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We investigate and prove analytic stability for strongly coupled critical elliptic systems in the inhomogeneous context of a compact Riemannian manifold.

Coupled systems of nonlinear Schrödinger equations are now a part of several important branches of mathematical physics. They appear in the Hartree–Fock theory for Bose–Einstein double condensates, in fiber-optic theory, in the theory of Langmuir waves in plasma physics, and in the behavior of deep water waves and freak waves in the ocean. A general reference book on such systems and their role in physics has been written by Ablowitz et al. [2004]. We focus here on coupled Gross–Pitaevskii type equations. These systems of equations are strongly related to two branches of mathematical physics. They arise [Burke et al. 1997] in the Hartree–Fock theory for double condensates, which are binary mixtures of Bose–Einstein condensates in two different hyperfine states. They also arise in the study of incoherent solitons in nonlinear optics, as describe in [Akhmediev and Ankiewicz 1998; Christodoulides et al. 1997; Hioe 1999; Hioe and Salter 2002; Kanna and Lakshmanan 2001]. Looking for standing wave solutions for these time evolution systems gives rise to their elliptic analogues that we investigate here. We consider these elliptic systems of equations in arbitrary dimensions $n \geq 3$, in the critical energy regime, and in a fully inhomogeneous medium that we model by an arbitrary compact Riemannian manifold, thus breaking the various symmetries that these systems enjoy in the Euclidean setting.

In what follows we let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. For $p \geq 1$ an integer, we let $M_p^s(\mathbb{R})$ denote the vector space of symmetrical $p \times p$ real matrices, and A be a C^1 map from M to $M_p^s(\mathbb{R})$. We can write $A = (A_{ij})_{i,j}$, where the A_{ij}'s are C^1 real valued functions in M. Let $\Delta_g = -\operatorname{div}_g \nabla$ be the Laplace–Beltrami operator on M. Let also $H^1(M)$ be the Sobolev space of functions in $L^2(M)$ with one derivative in $L^2(M)$. A p-map $\mathcal{U} = (u_1, \ldots, u_p)$ from M to $\mathbb{R}^p$ is said to be nonnegative if $u_i \geq 0$ for all i. The coupled system of nonlinear Schrödinger equations we consider here is written as

$$\Delta_g u_i + \sum_{j=1}^p A_{ij}(x) u_j = |\mathcal{U}|^{2^\star - 2} u_i \tag{0-1}$$

in M for all i, where $|\mathcal{U}|^2 = \sum_{i=1}^p u_i^2$, and $2^\star = 2n/(n-2)$ is the critical Sobolev exponent for the embeddings of the Sobolev space $H^1(M)$ into Lebesgue's spaces. The systems (0-1) are weakly coupled by the linear matrix A, and strongly coupled by the Gross–Pitaevskii type nonlinearity in the right hand side of (0-1). Besides, (0-1) is critical for Sobolev embeddings. From the viewpoint of conformal

geometry, our systems are pure extensions of Yamabe type equations in the strongly coupled regime. As a by-product (0-1) inherits a conformal structure.

Our aim in this paper is to discuss stability for systems like (0-1). Contrary to time evolution equations, where perturbations of the initial data together with perturbations of the equations are used to measure stability, stability for elliptic equations has to do solely with perturbations of the equations. In the framework of systems such as (0-1), stability is naturally measured with respect to perturbations of the map A. In what follows, a system like (0-1) is said to be analytically stable if for any sequence $(A_\alpha)_\alpha$ of maps from M to $M_p^s(\mathbb{R})$, $\alpha \in \mathbb{N}$, and for any bounded sequence in H^1 of nonnegative nontrivial solutions $\mathcal{U}_\alpha$ of the associated systems (0-1), if $A_\alpha \to A$ in C^1 as $\alpha \to +\infty$, then, up to a subsequence, $\mathcal{U}_\alpha \to \mathcal{U}$ in C^2 as $\alpha \to +\infty$ for some nonnegative nontrivial solution $\mathcal{U}$ of (0-1). When the strong convergence in C^2 is replaced by a weak convergence $\mathcal{U}_\alpha \rightharpoonup \mathcal{U}$ in H^1, the system (0-1) is said to be weakly stable. We refer to Section 1 for more precise definitions.

Before stating our theorem we need to introduce two assumptions. Let $\mathbf{\Delta}_g$ be the Laplace–Beltrami operator acting on p-maps by acting on each of the components of the map, and let $\mathrm{Vect}_+(\mathbb{R}^p)$ be the set of vectors in $\mathbb{R}^p$ with nonnegative components. The first assumption we may impose is

$$\mathrm{Ker}(\mathbf{\Delta}_g + A) \cap L^2(M, \mathrm{Vect}_+(\mathbb{R}^p)) = \{0\}, \tag{H}$$

where $\mathrm{Ker}(\mathbf{\Delta}_g + A)$ is the kernel of $\mathbf{\Delta}_g + A$, and $L^2(M, \mathrm{Vect}_+(\mathbb{R}^p))$ stands for the set of L^2 maps from M to $\mathrm{Vect}_+(\mathbb{R}^p)$. In order to introduce our second assumption we let $A_n = A_n(A)$ be given by

$$A_n = A - \frac{n-2}{4(n-1)} S_g \, \mathrm{Id}_p, \tag{0-2}$$

where S_g is the scalar curvature of g, and Id_p is the identity $p \times p$ matrix. For $x \in M$, let also $\mathrm{Is}_{A_n(x)}$ be the set consisting of the isotropic vectors for $A_n(x)$, namely of the vectors $X \in \mathbb{R}^p$ which are such that $\langle A_n(x).X, X \rangle_{\mathbb{R}^p} = 0$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}^p}$ is the Euclidean scalar product in $\mathbb{R}^p$. The second assumption we introduce is that for any $x \in M$, $A_n(x)$ should not possess stable subspaces with an orthonormal basis consisting of isotropic nonnegative vectors. More precisely, it is this:

> For any $x \in M$ and any $k \in \{1, \ldots, p\}$, there does not exist an orthonormal family $(e_1, \ldots, e_k)$ of vectors in $\mathrm{Is}_{A_n(x)} \cap \mathrm{Vect}_+(\mathbb{R}^p)$ such that $A_n(x)V \subset V$, where V is the k-dimensional subspace of $\mathbb{R}^p$ with basis $(e_1, \ldots, e_k)$. $\tag{H$'$}$

The case $k = 1$ in (H$'$) reduces to the nonexistence of a nontrivial vector in $\mathrm{Vect}_+(\mathbb{R}^p) \cap \mathrm{Ker}\, A_n(x)$, where $\mathrm{Ker}\, A_n(x)$ is the kernel of $A_n(x)$. An assumption like (H$'$) is automatically satisfied in several simple situations. This is the case if we prevent the existence of isotropic vectors for A_n. In particular, (H$'$) holds true if $A_n(x) > 0$ or $A_n(x) < 0$ for all x in the sense of bilinear forms. Clearly there are other cases where (H$'$) holds true. Assumption (H) is analytic in nature. Assumption (H$'$) is algebraic in nature and related to the underlying geometric conformal structure of the equations. Our main result, establishing analytic stability for (0-1), is stated as follows.

Theorem 0.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 4$ and $p \geq 1$ be an integer. For any C^1-map $A : M \to M_p^s(\mathbb{R})$ satisfying (H) and (H$'$), the system (0-1) is analytically stable when $n \neq 6$, and weakly stable when $n = 6$. Besides, there are examples of six-dimensional manifolds and C^1-maps A satisfying (H) and (H$'$) for which (0-1) is analytically unstable.*

A closely related notion to stability is that of compactness. A system like (0-1) is said to be compact if bounded sequences in H^1 of nonnegative solutions of (0-1) converge, up to a subsequence, in the C^2-topology. As is easily seen, analytic stability implies compactness. In particular, as a direct consequence of the analytic stability part in our theorem, we get that systems like (0-1) are compact when $n \neq 6$ as soon as (H) and (H′) are satisfied. Assumptions (H) and (H′) in Theorem 0.1 are sharp, as discussed in Section 1 below.

Most of the asymptotic analysis developed in this paper carries over to the $n = 3$ case. However, the concluding argument needs to be changed when $n = 3$. In this dimension the mass of the Green's expansion of the Schrödinger operator $\Delta_g + A$ leads over A_n. We can conclude when the mass is positive. The analysis is developed in [Druet et al. 2009].

The paper is organized as follows. We discuss general properties of stability and compactness in Section 1. We prove the $n = 6$ part of Theorem 0.1 in Section 2. We provide a complete classification of H^1-nonnegative solutions of the strongly coupled critical limit Euclidean system $\Delta u_i = |\mathcal{U}|^{2^\star - 2} u_i$, $i = 1, \ldots, p$, in Section 3. We prove Theorem 0.1 in its $n \neq 6$ part in Sections 4 to 10. In the process we establish in Sections 5, 6, and 8 the full C^0-theory for the blow-up of arbitrary sequences of solutions of strongly coupled systems like (0-1).

1. General considerations on stability and compactness

We start with the precise definition of elliptic stability we use for our systems (0-1). As already mentioned stability is here measured with respect to perturbations of the parameter A in (0-1). In doing so we preserve the conformal structure of the equation. Historically speaking such type of perturbations were first considered in the early work of Aubin [1976] on the Yamabe equation. Given $(A_\alpha)_\alpha$ a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$, with $A_\alpha = (A_{ij}^\alpha)_{i,j}$ for all α integer, we consider the systems

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij}^\alpha(x) u_j = |\mathcal{U}|^{2^\star - 2} u_i. \tag{1-1}$$

A sequence $(\mathcal{U}_\alpha)_\alpha$ of C^2 maps from M to $\mathbb{R}^p$ is said to be a sequence of nonnegative solutions of (1-1) if for any $\alpha \in \mathbb{N}$, $\mathcal{U}_\alpha = (u_1^\alpha, \ldots, u_p^\alpha)$ solves (0-1) and $u_i^\alpha \geq 0$ for all i. The sequence is said to be bounded in $H^1(M)$, or to have finite energy, if its components u_α^i are all bounded in $H^1(M)$ with respect to α. Given $\Lambda > 0$, we define the slice $\mathcal{S}_A^\Lambda$ to be the set of p-maps $\mathcal{U} \in H^1$ such that $\mathcal{U}$ solves (0-1), $\mathcal{U}$ is nonnegative and the H^1-norm of $\mathcal{U}$ is less than or equal to Λ. By standard regularity, adapting classical arguments from Trudinger [1968], weak solutions in H^1 of systems like (0-1) are always of class C^2. In particular, $\mathcal{S}_A^\Lambda \subset C^2$ for all $\Lambda > 0$. For $X, Y \subset C^2$ we let $d_{C^2}^\hookrightarrow(X; Y)$ be the C^2-pointed distance from X to Y defined by

$$d_{C^2}^\hookrightarrow(X; Y) = \sup_{\mathcal{U} \in X} \inf_{\mathcal{V} \in Y} \|\mathcal{V} - \mathcal{U}\|_{C^2}, \tag{1-2}$$

where $\|\mathcal{V} - \mathcal{U}\|_{C^2} = \sum_i \|v_i - u_i\|_{C^2}$ and the u_i's and v_i's are the components of $\mathcal{U}$ and $\mathcal{V}$. Stability in the elliptic regime is defined in Definition 1.1 below. The C^1 convergence $A_\alpha \to A$ in Definition 1.1 refers to the C^1 convergence of the components A_{ij}^α of A_α to the components A_{ij} of A. Similarly, the C^2 convergences, and the weak convergences in H^1, of the $\mathcal{U}_\alpha$'s in Definition 1.1 refer to the convergences of the components of the maps.

Definition 1.1. Let (M, g) be a smooth compact Riemannian manifold, let $p \geq 1$ be an integer, and let $A : M \to M_p^s(\mathbb{R})$ be a C^1 map. The system (0-1) is said to be

(i) analytically stable if for any sequence $(A_\alpha)_\alpha$ of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$, and for any bounded sequence $(\mathcal{U}_\alpha)_\alpha$ in $H^1(M)$ of nonnegative nontrivial solutions of (1-1), there exists a nonnegative nontrivial solution $\mathcal{U}$ of (0-1) such that, up to a subsequence, the $\mathcal{U}_\alpha$'s converge strongly to $\mathcal{U}$ in $C^2(M)$ as $\alpha \to +\infty$, and

(ii) weakly stable if for any sequence $(A_\alpha)_\alpha$ of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ and for any bounded sequence $(\mathcal{U}_\alpha)_\alpha$ in $H^1(M)$ of nonnegative nontrivial solutions of (1-1), there exists a nonnegative nontrivial solution $\mathcal{U}$ of (0-1) such that, up to a subsequence, the $\mathcal{U}_\alpha$'s converge weakly to $\mathcal{U}$ in $H^1(M)$ as $\alpha \to +\infty$.

The system is said to be geometrically stable, if the slices $\mathcal{S}_A^\Lambda$ are stable for all $\Lambda > 0$, where $\mathcal{S}_A^\Lambda$ is said to be stable, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any C^1 map A' from M to $M_p^s(\mathbb{R})$, we have $d_{C^2}^{\hookrightarrow}(\mathcal{S}_{A'}^\Lambda; \mathcal{S}_A^\Lambda) < \varepsilon$ when $\|A' - A\|_{C^1} < \delta$.

As already mentioned, a classical notion in the study of critical elliptic equations is that of compactness. A system like (0-1) is said to be *compact* if any bounded sequence $(\mathcal{U}_\alpha)_\alpha$ in $H^1(M)$ of nonnegative nontrivial solutions of (0-1) converges in $C^2(M)$ as $\alpha \to +\infty$ to a nonnegative nontrivial solution $\mathcal{U}$ of (0-1). This corresponds to the particular situation where $A_\alpha = A$ for all α in (i). Analytic stability as defined in (i) implies weak stability, geometric stability, and compactness. More precisely:

Proposition 1.2. *Assume* (H). *If the system* (0-1) *is analytically stable, it is weakly stable, geometrically stable, and compact. A compact system is analytically stable if and only if it is geometrically stable.*

Proof. It is obvious that analytic stability implies weak stability, geometric stability, and compactness. The only assertion, which deserves to be proved, is that a compact geometrically stable system like (0-1) is analytically stable. Let $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$, and let $(\mathcal{U}_\alpha)_\alpha$ be a bounded sequence in H^1 of nonnegative nontrivial solutions of (1-1). Since (0-1) is geometrically stable there exists $(\mathcal{V}_\alpha)_\alpha$, a bounded sequence in H^1 of nonnegative solutions of (0-1), such that, up to a subsequence, $\mathcal{U}_\alpha - \mathcal{V}_\alpha$ converges to zero in C^2 as $\alpha \to +\infty$. Since (0-1) is compact, up to a subsequence, $\mathcal{V}_\alpha \to \mathcal{V}$ in C^2 as $\alpha \to +\infty$, where $\mathcal{V}$ is a nonnegative solution of (0-1). In particular, up to a subsequence, $\mathcal{U}_\alpha \to \mathcal{V}$ in C^2 as $\alpha \to +\infty$. It remains to prove that $\mathcal{V}$ is nontrivial, and this is given by Lemma 1.3 below. Proposition 1.2 is proved. $\square$

The following lemma, which we derive as a direct consequence of (H), was used in the proof of Proposition 1.2. By standard elliptic theory, moreover, when A satisfies (H), we have $\mathcal{U}_\alpha \not\to 0$ in H^1 as $\alpha \to +\infty$.

Lemma 1.3. *Let (M, g) be a smooth compact Riemannian manifold, let $p \geq 1$ be an integer, and let $A : M \to M_p^s(\mathbb{R})$ be a C^1 map satisfying* (H). *Let $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$, and let $(\mathcal{U}_\alpha)_\alpha$ be a bounded sequence in H^1 of nonnegative nontrivial solutions of (1-1). Then $\mathcal{U}_\alpha \not\to 0$ in $L^\infty(M)$ as $\alpha \to +\infty$.*

Proof. By contradiction we assume that there exists $(\mathcal{U}_\alpha)_\alpha$, a bounded sequence in H^1 of nonnegative nontrivial solutions of (1-1), such that $\max_M |\mathcal{U}_\alpha|_\Sigma \to 0$ as $\alpha \to +\infty$, where $|\mathcal{U}_\alpha|_\Sigma = \sum_i u_{i,\alpha}$ is the sum of the components of the $\mathcal{U}_\alpha$'s. Let $\varepsilon_\alpha = |\mathcal{U}_\alpha|_\Sigma$ and define $v_{i,\alpha}$ by $v_{i,\alpha} = \varepsilon_\alpha^{-1} u_{i,\alpha}$ for all i and α. Then

$$\Delta_g v_{i,\alpha} + \sum_{j=1}^{p} A_{ij}^{\alpha}(x) v_{j,\alpha} = \varepsilon_{\alpha}^{4/(n-2)} |\mathcal{V}_{\alpha}|^{2^\star - 2} v_{i,\alpha} \tag{1-3}$$

for all i and α, where $\mathcal{V}_{\alpha}$ is the p-map whose components are the $v_{i,\alpha}$'s for $i = 1, \dots, p$. By construction the $v_{i,\alpha}$'s are bounded in $L^{\infty}(M)$. By standard elliptic theory it follows that, up to a subsequence, they converge in $C^2(M)$ to v_i's as $\alpha \to +\infty$. Let $\mathcal{V}$ be the p-map with components v_i for $i = 1, \dots, p$. By construction $\mathcal{V}$ is nonnegative and nontrivial, since there is one point, where $|\mathcal{V}|_{\Sigma}$ equals one. Letting $\alpha \to +\infty$ in (1-3) it follows that $\mathcal{V} \in \operatorname{Ker}(\Delta_g + A)$, and we get a contradiction with (H). □

Proposition 1.2 leaves open the question of whether or not there exist geometrically stable noncompact systems like (0-1). However, we can have noncompact systems with geometrically stable specific slices as discussed below. The most well-known example of a noncompact critical system like (0-1) is given by the Yamabe equation on the sphere. The Yamabe equation on the n-sphere possesses a $(n+1)$-parameter noncompact family of solutions and it turns out that it is also geometrically unstable. This is a direct consequence of the constructions in [Druet and Hebey 2005a], where arbitrarily high energy solutions of approximated equations are constructed, together with the property that all nonnegative nontrivial solutions of the Yamabe equation on the sphere have the same energy. On the other hand, the first blow-up slice for this equation is geometrically H^1-stable in the sense of Definition 1.1 when we replace the C^2-pointed distance and the C^2-norm in (1-2) by a H^1-pointed distance and a H^1-norm, where the first blow-up slice is given by $\Lambda = K_n^2$, and K_n is as in (3-8). This geometric H^1-stability of the first blow-up slice follows from H^1-decompositions as in Proposition 4.2. As a direct consequence, noncompact equations may have stable slices.

In the subcritical regime, compactness goes back to [Gidas and Spruck 1981]. In the more involved critical regime, it goes back to Schoen's conjecture [Schoen 1989; 1991] that compactness holds true for the geometric Yamabe equation as soon as the background manifold is distinct from the sphere. His conjecture has been a source of motivations for several years. The conjecture was proved to be true for conformally flat manifolds by Schoen [1989; 1991]. The nonconformally flat case turned out to be more intricate. The case of low-dimensional manifolds was recently addressed in [Druet 2004; Marques 2005; Li and Zhu 1999; Li and Zhang 2004; 2005], and compactness up to dimension 24 was finally proved recently [Khuri et al. 2009]. On the other hand, Brendle [2008a] and Brendle and Marques [2009] exhibited counterexamples to the conjecture in dimensions $n \geq 25$. For any $n \geq 25$ they constructed examples of nonconformally flat n-manifolds with the striking property that their associated Yamabe equations possess sequences of solutions with minimal type energy and unbounded L^{∞}-norms. In particular, they proved the very surprising result that the compactness conjecture is false for nonconformally flat manifolds in any dimension $n \geq 25$. A very interesting survey on the subject is [Brendle 2008b]. We refer also to [Druet and Hebey 2005b].

An easy remark is that if u is a solution of a scalar Yamabe type equation with linear term h, that is, an equation of the form

$$\Delta_g u + h(x) u = u^{2^\star - 1}, \tag{1-4}$$

then $\mathcal{U} = \left(\frac{1}{\sqrt{p}} u, \dots, \frac{1}{\sqrt{p}} u \right)$ is a solution of (0-1) when $A_{ij} = h a_{ij}$ for all i, j, and $\sum_{j=1}^{p} a_{ij} = 1$ for all i. In what follows we let $(a_{ij})_{i,j}$ be a symmetrical matrix of C^1 functions $a_{ij} : M \to \mathbb{R}$ such that $\sum_{j=1}^{p} a_{ij}(x) = 1$ for all $i = 1, \dots, p$ and all $x \in M$. A possible choice is $a_{ij} = \delta_{ij}$ for all i, j. Then we

define $A(g)$ and $A'(g)$ to be the C^1 maps from M to $M_p^s(\mathbb{R})$ given by

$$A(g)_{ij} = \frac{n-2}{4(n-1)} S_g a_{ij} \quad \text{and} \quad A'(g)_{ij} = \frac{n-2}{4(n-1)} (\max_M S_g) a_{ij}$$

for all $i, j = 1, \ldots, p$, where S_g is the scalar curvature of g. By combining results in [Brendle 2008a; Brendle and Marques 2009], where noncompactness of the Yamabe equation in the nonconformally flat case is investigated, and those in [Druet and Hebey 2005a; Hebey and Vaugon 2001], where unstability of Yamabe type equations in the conformally flat case is investigated, we obtain the following theorem, in view of the remark above.

Theorem 1.4. *The system* (0-1) *associated with* $A(g)$ *is analytically unstable when posed on spherical spaces forms in any dimension* $n \geq 6$, *and even noncompact when posed on the sphere in any dimension* $n \geq 3$. *For any conformally flat manifold* (M, g) *of dimension* $n \geq 4$ *there exists a conformal metric* $\tilde{g}$ *to* g *of nonconstant scalar curvature having one and only one maximum point such that the system* (0-1) *associated with* $A'(\tilde{g})$ *is analytically unstable. In any dimension* $n \geq 25$ *there are examples of nonconformally flat manifolds such that the system* (0-1) *associated with* $A(g)$ *is noncompact, and thus also analytically unstable.*

The examples in Theorem 1.4 do not satisfy (H′). This can be checked by noting that $(1, \ldots, 1) \in$ Ker $A_n(x)$ for all x, where A_n is as in (0-2). Theorem 0.1 and Theorem 1.4 complement each other. As a remark, the Yamabe equation on quotients of the sphere is obviously compact since it possesses a unique solution. In particular, there are compact equations which are neither analytically nor geometrically stable. Compactness does not imply stability. We concentrate in the rest of this section on the subcritical regime for systems like (0-1) and prove that analytic stability holds true in the subcritical regime without assuming (H′). Let $q \in (2, 2^\star)$ and let us consider the subcritical system

$$\Delta_g u_i + \sum_{j=1}^p A_{ij}(x) u_j = |\mathcal{U}|^{q-2} u_i \tag{1-5}$$

in M for all i, where $A = (A_{ij})_{i,j}$ is a C^1 map from M to $M_p^s(\mathbb{R})$. We define the notions of analytic stability, weak stability, and geometric stability for (1-5) as in Definition 1.1.

