S\tilde{\chi} = \Delta_D\tilde{\chi}$.

Now let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on the support of χ and be supported in a neighborhood of $\partial\Omega$ such that, on its support, the operator $-\Delta_D$ coincides with $-\Delta_S$. From their respective definitions, we know that $v_{h,l} = \tilde{\chi}v_{h,l}$ and $V_{h,l} = \tilde{\chi}V_{h,l}$; consequently $v_{h,l}$ will also solve, on the compact domain S, the equation

$$\begin{cases} i\partial_t v_{h,l} + \Delta_S v_{h,l} = V_{h,l}, \\ v_{h,l}|_{t<h(l-1/2)\pi} = 0, \quad v_{h,l}|_{t>h(l+1)\pi} = 0. \end{cases} \quad (3\text{-}11)$$

Writing the Duhamel formula for the last equation in (3-11) on S, applying $\tilde{\Psi}(-h^2\Delta_D)$, and using that $\tilde{\chi}v_{h,l} = v_{h,l}$, $\tilde{\chi}V_{h,l} = V_{h,l}$ and writing

$$\tilde{\Psi}(-h^2\Delta_D)\tilde{\chi} = \chi_1\tilde{\Psi}(-h^2\Delta_S)\tilde{\chi} + (1-\chi_1)\tilde{\Psi}(-h^2\Delta_D)\tilde{\chi} + \chi_1\big(\tilde{\Psi}(-h^2\Delta_D) - \tilde{\Psi}(-h^2\Delta_S)\big)\tilde{\chi} \tag{3-12}$$

for some $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on the support of $\tilde{\chi}$, we obtain

$$\begin{aligned}\tilde{\Psi}(-h^2\Delta_D)v_{h,l}(x,t) = \chi_1 \int_{hl-l}^{t} e^{i(t-s)\Delta_S}\tilde{\Psi}(-h^2\Delta_S)V_{h,l}(x,s)\,ds \\ + (1-\chi_1)\int_{hl-l}^{t} \tilde{\Psi}(-h^2\Delta_D)e^{i(t-s)\Delta_S}V_{h,l}(x,s)\,ds \\ + \chi_1\big(\tilde{\Psi}(-h^2\Delta_D) - \tilde{\Psi}(-h^2\Delta_S)\big)v_{h,l}.\end{aligned} \tag{3-13}$$

Denote by $v_{h,l,m}$ the first term of (3-13), by $v_{h,l,f}$ the second one, and by $v_{h,l,s}$ the last one. We deal with them separately. To estimate the $L_t^q L^r(\Omega)$ norm of $v_{h,l,f}$ we notice that it is supported away from the boundary and therefore the estimates will follow as in the previous part of this section. Indeed, notice that since $v_{h,l}$ also solves (3-7) on Ω, we can use the Duhamel formula on Ω so that in the integral we can define $v_{h,l,f}$ to have Δ_D instead of Δ_S. We then estimate the $L_t^q L^r(\Omega)$ norm of $v_{h,l,f}$ by applying the Minkowski inequality and using the sharp Strichartz estimates for $(1-\chi_1)\tilde{\Psi}(-h^2\Delta_D)e^{i(t-s)\Delta_D}V_{h,l}$ deduced in the first part of the proof of Proposition 3.1 and obtain, denoting $I_l^h = [hl-h, hl+2h]$,

$$\|v_{h,l,f}\|_{L^q(I_l^h, L^r(\Omega))} \le C \int_{I_l^h} \|V_{h,l}(x,s)\|_{L^2(\Omega)}\,ds. \tag{3-14}$$

For the last term $v_{h,l,s}$ we use the following lemma, which will be proved in Appendix B:

Lemma 3.4. *Let $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on a fixed neighborhood of the support of $\tilde{\chi}$. Then we have*

$$\|v_{h,l,s}\|_{L^q(I_l^h, L^r(\Omega))} \le C_N h^N \|V_{h,l}(x,s)\|_{L^2(I_l^h, H_0^{n(\frac12-\frac1r)-\frac12}(\Omega))} \quad \text{for all } N \in \mathbb{N}. \tag{3-15}$$