Proposition 1.5. *Let* (M, g) *be a smooth compact Riemannian manifold of dimension* $n \geq 3$, $p \geq 1$ *be an integer, and* $A : M \to M_p^s(\mathbb{R})$ *be a* C^1 *map satisfying* (H). *For any* $q \in (2, 2^\star)$ *the subcritical system* (1-5) *is analytically stable.*

Proof. Let $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$, and let $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in H^1 of nonnegative nontrivial solutions of

$$\Delta_g u_{i,\alpha} + \sum_{j=1}^p A_{ij}^\alpha(x) u_{j,\alpha} = |\mathcal{U}_\alpha|^{q-2} u_{i,\alpha} \tag{1-6}$$

for all i and all α. We aim in proving that a subsequence of $(\mathcal{U}_\alpha)_\alpha$ converges in C^2 to a nonnegative nontrivial solution of (1-6). The nontriviality of any strong limit follows from (H) mimicking the proof of Lemma 1.3. Then, as is easily checked, it suffices to prove that the $\mathcal{U}_\alpha$'s are L^∞-bounded in M. By contradiction we assume that there exists a sequence $(x_\alpha)_\alpha$ of points, where the $|\mathcal{U}_\alpha|$'s are maximum

such that, up to a subsequence, $|\mathcal{U}_\alpha(x_\alpha)| \to +\infty$ as $\alpha \to +\infty$. Let $\mu_\alpha = |\mathcal{U}_\alpha(x_\alpha)|^{-(q-2)/2}$. Then $\mu_\alpha \to 0$ as $\alpha \to +\infty$. Let $\tilde{\mathcal{U}}_\alpha$ be given for $x \in \mathbb{R}^n$ by

$$\tilde{\mathcal{U}}_\alpha(x) = \mu_\alpha^{2/(q-2)} \mathcal{U}_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right)$$

and let g_α be the metric given by $g_\alpha(x) = (\exp_{x_\alpha}^\star g)(\mu_\alpha x)$. We have $g_\alpha \to \xi$ in $C_{\text{loc}}^2(\mathbb{R}^n)$ as $\alpha \to +\infty$, where ξ is the Euclidean metric. Noting that

$$\Delta_{g_\alpha} \tilde{u}_{i,\alpha} + \mu_\alpha^2 \sum_{j=1}^p A_{ij}^\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right)\tilde{u}_{j,\alpha} = |\tilde{\mathcal{U}}_\alpha|^{q-2}\tilde{u}_{i,\alpha}$$

for all i and α, since $|\tilde{\mathcal{U}}_\alpha| \le 1$ for all α by construction, it follows from standard elliptic theory that there exists $\tilde{\mathcal{U}} \in C^2(\mathbb{R}^n)$ such that, up to a subsequence, $\tilde{\mathcal{U}}_\alpha \to \tilde{\mathcal{U}}$ in $C_{\text{loc}}^2(\mathbb{R}^n)$. We have $|\tilde{\mathcal{U}}_\alpha(0)| = 1$ for all α. Hence, $|\tilde{\mathcal{U}}(0)| = 1$. Moreover, for any $R > 0$, and for α sufficiently large,

$$\int_{B_0(R)} |\tilde{\mathcal{U}}|^q dx \le C \int_{B_0(R)} |\tilde{\mathcal{U}}_\alpha|^q dv_{g_\alpha}$$

$$\le C \int_{B_0(1/\mu_\alpha)} |\tilde{\mathcal{U}}_\alpha|^q dv_{g_\alpha} = C\mu_\alpha^{2q/(q-2)-n} \int_{B_{x_\alpha}(1)} |\mathcal{U}_\alpha|^q dv_g \le C\mu_\alpha^{2q/(q-2)-n},$$

since the $\mathcal{U}_\alpha$'s have bounded energy. Noting that $2q/(q-2) > n$ as soon as $q < 2^\star$ and letting $\alpha \to +\infty$ in the inequality above, we get $\int_{B_0(R)} |\tilde{\mathcal{U}}|^q dx = 0$. This is in contradiction with $|\tilde{\mathcal{U}}(0)| = 1$. The proposition is proved. □

Analytic stability for critical equations like (1-4) has been investigated in [Druet 2003]. The case $p = 1$ in Theorem 0.1, in its $n \ne 6$ part and when considering $C^{2,\theta}$-perturbations of h, was proved in the same paper. The proof we propose here extends to the case of systems, allows us to consider $C^{0,\eta}$-perturbations of h, see the remark at the end of Section 10, and is more direct. At the time of [Druet 2003], analytic stability was still referred to as compactness. The confusion in the terminology has been the source of several misunderstandings.

2. The six-dimensional case

We discuss and prove the six-dimensional last assertion in Theorem 0.1 concerning the existence of systems like (0-1) in dimension $n = 6$, which satisfy (H) and (H′), but which, contrary to what happens when $n \ne 6$, are not analytically stable. We restrict ourselves to a very explicit construction in the case of the unit sphere (S^6, g_0). A more general discussion could have been developed. We let $(a_{ij})_{i,j}$ be a symmetrical matrix of C^1 functions $a_{ij} : S^6 \to \mathbb{R}$ such that $\sum_{j=1}^p a_{ij}(x) = 1$ for all $i = 1, \ldots, p$ and all $x \in S^6$. If $h : S^6 \to \mathbb{R}$ is of class C^1, we define $A(h)$ to be the C^1 map from S^6 to $M_p^s(\mathbb{R})$ with components $A(h)_{ij}$ given by $A(h)_{ij} = ha_{ij}$ for all $i, j = 1, \ldots, p$. When $n = 6$, we have $2^\star = 3$. For the unit sphere (S^6, g_0), we also have

$$\frac{n-2}{4(n-1)} S_{g_0} = 6.$$

Proposition 2.1. *Let (S^6, g_0) be the unit six-dimensional sphere in $\mathbb{R}^7$. There exists $h : S^6 \to \mathbb{R}$, $h > 6$ everywhere and of class C^1, such that the system (0-1) associated with $A = A(h)$ is analytically unstable.*

Proof. We fix $x_0 \in S^6$ and let r be the distance to x_0. We let also θ be given by $\theta = \cos r$. First we claim that there exist smooth positive functions h and u in S^6, which we write into the form $h(x) = \hat{h}(\theta)$ and $u(x) = \hat{u}(\theta)$, such that

$$\Delta_{g_0} u + hu = u^2 \quad \text{and} \quad h > 6 \tag{2-1}$$

in S^6, and such that

$$\hat{h}(1) = 3\hat{u}(1), \quad \hat{u}(1) = 6 \quad \text{and} \quad \hat{h}'(1) = 2\hat{u}'(1). \tag{2-2}$$

To prove the claim we let $\hat{u}$ be given by

$$\hat{u}(\theta) = 6\big(1 - 2(\theta-1) + 3(\theta-1)^2\big). \tag{2-3}$$

Clearly, $\hat{u}(1) = 6$ and $\hat{u}'(1) = -12$. Since $\Delta_{g_0}\theta = 6\theta$ and $|\nabla\theta|^2 = 1 - \theta^2$, we get

$$\tfrac{1}{6}\Delta_{g_0} u = 6(7\theta^2 - 8\theta - 1).$$

In particular, the first equation in (2-1) is satisfied if we let $\hat{h}$ be given by

$$\tfrac{1}{6}\hat{h}(\theta) = 3\theta^2 - 8\theta + 6 - \frac{7\theta^2 - 8\theta - 1}{3\theta^2 - 8\theta + 6}. \tag{2-4}$$

As is easily checked from (2-4), $\hat{h}(1) = 3\hat{u}(1)$ and $\hat{h}'(1) = 2\hat{u}'(1)$. In particular, (2-2) holds true. Noting that $\hat{h} > 6$ for all $\theta \in [-1, +1]$, we get two explicit smooth positive functions h and u in S^6, given by (2-3) and (2-4), such that (2-1) and (2-2) hold true. This proves the above claim. Now, for $\beta > 1$, we define B_β by $B_\beta(x) = \hat{B}_\beta(\theta)$, where

$$\hat{B}_\beta(\theta) = 6(\beta^2 - 1)(\beta - \theta)^{-2}.$$

We have $\Delta_{g_0} B_\beta + 6B_\beta = B_\beta^2$ in S^6. Let

$$u_\beta = u + B_\beta \tag{2-5}$$

and $\hat{u}_\beta = \hat{u} + \hat{B}_\beta$, where u and $\hat{u}$ are as in (2-1) and (2-2). As is easily checked from (2-1) and the equation satisfied by B_β, we have

$$\Delta_{g_0} u_\beta + h_\beta u_\beta = u_\beta^2 \tag{2-6}$$

in S^6 for all $\beta > 1$, where $h_\beta = \hat{h}_\beta(\theta)$ is given by

$$\hat{h}_\beta = \hat{h} - \frac{(12\hat{u} + 6 - \hat{h})\hat{B}_\beta}{\hat{u} + \hat{B}_\beta}. \tag{2-7}$$

Noting, thanks to (2-2), that $h_\beta \to h$ in $C^0_{\text{loc}}(S^6)$ as $\beta \to 1$, while $\hat{h}'_\beta \to \hat{h}'$ in $L^\infty([-1, +1])$ as $\beta \to 1$, we conclude that $h_\beta \to h$ in $C^1(S^6)$. Now we let $(\beta_\alpha)_\alpha$ be a sequence of positive real numbers such that $\beta_\alpha > 1$ for all α and $\beta_\alpha \to 1$ as $\alpha \to +\infty$. We let $\mathcal{U}_\alpha = p^{-1/2}(u_{\beta_\alpha}, \ldots, u_{\beta_\alpha})$, $A_\alpha = A(h_{\beta_\alpha})$, and $A = A(h)$ where u_β is given by (2-5), $h_\beta = \hat{h}_\beta(\theta)$ is given by (2-7), and $h = \hat{h}(\theta)$ is given by (2-4). The $\mathcal{U}_\alpha$'s solve (1-1), they have bounded energy, and $A_\alpha \to A$ in C^1. Noting that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$, this proves the proposition. $\qquad \square$

It is easily checked that $A = A(h)$ satisfies (H). If $\mathcal{U} \in L^2(M, \text{Vect}_+(\mathbb{R}^p))$ is in the kernel of the vector Schrödinger operator associated with $A(h)$, we conclude by summing over the components that $|\mathcal{U}|_\Sigma = \sum_{i=1}^p u_i$ belongs to $\text{Ker}(\Delta_{g_0} + h)$. This is impossible unless $\mathcal{U} \equiv 0$ since $h > 0$. It is also easily seen that, at least for small perturbations a_{ij} of δ_{ij}, the map $A = A(h)$ satisfies (H′).

3. The limit system

Of importance in blow-up theory, when discussing critical equations, is the classification of the solutions of the critical limit Euclidean system we get by blowing up the original equations. In our case, we need to classify the $\dot{H}^1$-nonnegative solutions of the limit system

$$\Delta u_i = |\mathcal{U}|^{2^\star - 2} u_i, \tag{3-1}$$

where $|\mathcal{U}|^2 = \sum_{i=1}^{p} u_i^2$, and $\Delta = -\sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Euclidean Laplace–Beltrami operator . Depending on the context, we let $\dot{H}^1(\mathbb{R}^n)$ be the homogeneous Sobolev space defined as the completion of functions with compact supports, or of p-maps with compact supports, with respect to the L^2-norm of their gradient. The classification result we prove here is stated as follows.

Proposition 3.1. *Let $p \geq 1$ and $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of (3-1). Then there exist $a \in \mathbb{R}^n$, $\lambda > 0$, and $\Lambda \in S_+^{p-1}$, such that*

$$\mathcal{U}(x) = \left(\frac{\lambda}{\lambda^2 + \dfrac{|x-a|^2}{n(n-2)}} \right)^{(n-2)/2} \Lambda \tag{3-2}$$

for all $x \in \mathbb{R}^n$, where S_+^{p-1} consists of those elements $(\Lambda_1, \ldots, \Lambda_p)$ in the unit sphere S^{p-1} (in $\mathbb{R}^p$) that satisfy $\Lambda_i \geq 0$ for all i.

We prove Proposition 3.1 in several steps. Let $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of (3-1). Regularity theory and the maximum principle apply to (3-1). In particular, $\mathcal{U}$ is necessarily smooth with the property that for any i, either $u_i \equiv 0$ or $u_i > 0$ in $\mathbb{R}^n$. We may therefore assume that there exists $p' \leq p$ such that $u_i > 0$ in $\mathbb{R}^n$ for all $i = 1, \ldots, p'$. A first step in the proof of Proposition 3.1 is as follows.

Step 1. *Let $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of (3-1) such that $u_i > 0$ in $\mathbb{R}^n$ for all $i = 1, \ldots, p'$, $p' \leq p$. Then, for any $R > 0$,*

$$\min_{\partial B_0(R)} \frac{u_i}{u_j} \leq \frac{u_i}{u_j} \leq \max_{\partial B_0(R)} \frac{u_i}{u_j} \tag{3-3}$$

in $B_0(R)$ for all $i, j \in \{1, \ldots, p'\}$.

Proof of Step 1. By (3-1),

$$\Delta\left(\frac{u_i}{u_j}\right) = 2\left(\nabla\left(\frac{u_i}{u_j}\right), \nabla u_j\right) u_j^{-1}.$$

Applying the maximum principle we get (3-3). $\qquad\square$

The main objective now is to prove that

$$\min_{\partial B_0(R)} \frac{u_i}{u_j} \to \lambda_{i,j} \quad \text{and} \quad \max_{\partial B_0(R)} \frac{u_i}{u_j} \to \lambda_{i,j} \tag{3-4}$$

as $R \to +\infty$ for some $\lambda_{i,j} > 0$ so that, together with Step 1, we obtain $u_i = \lambda_{i,j} u_j$ in $\mathbb{R}^n$ for all $i, j = 1, \ldots p'$. To prove (3-4) we first observe that

$$|x|^{(n-2)/2} u_i(x) \to 0 \tag{3-5}$$

as $|x| \to +\infty$ for all $i \in \{1, \ldots, p'\}$. Indeed, let $r > 0$, and $V_r = r^{(n-2)/2} U(rx)$. We have $\Delta V_r = |V_r|^{2^\star - 2} V_r$ and

$$\int_{B_0(2) \setminus B_0(1/2)} |V_r|^{2^\star} \, dx \to 0 \quad \text{as } r \to +\infty,$$

since $u_i \in L^{2^\star}(\mathbb{R})$ for all i. Then $v_r^i \to 0$ in $C_{\mathrm{loc}}^0\left(B_0\left(\frac{3}{2}\right) \setminus B_0\left(\frac{3}{4}\right)\right)$ as $r \to +\infty$ for all i, where the v_r^i's are the components of V_r. This proves (3-5). Now, in order to prove (3-4), we prove that the following step holds true.

Step 2. *Let* $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ *be a nonnegative solution of* (3-1) *such that* $u_i > 0$ *in* $\mathbb{R}^n$ *for all* $i = 1, \ldots, p'$, $p' \leq p$. *For any* $0 < \varepsilon < \frac{1}{2}$, *there exists* $C_\varepsilon > 0$ *such that*

$$u_i(x) \leq C_\varepsilon |x|^{(2-n)(1-\varepsilon)}$$

for all $x \in \mathbb{R}^n$ *and all* $i \in \{1, \ldots, p'\}$.

Proof of Step 2. Let $0 < \varepsilon < \frac{1}{2}$ and $R_\varepsilon > 0$ be such that

$$\sup_{x \in \mathbb{R}^n \setminus B_0(R_\varepsilon)} |x|^2 |U(x)|^{2^\star - 2} \leq \frac{(n-2)^2}{2} \varepsilon (1 - \varepsilon).$$

It is always possible to find such a R_ε thanks to (3-5). For $R \geq R_\varepsilon$, we let

$$\eta(R) = \max_{i=1,\ldots,p'} \max_{\partial B_0(R)} u_i$$

and

$$G_\varepsilon(x) = \eta(R_\varepsilon) \left(\frac{|x|}{R_\varepsilon}\right)^{(2-n)(1-\varepsilon)} + \eta(R) \left(\frac{|x|}{R}\right)^{(2-n)\varepsilon}.$$

It is clear that $u_i \leq G_\varepsilon$ on $\partial B_0(R_\varepsilon) \bigcup \partial B_0(R)$. Let us assume that $\frac{u_i}{G_\varepsilon}$ possesses a local maximum at $x \in B_0(R) \setminus \overline{B_0(R_\varepsilon)}$. Then

$$\frac{\Delta u_i(x)}{u_i(x)} \geq \frac{\Delta G_\varepsilon(x)}{G_\varepsilon(x))}.$$

Since

$$\frac{\Delta G_\varepsilon(x)}{G_\varepsilon(x)} = \varepsilon(1-\varepsilon)(n-2)^2 |x|^{-2},$$

we get

$$|x|^2 |U(x)|^{2^\star - 2} \geq \varepsilon(1-\varepsilon)(n-2)^2.$$

But this is absurd by the choice of R_ε we made. Thus we can write, for any $R > R_\varepsilon$ and any $i \in \{1, \ldots, p'\}$,

$$u_i(x) \leq \eta(R_\varepsilon) \left(\frac{|x|}{R_\varepsilon}\right)^{(2-n)(1-\varepsilon)} + \eta(R) \left(\frac{|x|}{R}\right)^{(2-n)\varepsilon} \tag{3-6}$$

in $B_0(R) \setminus B_0(R_\varepsilon)$. Fix $x \in \mathbb{R}^n \setminus B_0(R_\varepsilon)$. Passing to the limit as $R \to +\infty$ in (3-6), since, by (3-5), $R^{(n-2)/2} \eta(R) \to 0$ as $R \to +\infty$, we get

$$u_i(x) \leq \eta(R_\varepsilon) \left(\frac{|x|}{R_\varepsilon}\right)^{(2-n)(1-\varepsilon)}.$$

This ends the proof of Step 2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Step 3. *Let $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of (3-1) such that $u_i > 0$ in $\mathbb{R}^n$ for all $i = 1, \dots, p'$, $p' \le p$. Then $u_i \in L^{2^\star-1}(\mathbb{R}^n)$ and*

$$\lim_{|x| \to +\infty} |x|^{n-2} u_i(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} |U|^{2^\star-2} u_i \, dx$$

for all $i \in \{1, \dots, p'\}$.

Proof of Step 3. We apply Green's representation formula in $B_x(R)$ and get

$$u_i(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{B_x(R)} \left(|x-y|^{2-n} - R^{2-n} \right) |U(y)|^{2^\star-2} u_i(y) \, dy + \frac{1}{\omega_{n-1} R^{n-1}} \int_{\partial B_x(R)} u_i \, d\sigma.$$

Thanks to the estimate of Step 2 with $0 < \varepsilon < 2/(n+2)$, we have $u_i \in L^{2^\star-1}(\mathbb{R}^n)$ for all i. Passing to the limit as $R \to +\infty$ we obtain

$$u_i(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} |x-y|^{2-n} |U(y)|^{2^\star-2} u_i(y) \, dy.$$

Thus

$$|x|^{n-2} u_i(x)$$

$$= \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|x|^{n-2}}{|x-y|^{n-2}} |U(y)|^{2^\star-2} u_i(y) \, dy$$

$$= \frac{1}{(n-2)\omega_{n-1}} \left(\int_{B_0(R)} |U(y)|^{2^\star-2} u_i(y) \, dy + o_R(1) + \int_{\mathbb{R}^n \setminus B_0(R)} \frac{|x|^{n-2}}{|x-y|^{n-2}} |U(y)|^{2^\star-2} u_i(y) \, dy \right),$$

where $o_R(1) \to 0$ as $|x| \to +\infty$. Now, using Step 2, we write

$$\int_{\mathbb{R}^n \setminus B_0(R)} \frac{|x|^{n-2}}{|x-y|^{n-2}} |U(y)|^{2^\star-2} u_i(y) \, dy$$

$$\le N^{(2^\star-2)/2} C_\varepsilon^{2^\star-1} \int_{B_x(|x|/2)} \frac{|x|^{n-2}}{|x-y|^{n-2}} dy \left(\frac{|x|}{2} \right)^{-(n+2)(1-\varepsilon)} + 2^{n-2} \int_{\mathbb{R}^n \setminus B_0(R)} |U(y)|^{2^\star-2} u_i(y) \, dy$$

$$\le N^{(2^\star-2)/2} C_\varepsilon^{2^\star-1} 2^{(n+2)(1-\varepsilon)-2} \omega_{n-1} |x|^{n-(n+2)(1-\varepsilon)} + 2^{n-2} \int_{\mathbb{R}^n \setminus B_0(R)} |U(y)|^{2^\star-2} u_i(y) \, dy.$$

Choosing $0 < \varepsilon < \dfrac{2}{n+2}$, we thus obtain that

$$\lim_{R \to +\infty} \limsup_{|x| \to +\infty} \int_{\mathbb{R}^n \setminus B_0(R)} \frac{|x|^{n-2}}{|x-y|^{n-2}} |U(y)|^{2^\star-2} u_i(y) \, dy = 0.$$

This ends the proof of Step 3. $\qquad\square$

Using Steps 1 and 3 we are now in a position to prove (3-4), and then Proposition 3.1.

Proof of Proposition 3.1. Let $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of (3-1) such that $u_i > 0$ in $\mathbb{R}^n$ for all $i = 1, \dots, p'$, $p' \le p$. Since the u_i's are all positive for $i = 1, \dots, p'$, we get from Step 3 that

$$\min_{\partial B_0(R)} \frac{u_i}{u_j}, \quad \max_{\partial B_0(R)} \frac{u_i}{u_j} \to \lambda_{i,j}$$

as $R \to +\infty$, where $\lambda_{i,j} > 0$ is given by

$$\lambda_{i,j} = \frac{\int_{\mathbb{R}^n} |U|^{2^\star - 2} u_i \, dx}{\int_{\mathbb{R}^n} |U|^{2^\star - 2} u_j \, dx}.$$

In particular, (3-4) holds true. Thanks to Step 1, we thus get

$$u_i = \lambda_i u_1$$

for all $i \in \{1, \ldots, p'\}$ where $\lambda_i = \lambda_{i,1}$. By (3-1) we then get

$$\Delta u_1 = |\Lambda'|^{2^\star - 2} u_1^{2^\star - 1}$$

in $\mathbb{R}^n$ where $\Lambda' = (\lambda_i)_{i=1,\ldots,p'}$. By [Caffarelli et al. 1989] we can write

$$u_1(x) = |\Lambda'|^{-1} \left(\frac{\mu}{\mu^2 + \dfrac{|x - x_0|^2}{n(n-2)}} \right)^{(n-2)/2} \tag{3-7}$$

for some $x_0 \in \mathbb{R}^n$ and some $\mu > 0$. In particular, since $u_i = \lambda_i u_1$, we get with (3-7) that (3-2) holds true with $\Lambda = (\Lambda_i)_i$, where $\Lambda_i = |\Lambda'|^{-1} \lambda_i$ for all $i = 1, \ldots, p'$, and $\lambda_i = 0$ for all $i > p'$. Clearly, $|\Lambda| = 1$. This ends the proof of Proposition 3.1. $\qquad\square$

Let K_n be the sharp constant for the Sobolev inequality $\|u\|_{2^\star} \leq K \|\nabla u\|_2$ corresponding to the embedding $\dot{H}^1(\mathbb{R}^n) \subset L^{2^\star}(\mathbb{R}^n)$. Then, as is well known,

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}, \tag{3-8}$$

where ω_n is the volume of the unit sphere. The multipliers in (3-2), which we get by taking the Euclidean norm $|\mathcal{U}|$ of $\mathcal{U}$ in (3-2), turn out to be extremal functions for the sharp Euclidean Sobolev inequality $\|u\|_{2^\star} \leq K_n \|\nabla u\|_2$. As a direct consequence of Proposition 3.1 we then get

$$\int_{\mathbb{R}^n} |\mathcal{U}|^{2^\star} dx = K_n^{-n} \tag{3-9}$$

for all nonnegative solutions $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ of (3-1), where K_n is as in (3-8). Proposition 3.1, combined with the moving sphere approach, gives the full classification of nonnegative solutions of (3-1), namely without the requirement that $\mathcal{U} \in \dot{H}^1$. This is carried out in [Druet et al. 2009].

4. Weak pointwise estimates

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, and $(x_\alpha)_\alpha$ be a converging sequence of points in M. Let also $(\lambda_\alpha)_\alpha$ be a sequence of positive real numbers. For $\mathcal{U} : M \to \mathbb{R}^p$ and $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}^p$, we define the direct $\hat{R}_{x_\alpha}^{\lambda_\alpha}$-rescalings and the inverse $\check{R}_{x_\alpha}^{\lambda_\alpha}$-rescalings by

$$\left(\hat{R}_{x_\alpha}^{\lambda_\alpha} \mathcal{U} \right)(x) = \lambda_\alpha^{(n-2)/2} \mathcal{U}(\exp_{x_\alpha}(\lambda_\alpha x)) \quad \text{and} \quad \left(\check{R}_{x_\alpha}^{\lambda_\alpha} \mathcal{V} \right)(x) = \lambda_\alpha^{(n-2)/2} \mathcal{V}(\lambda_\alpha \exp_{x_\alpha}^{-1}(x)), \tag{4-1}$$

where x in the first equation is a variable in $\mathbb{R}^n$, x in the second equation is a variable in M, localized around the limit of the x_α's, and $\exp_{x_\alpha}$ is the exponential map at x_α.

Definition 4.1. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ and $p \geq 1$ be an integer. A p-vector bubble is a sequence $(\mathcal{B}_\alpha)_\alpha$ of p-maps from M to $\mathbb{R}^p$ given by

$$\mathcal{B}_\alpha(x) = \left(\frac{\mu_\alpha}{\mu_\alpha^2 + \dfrac{d_g(x_\alpha, x)^2}{n(n-2)}} \right)^{(n-2)/2} \Lambda \tag{4-2}$$

for all $x \in M$ and all α, where $(x_\alpha)_\alpha$ is a converging sequence of points in M, $(\mu_\alpha)_\alpha$ is a sequence of positive real numbers converging to 0, and $\Lambda \in S_+^{p-1}$. The x_α's are the centers of the bubble, the μ_α's are the weights of the bubble, and Λ is the S^{p-1}-projection of the bubble.

The right-hand side in (4-2) can be seen as the Riemannian extension of the right-hand side in (3-2). At last we let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be the function given by

$$u_0(x) = \left(1 + \frac{|x|^2}{n(n-2)} \right)^{-(n-2)/2} \tag{4-3}$$

for all $x \in \mathbb{R}^n$. Another possible definition of u_0 is that it is the unique nonnegative solution of $\Delta u = u^{2^\star - 1}$ which achieves its maximum at 0 and which is such that $u_0(0) = 1$. The result we prove in this section provides a complete description of the blow-up in Sobolev spaces and very useful pointwise estimates.

Proposition 4.2. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, and $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$. Let also $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of (1-1) such that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. Then there exist $N \in \mathbb{N}^\star$, a nonnegative solution U_∞ of (0-1), and vector bubbles $(\mathcal{B}_\alpha^i)_\alpha$ as in (4-2) for $i = 1, \ldots, N$, such that, up to a subsequence,*

$$\mathcal{U}_\alpha = \mathcal{U}_\infty + \sum_{i=1}^N \mathcal{B}_\alpha^i + \mathcal{R}_\alpha \qquad \text{for all } \alpha,$$

$$\int_M |\mathcal{U}_\alpha|^{2^\star} dv_g = \int_M |\mathcal{U}_\infty|^{2^\star} dv_g + N K_n^{-n} + o(1) \quad \text{for all } \alpha, \text{ and} \tag{4-4}$$

$$\mathcal{D}_\alpha^{(n-2)/2} \left| \mathcal{U}_\alpha - \mathcal{U}_\infty - \sum_{i=1}^N \mathcal{B}_\alpha^i \right| \to 0 \quad \text{in } L^\infty(M) \quad \text{as } \alpha \to +\infty,$$

where $\mathcal{R}_\alpha \to 0$ in $H^1(M)$ as $\alpha \to +\infty$, $o(1) \to 0$ as $\alpha \to +\infty$, K_n is as in (3-8), $\mathcal{D}_\alpha : M \to \mathbb{R}^+$ is given by

$$\mathcal{D}_\alpha(x) = \min_{i=1,\ldots,N} \left(d_g(x_{i,\alpha}, x) + \mu_{i,\alpha} \right),$$

and the $x_{i,\alpha}$'s and $\mu_{i,\alpha}$'s are the centers and weights of the vector bubbles $(\mathcal{B}_\alpha^i)_\alpha$. Moreover, as $\alpha \to +\infty$,

$$\frac{d_g(x_{i,\alpha}, x_{j,\alpha})^2}{\mu_{i,\alpha} \mu_{j,\alpha}} + \frac{\mu_{i,\alpha}}{\mu_{j,\alpha}} + \frac{\mu_{j,\alpha}}{\mu_{i,\alpha}} \to +\infty \quad \text{for all } i \neq j \quad \text{and}$$

$$\hat{R}_{x_{i,\alpha}}^{\mu_{i,\alpha}} \mathcal{U}_\alpha \to u_0 \Lambda_i \text{ in } C_{\text{loc}}^2(\mathbb{R}^n \setminus \mathcal{S}_i) \qquad \text{for all } i, \tag{4-5}$$

where the $\hat{R}_{x_\alpha}^{\mu_\alpha}$-rescaling procedure is defined in (4-1), u_0 is as in (4-3), the Λ_i's are the S^{p-1}-projections of the $(\mathcal{B}_\alpha^i)_\alpha$'s,

$$\mathcal{S}_i = \left\{ \lim_{\alpha \to +\infty} \mu_{i,\alpha}^{-1} \exp_{x_{i,\alpha}}^{-1}(x_{j,\alpha}), \, j \in I_i \right\},$$

the limits in the definition of $\mathcal{S}_i$ are as $\alpha \to +\infty$, and I_i consists of the j's such that $d_g(x_{i,\alpha}, x_{j,\alpha}) = O(\mu_{i,\alpha})$ and $\mu_{j,\alpha} = o(\mu_{i,\alpha})$ for all α.

Proof. Let I_α be the free functionals associated with (0-1). They are defined for $\mathcal{U} \in H^1(M)$ by

$$I_\alpha(\mathcal{U}) = \frac{1}{2} \int_M \left(|\nabla \mathcal{U}|^2 + A_\alpha(\mathcal{U}, \mathcal{U}) \right) dv_g - \frac{1}{2^\star} \int_M |\mathcal{U}|^{2^\star} dv_g.$$

The $\mathcal{U}_\alpha$'s in Proposition 4.2 solve (0-1) and are bounded in H^1. In particular, $(\mathcal{U}_\alpha)_\alpha$ is a Palais–Smale sequence for the I_α's in the sense that the sequence $(I_\alpha(\mathcal{U}_\alpha))_\alpha$ is bounded and $DI_\alpha(\mathcal{U}_\alpha) \to 0$ in $H^1(M)'$ as $\alpha \to +\infty$. Let η be a smooth cutoff function in $\mathbb{R}^n$ with small support around 0. Mimicking the proof in [Struwe 1984] (see also [Druet et al. 2004] for its Riemannian analogue), we get that there exist $N \in \mathbb{N}^\star$, a nonnegative solution U_∞ of (0-1), converging sequences $(x_{i,\alpha})_\alpha$ in M, sequences $(\mu_{i,\alpha})_\alpha$ of positive real numbers converging to 0, and nonnegative solutions $\mathcal{U}_i \in \dot{H}^1(\mathbb{R}^n)$ of (3-1) in $\mathbb{R}^n$, $i = 1, \ldots, N$, such that, up to a subsequence, the first equation in (4-5) holds true, such that

$$\mathcal{U}_\alpha = \mathcal{U}_\infty + \sum_{i=1}^{N} \eta_\alpha^i \check{R}_{x_{i,\alpha}}^{1/\mu_{i,\alpha}} \mathcal{U}_i + \mathcal{R}_\alpha \tag{4-6}$$

for all α, and such that

$$\int_M |\mathcal{U}_\alpha|^{2^\star} dv_g = \int_M |\mathcal{U}_\infty|^{2^\star} dv_g + \sum_{i=1}^{N} \int_{\mathbb{R}^n} |\mathcal{U}_i|^{2^\star} dx + o(1) \tag{4-7}$$

for all α, where $\eta_\alpha^i(x) = \eta(\exp_{x_{i,\alpha}}^{-1}(x))$, the $\check{R}_{x_{i,\alpha}}^{1/\mu_{i,\alpha}}$-rescalings are defined in (4-1), $\mathcal{R}_\alpha \to 0$ in $H^1(M)$ as $\alpha \to +\infty$, and $o(1) \to 0$ as $\alpha \to +\infty$. By Proposition 3.1,

$$\mathcal{U}_i(x) = \left(\frac{\lambda_i}{\lambda_i^2 + \dfrac{|x - a_i|}{n(n-2)}} \right)^{(n-2)/2} \Lambda_i \tag{4-8}$$

for some $a_i \in \mathbb{R}^n$, $\lambda_i > 0$, $\Lambda_i \in S_+^{p-1}$, and all $x \in \mathbb{R}^n$. Up to changing the $x_{i,\alpha}$'s and $\mu_{i,\alpha}$'s, letting $\tilde{x}_{i,\alpha} = \exp_{x_{i,\alpha}}(\mu_{i,\alpha} a_i)$ and $\tilde{\mu}_{i,\alpha} = \lambda_i \mu_{i,\alpha}$, we can write, as in [Druet and Hebey 2005b], that

$$\eta_\alpha^i \check{R}_{x_{i,\alpha}}^{1/\mu_{i,\alpha}} \mathcal{U}_i = \mathcal{B}_\alpha^i + \mathcal{R}_\alpha \tag{4-9}$$

for all α, where $\mathcal{U}_i$ is as in (4-8), $\mathcal{R}_\alpha \to 0$ in $H^1(M)$ as $\alpha \to +\infty$, and $(\mathcal{B}_\alpha^i)_\alpha$ is the vector bubble with center $\tilde{x}_{i,\alpha}$, weight $\tilde{\mu}_{i,\alpha}$, and S^{p-1}-projection Λ_i. Noting that the changes $x_{i,\alpha} \to \tilde{x}_{i,\alpha}$ and $\mu_{i,\alpha} \to \tilde{\mu}_{i,\alpha}$ do not affect the first equation in (4-5), it follows from the above discussion, from (3-9), and from (4-6), (4-7), and (4-9), that the two first equations in (4-4) and the first equation in (4-5) hold true. Now we forget about the tilde notation for the centers and weights of the bubbles and, for $i = 1, \ldots, N$, we let

$\mathcal{S}_i$ be as in Proposition 4.2. As one can check from the first equations in (4-4) and (4-5), for any i,

$$\hat{R}^{\mu_{i,a}}_{x_{i,a}} \sum_{j=1}^N \mathcal{B}^j_a \to u_0 \Lambda_i \text{ in } L^\infty_{\text{loc}}(\mathbb{R}^n \backslash \mathcal{S}_i) \quad \text{and} \quad \hat{R}^{\mu_{i,a}}_{x_{i,a}} \mathcal{U}_a - \hat{R}^{\mu_{i,a}}_{x_{i,a}} \sum_{j=1}^N \mathcal{B}^j_a \to 0 \text{ in } L^{2^\star}_{\text{loc}}(\mathbb{R}^n) \quad (4\text{-}10)$$

as $a \to +\infty$, where the $\hat{R}^{\mu_{i,a}}_{x_{i,a}}$-rescalings are defined in (4-1), Λ_i is the S^{p-1}-projection of $(\mathcal{B}^i_a)_a$, and u_0 is as in (4-3). Moreover, in any compact subset of $\mathbb{R}^n$, and for a sufficiently large,

$$\Delta_{g_a} \tilde{u}_{i,a} + \mu^2_{i,a} \sum_{j=1}^p \tilde{A}^a_{ij}(x) \tilde{u}_{j,a} = |\tilde{\mathcal{U}}_a|^{2^\star-2} \tilde{u}_{i,a} \quad (4\text{-}11)$$

for all a and all i, where the $\tilde{u}_{i,a}$'s are the components of $\tilde{\mathcal{U}}_a = \hat{R}^{\mu_{i,a}}_{x_{i,a}} \mathcal{U}_a$,

$$\tilde{A}^a_{ij}(x) = A^a_{ij}(\exp_{x_{i,a}}(\mu_{i,a}x)),$$

and g_a is the Riemannian metric in $\mathbb{R}^n$ given by $g_a(x) = (\exp^\star_{x_{i,a}} g)(\mu_{i,a}x)$. Since $\mu_{i,a} \to 0$ as $a \to +\infty$, we get that $g_a \to \xi$ in $C^2_{\text{loc}}(\mathbb{R}^n)$ as $a \to +\infty$, where ξ is the Euclidean metric. By (4-10), for any $x \in \mathbb{R}^n \backslash \mathcal{S}_i$,

$$\lim_{\delta \to 0} \limsup_{a \to +\infty} \int_{B_x(\delta)} |\hat{R}^{\mu_{i,a}}_{x_{i,a}} \mathcal{U}_a|^{2^\star} dx = 0. \quad (4\text{-}12)$$

In particular, the $L^{2^\star}$-norm of $\hat{R}^{\mu_{i,a}}_{x_{i,a}} \mathcal{U}_a$ can be made uniformly arbitrarily small in small regions of $\mathbb{R}^n \backslash \mathcal{S}_i$, and by adapting and transposing the classical regularity argument [Trudinger 1968] to the present situation (see also [Struwe 1990]) we get from (4-11) and (4-12) that the $\tilde{\mathcal{U}}_a$'s are uniformly bounded in $C^{2,\theta}_{\text{loc}}(\mathbb{R}^n \backslash \mathcal{S}_i)$. It easily follows that, up to a subsequence, the second equation in (4-5) also holds true. Now it remains to prove that the third equation in (4-4) holds true. We proceed by contradiction and assume that there exists $\varepsilon_0 > 0$ and a sequence $(x_a)_a$ in M such that, up to a subsequence,

$$\mathcal{D}_a(x_a)^2 \left| \mathcal{U}_a(x_a) - \mathcal{U}_\infty(x_a) - \sum_{i=1}^N \mathcal{B}^i_a(x_a) \right|^{2^\star-2} = \max_M \mathcal{D}^2_a \left| \mathcal{U}_a - \mathcal{U}_\infty - \sum_{i=1}^N \mathcal{B}^i_a \right|^{2^\star-2} \geq 4\varepsilon_0 \quad (4\text{-}13)$$

for all a. First we claim that

$$\mathcal{D}_a(x_a)^2 |\mathcal{B}^i_a(x_a)|^{2^\star-2} \to 0 \quad (4\text{-}14)$$

as $a \to +\infty$, for all $i = 1, \dots, N$. In order to prove (4-14) we proceed by contradiction and assume that there exists $i = 1, \dots, N$ and $\varepsilon_1 > 0$ such that, up to a subsequence,

$$\mathcal{D}_a(x_a)^2 |\mathcal{B}^i_a(x_a)|^{2^\star-2} \geq \varepsilon_1 \quad (4\text{-}15)$$

for all a. Up to passing to another subsequence we may then assume that there is $\lambda \in [0, +\infty)$ such that

$$\frac{d_g(x_{i,a}, x_a)}{\mu_{i,a}} \to \lambda \text{ as } a \to +\infty, \quad \text{and} \quad \frac{\mu_{j,a}}{\mu_{i,a}} + \frac{d_g(x_{j,a}, x_a)}{\mu_{i,a}} \geq \sqrt{\varepsilon_1} \text{ for all } a \text{ and } j. \quad (4\text{-}16)$$

Then, letting $y_a = \mu^{-1}_{i,a} \exp^{-1}_{x_{i,a}}(x_a)$, we get from the second equation in (4-16) that there exists $\varepsilon > 0$ such that $d(y_a, \mathcal{S}_i) \geq \varepsilon$ for all a, and it follows from the second equation in (4-5) that

$$\mathcal{D}_a(x_a)^2 \left| \mathcal{U}_a(x_a) - \mathcal{U}_\infty(x_a) - \mathcal{B}^i_a(x_a) \right|^{2^\star-2} \to 0 \quad \text{as } a \to +\infty. \quad (4\text{-}17)$$

By the first equation in (4-5), and by (4-16), we can also write

$$\mathcal{D}_\alpha(x_\alpha)^2 \big| \mathcal{B}_\alpha^j(x_\alpha) \big|^{2^\star-2} \to 0 \tag{4-18}$$

as $\alpha \to +\infty$, for all $j \neq i$. Combining (4-17) and (4-18) we get a contradiction with (4-13). It follows that (4-14) holds true. Next we claim that

$$|\mathcal{U}_\alpha(x_\alpha)| \to +\infty \quad \text{as } \alpha \to +\infty. \tag{4-19}$$

By (4-13) and (4-14), we see that (4-19) holds if $\mathcal{D}_\alpha(x_\alpha) \to 0$ as $\alpha \to +\infty$. Suppose on the contrary that, up to a subsequence, $\mathcal{D}_\alpha(x_\alpha) \to \delta$ as $\alpha \to +\infty$ for some $\delta > 0$. Then, by (4-13) and (4-14),

$$\big| \mathcal{U}_\alpha(x) - \mathcal{U}_\infty(x) \big|^{2^\star-2} + o(1) \leq 8 \big| \mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha) \big|^{2^\star-2} + o(1) \tag{4-20}$$

for all $x \in B_{x_\alpha}(\delta/2)$ and all α sufficiently large. Now, if we assume that (4-19) is false, then we get from (4-20) that the $\mathcal{U}_\alpha$'s are bounded in a neighbourhood of the x_α's, and it follows from standard elliptic theory that $\mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha) \to 0$ as $\alpha \to +\infty$. Noting that this convergence of the $(\mathcal{U}_\alpha - \mathcal{U}_\infty)(x_\alpha)$'s is in contradiction with (4-13) and (4-14), we obtain (4-19).

Now let the μ_α's be given by $\mu_\alpha^{1-(n/2)} = |\mathcal{U}_\alpha(x_\alpha)|$ for all α, and define the $\mathcal{V}_\alpha$'s by $\mathcal{V}_\alpha = \hat{R}_{x_\alpha}^{\mu_\alpha} \mathcal{U}_\alpha$, where the $\hat{R}_{x_\alpha}^{\mu_\alpha}$-rescalings are defined in (4-1). Then,

$$\Delta_{g_\alpha} v_{i,\alpha} + \mu_\alpha^2 \sum_{j=1}^p \hat{A}_{ij}^\alpha(x) v_{j,\alpha} = |\mathcal{V}_\alpha|^{2^\star-2} v_{i,\alpha} \tag{4-21}$$

in $B_0(\delta/\mu_\alpha)$ for all α, where the $v_{i,\alpha}$'s are the components of $\mathcal{V}_\alpha$, the $\hat{A}_{ij}^\alpha$'s are given by $\hat{A}_{ij}^\alpha(x) = A_{ij}^\alpha(\exp_{x_\alpha}(\mu_\alpha x))$, and g_α is given by $g_\alpha(x) = (\exp_{x_\alpha}^\star g)(\mu_\alpha x)$. From (4-19) we have $\mu_\alpha \to 0$ as $\alpha \to +\infty$. In particular, $g_\alpha \to \xi$ in $C_{\text{loc}}^2(\mathbb{R}^n)$ as $\alpha \to +\infty$. We also have $|\mathcal{V}_\alpha(0)| = 1$ for all α. Noting that the $\mathcal{V}_\alpha$'s are bounded in $\dot{H}^1(\mathbb{R}^n)$, we may assume that, up to a subsequence, $\mathcal{V}_\alpha \rightharpoonup \mathcal{V}_\infty$ weakly in $H_{\text{loc}}^1(\mathbb{R}^n)$ as $\alpha \to +\infty$ for some $\mathcal{V}_\infty \in H^1(\mathbb{R}^n)$ that solves (3-1). Let $\tilde{\mathcal{S}}$ be given by

$$\tilde{\mathcal{S}} = \left\{ \lim_{\alpha \to +\infty} \frac{1}{\mu_\alpha} \exp_{x_\alpha}^{-1}(x_{i,\alpha}) : i \in J \right\},$$

where J consists of the $i = 1, \ldots, N$ which are such that $d_g(x_{i,\alpha}, x_\alpha) = O(\mu_\alpha)$ and $\mu_{i,\alpha} = o(\mu_\alpha)$ for all α. In what follows we let $K \Subset \mathbb{R}^n \setminus \tilde{\mathcal{S}}$ be a compact subset of $\mathbb{R}^n \setminus \tilde{\mathcal{S}}$, and let $x \in K$. By (4-13) and (4-14) we have

$$\left| \mathcal{V}_\alpha(x) - \mu_\alpha^{(n-2)/2} \mathcal{U}_\infty(y_\alpha) - \mu_\alpha^{(n-2)/2} \sum_{i=1}^N \Lambda_i B_{i,\alpha}(y_\alpha) \right|^{2^\star-2} \leq \left(\frac{\mathcal{D}_\alpha(x_\alpha)}{\mathcal{D}_\alpha(y_\alpha)} \right)^2 (1 + o(1)) + o(1), \tag{4-22}$$

where $y_\alpha = \exp_{x_\alpha}(\mu_\alpha x)$ for all α, Λ_i is the S^{p-1}-projection of $(\mathcal{B}_\alpha^i)_\alpha$ for all i, and $B_{i,\alpha} = |\mathcal{B}_\alpha^i|$ for all α and i. Now we claim that

$$\mu_\alpha^{(n-2)/2} |B_{i,\alpha}(y_\alpha)| \to 0 \tag{4-23}$$

as $\alpha \to +\infty$, for all $i = 1, \ldots, N$. Equation (4-23) is obvious if $\mu_\alpha = o(\mu_{i,\alpha})$. On the other hand, if we assume that $\mu_{i,\alpha} = o(\mu_\alpha)$, then, since $d_\xi(x, \tilde{\mathcal{S}}) > 0$, we get $\mu_\alpha = O(d_g(x_{i,\alpha}, y_\alpha))$. Here again, (4-23) holds true. At last we may assume that there exists $C > 0$ such that $C^{-1} \mu_\alpha \leq \mu_{i,\alpha} \leq C \mu_\alpha$ for all α.