To estimate the main contribution $v_{h,l,m}$ we use the Minkowski inequality, which yields

$$\|v_{h,l,m}\|_{L^q(I_l^h, L^r(\Omega))} = \|v_{h,l,m}\|_{L^q(I_l^h, L^r(S))} \le C \int_{I_l^h} \big\|e^{i(t-s)\Delta_S}\tilde{\Psi}(-h^2\Delta_S)V_{h,l}(x,s)\big\|_{L^q(I_l^h, L^r(S))}\,ds. \tag{3-16}$$

Applying Theorem 1.3 for the linear semiclassical Schrödinger flow on S, the term to integrate in (3-16) is bounded by $C\|\tilde{\Psi}(-h^2\Delta_S)V_{h,l}(x,s)\|_{L^2(S)}$. Using [Ivanovici and Planchon 2008, Theorem 1.1] and the fact that $\tilde{\chi}V_{h,l} = V_{h,l}$ (so that taking the norm over Ω or S makes no difference) we obtain

$$\|v_{h,l,m}\|_{L^q(I_l^h, L^r(\Omega))} \le C \int_{I_l^h} \|V_{h,l}(x,s)\|_{L^2(\Omega)}\,ds. \tag{3-17}$$

After applying the Cauchy–Schwartz inequality in Equations (3-14) and (3-17) it remains to estimate the $L^2(I_l^h, H^\sigma(\Omega))$ norm of $V_{h,l}$, where $\sigma \in \big\{0, n\big(\frac12 - \frac1r\big) - \frac12\big\}$. We do this using the precise form (3-8) and obtain

$$\begin{aligned}\|V_{h,l}\|_{L^2(I_l^h, H^\sigma(\Omega))} \le C\big\|\varphi(t/h-l)[\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|_{L^2(I_l^h, H^\sigma(\Omega))} \\ + Ch^{-1}\big\|\varphi'(t/h-l)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|_{L^2(I_l^h, H^\sigma(\Omega))}.\end{aligned} \tag{3-18}$$

Since the operator $[\Delta_D,\chi]\Psi(-h^2\Delta_D)$ is bounded from $H^{\sigma+1}$ to H^σ, we deduce from (3-13), (3-14), (3-18), (3-19), and Lemma 3.4 the following bound (the last two lines differing only in the superscript of H_0):

$$\begin{aligned}\|\tilde\Psi(-h^2\Delta_D)v_{h,l}\|_{L^q(I_l^h,L^r(\Omega))} \le{}& Ch^{1/2}\|\tilde\varphi(t/h-l)\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(I_l^h,H_0^1(\Omega))}\\ &+Ch^{-1/2}\|\tilde\varphi(t/h-l)\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(I_l^h,L^2(\Omega))}\\ &+C_Nh^{N+1/2}\|\tilde\varphi(t/h-l)\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(I_l^h,H_0^{n(\frac12-\frac1r)+\frac12}(\Omega))}\\ &+C_Nh^{N-1/2}\|\tilde\varphi(t/h-l)\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(I_l^h,H_0^{n(\frac12-\frac1r)-\frac12}(\Omega))},\end{aligned}\tag{3-19}$$

where $\tilde\varphi\in C_0^\infty(\mathbb{R})$ is chosen equal to 1 on the support of φ. Since $q\ge 2$ we estimate

$$\begin{aligned}&\|\tilde\Psi(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^q_{L^q(\mathbb{R},L^r(\Omega))}\\ &\quad\le C\sum_{l=-\infty}^{\infty}\|\tilde\Psi(-h^2\Delta_D)v_{h,l}\|^q_{L^q(I_l^h,L^r(\Omega))}\\ &\quad\le Ch^{q/2}\Big(\sum_{l=-\infty}^{\infty}\|\tilde\varphi(t/h-l)\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^2_{L^2(I_l^h,H_0^1(\Omega))}\Big)^{q/2}\\ &\qquad\qquad+Ch^{-q/2}\Big(\sum_{l=-\infty}^{\infty}\|\tilde\varphi(t/h-l)\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^2_{L^2(I_l^h,L^2(\Omega))}\Big)^{q/2}\\ &\qquad\qquad+C_Nh^{q(N+1/2)}\Big(\sum_{l=-\infty}^{\infty}\|\tilde\varphi(t/h-l)\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^2_{L^2(I_l^h,H_0^{n(\frac12-\frac1r)+\frac12}(\Omega))}\Big)^{q/2}\\ &\qquad\qquad+C_Nh^{q(N-1/2)}\Big(\sum_{l=-\infty}^{\infty}\|\tilde\varphi(t/h-l)\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^2_{L^2(I_l^h,H_0^{n(\frac12-\frac1r)-\frac12}(\Omega))}\Big)^{q/2}.\end{aligned}\tag{3-20}$$