Then (4-23) holds true unless $d_g(x_{i,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$. In this case we have $d_g(x_{i,\alpha}, x_\alpha) = O(\mu_{i,\alpha})$, and it follows that $|B_{i,\alpha}(x_\alpha)|/|\mathcal{U}_\alpha(x_\alpha)| \not\to 0$ as $\alpha \to +\infty$. Combining (4-13) and (4-14) we get a contradiction, and it follows that (4-23) holds true. In particular, by (4-19), (4-22), and (4-23), we can write

$$|\mathcal{V}_\alpha(x)|^{2^\star - 2} \leq \left(\frac{\mathcal{D}_\alpha(x_\alpha)}{\mathcal{D}_\alpha(y_\alpha)} \right)^2 (1 + o(1)) + o(1). \tag{4-24}$$

At this point we claim that

$$\mathcal{D}_\alpha(x_\alpha) = O(\mathcal{D}_\alpha(y_\alpha)). \tag{4-25}$$

We prove (4-25) by contradiction and assume that

$$d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha} = o(\mathcal{D}_\alpha(x_\alpha)). \tag{4-26}$$

If $d_g(x_{i,\alpha}, x_\alpha)/\mu_\alpha \to +\infty$ as $\alpha \to +\infty$, then

$$d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha} \geq (1 + o(1))d_g(x_{i,\alpha}, x_\alpha) + \mu_{i,\alpha} \geq (1 + o(1))\mathcal{D}_\alpha(x_\alpha),$$

and this contradicts (4-26). Hence, $d_g(x_{i,\alpha}, x_\alpha) = O(\mu_\alpha)$. Then, by (4-26),

$$d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha} = o(\mu_\alpha) + o(\mu_{i,\alpha}). \tag{4-27}$$

In particular, $d_g(x_{i,\alpha}, y_\alpha) = o(\mu_\alpha)$. Since $x \in K$, this implies in turn that $\mu_\alpha = O(\mu_{i,\alpha})$, and we get with (4-27) that $\mu_{i,\alpha} + o(\mu_{i,\alpha}) = 0$, another contradiction. This proves (4-25). By (4-24) and (4-25), for any compact subset $K \Subset \mathbb{R}^n \backslash \tilde{\mathcal{S}}$, there exists $C_K > 0$ such that $|\mathcal{V}_\alpha| \leq C_K$ in K. In particular, by standard elliptic theory and (4-21), we get

$$\mathcal{V}_\alpha \to \mathcal{V}_\infty \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^n \backslash \tilde{\mathcal{S}}) \quad \text{as } \alpha \to +\infty. \tag{4-28}$$

Clearly $0 \notin \tilde{\mathcal{S}}$ since, if not the case, $\mathcal{D}_\alpha(x_\alpha) = o(\mu_\alpha)$ and we get a contradiction with (4-13). Thus, since $|\mathcal{V}_\alpha(0)| = 1$ for all α, we see that $|\mathcal{V}_\infty(0)| = 1$ and $\mathcal{V}_\infty \not\equiv 0$ is not identically zero. By Proposition 3.1 it follows that there exists $a \in \mathbb{R}^n$, $\lambda > 0$, and $\Lambda \in S^{p-1}_+$, such that

$$\mathcal{V}_\infty(x) = \left(\frac{\lambda}{\lambda^2 + \frac{|x-a|}{n(n-2)}} \right)^{(n-2)/2} \Lambda \tag{4-29}$$

for all $x \in \mathbb{R}^n$. Let $K \Subset \mathbb{R}^n \backslash \tilde{\mathcal{S}}$ be a nonempty compact subset of $\mathbb{R}^n \backslash \tilde{\mathcal{S}}$. By the first equation in (4-4) and by (4-23), we can write $\mathcal{V}_\alpha \to 0$ in $L^{2^\star}(K)$ as $\alpha \to +\infty$. Then, by (4-28), we get $\int_K |\mathcal{V}_\infty|^{2^\star} dx = 0$, a contradiction with (4-29). Proposition 4.2 is proved. $\qquad \square$

5. A first strong pointwise estimate

We prove pointwise estimates on the $\mathcal{U}_\alpha$'s which we use as the initial step in the induction argument we develop in the next section. First we fix some notations. We let $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of (1-1) such that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. Proposition 4.2 applies to the $\mathcal{U}_\alpha$'s. We let $\mathcal{S}$ be the set of the geometrical points of the $\mathcal{U}_\alpha$'s. Then,

$$\mathcal{S} = \left\{ \lim_{\alpha \to +\infty} x_{i,\alpha} : i = 1, \ldots, N \right\}, \tag{5-1}$$

where all the limits do exist, up to a subsequence. For $\delta > 0$ small enough, we let

$$\eta_\alpha(\delta) = \max_{M \setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(\delta)} |\mathcal{U}_\alpha|. \tag{5-2}$$

Thanks to the last equation in (4-4) of Proposition 4.2

$$\limsup_{\alpha \to +\infty} \eta_\alpha(\delta) \leq \|\mathcal{U}_\infty\|_\infty. \tag{5-3}$$

Moreover, by standard elliptic theory, for any $\delta' > \delta$,

$$\max_{M \setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(\delta')} |\nabla \mathcal{U}_\alpha|_g = O(\eta_\alpha(\delta)). \tag{5-4}$$

In what follows we let $R_0 > 0$ be such that for any $i = 1, \ldots, N$,

$$|x| \leq \frac{R_0}{2} \tag{5-5}$$

for all $x \in \mathcal{S}_i$, where $\mathcal{S}_i$ is as in Proposition 4.2. We also set

$$\mu_\alpha = \max_{i \in \{1,\ldots,N\}} \mu_{i,\alpha}, \quad \text{and} \quad r_\alpha(x) = \min_{i \in \{1,\ldots,N\}} d_g(x_{i,\alpha}, x). \tag{5-6}$$

The pointwise estimate we prove in this section is stated as follows.

Proposition 5.1. *Let* (M, g) *be a smooth compact Riemannian manifold of dimension* $n \geq 3$, $p \geq 1$ *be an integer, and* $(A_\alpha)_\alpha$ *be a sequence of* C^1 *maps from* M *to* $M_p^s(\mathbb{R})$ *such that* $A_\alpha \to A$ *in* $C^1(M)$ *as* $\alpha \to +\infty$ *for some* C^1 *map* A *from* M *to* $M_p^s(\mathbb{R})$ *satisfying* (H). *Let also* $(\mathcal{U}_\alpha)_\alpha$ *be an arbitrary bounded sequence in* $H^1(M)$ *of nonnegative solutions of* (1-1) *such that* $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ *as* $\alpha \to +\infty$. *There exists* $C_1 > 0$ *such that, up to passing to a subsequence on the* $\mathcal{U}_\alpha$'s, *there holds that for any sequence* $(x_\alpha)_\alpha$ *of points in* M,

$$\left| \mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha) \right| \leq C_1 \mu_\alpha^{(n-2)/2} \mathcal{D}_\alpha(x_\alpha)^{2-n} + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty, \tag{5-7}$$

where $\mathcal{D}_\alpha$ *and* $\mathcal{U}_\infty$ *are as in Proposition 4.2,* μ_α *is as in* (5-6), *and* $\varepsilon_\alpha \to 0$ *as* $\alpha \to +\infty$.

We divide the proof of Proposition 5.1 into two steps.

Step 1. *For any* $0 < \varepsilon < \frac{1}{2}$, *there exist* $R_\varepsilon > 0$, $\delta_\varepsilon > 0$ *and* $C_\varepsilon > 0$ *such that*

$$|\mathcal{U}_\alpha(x)| \leq C_\varepsilon \left(\mu_\alpha^{(1-2\varepsilon)(n-2)/2} r_\alpha(x)^{(2-n)(1-\varepsilon)} + \eta_\alpha(\delta_\varepsilon) r_\alpha(x)^{(2-n)\varepsilon} \right)$$

for all α *and all* $x \in M \setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha})$.

Proof of Step 1. Let $0 < \varepsilon < \frac{1}{2}$. Consider $\mathcal{G}$ the Green's function of the operator $u \mapsto \Delta_g u + u$. We know (see [Druet et al. 2004], for example) that there exist $\gamma_1 > 1$, $\gamma_2 > 0$ and $\gamma_3 > 0$ such that for any distinct $x, y \in M$,

$$\frac{1}{\gamma_1} \leq d_g(x, y)^{n-2} \mathcal{G}(x, y) \leq \gamma_1 \tag{5-8}$$

and

$$\frac{|\nabla \mathcal{G}(x, y)|^2}{\mathcal{G}(x, y)^2} \geq \gamma_2 d_g(x, y)^{-2} - \gamma_3, \tag{5-9}$$

where ∇ in (5-9) is with respect to one of the two variables, for instance y. We let

$$\Psi_{\alpha,\varepsilon}(x) = \mu_\alpha^{(1-2\varepsilon)(n-2)/2} \sum_{i=1}^N \mathcal{G}(x_{i,\alpha}, x)^{1-\varepsilon} + \eta_\alpha(\delta) \sum_{i=1}^N \mathcal{G}(x_{i,\alpha}, x)^\varepsilon,$$

and let $y_\alpha \in M \setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(R\mu_{i,\alpha})$ be such that

$$\max_{M\setminus\bigcup_{i=1}^p B_{x_{i,\alpha}}(R\mu_{i,\alpha})} \frac{\sum_{i=1}^p u_{i,\alpha}}{\Psi_{\alpha,\varepsilon}} = \frac{\sum_{i=1}^N u_{i,\alpha}}{\Psi_{\alpha,\varepsilon}}(y_\alpha) \tag{5-10}$$

for all α. We claim that, if $\delta > 0$ is chosen sufficiently small and $R > 0$ sufficiently large, then

$$y_\alpha \in \partial\Big(M \setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(R\mu_{i,\alpha})\Big) \quad \text{or} \quad r_\alpha(y_\alpha) > \delta \tag{5-11}$$

for α large. We prove the claim by contradiction. Indeed, assume that (5-11) fails for all α. We can write

$$\frac{\Delta_g\big(\sum_{i=1}^p u_{i,\alpha}\big)}{\sum_{i=1}^p u_{i,\alpha}}(y_\alpha) \geq \frac{\Delta_g \Psi_{\alpha,\varepsilon}}{\Psi_{\alpha,\varepsilon}}(y_\alpha). \tag{5-12}$$

Thanks to (1-1),

$$\frac{\Delta_g\big(\sum_{i=1}^p u_{i,\alpha}\big)}{\sum_{i=1}^p u_{i,\alpha}}(y_\alpha) \leq |\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} + p\|A_\alpha\|_\infty,$$

where $\|A_\alpha\|_\infty = \max_{i,j} \|A_{ij}^\alpha\|_\infty$ for all α. By (5-12) we then get $\frac{\Delta_g \Psi_{\alpha,\varepsilon}}{\Psi_{\alpha,\varepsilon}}(y_\alpha) \leq |\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} + p\|A_\alpha\|_\infty$.
Since $r_\alpha(y_\alpha) \leq \delta$, this yields

$$r_\alpha(y_\alpha)^2 \frac{\Delta_g \Psi_{\alpha,\varepsilon}}{\Psi_{\alpha,\varepsilon}}(y_\alpha) \leq r_\alpha(y_\alpha)^2 |\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} + \delta^2 p\|A\|_\infty + o(1). \tag{5-13}$$

Now we write

$$\Delta_g \Psi_{\alpha,\varepsilon}(y_\alpha) = \varepsilon(1-\varepsilon)\mu_\alpha^{(1-2\varepsilon)(n-2)/2} \sum_{i=1}^N \frac{\big|\nabla\mathcal{G}(x_{i,\alpha}, y_\alpha)\big|_g^2}{\mathcal{G}(x_{i,\alpha}, y_\alpha)^2} \mathcal{G}(x_{i,\alpha}, y_\alpha)^{1-\varepsilon}$$

$$+ \varepsilon(1-\varepsilon)\eta_\alpha(\delta) \sum_{i=1}^N \frac{\big|\nabla\mathcal{G}(x_{i,\alpha}, y_\alpha)\big|_g^2}{\mathcal{G}(x_{i,\alpha}, y_\alpha)^2} \mathcal{G}(x_{i,\alpha}, y_\alpha)^\varepsilon$$

$$- \varepsilon\eta_\alpha(\delta) \sum_{i=1}^N \mathcal{G}(x_{i,\alpha}, y_\alpha)^\varepsilon - (1-\varepsilon)\mu_\alpha^{(1-2\varepsilon)(n-2)/2} \sum_{i=1}^N \mathcal{G}(x_{i,\alpha}, y_\alpha)^{1-\varepsilon}.$$

Using (5-8) and (5-9), and since $0 < \varepsilon < \frac{1}{2}$, it follows that

$$\Delta_g \Psi_{\alpha,\varepsilon}(y_\alpha)$$
$$\geq -(1-\varepsilon)\Psi_{\alpha,\varepsilon}(y_\alpha) - \gamma_3\varepsilon(1-\varepsilon)\Psi_{\alpha,\varepsilon}(y_\alpha) + \varepsilon(1-\varepsilon)\gamma_2\mu_\alpha^{(1-2\varepsilon)(n-2)/2}\sum_{i=1}^N d_g(x_{i,\alpha}, y_\alpha)^{-2}\mathcal{G}(x_{i,\alpha}, y_\alpha)^{1-\varepsilon}$$
$$+ \varepsilon(1-\varepsilon)\gamma_2\eta_\alpha(\delta)\sum_{i=1}^N d_g(x_{i,\alpha}, y_\alpha)^{-2}\mathcal{G}(x_{i,\alpha}, y_\alpha)^\varepsilon$$
$$\geq -(1-\varepsilon)(1+\gamma_3\varepsilon)\Psi_{\alpha,\varepsilon}(y_\alpha) + \varepsilon(1-\varepsilon)\gamma_2\gamma_1^{\varepsilon-1}\mu_\alpha^{(1-2\varepsilon)(n-2)/2}r_\alpha(y_\alpha)^{-2-(n-2)(1-\varepsilon)}$$
$$+ \varepsilon(1-\varepsilon)\gamma_2\gamma_1^{-\varepsilon}\eta_\alpha(\delta)r_\alpha(y_\alpha)^{-2-(n-2)\varepsilon}.$$

From (5-8) we obtain $\Psi_{\alpha,\varepsilon}(y_\alpha) \leq N\gamma_1^{1-\varepsilon}\mu_\alpha^{(1-2\varepsilon)(n-2)/2}r_\alpha(y_\alpha)^{-(n-2)(1-\varepsilon)} + N\gamma_1^\varepsilon\eta_\alpha(\delta)r_\alpha(y_\alpha)^{-(n-2)\varepsilon}$, and we can write

$$r_\alpha(y_\alpha)^2\Delta_g\Psi_{\alpha,\varepsilon}(y_\alpha) \geq -(1-\varepsilon)(1+\gamma_3\varepsilon)r_\alpha(y_\alpha)^2\Psi_{\alpha,\varepsilon}(y_\alpha) + \frac{1}{N}\varepsilon(1-\varepsilon)\gamma_2\gamma_1^{2(\varepsilon-1)}\Psi_{\alpha,\varepsilon}(y_\alpha).$$

Coming back to (5-13), we thus get

$$\frac{1}{N}\varepsilon(1-\varepsilon)\gamma_2\gamma_1^{2(\varepsilon-1)} \leq r_\alpha(y_\alpha)^2|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} + \delta^2 p\|A\|_\infty + o(1) + (1-\varepsilon)(1+\gamma_3\varepsilon)\delta^2,$$

since we assumed that $r_\alpha(y_\alpha) \leq \delta$. By the last equation in (4-4) of Proposition 4.2 we can choose $\delta > 0$ and $R > 0$ so as to get a contradiction. Thus (5-11) is proved. Up to choosing R a little bit larger, we deduce from the second equation in (4-5) of Proposition 4.2, and the definitions of μ_α and $\eta_\alpha(\delta)$, that there exists $C > 0$ such that

$$\sup_{M\setminus\bigcup_{i=1}^N B_{x_{i,\alpha}}(R\mu_{i,\alpha})} \frac{\sum_{i=1}^p u_{i,\alpha}}{\Psi_{\alpha,\varepsilon}} \leq C_\varepsilon.$$

Using (5-8), we obtain the existence of $\delta_\varepsilon > 0$, $R_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$\sum_{i=1}^p u_{i,\alpha}(x) \leq C_\varepsilon\Big(\mu_\alpha^{(1-2\varepsilon)(n-2)/2}r_\alpha(x)^{(2-n)(1-\varepsilon)} + \eta_\alpha(\delta_\varepsilon)r_\alpha(x)^{(2-n)\varepsilon}\Big)$$

for all α and all $x \in M \setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})$. This proves Step 1. $\qquad\square$

Step 2. *There exists $C_0 > 0$ such that $|\mathcal{U}_\alpha(x)| \leq C_0\big(\mu_\alpha^{(n-2)/2}\mathcal{D}_\alpha(x)^{2-n} + \|\mathcal{U}_\infty\|_\infty\big)$ for all α and all $x \in M$.*

Proof of Step 2. First we prove that there is $\delta > 0$ small such that for any sequence (y_α) of points in M,

$$\limsup_{\alpha\to+\infty} \frac{|\mathcal{U}_\alpha(y_\alpha)|}{\mu_\alpha^{(n-2)/2}\mathcal{D}_\alpha(y_\alpha)^{2-n} + \eta_\alpha(\delta)} < +\infty. \tag{5-14}$$

By the definition of $\eta_\alpha(\delta)$, it is clear that (5-14) holds if $r_\alpha(y_\alpha) \geq \delta$. Now assume that $r_\alpha(y_\alpha) = O(\mu_\alpha)$. Then $\mathcal{D}_\alpha(y_\alpha) = O(\mu_\alpha)$. We can use the last equation in (4-4) of Proposition 4.2 to obtain

$$\mathcal{D}_\alpha(y_\alpha)^2\mu_\alpha^{-1}|\mathcal{U}_\alpha(y_\alpha)|^{2/n-2} = O(\mathcal{D}_\alpha(y_\alpha)\mu_\alpha^{-1}) + O\bigg(\sum_{i=1}^N \mathcal{D}_\alpha(y_\alpha)^2\mu_\alpha^{-1}\mu_{i,\alpha}^{-1}\Big(1 + \frac{d_g(x_{i,\alpha}, y_\alpha)^2}{n(n-2)\mu_{i,\alpha}^2}\Big)^{-1}\bigg)$$

$$= O(1),$$

since $\mathcal{D}_\alpha(y_\alpha) \leq d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha}$ for all $i \in \{1, \dots, N\}$. In particular, (5-14) holds true also in this case. Thus we may assume from now on that

$$r_\alpha(y_\alpha) \leq \delta \quad\text{and}\quad \frac{r_\alpha(y_\alpha)}{\mu_\alpha} \to +\infty \quad\text{as } \alpha \to +\infty. \tag{5-15}$$

We let $\lambda > 1$ be such that $\lambda p\|A\|_\infty \notin Sp(\Delta_g)$ and we let G be the Green's function of $\Delta_g - \lambda p\|A\|_\infty$. Here again, there exist $C_1 > 1$, $C_2 > 0$ and $C_3 > 0$ such that

$$\frac{1}{C_1}d_g(x, y)^{2-n} - C_2 \leq G(x, y) \leq C_1 d_g(x, y)^{2-n} \tag{5-16}$$

and

$$|\nabla G(x, y)|_g \leq C_3 d_g(x, y)^{1-n} \tag{5-17}$$

for all $x, y \in M$, $x \neq y$. We let $x_0 \in \mathcal{S}$ be such that $d_g(y_\alpha, x_0) \leq \delta + o(1)$; such an x_0 does exist thanks to (5-15). We choose $\delta > 0$ such that

$$d_g(x, y) \geq 4\delta \tag{5-18}$$

for all $x, y \in \mathcal{S}$, $x \neq y$, and such that

$$\delta \leq \tfrac{1}{4}(C_1 C_2)^{-1/(n-2)}, \tag{5-19}$$

where C_1 and C_2 are as in (5-16). We write with Green's representation formula that

$$\sum_{i=1}^{p} u_{i,\alpha}(y_\alpha) = \int_{B_{x_0}(2\delta)} G(y_\alpha, x)\left(\Delta_g\left(\sum_{i=1}^{p} u_{i,\alpha}\right) - \lambda p\|A\|_\infty \sum_{i=1}^{p} u_i^\alpha\right)(x)\, dv_g(x)$$

$$+ \int_{\partial B_{x_0}(2\delta)} G(y_\alpha, x)\partial_\nu\left(\sum_{i=1}^{p} u_{i,\alpha}\right)(x)\, d\sigma_g(x) - \int_{\partial B_{x_0}(2\delta)} \partial_\nu G(y_\alpha, x)\left(\sum_{i=1}^{p} u_{i,\alpha}\right)(x)\, d\sigma_g(x). \tag{5-20}$$

Since $\lambda > 1$, we get with (1-1) that

$$\Delta_g\left(\sum_{i=1}^{p} u_{i,\alpha}\right) - \lambda p\|A\|_\infty \sum_{i=1}^{p} u_{i,\alpha} \leq |\mathcal{U}_\alpha|^{2^\star-2} \sum_{i=1}^{p} u_{i,\alpha}.$$

We have $G(y_\alpha, x) \geq 0$ in $B_{x_0}(2\delta)$ for α large, thanks to (5-16) and (5-19). Thus we can write

$$\int_{B_{x_0}(2\delta)} G(y_\alpha, x)\left(\Delta_g\left(\sum_{i=1}^{p} u_{i,\alpha}\right) - \lambda p\|A\|_\infty \sum_{i=1}^{p} u_{i,\alpha}\right)(x)\, dv_g(x)$$

$$\leq C_1 \int_M d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2} \sum_{i=1}^{p} u_{i,\alpha}(x)\, dv_g(x). \tag{5-21}$$

From (5-18), we also know that $d_g(x_{i,\alpha}, \partial B_{x_0}(2\delta)) \geq \delta$ for all $i = 1, \ldots, N$ and for α large so that we can control the boundary terms in (5-20) thanks to (5-4), (5-16) and (5-17). We thus obtain that

$$|\mathcal{U}_\alpha(y_\alpha)| = O(\eta_\alpha(\delta)) + O\left(\int_M d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x)\right). \tag{5-22}$$

We fix $0 < \varepsilon < \dfrac{1}{n+2}$ and we let $R_\varepsilon > 0$, $\delta_\varepsilon > 0$ and $C_\varepsilon > 0$ be given by Step 1. We write

$$\int_M d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x)$$

$$\leq \int_{M_{\alpha,\varepsilon}} d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x) + \sum_{i=1}^{p} \int_{B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)} d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x),$$

where $M_{\alpha,\varepsilon} = M \setminus \bigcup_{i=1}^{N} B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)$. From (5-15) and Hölder's inequalities we obtain

$$\int_{B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)} d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x) = O\left(\mu_\alpha^{(n-2)/2}d_g(x_{i,\alpha}, y_\alpha)^{2-n}\right)$$

for all $i \in \{1, \ldots, N\}$. Thus we get

$$\int_M d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x)$$
$$\leq \int_{M_{\alpha,\varepsilon}} d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x) + O\big(\mu_\alpha^{(n-2)/2} r_\alpha(y_\alpha)^{2-n}\big). \quad (5\text{-}23)$$

Using Step 1, we know that for any $x \in M_{\alpha,\varepsilon}$,

$$|\mathcal{U}_\alpha(x)|^{2^\star-1} \leq 2^{2^\star-2} C_\varepsilon^{2^\star-1} \bigg(\frac{\mu_\alpha^{(1-2\varepsilon)(n+2)/2}}{r_\alpha(x)^{(n+2)(1-\varepsilon)}} + \frac{\eta_\alpha(\delta_\varepsilon)^{2^\star-1}}{r_\alpha(x)^{(n+2)\varepsilon}} \bigg)$$

so that

$$\int_{M_{\alpha,\varepsilon}} d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x)$$
$$\leq 2^{2^\star-2} C_\varepsilon^{2^\star-1} \mu_\alpha^{(1-2\varepsilon)(n+2)/2} \int_{M_{\alpha,\varepsilon}} d_g(y_\alpha, x)^{2-n} r_\alpha(x)^{-(n+2)(1-\varepsilon)} dv_g(x)$$
$$+ 2^{2^\star-2} C_\varepsilon^{2^\star-1} \eta_\alpha(\delta_\varepsilon)^{2^\star-1} \int_{M_{\alpha,\varepsilon}} d_g(y_\alpha, x)^{2-n} r_\alpha(x)^{-(n+2)\varepsilon} dv_g(x)$$
$$\leq 2^{2^\star-2} C_\varepsilon^{2^\star-1} \mu_\alpha^{(1-2\varepsilon)(n+2)/2} \sum_{i=1}^{N} \int_{M \setminus B_{x_{i,\alpha}}(R_\varepsilon \mu_\alpha)} d_g(y_\alpha, x)^{2-n} d_g(x_{i,\alpha}, x)^{-(n+2)(1-\varepsilon)} dv_g(x)$$
$$+ 2^{2^\star-2} C_\varepsilon^{2^\star-1} \eta_\alpha(\delta_\varepsilon)^{2^\star-1} \sum_{i=1}^{N} \int_{M \setminus B_{x_{i,\alpha}}(R_\varepsilon \mu_\alpha)} d_g(y_\alpha, x)^{2-n} d_g(x_{i,\alpha}, x)^{-(n+2)\varepsilon} dv_g(x).$$