The almost-orthogonality of the supports of $\tilde\varphi(\cdot-l)$ in time allows us to estimate the term on the third line of (3-20) by

$$Ch^{q/2}\|\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^q_{L^2(\mathbb{R},H_0^1(\Omega))},\tag{3-21}$$

the one on the fourth line by

$$Ch^{-q/2}\|\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^q_{L^2(\mathbb{R},L^2(\Omega))},\tag{3-22}$$

the term on the fifth line by

$$C_Nh^{q(N+1/2)}\|\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^q_{L^2(\mathbb{R},H_0^{n(\frac12-\frac1r)+\frac12}(\Omega))},\tag{3-23}$$

and the one on the last line of (3-20) by

$$C_Nh^{q(N-1/2)}\|\tilde\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^q_{L^2(\mathbb{R},H_0^{n(\frac12-\frac1r)-\frac12}(\Omega))}.\tag{3-24}$$

We need the following smoothing effect on a nontrapping domain:

Proposition 3.5 [Burq et al. 2004a, Proposition 2.7]. *Assume that* $\Omega = \mathbb{R}^n \setminus \mathbb{O}$, *where* $\mathbb{O} \neq \varnothing$ *is a compact nontrapping obstacle. For every* $\tilde\chi \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, $\sigma \in [-1/2, 1]$, *one has*

$$\big\|\tilde\chi \Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|_{L^2(\mathbb{R}, H_0^{\sigma+1/2}(\Omega))} \leq C\|\Psi(-h^2\Delta_D)u_0\|_{H^\sigma(\Omega)}. \tag{3-25}$$

Remark 3.6. This is proved in [Burq et al. 2004a] for $\sigma \in [0, 1]$, but for spectrally localized data the result also follows using the estimates (2.15) of [Burq et al. 2004a, Proposition 2.7].

We apply Proposition 3.5 with $\sigma = \frac{1}{2}$ in (3-21), with $\sigma = -\frac{1}{2}$ in (3-22) and with $\sigma = n\big(\frac{1}{2} - \frac{1}{r}\big) = \frac{2}{q} \in [0, 1]$ in (3-23). In (3-24) we use that $n\big(\frac{1}{2} - \frac{1}{r}\big) - \frac{1}{2} \leq \frac{1}{2}$ to estimate the $L^2\big(\mathbb{R}, H^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega)\big)$ norm by the $L^2(\mathbb{R}, H^{1/2}(\Omega))$ norm and use Proposition 3.5 with $\sigma = 0$. This yields

$$\big\|\tilde\Psi(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq C\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}. \tag{3-26}$$

Here we used the spectral localization Ψ to estimate $\|\Psi(-h^2\Delta_D)u_0\|_{H^\sigma(\Omega)}$ by $h^{-\sigma}\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}$. This achieves the proof of Proposition 3.1. □

In the rest of this section we show how Proposition 3.1 implies Theorem 1.7.

Lemma 3.7 [Ivanovici and Planchon 2008, Theorem 1.1]. *Let* $\Psi_0 \in C_0^\infty(\mathbb{R})$, $\Psi \in C_0^\infty((1/2, 2))$ *satisfy*

$$\Psi_0(\lambda) + \sum_{j\geq 1}\Psi(2^{-2j}\lambda) = 1, \quad \textit{for all } \lambda \in \mathbb{R}.$$

Then for all $r \in [2, \infty)$ *we have*

$$\|f\|_{L^r(\Omega)} \leq C_r\bigg(\|\Psi_0(-\Delta_D)f\|_{L^r(\Omega)} + \bigg(\sum_{j=1}^{\infty}\|\Psi(-2^{-2j}\Delta_D)f\|^2_{L^r(\Omega)}\bigg)^{1/2}\bigg). \tag{3-27}$$

Applying Lemma 3.7 to $f = e^{it\Delta_D}u_0$ and taking the L^q norm in time yields

$$\|e^{it\Delta_D}u_0\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq \bigg\|\big\|e^{it\Delta_D}\Psi_0(-\Delta_D)u_0\big\|_{L^r(\Omega)} + \bigg(\sum_{j\geq 1}\big\|e^{it\Delta_D}\Psi(-2^{-2j}\Delta_D)u_0\big\|^2_{L^r(\Omega)}\bigg)^{1/2}\bigg\|_{L^q(\mathbb{R})}$$

which, by the Minkowski inequality, leads to $\|e^{it\Delta_D}u_0\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}$. The proof of Theorem 1.7 is complete.

4. Applications

In this section we sketch the proofs of Theorem 1.8 and Theorem 1.9.

We start with Theorem 1.8. From Theorem 1.7 we have an estimate of the linear flow of the Schrödinger equation

$$\|e^{-it\Delta_D}u_0\|_{L^5(\mathbb{R}, L^{30/11}(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}. \tag{4-1}$$

One may shift the regularity by 1 and obtain

$$\|e^{-it\Delta_D}u_0\|_{L^5(\mathbb{R}, W^{1,30/11}(\Omega))} \leq C\|u_0\|_{H_0^1(\Omega)}. \tag{4-2}$$

Hence for small $T > 0$ the left-hand side of (4-1) and (4-2) will be small; for such T let $X_T := L^5\big((0, T], W^{1,30/11}(\Omega)\big)$. One may then set up the usual fixed point argument in X_T, as if $u \in X_T$ then $u^5 \in L^1\big([0, T], H^1(\Omega)\big)$.

Let us proceed with Theorem 1.9. From [Planchon and Vega 2009], one has a time-global control on the solution u, at the level of $\dot{H}^{\frac{1}{4}}$ regularity:

$$u \in L^4\big((0,+\infty), L^4(\Omega)\big).$$

By interpolation with either mass or energy conservation, combined with the local existence theory, one may bootstrap this time-global control into

$$u \in L^{p-1}\big((0,+\infty), L^\infty(\Omega)\big),$$

from which scattering in $H^1_0(\Omega)$ follows immediately.

Appendices

A. ***Finite speed of propagation for the semiclassical equation.*** In this appendix we recall some properties of the semiclassical Schrödinger flow. For further discussion and proofs, see [Lebeau 1992].

Let S be a compact manifold with smooth boundary ∂S.

Definition A.1. We say that a symbol $q(y,\eta) \in S^m_{\rho,\delta}$ is of type (ρ,δ) and of order m if, for any α and β, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_y^\beta \partial_\eta^\alpha q(y,\eta)| \le C_{\alpha,\beta}(1+|\eta|)^{m-\rho|\alpha|+\delta|\beta|}.$$

For $q \in S^m_{1,0}$ we let $Op_h(q) = Q(y, hD, h)$ be the h-pseudodifferential operator defined by

$$Op_h(q) f(y) = \frac{1}{(2\pi h)^n} \int e^{(i/h)(y-\tilde{y})\eta} q(y,\eta,h) f(\tilde{y})\, d\tilde{y}.$$

We set $y = (x,t) \in S \times \mathbb{R}$ and denote $\eta = (\xi,\tau)$ the dual variable of y. Near a point $x_0 \in \partial S$ we can choose a system of local coordinates such that S is given by $S = \{x = (x', x_n) : x_n > 0\}$. We define the tangential operators

$$Op_{h,\text{tang}}(q) f(y) = \frac{1}{(2\pi h)^{n-1}} \int e^{(i/h)(y'-\tilde{y}')\eta'} q(y,\eta',h) f(\tilde{x}', x_n, \tilde{t})\, d\tilde{y}'\, d\eta',$$

where $y = (x', x_n, t)$, $y' = (x', t)$, $\tilde{y}' = (\tilde{x}', \tilde{t})$, $\eta = (\xi', \xi_n, \tau)$, $\eta' = (\xi', \tau)$, and where the symbol $q(y, \eta', h)$ lies in $S^m_{1,0,\text{tang}}$; in other words, for any α and β, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_y^\alpha \partial_{\eta'}^\beta q(y,\eta',h)| \le C_{\alpha,\beta}(1+|\eta'|)^{m-|\beta|}.$$

Let g be a Riemannian metric on S such that ∂S is strictly concave and (S,g) satisfies Assumption 1.1. Let $v_0 \in L^2(S)$ be compactly supported outside a small neighborhood of the boundary, take $\Psi \in C_0^\infty((\alpha_0, \beta_0))$, and let $v(x,t) = e^{iht\Delta_g}\Psi(-h^2\Delta_g)v_0$ denote the linear semiclassical Schrödinger flow with initial data at time $t=0$ equal to $\Psi(-h^2\Delta_g)v_0$ and such that $\|\Psi(-h^2\Delta_g)v_0\|_{L^2(S)} \lesssim 1$.