From (5-15), straightforward computations yield

$$\int_{M_{\alpha,\varepsilon}} d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x) = O\big(\mu_\alpha^{(n-2)/2} r_\alpha(y_\alpha)^{2-n}\big) + O\big(\eta_\alpha(\delta_\varepsilon)^{2^\star-1}\big).$$

Coming back to (5-23), using (5-3), we finally obtain that

$$\int_M d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x) = O\big(\mu_\alpha^{(n-2)/2} r_\alpha(y_\alpha)^{2-n}\big) + O(\eta_\alpha(\delta_\varepsilon)).$$

Coming back to (5-22), taking $0 < \delta < \delta_\varepsilon$ such that (5-18) and (5-19) hold, we get (5-14) under assumption (5-15). In particular, if δ is chosen sufficiently small, (5-14) holds. Now we claim that if $\mathcal{U}_\infty \equiv 0$, then

$$\eta_\alpha(\delta) = O\big(\mu_\alpha^{(n-2)/2}\big). \quad (5\text{-}24)$$

As a consequence of (5-14), there exists $C_0 > 0$ such that in any compact subset K of $M \setminus \mathcal{S}$,

$$|\mathcal{U}_\alpha(x)| \leq C_0 \Big(\mu_\alpha^{(n-2)/2} C_K + \eta_\alpha(\delta) \Big)$$

for some $C_K > 0$. If (5-24) were false, we would get by standard elliptic theory that

$$\frac{\mathcal{U}_\alpha}{\eta_\alpha(\delta)} \to H \quad \text{in } C^2_{\text{loc}}(M \setminus \mathcal{S}) \quad \text{as } \alpha \to +\infty,$$

where H satisfies $\Delta_g H + AH = 0$ in $M \setminus \mathcal{S}$ and $|H| \leq C_0$ in $M \setminus \mathcal{S}$. This implies that H is in the kernel of $\Delta_g + A$. Since all the components of H are nonnegative and H is not identically zero by the definition of $\eta_\alpha(\delta)$, this would contradict assumption (H). In particular, (5-24) is proved. Noting that if $\mathcal{U}_\infty \not\equiv 0$, then, by (5-3), $\eta_\alpha(\delta) = O(\|\mathcal{U}_\infty\|_\infty)$, we get with (5-14) that Step 2 holds true. □

Conclusion of the proof of Proposition 5.1. If $\mathcal{U}_\infty \equiv 0$, the proposition is a direct consequence of Step 2. Assume now that $\mathcal{U}_\infty \not\equiv 0$. We let $\mathcal{H}$ be the Green's function of the Laplacian on M normalized such that $\mathcal{H}(x, y) \geq 1$ for all $x, y \in M$, $x \neq y$. There exists $\Theta_1 > 1$ such that

$$\frac{1}{\Theta_1} d_g(x, y)^{2-n} \leq \mathcal{H}(x, y) \leq \Theta_1 d_g(x, y)^{2-n} \tag{5-25}$$

for all $x, y \in M$, $x \neq y$. We let (x_α) be a sequence of points in M and prove that

$$|\mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha)| = O\left(\mu_\alpha^{(n-2)/2} \mathcal{D}_\alpha(x_\alpha)^{2-n}\right) + o(1). \tag{5-26}$$

If $\mathcal{D}_\alpha(x_\alpha) = O(\mu_\alpha)$, then (5-26) is a direct consequence of the last equation in (4-4) of Proposition 4.2. We may therefore assume that

$$\frac{\mathcal{D}_\alpha(x_\alpha)}{\mu_\alpha} \to +\infty \quad \text{as } \alpha \to +\infty. \tag{5-27}$$

By standard elliptic theory,

$$\mathcal{U}_\alpha \to \mathcal{U}_\infty \text{ in } C^2_{\text{loc}}(M \setminus \mathcal{S}) \quad \text{as } \alpha \to +\infty, \tag{5-28}$$

where $\mathcal{S}$ is as in (5-1). We write using Green's representation formula that

$$\sum_{i=1}^{p} u_{i,\alpha}(x_\alpha) - \sum_{i=1}^{p} u_{i,\infty}(x_\alpha) = \frac{1}{V_g} \sum_{i=1}^{p} \int_M \left(u_{i,\alpha}(x_\alpha) - u_{i,\infty}(x_\alpha)\right) dv_g$$
$$+ \sum_{i=1}^{p} \int_M \mathcal{H}(x_\alpha, x) \Delta_g(u_{i,\alpha} - u_{i,\infty})(x) \, dv_g(x),$$

where V_g is the volume of (M, g), and the $u_{i,\infty}$'s are the components of $\mathcal{U}_\infty$. Then we get

$$\sum_{i=1}^{p} u_{i,\alpha}(x_\alpha) - \sum_{i=1}^{p} u_{i,\infty}(x_\alpha) = \sum_{i=1}^{p} \int_M \mathcal{H}(x_\alpha, x) \Delta_g(u_{i,\alpha} - u_{i,\infty})(x) \, dv_g(x) + o(1). \tag{5-29}$$

Thanks to (5-28) there exists $\delta_\alpha > 0$, $\delta_\alpha \to 0$ as $\alpha \to +\infty$, such that, up to a subsequence,

$$\|\mathcal{U}_\alpha - \mathcal{U}_\infty\|_{C^2(\{\mathcal{D}_\alpha > \delta_\alpha\})} = o(1), \tag{5-30}$$

where $\|\mathcal{U}\|_{C^2} = \sum_{i=1}^{p} \|u_i\|_{C^2}$, and $\{\mathcal{D}_\alpha > \delta_\alpha\}$ is the subset of M consisting of the $x \in M$ such that $\mathcal{D}_\alpha(x) > \delta_\alpha$. In particular, it follows from (5-25), (5-29) and (5-30) that

$$\sum_{i=1}^{p} u_{i,\alpha}(x_\alpha) - \sum_{i=1}^{p} u_{i,\infty}(x_\alpha) = \sum_{i=1}^{p} \int_{\{\mathcal{D}_\alpha(x) \leq \delta_\alpha\}} \mathcal{H}(x_\alpha, x) \Delta_g u_{i,\alpha}(x) \, dv_g(x) + o(1).$$

The proof of (5-26) then follows the lines of the proof of Step 2, using the estimate of that step. This ends the proof of Proposition 5.1. □

6. Strong pointwise estimates and sharp asymptotics

We now turn to pointwise estimates and sharp asymptotics. Our main result in this section is this:

Proposition 6.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, and $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$ satisfying (H). Let also $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of (1-1) such that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. Up to passing to a subsequence on the $\mathcal{U}_\alpha$'s, there holds that for any sequence $(x_\alpha)_\alpha$ of points in M,*

$$\left| \mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha) - \sum_{i=1}^N B_{i,\alpha}(x_\alpha)\Lambda_i \right| = \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty + O\left(\mu_\alpha^{(n-2)/2}\right) + o\left(\sum_{i=1}^N B_{i,\alpha}(x_\alpha)\right), \quad (6\text{-}1)$$

where $\mathcal{B}_\alpha^i = B_{i,\alpha}\Lambda_i$ for all α and all i, where $\mathcal{U}_\infty$, N, and the $\mathcal{B}_\alpha^i$'s are as in Proposition 4.2, where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, and $\mu_\alpha = \max_i \mu_{i,\alpha}$ is the maximum weight of the weights of the $B_{i,\alpha}$'s as in (5-6).

We prove Proposition 6.1 in several steps, based on induction on the following statement, defined for $1 \leq k \leq N+1$:

> There exists $C_k > 0$ such that, up to a subsequence on the $\mathcal{U}_\alpha$'s, for any sequence (x_α) of points in M,
>
> $$\left| \mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha) - \sum_{i=1}^{k-1} B_{i,\alpha}(x_\alpha)\Lambda_i \right|$$
> $$\leq C_k\left(\mu_\alpha^{(n-2)/2} + \chi_k \mu_{k,\alpha}^{(n-2)/2} R_{k,\alpha}(x_\alpha)^{2-n}\right) + \varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty + o\left(\sum_{i=1}^{k-1} B_{i,\alpha}(x_\alpha)\right), \quad (\mathcal{I}_k)$$
>
> where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, $\chi_k = 1$ if $k \leq N$, and $\chi_{N+1} = 0$.

Here we have reordered the blow-up points in such a way that $\mu_\alpha = \mu_{1,\alpha} \geq \mu_{2,\alpha} \geq \cdots \geq \mu_{N,\alpha}$, and we have defined

$$r_{i,\alpha}(x) = \min_{i \leq j \leq N} d_g(x_{j,\alpha}, x), \quad (6\text{-}2)$$

$$R_{i,\alpha}(x) = \min_{i \leq j \leq N} \left(d_g(x_{j,\alpha}, x) + \mu_{j,\alpha}\right). \quad (6\text{-}3)$$

We have $R_{1,\alpha}(x) = \mathcal{D}_\alpha(x)$ and $r_{1,\alpha}(x) = r_\alpha(x)$, where $\mathcal{D}_\alpha$ is as in Proposition 4.2 and r_α is as in (5-6).

We will refer to the whole indented statement above as $(\mathcal{I}_k)$, as well as the inequality so labeled. Clearly, Proposition 6.1 is equivalent to $(\mathcal{I}_{N+1})$, while Proposition 5.1 implies $(\mathcal{I}_1)$.

We apply induction on k to pass from $(\mathcal{I}_1)$ to $(\mathcal{I}_N)$, and then we use a slightly distinct argument to pass from $(\mathcal{I}_N)$ to $(\mathcal{I}_{N+1})$. In the following, we fix $1 \leq \kappa \leq N - 1$ and assume that $(\mathcal{I}_\kappa)$ holds true. We proceed in several steps, but first we fix some notation. As in the preceding section we let $\mathcal{G}$ be the Green's function of the operator $u \mapsto \Delta_g u + u$. Then (5-8) and (5-9) hold. We fix $0 < \varepsilon < 1/(n+2)$ and fix R_0 as in (5-5). For any $1 \leq i \leq \kappa$, we define

$$\Phi_{i,\alpha}^\varepsilon(x) = \min\left\{\mu_{i,\alpha}^{(1-2\varepsilon)(n-2)/2}\mathcal{G}(x_{i,\alpha}, x)^{1-\varepsilon}; D_0\mu_{i,\alpha}^{-(1-2\varepsilon)(n-2)/2}\mathcal{G}(x_{i,\alpha}, x)^\varepsilon\right\}, \quad (6\text{-}4)$$

where $x \in M \setminus \{x_{i,\alpha}\}$,

$$D_0 = \gamma_1^{2\varepsilon-1}(4R_0)^{(2-n)(1-2\varepsilon)},$$

and γ_1 is as in (5-8). This choice of D_0, together with (5-8), implies that

$$\Phi_{i,\alpha}^{\varepsilon}(x) = D_0 \mu_{i,\alpha}^{-(1-2\varepsilon)(n-2)/2} \mathcal{G}(x_{i,\alpha}, x)^{\varepsilon},$$

if $d_g(x_{i,\alpha}, x) \leq 2R_0 \mu_{i,\alpha}$. We also let

$$\varphi_{\alpha}^{\varepsilon}(x) = \max\{\|\mathcal{U}_{\infty}\|_{\infty}; \mu_{\alpha}^{(1-2\varepsilon)(n-2)/2}\} \sum_{i=1}^{N} \mathcal{G}(x_{i,\alpha}, x)^{\varepsilon} \tag{6-5}$$

and

$$\Psi_{\alpha}^{\varepsilon}(x) = \sum_{i=\kappa+1}^{N} \mathcal{G}(x_{i,\alpha}, x)^{1-\varepsilon}. \tag{6-6}$$

For $1 \leq i \leq \kappa$, we set

$$\Omega_{i,\alpha}^{\varepsilon} = \{x \in M \text{ s.t. } \Phi_{i,\alpha}^{\varepsilon}(x) \geq \Phi_{j,\alpha}^{\varepsilon}(x) \text{ for all } 1 \leq j \leq \kappa\}. \tag{6-7}$$

We also set

$$D(\varepsilon) = \frac{\gamma_2 \varepsilon(1-\varepsilon)}{2N}, \tag{6-8}$$

where γ_2 is as in (5-8), and we define $v_{\kappa,\alpha}$ by

$$v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} = \max\left\{\mu_{\kappa+1,\alpha}^{(1-2\varepsilon)(n-2)/2}; \max_{1 \leq i \leq \kappa} \sup_{\tilde{\Omega}_{i,\alpha}^{\varepsilon}} \frac{\Phi_{i,\alpha}^{\varepsilon}(x)}{\Psi_{\alpha}^{\varepsilon}(x)}\right\}, \tag{6-9}$$

where

$$\tilde{\Omega}_{i,\alpha}^{\varepsilon} = \{x \in \Omega_{i,\alpha}^{\varepsilon} \text{ s.t. } d_g(x_{i,\alpha}, x)^2 \mathcal{D}_{\kappa,\alpha}(x)^{2^{\star}-2} \geq D(\varepsilon)\}, \tag{6-10}$$

and $\mathcal{D}_{\kappa,\alpha}(x) = |\mathcal{U}_{\alpha}(x) - \mathcal{U}_{\infty}(x) - \sum_{i=1}^{\kappa} B_{i,\alpha}(x)\Lambda_i|$. By convention, the suprema in (6-9) are $-\infty$ if the sets $\tilde{\Omega}_{i,\alpha}^{\varepsilon}$ are empty. We can now start the proof of Proposition 6.1.

Step 1. $v_{\kappa,\alpha} = O(\mu_{\kappa,\alpha})$.

Proof of Step 1. We let $y_\alpha \in \tilde{\Omega}_{i,\alpha}^{\varepsilon}$ and assume that

$$v_{\kappa,\alpha}^{1-2\varepsilon} \Psi_{\alpha}^{\varepsilon}(y_\alpha)^{2/(n-2)} = \Phi_{i,\alpha}^{\varepsilon}(y_\alpha)^{2/(n-2)}.$$

This and (5-8) imply that

$$v_{\kappa,\alpha}^{1-2\varepsilon} = O\big(R_{\kappa+1,\alpha}(y_\alpha)^{2(1-\varepsilon)} \Phi_{i,\alpha}^{\varepsilon}(y_\alpha)^{2/(n-2)}\big). \tag{6-11}$$

Since $(\mathcal{I}_k)$ holds and $y_\alpha \in \tilde{\Omega}_{i,\alpha}^{\varepsilon}$, we also have

$$D(\varepsilon) \leq o(1) + o\left(\sum_{j=1}^{\kappa-1} d_g(x_{i,\alpha}, y_\alpha)^2 B_{j,\alpha}(y_\alpha)^{2^{\star}-2}\right) + O\big(\mu_{\kappa,\alpha}^2 d_g(x_{i,\alpha}, y_\alpha)^2 R_{\kappa,\alpha}(y_\alpha)^{-4}\big).$$

Since $y_\alpha \in \tilde{\Omega}_{i,\alpha}^{\varepsilon}$ and $\tilde{\Omega}_{i,\alpha}^{\varepsilon} \subset \Omega_{i,\alpha}^{\varepsilon}$, we can write

$$\sum_{j=1}^{\kappa-1} d_g(x_{i,\alpha}, y_\alpha)^2 B_{j,\alpha}(y_\alpha)^{2^{\star}-2} = O(1),$$

and we thus get

$$R_{\kappa,\alpha}(y_\alpha)^2 = O\big(\mu_{\kappa,\alpha} d_g(x_{i,\alpha}, y_\alpha)\big). \tag{6-12}$$

If $R_{\kappa+1,\alpha}(y_\alpha) = O(R_{\kappa,\alpha}(y_\alpha))$, we get from (6-11) and (6-12) that

$$v_{\kappa,\alpha}^{1-2\varepsilon} = O\big(\mu_{\kappa,\alpha}^{1-\varepsilon} d_g(x_{i,\alpha}, y_\alpha)^{1-\varepsilon} \Phi_{i,\alpha}^{\varepsilon}(y_\alpha)^{2/(n-2)}\big) = O(\mu_{\kappa,\alpha}^{1-\varepsilon} \mu_{i,\alpha}^{-\varepsilon}) = O(\mu_{\kappa,\alpha}^{1-2\varepsilon}),$$

and Step 1 is proved. Assume now that $R_{\kappa,\alpha}(y_\alpha) = o(R_{\kappa+1,\alpha}(y_\alpha))$. Then (6-12) becomes

$$\left(d_g(x_{\kappa,\alpha}, y_\alpha) + \mu_{\kappa,\alpha}\right)^2 = O(\mu_{\kappa,\alpha} d_g(x_{i,\alpha}, y_\alpha)). \tag{6-13}$$

If $i = \kappa$ we obtain $d_g(x_{i,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$; using the last equation in (4-4), and since $\mu_{i,\alpha} = o(R_{\kappa+1,\alpha}(y_\alpha))$, we obtain that

$$d_g(x_{i,\alpha}, y_\alpha)^2 \left| \mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \sum_{j=1}^{\kappa} B_{j,\alpha}(y_\alpha)\Lambda_j \right|^{2^\star-2} \to 0 \quad \text{as } \alpha \to +\infty.$$

This contradicts the fact that $y_\alpha \in \tilde{\Omega}_{i,\alpha}^\varepsilon$. Thus we must have $1 \le i \le \kappa - 1$. Since $\Phi_{i,\alpha}^\varepsilon(y_\alpha) \ge \Phi_{\kappa,\alpha}^\varepsilon(y_\alpha)$, because of (5-8), we can write

$$\mu_{\kappa,\alpha}^{1-2\varepsilon} d_g(x_{\kappa,\alpha}, y_\alpha)^{-2\varepsilon} \left(\mu_{\kappa,\alpha} + d_g(x_{\kappa,\alpha}, y_\alpha)\right)^{-2(1-2\varepsilon)} = O\left(\mu_{i,\alpha}^{1-2\varepsilon} d_g(x_{i,\alpha}, y_\alpha)^{-2\varepsilon} \left(\mu_{i,\alpha} + d_g(x_{i,\alpha}, y_\alpha)\right)^{-2(1-2\varepsilon)}\right).$$

In particular we obtain with (6-13) that

$$\left(\mu_{i,\alpha} + d_g(x_{i,\alpha}, y_\alpha)\right)^{1-\varepsilon} = O(\mu_{\kappa,\alpha}^\varepsilon \mu_{i,\alpha}^{1-2\varepsilon}).$$

Since $\mu_{k,\alpha} \le \mu_{i,\alpha}$, this implies that $d_g(x_{i,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$. We also get $\mu_{i,\alpha} = O(\mu_{\kappa,\alpha})$. Then we obtain with (6-13) that $d_g(x_{\kappa,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$, and this contradicts the first equation in (4-5) of Proposition 4.2. Step 1 is proved. □

Step 2. *There exists $C_\varepsilon > 0$ such that*

$$|\mathcal{U}_\alpha(x)| \le C_\varepsilon \left(\sum_{i=1}^{\kappa} \Phi_{i,\alpha}^\varepsilon(x) + v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} r_{\kappa+1,\alpha}(x)^{(2-n)(1-\varepsilon)} + \max\left\{\|\mathcal{U}_\infty\|_\infty; \mu_\alpha^{(1-2\varepsilon)(n-2)/2}\right\} r_\alpha(x)^{(2-n)\varepsilon}\right)$$

for all $x \in M \setminus \bigcup_{i=\kappa+1}^{N} B_{x_{i,\alpha}}(R_0\mu_{i,\alpha})$.

Proof of Step 2. We let $y_\alpha \in M \setminus \bigcup_{i=\kappa+1}^{N} B_{x_{i,\alpha}}(R_0\mu_{i,\alpha})$ be such that

$$\frac{\sum_{i=1}^{p} u_{i,\alpha}}{\sum_{i=1}^{\kappa} \Phi_{i,\alpha}^\varepsilon + v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}\Psi_{\alpha,\varepsilon} + \varphi_\alpha^\varepsilon}(y_\alpha) = \sup_{M \setminus \bigcup_{i=\kappa+1}^{N} B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})} \frac{\sum_{i=1}^{p} u_{i,\alpha}}{\sum_{i=1}^{\kappa} \Phi_{i,\alpha}^\varepsilon + v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}\Psi_{\alpha,\varepsilon} + \varphi_\alpha^\varepsilon}, \tag{6-14}$$

and we assume by contradiction that

$$\frac{\sum_{i=1}^{p} u_i^\alpha}{\sum_{i=1}^{\kappa} \Phi_{i,\alpha}^\varepsilon + v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}\Psi_{\alpha,\varepsilon} + \varphi_\alpha^\varepsilon}(y_\alpha) \to +\infty \quad \text{as } \alpha \to +\infty. \tag{6-15}$$

From $(\mathcal{I}_\kappa)$ and (6-15) we get

$$r_\alpha(y_\alpha) \to 0 \quad \text{as } \alpha \to +\infty. \tag{6-16}$$

We also have, using the second equation in (4-5),

$$\frac{d_g(x_{j,\alpha}, y_\alpha)}{\mu_{j,\alpha}} \to +\infty \tag{6-17}$$

as $\alpha \to +\infty$ for all $\kappa + 1 \le j \le N$. Here we used the fact that, by (6-9), $v_{\kappa,\alpha} \ge \mu_{\kappa+1,\alpha}$. Thanks to (6-15) and the second equation in (4-5), we also know that, for any $1 \le j \le \kappa$, either

$$d_g(x_{j,\alpha}, y_\alpha) \le R_0\mu_{j,\alpha} \quad \text{or} \quad \frac{d_g(x_{j,\alpha}, y_\alpha)}{\mu_{j,\alpha}} \to +\infty \quad \text{as } \alpha \to +\infty. \tag{6-18}$$

In particular, thanks to (6-14) we can write

$$\frac{\Delta_g \sum_{i=1}^{p} u_{i,\alpha}}{\sum_{i=1}^{p} u_{i,\alpha}}(y_\alpha) \geq \frac{\Delta_g \left(\sum_{i=1}^{\kappa} \Phi_{i,\alpha}^\varepsilon + v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon} + \varphi_\alpha^\varepsilon \right)}{\sum_{i=1}^{\kappa} \Phi_{i,\alpha}^\varepsilon + v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon} + \varphi_\alpha^\varepsilon}(y_\alpha). \tag{6-19}$$

From (1-1), (5-8), and (5-9), we then get

$$0 \geq \sum_{i=1}^{\kappa} \left(d_g(x_{i,\alpha}, y_\alpha)^{-2} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2 \varepsilon(1-\varepsilon)} - A_\varepsilon \right) \Phi_{i,\alpha}^\varepsilon(y_\alpha)$$

$$+ \left(\frac{r_{\kappa+1,\alpha}(y_\alpha)^{-2}}{N\gamma_1^{2(1-\varepsilon)}} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2 \varepsilon(1-\varepsilon)} - A_\varepsilon \right) v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon}(y_\alpha)$$

$$+ \left(\frac{r_\alpha(y_\alpha)^{-2}}{N\gamma_1^{2\varepsilon}} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2 \varepsilon(1-\varepsilon)} - A_\varepsilon \right) \varphi_{\alpha,\varepsilon}(y_\alpha), \tag{6-20}$$

where

$$A_\varepsilon = \frac{p\|A_\alpha\|_\infty + (1+\gamma_3\varepsilon)(1-\varepsilon)}{\gamma_2 \varepsilon(1-\varepsilon)}.$$

We let in the following $1 \leq i \leq \kappa$ be such that $y_\alpha \in \Omega_{i,\alpha}^\varepsilon$. Then we deduce from (6-20) that

$$0 \geq \left(d_g(x_{i,\alpha}, y_\alpha)^{-2} - \frac{\kappa}{\gamma_2 \varepsilon(1-\varepsilon)} |\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} - \kappa A_\varepsilon \right) \Phi_{i,\alpha}^\varepsilon(y_\alpha)$$

$$+ \left(\frac{r_{\kappa+1,\alpha}(y_\alpha)^{-2}}{N\gamma_1^{2(1-\varepsilon)}} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2 \varepsilon(1-\varepsilon)} - A_\varepsilon \right) v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon}(y_\alpha)$$

$$+ \left(\frac{r_\alpha(y_\alpha)^{-2}}{N\gamma_1^{2\varepsilon}} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2 \varepsilon(1-\varepsilon)} - A_\varepsilon \right) \varphi_{\alpha,\varepsilon}(y_\alpha). \tag{6-21}$$

From (6-15), we know that

$$\|\mathcal{U}_\infty\|_\infty = o(|\mathcal{U}_\alpha(y_\alpha)|) \quad \text{and} \quad \mu_\alpha^{(n-2)/2} = o(|\mathcal{U}_\alpha(y_\alpha)|), \tag{6-22}$$

and that

$$B_{j,\alpha}(y_\alpha) = o(|\mathcal{U}_\alpha(y_\alpha)|) \tag{6-23}$$

for all $1 \leq j \leq \kappa$ since

$$B_{j,\alpha}(y_\alpha) = O(\Phi_{j,\alpha}^\varepsilon(y_\alpha)) \tag{6-24}$$

for all $1 \leq j \leq \kappa$. From (6-17), we have

$$R_{\kappa+1,\alpha}(y_\alpha)^2 B_{j,\alpha}(y_\alpha)^{2^\star-2} = o(1) \tag{6-25}$$

for all $\kappa + 1 \leq j \leq N$. Thus we can deduce from the last equation in (4-4) of Proposition 4.2 together with (6-22), (6-23), and (6-25), that

$$\mathcal{D}_\alpha(y_\alpha)^2 |\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} = o(1). \tag{6-26}$$

Using (6-16) and (6-26), we can transform (6-21) into

$$0 \geq \left(d_g(x_{i,\alpha}, y_\alpha)^{-2} - \frac{\kappa}{\gamma_2 \varepsilon (1-\varepsilon)} |\mathcal{U}_\alpha(y_\alpha)|^{2^*-2} - \kappa A_\varepsilon \right) \Phi_{i,\alpha}^\varepsilon(y_\alpha)$$
$$+ \left(\frac{r_{\kappa+1,\alpha}(y_\alpha)^{-2}}{N\gamma_1^{2(1-\varepsilon)}} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^*-2}}{\gamma_2 \varepsilon (1-\varepsilon)} - A_\varepsilon \right) v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon}(y_\alpha)$$
$$+ \left(\frac{1}{N\gamma_1^{2\varepsilon}} + o(1) \right) r_\alpha(y_\alpha)^{-2} \varphi_{\alpha,\varepsilon}(y_\alpha). \tag{6-27}$$

Since $(\mathcal{I}_\kappa)$ holds true, we can prove with (6-22) and (6-23) that

$$|\mathcal{U}_\alpha(y_\alpha)|^{2^*-2} = O\left(\mu_{\kappa,\alpha}^2 R_{\kappa+1,\alpha}(y_\alpha)^{-4} \right). \tag{6-28}$$

This implies that

$$R_{\kappa+1,\alpha}(y_\alpha)^2 |\mathcal{U}_\alpha(y_\alpha)|^{2^*-2} \to 0 \quad \text{as } \alpha \to +\infty. \tag{6-29}$$

Indeed, if it is not the case, we would have from (6-28) that

$$R_{\kappa+1,\alpha}(y_\alpha) = O(\mu_{k,\alpha})$$

and thanks to (6-26) that there exists $j \in \{1, \ldots, \kappa\}$ such that

$$d_g(x_{j,\alpha}, y_\alpha) + \mu_{j,\alpha} = o(R_{\kappa+1,\alpha}(y_\alpha)).$$

In particular, we get a contradiction since $\mu_{j,\alpha} \geq \mu_{\kappa,\alpha}$. As a remark, (6-28) also implies that

$$R_{\kappa+1,\alpha}(y_\alpha) \to 0 \quad \text{as } \alpha \to +\infty, \tag{6-30}$$

due to (6-23). Now, thanks to (6-29) and (6-30), we deduce from (6-27) that

$$0 \geq \left(d_g(x_{i,\alpha}, y_\alpha)^{-2} - \frac{\kappa}{\gamma_2 \varepsilon (1-\varepsilon)} |\mathcal{U}_\alpha(y_\alpha)|^{2^*-2} - \kappa A_\varepsilon \right) \Phi_{i,\alpha}^\varepsilon(y_\alpha)$$
$$+ \left(\frac{1}{N\gamma_1^{2(1-\varepsilon)}} + o(1) \right) v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} r_{\kappa+1,\alpha}(y_\alpha)^{-2} \Psi_{\alpha,\varepsilon}(y_\alpha)$$
$$+ \left(\frac{1}{N\gamma_1^{2\varepsilon}} + o(1) \right) r_\alpha(y_\alpha)^{-2} \varphi_{\alpha,\varepsilon}(y_\alpha). \tag{6-31}$$

If $y_\alpha \notin \tilde{\Omega}_{i,\alpha}^\varepsilon$, we transform (6-31) into

$$0 \geq \left(1 + o(1) - \frac{\kappa D(\varepsilon)}{\gamma_2 \varepsilon (1-\varepsilon)} - \kappa A_\varepsilon d_g(x_{i,\alpha}, y_\alpha)^2 \right) d_g(x_{i,\alpha}, y_\alpha)^{-2} \Phi_{i,\alpha}^\varepsilon(y_\alpha)$$
$$+ \left(\frac{1}{N\gamma_1^{2(1-\varepsilon)}} + o(1) \right) v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} r_{\kappa+1,\alpha}(y_\alpha)^{-2} \Psi_{\alpha,\varepsilon}(y_\alpha) + \left(\frac{1}{N\gamma_1^{2\varepsilon}} + o(1) \right) r_\alpha(y_\alpha)^{-2} \varphi_{\alpha,\varepsilon}(y_\alpha)$$

by using (6-22) and (6-23). This leads to

$$\left(\frac{1}{N\gamma_1^{2\varepsilon}} + o(1) \right) r_\alpha(y_\alpha)^{-2} \varphi_{\alpha,\varepsilon}(y_\alpha) = O\left(\mu_{i,\alpha}^{(1-2\varepsilon)(n-2)/2} \right),$$

thanks to our choice of $D(\varepsilon)$. From (5-8), (6-16), and the definition of $\varphi_{\alpha,\varepsilon}$, we clearly get a contradiction.

Thus $y_\alpha \in \tilde{\Omega}_{i,\alpha}^\varepsilon$. Coming back to (6-31), we obtain in this situation that

$$\left(\frac{1}{N\gamma_1^{2(1-\varepsilon)}} + o(1)\right) v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon}(y_\alpha) \leq \left(\frac{\kappa}{\gamma_2 \varepsilon(1-\varepsilon)} |\mathcal{U}_\alpha(y_\alpha)|^{2^\star - 2} + \kappa A_\varepsilon\right) r_{\kappa+1,\alpha}(y_\alpha)^2 \Phi_{i,\alpha}^\varepsilon(y_\alpha).$$

Using (6-29), (6-30), and the definition of $v_{\kappa,\alpha}$, this leads to

$$\left(\frac{1}{N\gamma_1^{2(1-\varepsilon)}} + o(1)\right) v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon}(y_\alpha) = o(\Phi_{i,\alpha}^\varepsilon(y_\alpha)) = o\left(v_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon}(y_\alpha)\right)$$

and this is again a contradiction. Thus (6-15) cannot hold true and we get the equation in Step 2 from (5-8). This ends the proof of Step 2. $\qquad\square$

Step 3. *There exists $C_0 > 0$ such that*

$$|\mathcal{U}_\alpha(x)| \leq C_0 \left(\sum_{i=1}^\kappa B_{i,\alpha}(x) + \|\mathcal{U}_\infty\|_\infty + v_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(x)^{2-n}\right) \quad \textit{for all } x \in M \textit{ and all } \alpha > 0.$$

Proof of Step 3. We let (y_α) be a sequence of points in M and we aim to prove that

$$\limsup_{\alpha \to +\infty} \frac{|\mathcal{U}_\alpha(y_\alpha)|}{\sum_{i=1}^\kappa B_{i,\alpha}(y_\alpha) + \|\mathcal{U}_\infty\|_\infty + v_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}} < +\infty. \tag{6-32}$$

Since $(\mathcal{I}_\kappa)$ holds true, it is clear that (6-32) also holds true as soon as

$$\mu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n} = O(B_{i,\alpha}(y_\alpha))$$

for some $1 \leq i \leq \kappa$. By contradiction we assume in what follows that (6-32) does not hold true. Thus we assume from now on that

$$R_{\kappa+1,\alpha}(y_\alpha)^2 = o(\mu_{i,\alpha} \mu_{\kappa,\alpha}) + o\left(\frac{\mu_{\kappa,\alpha}}{\mu_{i,\alpha}} d_g(x_{i,\alpha}, y_\alpha)^2\right) \tag{6-33}$$

for all $1 \leq i \leq \kappa$. This implies in particular that

$$R_{\kappa+1,\alpha}(y_\alpha) \to 0 \quad \text{as } \alpha \to +\infty. \tag{6-34}$$

Thanks to the last equation in (4-4) and to (6-33), we can assume that

$$R_\alpha(y_\alpha) = R_{\kappa+1,\alpha}(y_\alpha) \quad \text{and} \quad \frac{R_{\kappa+1,\alpha}(y_\alpha)}{v_{\kappa,\alpha}} \to +\infty \quad \text{as } \alpha \to +\infty. \tag{6-35}$$

Indeed, otherwise, (6-32) holds true. We let $\lambda > 1$ be such that $\lambda p \|A\|_\infty \notin \mathrm{Sp}(\Delta_g)$, where $\mathrm{Sp}(\Delta_g)$ is the spectrum of Δ_g, and let G be the Green's function of $\mathfrak{L}_g = \Delta_g - \lambda p \|A\|_\infty$. There exist $C_1 > 1$, $C_2 > 0$ and $C_3 > 0$ such that

$$\frac{1}{C_1} d_g(x, y)^{2-n} - C_2 \leq G(x, y) \leq C_1 d_g(x, y)^{2-n} \tag{6-36}$$

and

$$|\nabla G(x, y)|_g \leq C_3 d_g(x, y)^{1-n} \tag{6-37}$$

for all $x, y \in M$, $x \neq y$. We let $x_0 \in \mathcal{S}$ be such that $d_g(y_\alpha, x_0) \leq \delta + o(1)$; such an x_0 does exist thanks to (6-34). We choose $\delta > 0$ such that

$$d_g(x, y) \geq 4\delta \tag{6-38}$$

for all distinct $x, y \in \mathcal{S}$, and such that

$$\delta \leq \tfrac{1}{4}(C_1 C_2)^{-1/(n-2)}, \tag{6-39}$$

where C_1 and C_2 are as in (6-36). We write with Green's representation formula that

$$\sum_{i=1}^{p} u_{i,\alpha}(y_\alpha) = \int_{B_{x_0}(2\delta)} G(y_\alpha, x) \mathcal{L}_g \left(\sum_{i=1}^{p} u_{i,\alpha} \right)(x) \, dv_g(x)$$

$$+ \int_{\partial B_{x_0}(2\delta)} G(y_\alpha, x) \partial_\nu \left(\sum_{i=1}^{p} u_{i,\alpha} \right)(x) \, d\sigma_g(x) - \int_{\partial B_{x_0}(2\delta)} \partial_\nu G(y_\alpha, x) \left(\sum_{i=1}^{p} u_{i,\alpha} \right)(x) \, d\sigma_g(x). \tag{6-40}$$

Since $\lambda > 1$, we get with (1-1) that

$$\mathcal{L}_g \left(\sum_{i=1}^{p} u_{i,\alpha} \right) \leq |\mathcal{U}_\alpha|^{2^\star - 2} \sum_{i=1}^{p} u_{i,\alpha}.$$

We have $G(y_\alpha, x) \geq 0$ in $B_{x_0}(2\delta)$ for α large by (6-36) and (6-39). Thus we can write

$$\int_{B_{x_0}(2\delta)} G(y_\alpha, x) \mathcal{L}_g \left(\sum_{i=1}^{p} u_{i,\alpha} \right)(x) \, dv_g(x) \leq C_1 \int_M d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star - 2} \sum_{i=1}^{p} u_{i,\alpha}(x) \, dv_g(x). \tag{6-41}$$

From (6-38), we also know that

$$d_g \left(x_{i,\alpha}, \partial B_{x_0}(2\delta) \right) \geq \delta$$

for α large. In particular, we can control the boundary terms in (6-40) thanks to Proposition 5.1 and standard elliptic theory. We thus obtain that

$$|\mathcal{U}_\alpha(y_\alpha)| = O \left(\max\{\mu_\alpha^{(n-2)/2}; \|\mathcal{U}_\infty\|_\infty\} \right) + O \left(\int_M d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star - 1} \, dv_g(x) \right). \tag{6-42}$$

We can now write thanks to Step 2 that

$$\int_M d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star - 1} dv_g(x) = O \left(\sum_{i=1}^{\kappa} \int_M d_g(y_\alpha, x)^{2-n} \Phi_{i,\alpha}^\varepsilon(x)^{2^\star - 1} dv_g(x) \right)$$

$$+ O \left(\max\{\|\mathcal{U}_\infty\|_\infty^{2^\star - 1}; \mu_\alpha^{(1-2\varepsilon)(n+2)/2}\} \int_M \frac{d_g(y_\alpha, x)^{2-n}}{r_\alpha(x)^{(n+2)\varepsilon}} dv_g(x) \right)$$

$$+ O \left(\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n+2)/2} \int_{\{r_{\kappa+1,\alpha}(x) \geq R_0 \nu_{\kappa,\alpha}\}} \frac{d_g(y_\alpha, x)^{2-n}}{r_{\kappa+1,\alpha}(x)^{(n+2)(1-\varepsilon)}} dv_g(x) \right)$$

$$+ O \left(\int_{\{r_{\kappa+1,\alpha}(x) \leq R_0 \nu_{\kappa,\alpha}\}} d_g(y_\alpha, x)^{2-n} |\mathcal{U}^\alpha|^{2^\star - 1} dv_g(x) \right). \tag{6-43}$$

Since $0 < \varepsilon < \dfrac{1}{n+2}$, it follows from Giraud's lemma that

$$\int_M d_g(y_\alpha, x)^{2-n} r_\alpha(x)^{-(n+2)\varepsilon} \, dv_g(x) = O(1). \tag{6-44}$$

We can also write, for $1 \le i \le \kappa$,

$$\int_M d_g(y_\alpha, x)^{2-n} \Phi_{i,\alpha}^\varepsilon(x)^{2^\star-1} dv_g(x)$$

$$= O\left(\mu_{i,\alpha}^{-(1-2\varepsilon)(n+2)/2} \int_{\{d_g(x_{i,\alpha}, x) \le \mu_{i,\alpha}\}} d_g(y_\alpha, x)^{2-n} d_g(x_{i,\alpha}, x)^{-(n+2)\varepsilon} dv_g(x)\right)$$

$$+ O\left(\mu_{i,\alpha}^{(1-2\varepsilon)(n+2)/2} \int_{\{d_g(x_{i,\alpha}, x) \ge \mu_{i,\alpha}\}} d_g(y_\alpha, x)^{2-n} d_g(x_{i,\alpha}, x)^{-(n+2)(1-\varepsilon)} dv_g(x)\right)$$

thanks to (5-8) and (6-4). Direct computations, using Giraud's lemma and the inequalities $0 < \varepsilon < \dfrac{1}{n+2}$, lead then to

$$\int_M d_g(y_\alpha, x)^{2-n} \Phi_{i,\alpha}^\varepsilon(x)^{2^\star-1} dv_g(x) = O(B_{i,\alpha}(y_\alpha)). \tag{6-45}$$

By direct computations, using Giraud's lemma, the inequalities $0 < \varepsilon < \dfrac{1}{n+2}$ and (6-35), we also get

$$\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n+2)/2} \int_{\{r_{\kappa+1,\alpha}(x) \ge R_0 \nu_{\kappa,\alpha}\}} d_g(y_\alpha, x)^{2-n} r_{\kappa+1,\alpha}(x)^{-(n+2)(1-\varepsilon)} dv_g(x)$$

$$= O\left(\nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\right), \tag{6-46}$$

while, using (6-35), the fact that $\nu_{\kappa,\alpha} \ge \mu_{\kappa+1,\alpha}$, and Hölder's inequalities, we also have

$$\int_{\{r_{\kappa+1,\alpha}(x) \le R_0 \nu_{\kappa,\alpha}\}} d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha|^{2^\star-1} dv_g(x) = O\left(\nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\right). \tag{6-47}$$

Coming back to (6-42) with (6-43)-(6-47), we obtain a contradiction with the assumption that (6-32) does not hold true. This proves Step 3.   □

The fourth step in the proof of Proposition 6.1 is as follows. The constants $C > 0$ in the statement of this step and its proof are independent of α and built on C_κ. They may change from line to line.

Step 4. *There exists $C > 0$ such that for any sequence (y_α) of points in M,*

$$\left|\mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \sum_{i=1}^\kappa B_{i,\alpha}(y_\alpha) \Lambda_i\right|$$

$$\le \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty + o\left(\sum_{i=1}^\kappa B_{i,\alpha}(y_\alpha)\right) + C\left(\mu_\alpha^{(n-2)/2} + \nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\right),$$

where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$.

Proof of Step 4. Let (y_α) be a sequence of points in M. Assume first that

$$R_{\kappa+1,\alpha}(y_\alpha) = O(\nu_{\kappa,\alpha}). \tag{6-48}$$

If $R_{\kappa+1,\alpha}(y_\alpha) = \mathcal{D}_\alpha(y_\alpha)$, we can apply the last equation in (4-4) of Proposition 4.2 to obtain

$$\left|\mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \sum_{i=1}^\kappa B_{i,\alpha}(y_\alpha) \Lambda_i\right| \le C \nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}.$$

In particular, the estimate of Step 4 holds true. If $\mathcal{D}_\alpha(y_\alpha) < R_{\kappa+1,\alpha}(y_\alpha)$, then from Step 1 and (6-48) we obtain the existence of some $1 \le i \le \kappa$ such that

$$d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha} < R_{\kappa+1,\alpha}(y_\alpha) = O(\mu_{\kappa,\alpha}).$$

This implies the following facts:

$$\mu_{i,\alpha} = O(\mu_{\kappa,\alpha}),$$
$$d_g(x_{i,\alpha}, y_\alpha) = O(\mu_{i,\alpha}),$$
$$R_{\kappa+1,\alpha}(y_\alpha) \geq \mu_{i,\alpha}.$$

Using (4-5) in Proposition 4.2 we get

$$\left| \mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \sum_{i=1}^{\kappa} B_{i,\alpha}(y_\alpha)\Lambda_i \right| = o(B_{i,\alpha}(y_\alpha)),$$

and the estimate of Step 4 holds also in this case. As a consequence, we may assume below that

$$\frac{R_{\kappa+1,\alpha}(y_\alpha)}{v_{\kappa,\alpha}} \to +\infty \quad \text{as } \alpha \to +\infty. \tag{6-49}$$

The rest of the proof is based on controlling the different terms we get from Green's representation formula. We let $\mathcal{H}$ be the Green's function of the Laplacian on M normalized such that $\mathcal{H}(x, y) \geq 1$ for all $x, y \in M$, $x \neq y$. Then (5-25) holds and moreover

$$(x, y) \overset{\Phi}{\mapsto} d_g(x, y)^{n-2}\mathcal{H}(x, y)$$

extends to a continuous function in $M \times M$ whose value on the diagonal is

$$\Phi(x, x) = \frac{1}{(n-2)\omega_{n-1}}$$

for all x. Now we write, for any $i \in \{1, \ldots, p\}$,

$$u_{i,\alpha}(y_\alpha) - u_{i,\infty}(y_\alpha) = \frac{1}{V_g} \int_M (u_{i,\alpha} - u_{i,\infty})\, dv_g + \int_M \mathcal{H}(x, y_\alpha)\Delta_g(u_{i,\alpha} - u_{i,\infty})(x)\, dv_g(x). \tag{6-50}$$

Since $(\mathcal{I}_\kappa)$ holds true, we can write

$$\left| \int_M (u_{i,\alpha} - u_{i,\infty})\, dv_g \right| \leq C\mu_\alpha^{(n-2)/2} + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty,$$

where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$. Thus we can transform (6-50) into

$$\left| u_{i,\alpha}(y_\alpha) - u_{i,\infty}(y_\alpha) - \int_M \mathcal{H}(x, y_\alpha)\Delta_g(u_{i,\alpha} - u_{i,\infty})(x)\, dv_g(x) \right| \leq C\mu_\alpha^{(n-2)/2} + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty. \tag{6-51}$$

In view of the equations satisfied by the $\mathcal{U}_\alpha$'s and $\mathcal{U}_\infty$, we can now write

$$\Delta_g(u_{i,\alpha} - u_{i,\infty}) = |\mathcal{U}_\alpha|^{2^\star-2}u_{i,\alpha} - |\mathcal{U}_\infty|^{2^\star-2}u_{i,\infty} - \sum_{j=1}^p A_{ij}^\alpha u_{j,\alpha} + \sum_{j=1}^p A_{ij} u_{j,\infty}$$

$$= |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}(u_{i,\alpha} - u_{i,\infty}) + (|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2})(u_{i,\alpha} - u_{i,\infty})$$

$$+ (|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\infty|^{2^\star-2})u_{i,\infty} - \sum_{j=1}^p A_{ij}^\alpha(u_{j,\alpha} - u_{j,\infty}) + \sum_{j=1}^p (A_{ij} - A_{ij}^\alpha)u_{j,\infty}.$$

Thus we obtain

$$
\left| u_{i,\alpha}(y_\alpha) - u_{i,\infty}(y_\alpha) - \int_M \mathcal{H}_{y_\alpha} |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star - 2} (u_{i,\alpha} - u_{i,\infty}) \, dv_g \right|
$$

$$
\leq \left| \int_M \mathcal{H}_{y_\alpha} \left(|\mathcal{U}_\alpha|^{2^\star - 2} - |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star - 2} \right) (u_{i,\alpha} - u_{i,\infty}) \, dv_g \right|
$$

$$
+ \left| \int_M \mathcal{H}_{y_\alpha} \left(|\mathcal{U}_\alpha|^{2^\star - 2} - |\mathcal{U}_\infty|^{2^\star - 2} \right) u_{i,\infty} dv_g \right| + \left| \sum_{j=1}^p \int_M \mathcal{H}_{y_\alpha} \left(A_{ij} - A_{ij}^\alpha \right)(x) u_{j,\infty} dv_g \right|
$$

$$
+ \left| \sum_{j=1}^p \int_M \mathcal{H}_{y_\alpha} A_{ij}^\alpha (u_{j,\alpha} - u_{j,\infty}) \, dv_g \right| + C \mu_\alpha^{(n-2)/2} + \varepsilon_\alpha |\mathcal{U}_\infty|_\infty, \tag{6-52}
$$

where $\mathcal{H}_{y_\alpha}(x) = \mathcal{H}(y_\alpha, x)$ for all x. The convergence of the A_α's to A, together with (5-25), implies that

$$
\sum_{j=1}^p \int_M \mathcal{H}_{y_\alpha} (A_{ij} - A_{ij}^\alpha) u_{j,\infty} dv_g = \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty, \tag{6-53}
$$

where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$. Now we get with (5-25) that

$$
\left| \sum_{j=1}^p \int_M \mathcal{H}_{y_\alpha} A_{ij}^\alpha \left(u_{j,\alpha} - u_{j,\infty} \right) \, dv_g \right| \leq pC \|A_\alpha\|_\infty \int_M d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x) - \mathcal{U}_\infty(x)| \, dv_g(x).
$$