Let $\pi : T^*(\bar{S}\times\mathbb{R}) \to T^*(\partial S \times \mathbb{R}) \cup T^*(S\times\mathbb{R})$ be the canonical projection, defined by

$$\pi|_{T^*(S\times\mathbb{R})} = \text{Id}, \qquad \pi(y,\eta) = (y, \eta|_{T^*(\partial S\times\mathbb{R})}) \quad \text{for } (y,\eta) \in T^*(\bar{S}\times\mathbb{R})|_{\partial S\times\mathbb{R}}.$$

Writing $y = (x,t)$ and $\eta = (\xi,\tau)$, we introduce the characteristic set

$$\Sigma_b := \pi\{(y,\eta) : \eta = (\xi,\tau),\ \tau + |\xi|_g^2 = 0,\ -\beta_0 \le \tau \le -\alpha_0\},$$

where $|\xi|_g^2 = \langle \xi, \xi \rangle_g =: \xi_n^2 + r(x, \xi')$ denotes the inner product given by the metric g and where, due to the strict concavity of the boundary we have $\partial_{x_n} r(x', 0, \eta') < 0$.

Definition A.2. We say that a point $\rho_0 = (y_0, \eta_0) \in T_b^*(\partial S \times \mathbb{R}) := T^*(\partial S \times \mathbb{R}) \cup T^*(S \times \mathbb{R})$ *does not* belong to the b-wave front set $WF_b(v)$ of v if there exists a h-pseudodifferential operator of symbol $q(y, \eta, h)$ [or $q(y, \eta', h)$ if $\rho_0 \in T^*(\partial S \times \mathbb{R})$] with compact support in (y, η), elliptic at ρ_0, and a smooth function $\phi \in C_0^\infty$ equal to 1 near y_0, such that for every $\sigma \geq 0$ and $N \geq 0$ there exists $C_N > 0$ such that

$$\|Op_h(q)\phi v\|_{H^\sigma(S\times\mathbb{R})} \leq C_N h^N.$$

We then write $\rho_0 \notin WF_b(v)$.

Proposition A.3 (elliptic regularity [Lebeau 1992, Theorem 3.1]). *Let $q(y, \eta)$ a symbol such that $q = 0$ on a neighborhood of Σ_b. Then for every $\sigma \geq 0$ and $N \geq 0$ there exists $C_N > 0$ such that*

$$\|Op_h(q)v\|_{H^\sigma(S)} \leq C_N h^N.$$

This is proved in [Lebeau 1992] for eigenfunctions of the Laplace operator, but the same arguments apply in this setting. From Proposition A.3 and [Lebeau 1992, Sections 2, 3] we have:

Corollary A.4. *There exists a constant $D > 0$ such that*

$$WF_b(v) \subset \Sigma_b \cap \big\{-\tau \in [\alpha_0, \beta_0], |\xi|_g \leq D\big\}.$$

Corollary A.5 [Lebeau 1992, Chapter 3]. *Let $\varphi \in C_0^\infty(\mathbb{R})$ be equal to 1 near the interval $[-\beta_0, -\alpha_0]$. Then for any bounded interval I and any $N \geq 1$ there exists $C_N > 0$ such that*

$$\big|(1-\varphi)(hD_t)v\big| \leq C_N h^N \quad \textit{for all } t \in I. \tag{A-1}$$

Corollary A.6 (elliptic regularity at "∞"). *Let $\vartheta \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on $\{|\xi|_g \leq D\}$. Then, for all $N \geq 1$, there exists $C_N > 0$ such that*

$$\big|(1-\vartheta)(hD_x)v\big| \leq C_N h^N. \tag{A-2}$$

Proposition A.7. *Let $y_0 \notin \mathrm{pr}_y(WF_b(v))$, where by pr_y we mean the projection on the variable $y = (x, t)$. Then there exists $\phi \in C_0^\infty$ with $\phi = 1$ near y_0 and such that for every $\sigma \geq 0$ and $N \geq 0$, there exists $C_N > 0$ such that*

$$\|\phi v\|_{H^\sigma(S)} \leq C_N h^N.$$

Proof. Let φ, ϑ be as defined in Corollaries A.5 and A.6. Using Proposition A.3 again, we get

$$v(x, t) = \varphi(hD_t)\vartheta(hD_x)v + O(h^\infty). \tag{A-3}$$

Now let $y_0 = (x_0, t_0) \notin \mathrm{pr}_y(WF_b(v))$. It follows that for every $\eta \neq 0$, $(y_0, \eta) \notin WF_b(v)$ and in particular for every $\eta_0 \in \operatorname{supp}\vartheta \times \operatorname{supp}\varphi$ there exists a symbols $q_0(y, \eta, h)$ with compact support in (y, η) near (y_0, η_0) and elliptic at (y_0, η_0), and there exists $\phi_0 \in C_0^\infty$ equal to 1 in a neighborhood U_0 of y_0 such that for every $\sigma \geq 0$ and every $N \geq 0$, there exists $C_N > 0$ such that

$$\|Op_h(q_0)\phi v\|_{H^\sigma(S)} \leq C_N h^N.$$

After shrinking U_0 if necessary, suppose that q_0 is elliptic on $U_0 \times W_0$, where W_0 is an open neighborhood of η_0. Then it follows that on U_0, for every $\sigma \geq 0$ and $N \geq 0$, there exists $C_N > 0$ such that

$$\|\phi v\|_{H^\sigma(U_0)} \leq C_N h^N.$$

Since the set $\operatorname{supp} \vartheta \times \operatorname{supp} \varphi$ is compact there exist η^α, $\alpha \in \{1, \dots, N\}$ for some fixed $N \geq 1$ and for each η^α there exist symbols q_α elliptic on some neighborhoods $U_\alpha \times W_\alpha$ of (y_0, η^α) and smooth functions $\phi_\alpha \in C_0^\infty$ equal to 1 on the neighborhoods U_α of y_0, such that $\operatorname{supp} \vartheta \times \operatorname{supp} \varphi \subset \bigcup_{j=1}^N W_\alpha$. Suppose that $\phi \in C_0^\infty$ is equal to 1 in an open neighborhood of y_0 strictly included in the intersection $\bigcap_{\alpha=1}^N U_\alpha$ (which has nonempty interior) and supported in the compact set $\bigcap_{\alpha=1}^N \operatorname{supp} \phi_\alpha$. Considering a partition of unity associated to $(U_\alpha \times W_\alpha)_\alpha$ and using (A-3) we deduce that ϕ satisfies Proposition A.7. □

Proposition A.8 [Burq 1993, Lemma B.7]. *Let $v(x,t) = e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0$ as before, $v_0 \in L^2(S)$ and let Q be a h-pseudodifferential operator of order 0, $t_0 > 0$ and $\tilde{\psi} \in C_0^\infty((-2t_0, -t_0))$. Let w denote the solution to*

$$\begin{cases} (ih\partial_t + h^2\Delta_g)w = ih\tilde{\psi}(t)Q(v) & \text{on } S \times \mathbb{R}, \\ w|_{\partial S} = 0, \quad w|_{t<-2t_0} = 0. \end{cases} \tag{A-4}$$

If $\rho_0 \in WF_b(w)$ then the broken bicharacteristic starting from ρ_0 has a nonempty intersection with $WF_b(v) \cap \{t \in \operatorname{supp} \tilde{\psi}\}$.

B. *Proof of Lemma 3.4.* In this section (M, Δ_M) denotes either (S, Δ_S) or (Ω, Δ_D), respectively. This notation will be used to refer both domains at the same time. Let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ be such that $\Delta_D \tilde{\chi} = \Delta_S \tilde{\chi}$.

Let $\varphi_0 \in C^\infty(\mathbb{R})$ be supported in the interval $[-4, 4]$ and $\varphi \in C^\infty(\mathbb{R})$ be supported in $[-4, -1] \cup [1, 4]$ such that for all $\xi \in \mathbb{R}$

$$\varphi_0(\xi) + \sum_{k \geq 1} \varphi(2^{-k}\xi) = 1.$$

If $\hat{\Psi}$ denotes the Fourier transform of Ψ, we write it using the preceding sum as

$$\hat{\Psi}(\xi) = \hat{\Psi}(\xi)\Big(\varphi_0(\xi) + \sum_{k \geq 1} \varphi(2^{-k}\xi)\Big)$$