Thanks to $(\mathcal{I}_\kappa)$, we can write

$$
|\mathcal{U}_\alpha(x) - \mathcal{U}_\infty(x)| \leq D_1 \left(\sum_{j=1}^\kappa B_{j,\alpha}(x) + \mu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(x)^{2-n} + \mu_\alpha^{(n-2)/2} \right) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty
$$

for some $D_1 > 0$, where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, while, thanks to Step 3, we have

$$
|\mathcal{U}_\alpha(x) - \mathcal{U}_\infty(x)| \leq D_2 \left(\sum_{j=1}^\kappa B_{j,\alpha}(x) + \nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(x)^{2-n} + \|\mathcal{U}_\infty\|_\infty \right)
$$

for some $D_2 > 0$. Thus we can write

$$
\left| \sum_{j=1}^p \int_M \mathcal{H}_{y_\alpha} A_{ij}^\alpha \left(u_{j,\alpha} - u_{j,\infty} \right) \, dv_g \right| \leq C \left(\sum_{j=1}^\kappa \int_M d_g(y_\alpha, x)^{2-n} B_{j,\alpha}(x) \, dv_g(x) \right) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty + C \mu_\alpha^{(n-2)/2}
$$

$$
+ C \mu_{\kappa,\alpha}^{(n-2)/2} \int_{\{R_{\kappa+1,\alpha}(x) \geq \eta_\alpha\}} d_g(y_\alpha, x)^{2-n} R_{\kappa+1,\alpha}(x)^{2-n} dv_g(x)
$$

$$
+ C \nu_{\kappa,\alpha}^{(n-2)/2} \int_{\{R_{\kappa+1,\alpha}(x) \leq \eta_\alpha\}} d_g(y_\alpha, x)^{2-n} R_{\kappa+1,\alpha}(x)^{2-n} dv_g(x),
$$

where $\eta_\alpha = 2 \mathrm{diam}_g M$ if $\mathcal{U}_\infty \equiv 0$, $\eta_\alpha = \mu_{\kappa,\alpha}^{1/2}$ otherwise, and $\mathrm{diam}_g M$ is the diameter of M with respect to g. Simple computations, using Giraud's lemma, then lead to the estimate

$$
\left| \sum_{j=1}^p \int_M \mathcal{H}_{y_\alpha} A_{ij}^\alpha (u_{j,\alpha} - u_{j,\infty}) \, dv_g \right|
$$

$$
\leq o \left(\sum_{j=1}^\kappa |B_j^\alpha(y_\alpha)| \right) + C \mu_\alpha^{(n-2)/2} + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty + C \nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}. \tag{6-54}
$$

If $\mathcal{U}_\infty \equiv 0$, we have

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g = 0,$$

while, if $\mathcal{U}_\infty \not\equiv 0$, we write, thanks to Proposition 5.1,

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2}-|\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g = o(1) + \int_{\{\mathcal{D}_\alpha \leq \mu_\alpha^{1/4}\}} \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g$$

$$= o(1) + O\bigg(\int_{\{\mathcal{D}_\alpha \leq \mu_\alpha^{1/4}\}} d_g(x, y_\alpha)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2}dv_g(x)\bigg). \quad \text{(6-55)}$$

Now we use Step 3 and we briefly distinguish the $n = 3, 4, 5$, and $n \geq 6$ cases in the forthcoming computations. We let $(R_\alpha)_\alpha$ be suitably chosen such that $R_\alpha \to +\infty$ as $\alpha \to +\infty$. Assuming that $n = 3, 4, 5$, we write with (6-55) that

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g$$

$$= o(1) + O\bigg(\int_{\{R_{\kappa+1,\alpha} \leq R_{\kappa+1,\alpha}(y_\alpha)/R_\alpha\}} d_g(x, y_\alpha)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2}\,dv_g(x)\bigg)$$

$$+ O\bigg(\int_{\{R_{\kappa+1,\alpha} \geq R_{\kappa+1,\alpha}(y_\alpha)/R_\alpha\}\cap\{\mathcal{D}_\alpha \leq \mu_\alpha^{1/4}\}} d_g(x, y_\alpha)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2}dv_g(x)\bigg)$$

$$= o(1) + O\bigg(\sum_{j=1}^{\kappa} \int_M d_g(x, y_\alpha)^{2-n} B_{j,\alpha}(x)^{2^\star-2}dv_g(x)\bigg)$$

$$+ O\bigg(v_{\kappa,\alpha}^2 \int_{\{R_{\kappa+1,\alpha} \geq R_{\kappa+1,\alpha}(y_\alpha)/R_\alpha\}} d_g(x, y_\alpha)^{2-n} R_{\kappa+1,\alpha}(x)^{-4}dv_g(x)\bigg)$$

$$= o(1) + o\bigg(\sum_{j=1}^{\kappa} B_{j,\alpha}(y_\alpha)\bigg) + o\big(v_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big),$$

and, assuming that $n \geq 6$, since $2^\star - 2 \in (0, 1]$ in this case, we get from (6-55) that

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2}-|\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g = o(1) + O\bigg(\int_{\{\mathcal{D}_\alpha \leq \mu_\alpha^{1/4}\}} d_g(x, y_\alpha)^{2-n}|\mathcal{U}_\alpha(x)|dv_g(x)\bigg)$$

$$= o(1) + O\bigg(\sum_{j=1}^{\kappa} \int_M d_g(x, y_\alpha)^{2-n} B_{j,\alpha}(x)\,dv_g(x)\bigg)$$

$$+ O\bigg(v_{\kappa,\alpha}^{(n-2)/2} \int_M d_g(x, y_\alpha)^{2-n} R_{\kappa+1,\alpha}(x)^{2-n}dv_g(x)\bigg)$$

$$= o(1) + o\bigg(\sum_{j=1}^{\kappa} B_{j,\alpha}(y_\alpha)\bigg) + o\big(v_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big).$$

Thus, in all cases,

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g$$
$$= o\bigg(\sum_{j=1}^\kappa B_{j,\alpha}(y_\alpha)\bigg) + o\big(v_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big) + \varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty. \quad (6\text{-}56)$$

Similarly, if $\mathcal{U}_\infty \equiv 0$, then

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}\big)(u_{i,\alpha} - u_{i,\infty})\,dv_g = 0,$$

while, if $\mathcal{U}^\infty \not\equiv 0$, we can write

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}\big)(u_{i,\alpha} - u_{i,\infty})\,dv_g$$
$$= o(1) + O\bigg(\int_{\{R_\alpha(x)\leq\mu_\alpha^{1/4}\}} d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2}\,dv_g(x)\bigg)$$
$$= o(1) + o\bigg(\sum_{j=1}^\kappa B_{j,\alpha}(y_\alpha)\bigg) + o\big(v_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big).$$

Thus we have obtained

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}\big)(u_{i,\alpha} - u_{i,\infty})\,dv_g$$
$$= o\bigg(\sum_{j=1}^\kappa B_{j,\alpha}(y_\alpha)\bigg) + o\big(v_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big) + \varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty. \quad (6\text{-}57)$$

Coming back to (6-52): thanks to (6-53)–(6-57), we now obtain

$$\bigg|\mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \int_M \mathcal{H}_{y_\alpha}|\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}\,(\mathcal{U}_\alpha - \mathcal{U}_\infty)\,dv_g\bigg|$$
$$\leq o\bigg(\sum_{j=1}^\kappa B_{j,\alpha}(y_\alpha)\bigg) + o\big(v_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big) + \varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty + C\mu_\alpha^{(n-2)/2}. \quad (6\text{-}58)$$

Using (4-5), (5-25), and the extension property of $d_g(x, y)^{n-2}\mathcal{H}(x, y)$ mentioned above, we can find a sequence $(R_\alpha)_\alpha$ such that $R_\alpha \to +\infty$ as $\alpha \to +\infty$, that

$$\bigg|\int_{M_{i,\alpha}} \mathcal{H}_{y_\alpha}|\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}(\mathcal{U}_\alpha - \mathcal{U}_\infty)\,dv_g - B_{i,\alpha}(y_\alpha)\Lambda_i\bigg| \leq C\mu_\alpha^{(n-2)/2} + o(B_{i,\alpha}(y_\alpha))$$

for all $i \in \{1, \ldots, \kappa\}$, and that the sets

$$M_{i,\alpha} = B_{x_{i,\alpha}}(R_\alpha\mu_{i,\alpha}) \setminus \bigcup_{i+1\leq j\leq N} B_{x_{j,\alpha}}(R_\alpha^{-1}\mu_{i,\alpha})$$

are disjoint.

Then we can write thanks to Proposition 5.1, Step 3, and (6-49), that

$$\int_{M\setminus\bigcup_{1\le i\le \kappa} M_{i,\alpha}} d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x) - \mathcal{U}_\infty(x)|^{2^*-1}dv_g(x)$$

$$\le \varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty + C\mu_\alpha^{(n-2)/2} + C\sum_{j=1}^{\kappa}\int_{M_\alpha} d_g(y_\alpha, x)^{2-n}B_{j,\alpha}(x)^{2^*-1}dv_g(x)$$

$$+ C\nu_{\kappa,\alpha}^{(n+2)/2}\int_{\{R_{\kappa+1,\alpha}\ge \nu_{\kappa,\alpha}\}}\frac{d_g(y_\alpha, x)^{2-n}}{R_{\kappa+1,\alpha}(x)^{(n+2)}}dv_g(x) + C\nu_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}$$

$$\le \varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty + C\mu_\alpha^{(n-2)/2} + o\left(\sum_{j=1}^{\kappa}B_{j,\alpha}(y_\alpha)\right) + C\nu_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}.$$

Coming back to (6-58), this ends the proof of Step 4. □

Step 5. $\nu_{\kappa,\alpha} = \mu_{\kappa+1,\alpha}$.

Proof of Step 5. We proceed by contradiction and thus assume that there exists $i \in \{1, \ldots, \kappa\}$ and a sequence (y_α) of points in $\tilde{\Omega}_{i,\alpha}^\varepsilon$ such that

$$\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}\Psi_\alpha^\varepsilon(y_\alpha) = \Phi_{i,\alpha}^\varepsilon(y_\alpha). \tag{6-59}$$

Since $y_\alpha \in \tilde{\Omega}_{i,\alpha}^\varepsilon$, we know that

$$\Phi_{i,\alpha}^\varepsilon(y_\alpha) \ge \Phi_{j,\alpha}^\varepsilon(y_\alpha) \tag{6-60}$$

for all $1 \le j \le \kappa$ and that

$$d_g(x_{i,\alpha}, y_\alpha)^2\left|\mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \sum_{j=1}^{\kappa}B_{j,\alpha}(y_\alpha)\Lambda_j\right|^{2^*-2} \ge D(\varepsilon). \tag{6-61}$$

Clearly,

$$d_g(x_{i,\alpha}, x)^2 B_{i,\alpha}(x)^{2^*-2} = O(1). \tag{6-62}$$

We now claim that

$$d_g(x_{i,\alpha}, y_\alpha)^2 B_{j,\alpha}(y_\alpha)^{2^*-2} \to 0 \quad \text{as } \alpha \to +\infty \tag{6-63}$$

for all $1 \le j \le \kappa$, $j \ne i$. In order to prove (6-63), we proceed by contradiction once again and assume that there exists $1 \le j \le \kappa$, $j \ne i$, such that

$$\left(d_g(x_{j,\alpha}, y_\alpha) + \mu_{j,\alpha}\right)^2 = O\left(\mu_{j,\alpha}d_g(x_{i,\alpha}, y_\alpha)\right). \tag{6-64}$$

Since $\Phi_{i,\alpha}^\varepsilon(y_\alpha) \ge \Phi_{j,\alpha}^\varepsilon(y_\alpha)$, we then get

$$\left(d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha}\right)^{1-\varepsilon} = O(\mu_{j,\alpha}^\varepsilon\mu_{i,\alpha}^{1-2\varepsilon}),$$

so $\mu_{i,\alpha} = O(\mu_{j,\alpha})$ and $d_g(x_{i,\alpha}, y_\alpha)^{1-\varepsilon} = O(\mu_{j,\alpha}^\varepsilon\mu_{i,\alpha}^{1-2\varepsilon})$. Coming back to (6-64), we also obtain $\mu_{j,\alpha} = O(\mu_{i,\alpha})$ and $d_g(x_{j,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$. This contradicts the first equation in (4-5). Thus (6-63) is proved.

Applying Step 4, we get from (6-61), (6-62), and (6-63) that

$$R_{\kappa+1,\alpha}(y_\alpha)^2 = O\left(\nu_{\kappa,\alpha}d_g(x_{i,\alpha}, y_\alpha)\right). \tag{6-65}$$

Using (5-8) and (6-59), we also have

$$\left(d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha}\right)^{2(1-\varepsilon)} = O\left(\mu_{i,\alpha}^{1-2\varepsilon} v_{\kappa,\alpha}^{2\varepsilon-1} R_{\kappa+1,\alpha}(y_\alpha)^{2(1-\varepsilon)}\right),$$

so that, with (6-65) and Step 1, we get that $\mu_{i,\alpha} = O(\mu_{\kappa,\alpha})$, that $d_g(x_{i,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$ and that $\mu_{i,\alpha} = O\left(R_{\kappa+1,\alpha}(y_\alpha)\right)$. Using the second equation in (4-5) of Proposition 4.2 we then obtain

$$d_g(x_{i,\alpha}, y_\alpha)^2 |\mathcal{U}_\alpha(y_\alpha) - B_{i,\alpha}\Lambda_i(y_\alpha)|^{2^\star - 2} \to 0 \quad \text{as } \alpha \to +\infty.$$

This contradicts (6-61) thanks to (6-63). Step 5 is proved. $\qquad\square$

Conclusion of the proof of Proposition 6.1. By Proposition 5.1 we know that $(\mathcal{I}_1)$ holds true. By Steps 4 and 5, and by induction, it follows that $(\mathcal{I}_N)$ holds true. It remains to prove that $(\mathcal{I}_{N+1})$ also holds true. For this we proceed with similar arguments to those developed in the proof of Step 4. We let (y_α) be a sequence of points in M and write, for any $i = 1, \ldots, p$,

$$u_{i,\alpha}(y_\alpha) - u_{i,\infty}(y_\alpha) = \frac{1}{V_g} \int_M (u_{i,\alpha} - u_{i,\infty})\, dv_g + \int_M \mathcal{H}_{y_\alpha} \Delta_g (u_{i,\alpha} - u_{i,\infty})\, dv_g, \qquad (6\text{-}66)$$

where $\mathcal{H}_{y_\alpha}(\cdot) = \mathcal{H}(\cdot, y_\alpha)$ and $\mathcal{H}$ is the Green's function of Δ_g normalized so that $\mathcal{H} \geq 1$. Since $(\mathcal{I}_N)$ holds true,

$$\int_M |u_{i,\alpha} - u_{i,\infty}|\, dv_g \leq C\mu_\alpha^{(n-2)/2} + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty,$$

where $C > 0$ is independent of α, and $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$. Using the equations satisfied by the $\mathcal{U}_\alpha$'s and $\mathcal{U}_\infty$, but also $(\mathcal{I}_N)$, mimicking what was done in the proof of Step 4, we get with (6-66) that

$$\left| \mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \int_M \mathcal{H}_{y_\alpha} |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star - 2}(u_{i,\alpha} - u_{i,\infty})\, dv_g \right|$$
$$\leq C\mu_\alpha^{(n-2)/2} + o\left(\sum_{i=1}^N B_{i,\alpha}(y_\alpha)\right) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty. \quad (6\text{-}67)$$

We also have

$$\left| \int_M \mathcal{H}_{y_\alpha} |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star - 2}(u_{i,\alpha} - u_{i,\infty})\, dv_g - \sum_{i=1}^N B_{i,\alpha}(y_\alpha)\Lambda_i \right|$$
$$\leq C\mu_\alpha^{(n-2)/2} + o\left(\sum_{i=1}^N B_{i,\alpha}(y_\alpha)\right) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty, \quad (6\text{-}68)$$

where $C > 0$ in (6-67), (6-68) is independent of α, and $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$. Combining (6-67) and (6-68), we get $(\mathcal{I}_{N+1})$. This ends the proof of Proposition 6.1. $\qquad\square$

7. A Pohozaev identity for systems

Let (M, g) be a smooth compact Riemannian manifold. Let also X be a smooth 1-form over M and $\mathcal{U}: M \to \mathbb{R}^p$ be a C^2-map. We define $X(\nabla\mathcal{U})$ by $X(\nabla\mathcal{U}) = (\nabla\mathcal{U}, X)$. This is a p-map with components $X(\nabla\mathcal{U})_i = (\nabla u_i, X)$ where the u_i's are the components of $\mathcal{U}$. We define also $|\nabla\mathcal{U}|$ and $(T_\nabla X)_\mathcal{U}$ by $|\nabla\mathcal{U}|^2 = \sum_{i=1}^p |\nabla u_i|^2$ and $(T_\nabla X)_\mathcal{U} = \sum_{i=1}^p S_X^\sharp(\nabla u_i, \nabla u_i)$, where $S_X^\sharp$ is the $(0, 2)$-tensor field we obtain

from the $(2, 0)$-tensor field S_X by the musical isomorphism, and

$$S_X = \nabla X - \frac{1}{n}(\text{div}_g X)g. \tag{7-1}$$

For Ω a smooth bounded domain in M we let ν be the unit outer normal to $\partial\Omega$. The Pohozaev-type identity for systems we prove is stated as follows.

Proposition 7.1. *Let (M, g) be a smooth compact Riemannian n-manifold, Ω be a smooth bounded domain in M, and $A : M \to M_p^s(\mathbb{R})$ be a C^1-map. Let X be a smooth 1-form over M and $\mathcal{U}$ be a solution of* (0-1). *Then*

$$\int_\Omega \langle A\mathcal{U}, X(\nabla\mathcal{U})\rangle_{\mathbb{R}^p} dv_g + \frac{n-2}{4n}\int_\Omega (\Delta_g(\text{div}_g X))|\mathcal{U}|^2 dv_g + \frac{n-2}{2n}\int_\Omega (\text{div}_g X)\langle A\mathcal{U}, \mathcal{U}\rangle_{\mathbb{R}^p} dv_g$$

$$= -\int_\Omega (T_\nabla X)_{\mathcal{U}} dv_g + \frac{n-2}{2n}\int_{\partial\Omega} X(\nu)|\mathcal{U}|^{2^\star} d\sigma_g - \frac{n-2}{4n}\int_{\partial\Omega} \partial_\nu(\text{div}_g X)|\mathcal{U}|^2 d\sigma_g$$

$$+ \frac{n-2}{2n}\int_{\partial\Omega} (\text{div}_g X)\langle\partial_\nu\mathcal{U}, \mathcal{U}\rangle_{\mathbb{R}^p} d\sigma_g - \int_{\partial\Omega} B_{\partial\Omega}(\mathcal{U}) d\sigma_g, \quad \text{(7-2)}$$

where $X(\nabla\mathcal{U})$ and $(T_\nabla X)_{\mathcal{U}}$ are as above, $B_{\partial\Omega}(\mathcal{U}) = \frac{1}{2}X(\nu)|\nabla\mathcal{U}|^2 - \langle X(\nabla\mathcal{U}), \partial_\nu\mathcal{U}\rangle_{\mathbb{R}^p}$ on $\partial\Omega$, and $\langle\cdot, \cdot\rangle_{\mathbb{R}^p}$ is the scalar product in $\mathbb{R}^p$.

Proof. Integrating by parts we easily see that for $u : M \to \mathbb{R}$ of class C^2,

$$\int_\Omega (\nabla u, X)\Delta_g u\, dv_g = \int_\Omega S_{X,2}^\sharp(\nabla u, \nabla u)\, dv_g + \int_{\partial\Omega}\left(\frac{1}{2}\int_\Omega X(\nu)|\nabla u|^2 - (\nabla u, X)\partial_\nu u\right) d\sigma_g, \quad \text{(7-3)}$$

where $S_{X,2} = \nabla X - \frac{1}{2}(\text{div}_g X)g$. If we assume now that $\mathcal{U}$ is a p-map, applying (7-3) to the components u_i of $\mathcal{U}$, and summing over i, we obtain

$$\int_\Omega \langle X(\nabla\mathcal{U}), \Delta_g\mathcal{U}\rangle_{\mathbb{R}^p} dv_g = \int_\Omega S_{X,2}^\sharp(\nabla\mathcal{U}, \nabla\mathcal{U})\, dv_g + \int_{\partial\Omega} B_{\partial\Omega}(\mathcal{U})\, d\sigma_g.$$

We assume now that $\mathcal{U}$ solves (0-1) and we use the equations satisfied by $\mathcal{U}$ to explicit the left-hand side in the preceding equation. We can write

$$\int_\Omega \langle X(\nabla\mathcal{U}), \Delta_g\mathcal{U}\rangle_{\mathbb{R}^p} dv_g = \int_\Omega |\mathcal{U}|^{2^\star-2}\langle X(\nabla\mathcal{U}), \mathcal{U}\rangle_{\mathbb{R}^p} dv_g - \int_\Omega \langle A\mathcal{U}, X(\nabla\mathcal{U})\rangle_{\mathbb{R}^p} dv_g$$

$$= \frac{1}{2^\star}\int_\Omega (\nabla|\mathcal{U}|^{2^\star}, X)\, dv_g - \int_\Omega \langle A\mathcal{U}, X(\nabla\mathcal{U})\rangle_{\mathbb{R}^p} dv_g$$

$$= -\frac{1}{2^\star}\int_\Omega (\text{div}_g X)|\mathcal{U}|^{2^\star} dv_g - \int_\Omega \langle A\mathcal{U}, X(\nabla\mathcal{U})\rangle_{\mathbb{R}^p} dv_g + \frac{1}{2^\star}\int_{\partial\Omega} X(\nu)|\mathcal{U}|^{2^\star} d\sigma_g.$$

Then we get

$$\int_\Omega \langle A\mathcal{U}, X(\nabla\mathcal{U})\rangle_{\mathbb{R}^p} dv_g + \frac{1}{2^\star}\int_\Omega (\text{div}_g X)|\mathcal{U}|^{2^\star} dv_g + \int_\Omega S_{X,2}^\sharp(\nabla\mathcal{U}, \nabla\mathcal{U})\, dv_g$$

$$= \frac{1}{2^\star}\int_{\partial\Omega} X(\nu)|\mathcal{U}|^{2^\star} d\sigma_g - \int_{\partial\Omega} B_{\partial\Omega}(\mathcal{U})\, d\sigma_g.$$

Using once again (0-1), we obtain

$$
\int_\Omega S_{X,2}^\sharp(\nabla\mathcal{U}, \nabla\mathcal{U})\, dv_g = \int_\Omega S_X^\sharp(\nabla\mathcal{U}, \nabla\mathcal{U})\, dv_g - \frac{1}{2^\star}\int_\Omega (\mathrm{div}_g X)|\nabla\mathcal{U}|^2 dv_g
$$

$$
= \int_\Omega S_X^\sharp(\nabla\mathcal{U}, \nabla\mathcal{U})\, dv_g - \frac{1}{2^\star}\int_{\partial\Omega}(\mathrm{div}_g X)\langle \partial_\nu \mathcal{U}, \mathcal{U}\rangle_{\mathbb{R}^p} d\sigma_g
$$

$$
+ \frac{n-2}{4n}\int_{\partial\Omega} \partial_\nu(\mathrm{div}_g X)|\mathcal{U}|^2 d\sigma_g + \frac{n-2}{4n}\int_\Omega (\Delta_g(\mathrm{div}_g X))|\mathcal{U}|^2 dv_g
$$

$$
- \frac{1}{2^\star}\int_\Omega (\mathrm{div}_g X)|\mathcal{U}|^{2^\star} dv_g + \frac{1}{2^\star}\int_\Omega (\mathrm{div}_g X)\langle A\mathcal{U}, \mathcal{U}\rangle_{\mathbb{R}^p} dv_g,
$$

and (7-2) easily follows. This ends the proof of the proposition.   □

The Pohozaev-type identity (7-2) is used repeatedly, with different choices of X, in the next section.