and denote by $\phi_k \in \mathcal{S}(\mathbb{R})$ the functions such that $\hat{\phi}_0(\xi) = \hat{\Psi}(\xi)\varphi_0(\xi)$, $\hat{\phi}_k(\xi) = \hat{\Psi}(\xi)\varphi(2^{-k}\xi)$. We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions. Hence we have

$$\Psi(\lambda) = \sum_{k \in \mathbb{N}} \phi_k(\lambda), \quad \text{where} \quad \|\hat{\phi}_k\|_{L^\infty} = \|\hat{\Psi}(\xi)\varphi(2^{-k}\xi)\|_{L^\infty} \leq C_N 2^{-kN} \quad \text{for all } N \in \mathbb{N}. \tag{B-5}$$

For $k \in \mathbb{N}$, write

$$\phi_k(h\sqrt{-\Delta_M})\tilde{\chi} v_{h,l} = \frac{1}{2\pi} \int_{\operatorname{supp} \hat{\phi}_k} e^{i\xi h\sqrt{-\Delta_M}} \tilde{\chi} v_{h,l} \hat{\phi}_k(\xi)\, d\xi. \tag{B-6}$$

On the support of $\hat{\phi}_k(\xi)$, $|\xi| \simeq 2^k$ and for $k \leq \frac{1}{2}\log_2(1/h)$, for example, we see, by the finite speed of propagation of the wave operator, that on a time interval of size $2^k h \leq h^{1/2}$ we remain in a fixed neighborhood of the boundary of Ω where Δ_D coincides with Δ_S, therefore we can introduce χ_1 equal to 1 on a fixed neighborhood of the support of $\tilde{\chi}$ (independent of k, h) such that, for every $k \le \frac{1}{2}\log_2(1/h)$,

$$\chi_1\phi_k(h\sqrt{-\Delta_S})\tilde{\chi}v_{h,l} = \chi_1\phi_k(h\sqrt{-\Delta_\Omega})\tilde{\chi}v_{h,l}. \tag{B-7}$$

Since $v_{h,l,s} = \chi_1\big(\tilde{\Psi}(-h^2\Delta_D) - \tilde{\Psi}(-h^2\Delta_S)\big)v_{h,l}$ and $v_{h,l} = \tilde{\chi}v_{h,l}$, we obtain, using (B-7)

$$v_{h,l,s} = \chi_1\Bigg(\sum_{k\ge\frac{1}{4}\log_2(1/h)}\big(\phi_k(h\sqrt{-\Delta_\Omega}) - \phi_k(h\sqrt{-\Delta_S})\big)\Bigg)\tilde{\chi}v_{h,l}. \tag{B-8}$$

To estimate the $L^q(I_l^h, L^r(\Omega))$ norm of $v_{h,l,s}$ it will be enough to estimate separately the norms of $\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}$ for $k \ge \frac{1}{4}\log_2(1/h)$ where $(M, \Delta_M) \in \{(\Omega, \Delta_D), (S, \Delta_S)\}$. Using the Cauchy–Schwartz inequality and the Sobolev embeddings gives

$$\begin{aligned}\|\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}\|_{L^q(I_l^h,L^r(\Omega))} &\le Ch^{1/q}\|\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,L^r(\Omega))}\\ &\le Ch^{1/q}\|\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})}(\Omega))}\\ &\le C_N h^{1/q}2^{-kN}\|\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})}(\Omega))} \quad \text{for all } N\in\mathbb{N},\end{aligned} \tag{B-9}$$

where in the last line we used (B-5). We estimate the last term in (B-9) writing the Duhamel formula for $v_{h,l}$ only on Ω using the Equation (3-7), since in this case the smoothing effect yields (see [Staffilani and Tataru 2002], [Burq et al. 2004a], or the dual estimates of (3-25) in Proposition 3.5)

$$\|\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})}(\Omega))} \le C\|V_{h,l}\|_{L^2(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))}. \tag{B-10}$$

Since we consider here only large values $k \ge \frac{1}{4}\log_2(1/h)$, each 2^{-k} is bounded by $h^{1/4}$, therefore, after summing over k we obtain

$$\|v_{h,l,s}\|_{L^q(I_l^h,L^r(\Omega))} \le C_N h^{1/q+N/4}\|V_{h,l}\|_{L^2(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))} \quad \text{for all } N\in\mathbb{N}. \tag{B-11}$$

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