8. The range of influence of blow-up points

We start with notations and the definition of the range of influence of blow-up points. The blow-up points $x_{i,\alpha}$ of Proposition 4.2 come with vector bubbles $(\mathcal{B}_\alpha^i)_\alpha$ as in the same proposition. We let Λ_i be the S^{p-1} projection of $(\mathcal{B}_\alpha^i)_\alpha$, and $B_{i,\alpha} = |\mathcal{B}_\alpha^i|$ for all i and all α. As above, $(A_\alpha)_\alpha$ is a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$ satisfying (H), and we order the blow-up points in such a way that

$$
\mu_\alpha = \mu_{1,\alpha} \geq \cdots \geq \mu_{N,\alpha}, \tag{8-1}
$$

where the $\mu_{i,\alpha}$'s are the weights of the vector bubble $(\mathcal{B}_\alpha^i)_\alpha$. Given $i, j \in \{1, \ldots, N\}$, $i \neq j$, we let $s_{i,j,\alpha}$ be given by

$$
s_{i,j,\alpha}^2 = \frac{\mu_{i,\alpha}}{\mu_{j,\alpha}}\frac{d_g(x_{i,\alpha}, x_{j,\alpha})^2}{n(n-2)} + \mu_{i,\alpha}\mu_{j,\alpha} = \mu_{i,\alpha}B_{j,\alpha}(x_{i,\alpha})^{-2/(n-2)} \tag{8-2}
$$

and we define the range of influence of the blow-up point $x_{i,\alpha}$ by

$$
r_{i,\alpha} = \begin{cases} \min_{j\in\mathcal{A}_i} s_{i,j,\alpha} & \text{if } \mathcal{U}_\infty \equiv 0, \\ \min\{\min_{j\in\mathcal{A}_i} s_{i,j,\alpha};\ \sqrt{\mu_{i,\alpha}}\} & \text{if } \mathcal{U}_\infty \not\equiv 0. \end{cases} \tag{8-3}
$$

where

$$
\mathcal{A}_i = \{j \in \{1, \ldots, N\},\ j \neq i \text{ s.t. } \mu_{i,\alpha} = O(\mu_{j,\alpha})\}. \tag{8-4}
$$

If $\mathcal{A}_i = \varnothing$ (so that, in particular, $i = 1$) and $\mathcal{U}_\infty \equiv 0$, we let by definition $r_{i,\alpha} = \frac{1}{2}i_g$, where i_g is the injectivity radius of (M, g). Using the first equation in (4-5) it is easily checked that

$$
\frac{s_{i,j,\alpha}}{\mu_{i,\alpha}} \to +\infty \text{ as } \alpha \to +\infty \text{ for all } i,\ j \in \{1, \ldots, N\} \text{ and all } j \in \mathcal{A}_i. \tag{8-5}
$$

This implies in particular that

$$
\frac{r_{i,\alpha}}{\mu_{i,\alpha}} \to +\infty \quad \text{as } \alpha \to +\infty. \tag{8-6}
$$

If $j \in \mathcal{A}_i$ and $i \in \mathcal{A}_j$, we let $\lambda_{i,j} \geq 0$ be given by

$$
\lambda_{i,j} = \left(\lim_{\alpha\to+\infty}\frac{\mu_{j,\alpha}}{\mu_{i,\alpha}}\right)^{(n-2)/2}. \tag{8-7}
$$

Given $i \in \{1, \ldots, N\}$, we also let

$$\mathcal{B}_i = \begin{cases} \{j \in \{1, \ldots, N\}, \ j \neq i \text{ s.t. } d_g(x_{i,\alpha}, x_{j,\alpha}) = O(r_{i,\alpha})\} & \text{if } r_{i,\alpha} \to 0, \\ \{j \in \{1, \ldots, N\}, \ j \neq i \text{ s.t. } x_j \in B_{x_i}(\frac{1}{2}i_g)\} & \text{if } r_{i,\alpha} \not\to 0. \end{cases} \tag{8-8}$$

and, for $j \in \mathcal{B}_i$,

$$z_{i,j} = \lim_{\alpha \to +\infty} r_{i,\alpha}^{-1} \exp_{x_{i,\alpha}}^{-1}(x_{j,\alpha}). \tag{8-9}$$

Up to a subsequence, all these limits exist. We let $\delta_i > 0$ be such that for any i and any $j \in \mathcal{B}_i$,

$$|z_{i,j}| \neq 0 \Rightarrow |z_{i,j}| \geq 10\delta_i. \tag{8-10}$$

We also define $\mathcal{C}_i$ to be the subset of $\mathcal{B}_i$ given by

$$\mathcal{C}_i = \{j \in \mathcal{B}_i \text{ s.t. } z_{i,j} = 0\} \cap \mathcal{A}_i^c. \tag{8-11}$$

It can be proved that there exists a subset $\mathcal{D}_i$ of $\mathcal{C}_i$ and a family $(R_{i,j})_{j \in \mathcal{D}_i}$ of positive real numbers such that the two following assertions hold true: for any $j, k \in \mathcal{D}_i$, $j \neq k$,

$$\frac{d_g(x_{j,\alpha}, x_{k,\alpha})}{s_{j,i,\alpha}} \to +\infty \tag{8-12}$$

as $\alpha \to +\infty$, and for any $j \in \mathcal{C}_i$ there exists a unique $k \in \mathcal{D}_i$ such that

$$\limsup_{\alpha \to +\infty} \frac{d_g(x_{j,\alpha}, x_{k,\alpha})}{s_{k,i,\alpha}} \leq \frac{R_{i,k}}{20} \quad \text{and} \quad \limsup_{\alpha \to +\infty} \frac{s_{j,i,\alpha}}{s_{k,i,\alpha}} \leq \frac{R_{i,k}}{20}. \tag{8-13}$$

We also introduce the subsets

$$\Omega_{i,\alpha} = B_{x_{i,\alpha}}(\delta_i r_{i,\alpha}) \setminus \bigcup_{j \in \mathcal{D}_i} \Omega_{i,j,\alpha} \tag{8-14}$$

of M, where

$$\Omega_{i,j,\alpha} = B_{x_{j,\alpha}}(R_{i,j} s_{i,i,\alpha}) \tag{8-15}$$

for all $j \in \mathcal{D}_i$. The $\Omega_{i,j,\alpha}$'s are disjoint for α sufficiently large.

We now prove two lemmas to be used in the proof of Theorem 0.1.

Lemma 8.1. *Let $i \in \{1, \ldots, N\}$. Up to passing to a subsequence,*

$$|\mathcal{U}_\alpha - \Lambda_i B_{i,\alpha}| = o(B_{i,\alpha}) + O\left(\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n}\right) + O\left(\sum_{j \in \mathcal{D}_i} B_{j,\alpha}\right) = O(B_{i,\alpha})$$

in $B_{x_{i,\alpha}}(4\delta_i r_{i,\alpha}) \setminus \bigcup_{j \in \mathcal{D}_i} B_{x_{j,\alpha}}(\frac{1}{10} R_{i,j} s_{j,i,\alpha})$, and so, in particular, in $\Omega_{i,\alpha}$.

Proof. Let $x_\alpha \in B_{x_{i,\alpha}}(4\delta_i r_{i,\alpha}) \setminus \bigcup_{j \in \mathcal{D}_i} B_{x_{j,\alpha}}(\frac{1}{10} R_{i,j} s_{j,i,\alpha})$. Thanks to Proposition 6.1 we can write

$$\mathcal{U}_\alpha(x_\alpha) = \mathcal{U}_\infty(x_\alpha) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty + O\left(\mu_\alpha^{(n-2)/2}\right) + \sum_{j=1}^{N} (\Lambda_j + o(1)) B_{j,\alpha}(x_\alpha). \tag{8-16}$$

By the definition of $r_{i,\alpha}$, we know that $r_{i,\alpha}^2 \leq \mu_{i,\alpha}$ if $\mathcal{U}_\infty \not\equiv 0$ so that

$$\mathcal{U}_\infty(x_\alpha) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty = \mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} \begin{cases} 0 & \text{if } \mathcal{U}_\infty \equiv 0, \\ \left(\lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-(n/2)}\right) \mathcal{U}_\infty(x_i) + o(1) & \text{if } \mathcal{U}_\infty \not\equiv 0. \end{cases} \tag{8-17}$$

We now estimate $B_{j,\alpha}(x_\alpha)$. Assume first that $j \notin \mathscr{C}_i$ and $j \neq i$. As one can check with a little bit of work from the above definitions, if $r_{i,\alpha} \to 0$ as $\alpha \to +\infty$, then

$$B_{j,\alpha}(x_\alpha) = \mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} \Lambda_{i,j,\alpha}, \tag{8-18}$$

where

$$\Lambda_{i,j,\alpha} = \begin{cases} \left(\dfrac{n(n-2)}{|z-z_{i,j}|^2} \dfrac{\mu_{j,\alpha}}{\mu_{i,\alpha}} \right)^{(n-2)/2} + o(1) & \text{if } j \in \mathscr{A}_i \cap \mathscr{B}_i \text{ and } i \in \mathscr{A}_j, \\ r_{i,\alpha}^{n-2}/s_{i,j,\alpha}^{n-2} + o(1) & \text{if } j \in \mathscr{A}_i \setminus \mathscr{B}_i \text{ or } j \in \mathscr{A}_i \cap \mathscr{B}_i \text{ and } i \notin \mathscr{A}_j, \\ o(1) & \text{if } j \in \mathscr{A}_i^c \setminus \mathscr{C}_i, \end{cases}$$

where, up to a subsequence,

$$z = \lim_{\alpha \to +\infty} r_{i,\alpha}^{-1} \exp_{x_{i,\alpha}}^{-1}(x_\alpha).$$

Note that $z_{i,j} \neq 0$ if $j \in \mathscr{A}_i \cap \mathscr{B}_i$ and $i \in \mathscr{A}_j$. This is a direct consequence of the definition of the $s_{i,j,\alpha}$'s and (8-6). Moreover, $|z - z_{i,j}| \geq 6\delta_i$ in this case. As a consequence we have proved that

$$\begin{aligned} \mathscr{U}_\alpha(x_\alpha) = {} & O\left(\mu_\alpha^{(n-2)/2}\right) + o\left(\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n}\right) + (\Lambda_i + o(1)) B_{i,\alpha}(x_\alpha) \\ & + \mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} \Lambda(1)_{i,\alpha} + \mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} \sum_{j \in \mathscr{A}_i} \Lambda(2)_{i,j,\alpha} \Lambda_j + \sum_{j \in \mathscr{C}_i} (\Lambda_j + o(1)) B_{j,\alpha}(x_\alpha), \end{aligned} \tag{8-19}$$

where

$$\Lambda(1)_{i,\alpha} = \begin{cases} 0 & \text{if } \mathscr{U}_\infty \equiv 0, \\ \left(\lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-(n/2)} \right) \mathscr{U}_\infty(x_i) & \text{if } \mathscr{U}_\infty \not\equiv 0, \end{cases}$$

and

$$\Lambda(2)_{i,j,\alpha} = \begin{cases} (n(n-2))^{(n-2)/2} \lambda_{i,j} / |z - z_{i,j}|^{n-2} & \text{if } j \in \mathscr{B}_i \text{ and } i \in \mathscr{A}_j, \\ \lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2}/s_{i,j,\alpha}^{n-2} & \text{if } j \notin \mathscr{B}_i \text{ or } i \notin \mathscr{A}_j. \end{cases}$$

Let $j \in \mathscr{C}_i$. We claim that, up to a subsequence,

$$\lim_{\alpha \to +\infty} \left(\frac{B_{j,\alpha}(x_\alpha)}{B_{i,\alpha}(x_\alpha)} \right)^{2/(n-2)} = n(n-2) \lim_{\alpha \to +\infty} \frac{s_{j,i,\alpha}^2}{d_g(x_{j,\alpha}, x_\alpha)^2}. \tag{8-20}$$

To prove (8-20), we first remark that $i \in \mathscr{A}_j$ since $j \in \mathscr{C}_i$ (and in particular $j \notin \mathscr{A}_i$). Thus, using (8-5), we obtain that

$$\begin{aligned} \left(\frac{B_{j,\alpha}(x_\alpha)}{B_{i,\alpha}(x_\alpha)} \right)^{2/(n-2)} &= (1+o(1)) \mu_{j,\alpha} \mu_{i,\alpha}^{-1} d_g(x_{j,\alpha}, x_\alpha)^{-2} \left(n(n-2)\mu_{i,\alpha}^2 + d_g(x_{i,\alpha}, x_\alpha)^2 \right) \\ &= n(n-2) \frac{s_{j,i,\alpha}^2}{d_g(x_{j,\alpha}, x_\alpha)^2} + o(1) + O\left(\frac{\mu_{j,\alpha}}{\mu_{i,\alpha}} \frac{\left| d_g(x_{i,\alpha}, x_\alpha)^2 - d_g(x_{i,\alpha}, x_{j,\alpha})^2 \right|}{d_g(x_{j,\alpha}, x_\alpha)^2} \right). \end{aligned}$$

From the triangle inequality, we easily get

$$\frac{\left| d_g(x_{i,\alpha}, x_\alpha)^2 - d_g(x_{i,\alpha}, x_{j,\alpha})^2 \right|}{d_g(x_{j,\alpha}, x_\alpha)^2} \leq 1 + 2\frac{d_g(x_{i,\alpha}, x_{j,\alpha})}{d_g(x_{j,\alpha}, x_\alpha)} = O(1) + O\left(\frac{s_{j,i,\alpha}}{d_g(x_{j,\alpha}, x_\alpha)} \sqrt{\frac{\mu_{i,\alpha}}{\mu_{j,\alpha}}} \right),$$

hence the estimate (8-20). Now, for $j \in \mathscr{C}_i$, we let $k \in \mathscr{D}_i$ be given by (8-13). By (8-20) it is easily checked that

$$B_{j,\alpha}(x_\alpha) = O(B_{k,\alpha}(x_\alpha)). \tag{8-21}$$

Since

$$\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} = O(B_{i,\alpha}(x_\alpha)), \tag{8-22}$$

the first estimate in the lemma clearly holds true thanks to (8-19) and (8-21). Here it can be noted that

$$\mu_\alpha^{(n-2)/2} = O\big(\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n}\big)$$

for all i. Applying (8-20) again we easily obtain the second estimate in the lemma. This ends the proof of Lemma 8.1. $\qquad\square$

Now we prove that the following elliptic type lemma holds true. Lemma 8.2 provides estimates on the $\mathcal{U}_\alpha$'s and $\nabla \mathcal{U}_\alpha$'s in small regions around the blow-up points $x_{i,\alpha}$.

Lemma 8.2. *There exists $C > 0$ such that, up to a subsequence,*

$$|\mathcal{U}_\alpha| \leq C\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} \quad and \quad |\nabla \mathcal{U}_\alpha| \leq C\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{1-n}$$

in $B_{x_{i,\alpha}}(2\delta_i r_{i,\alpha}) \setminus B_{x_{i,\alpha}}\big(\frac{1}{2}\delta_i r_{i,\alpha}\big)$. There also exists $C > 0$ such that, up to a subsequence, for any $j \in \mathcal{D}_i$,

$$|\mathcal{U}_\alpha| \leq C\mu_{j,\alpha}^{(n-2)/2} s_{j,i,\alpha}^{2-n} \quad and \quad |\nabla \mathcal{U}_\alpha| \leq C\mu_{j,\alpha}^{(n-2)/2} s_{j,i,\alpha}^{1-n}$$

in $B_{x_{j,\alpha}}(5R_{i,j} s_{j,i,\alpha}) \setminus B_{x_{j,\alpha}}\big(\frac{1}{5}R_{i,j} s_{j,i,\alpha}\big)$.

Proof. The lemma follows from standard elliptic theory and the estimates we proved in Lemma 8.1. Assuming first that $x_\alpha \in B_{x_{i,\alpha}}(4\delta_i r_{i,\alpha}) \setminus B_{x_{i,\alpha}}\big(\frac{1}{4}\delta_i r_{i,\alpha}\big)$, we easily get from Lemma 8.1 that

$$|\mathcal{U}_\alpha(x_\alpha)| = O\big(\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n}\big). \tag{8-23}$$

On the other hand, if we let $\tilde{\mathcal{U}}_\alpha$ be given by $\tilde{\mathcal{U}}_\alpha(x) = r_{i,\alpha}^{(n-2)/2} \mathcal{U}_\alpha\big(\exp_{x_{i,\alpha}}(r_{i,\alpha}x)\big)$, then

$$\Delta_{\tilde{g}_\alpha} \tilde{\mathcal{U}}_\alpha + r_{i,\alpha}^2 \tilde{A}_\alpha \tilde{\mathcal{U}}_\alpha = |\tilde{\mathcal{U}}_\alpha|^{2^\star - 2} \tilde{\mathcal{U}}_\alpha, \tag{8-24}$$

where $\tilde{g}_\alpha = \big(\exp_{x_{i,\alpha}}^\star g\big)(r_{i,\alpha}x)$ and $\tilde{A}_\alpha(x) = A_\alpha\big(\exp_{x_{i,\alpha}}(r_{i,\alpha}x)\big)$. The first two estimates in Lemma 8.2 follow from (8-23) and (8-24) by standard elliptic theory. Similarly, if we assume that

$$x_\alpha \in B_{x_{j,\alpha}}(10R_{i,j} s_{j,i,\alpha}) \setminus B_{x_{j,\alpha}}\big(\tfrac{1}{10}R_{i,j} s_{j,i,\alpha}\big),$$

noting that $s_{j,i,\alpha} = o(r_{i,\alpha})$ in this case, we get from Lemma 8.1 and (8-20) that

$$|\mathcal{U}_\alpha(x_\alpha)| = O\big(\mu_{j,\alpha}^{(n-2)/2} s_{j,i,\alpha}^{2-n}\big). \tag{8-25}$$

Letting $\hat{\mathcal{U}}_\alpha$ be given by $\hat{\mathcal{U}}_\alpha(x) = s_{j,i,\alpha}^{(n-2)/2} \mathcal{U}_\alpha\big(\exp_{x_{j,\alpha}}(s_{j,i,\alpha}x)\big)$, we also have

$$\Delta_{\hat{g}_\alpha} \hat{\mathcal{U}}_\alpha + s_{j,i,\alpha}^2 \hat{A}_\alpha \hat{\mathcal{U}}_\alpha = |\hat{\mathcal{U}}_\alpha|^{2^\star - 2} \hat{\mathcal{U}}_\alpha, \tag{8-26}$$

where $\hat{g}_\alpha = (\exp_{x_{j,\alpha}}^\star g)(s_{j,i,\alpha}x)$ and $\hat{A}_\alpha(x) = A_\alpha\big(\exp_{x_{j,\alpha}}(s_{j,i,\alpha}x)\big)$. The last two estimates in Lemma 8.2 follow from (8-25) and (8-26) here again by standard elliptic theory. This proves Lemma 8.2. $\qquad\square$

9. Sharp asymptotics for the range of influence

Our goal now is to prove the sharp asymptotics connecting the range of influence $r_{i,\alpha}$ of the blow-up points with the weights $\mu_{i,\alpha}$ of the bubbles in the decomposition of Proposition 4.2. This is the subject of Proposition 9.2. We adopt here the notations of the preceding section. In particular, $(A_\alpha)_\alpha$ is a sequence

of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$ satisfying (H), and we order the blow-up points in such a way that

$$\mu_\alpha = \mu_{1,\alpha} \geq \cdots \geq \mu_{N,\alpha},$$

where the $\mu_{i,\alpha}$'s are the weights of the vector bubble $(\mathcal{B}_\alpha^i)_\alpha$ in Proposition 4.2, and let the $r_{i,\alpha}$'s be given by (8-3). First we prove:

Lemma 9.1. *If $r_{i,\alpha} = o\left(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\right)$, then, up to a subsequence,*

$$r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-(n/2)} \mathcal{U}_\alpha\left(\exp_{x_{i,\alpha}}(r_{i,\alpha}z)\right) \to \left(n(n-2)\right)^{(n-2)/2}\left(\Lambda_i|z|^{2-n} + \mathcal{H}_i(z)\right)$$

in $C^2_{\mathrm{loc}}\left(B_0(2\delta_i) \setminus \{0\}\right)^p$ as $\alpha \to +\infty$, where

$$\mathcal{H}_i(z) = \sum_{\substack{j \in \mathcal{A}_i \cap \mathcal{B}_i, \\ i \in \mathcal{A}_j}} \frac{\lambda_{i,j}\Lambda_j}{|z - z_{i,j}|^{n-2}} + X_i$$

is a smooth function in $B_0(2\delta_i)$ satisfying that $\mathcal{H}_i(0) \neq 0$, the $\lambda_{i,j}$'s are as in (8-7), δ_i is as in (8-10), and the X_i's are nonnegative vectors in $\mathbb{R}^p$.

Proof. Let $z \in B_0(3\delta_i) \setminus \{0\}$ and set $x_\alpha = \exp_{x_{i,\alpha}}(r_{i,\alpha}z)$. Let also $\mathcal{W}_\alpha$ be given by

$$\overline{\mathcal{U}}_\alpha(x) = r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{-(n-2)/2} \mathcal{U}_\alpha\left(\exp_{x_{i,\alpha}}(r_{i,\alpha}x)\right).$$

Then

$$\Delta_{g_\alpha}\overline{\mathcal{U}}_\alpha + r_{i,\alpha}^2 \tilde{A}_\alpha \overline{\mathcal{U}}_\alpha = \left(\frac{\mu_{i,\alpha}}{r_{i,\alpha}}\right)^2 |\overline{\mathcal{U}}_\alpha|^{2^*-2}\overline{\mathcal{U}}_\alpha, \tag{9-1}$$

where $g_\alpha = (\exp_{x_{i,\alpha}}^\star g)(r_{i,\alpha}x)$ and $\tilde{A}_\alpha(x) = A_\alpha\left(\exp_{x_{i,\alpha}}(r_{i,\alpha}x)\right)$. In particular, we get by (8-6), (8-19) and (8-20) that, if $r_{i,\alpha}^2 = o\left(\mu_{i,\alpha}/\mu_\alpha\right)$, then

$$\lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-\frac{n}{2}} \mathcal{U}_\alpha(x_\alpha) = \left(n(n-2)\right)^{(n-2)/2}\left(\Lambda_i|z|^{2-n} + \mathcal{H}_i(z)\right), \tag{9-2}$$

where $\mathcal{H}_i(z)$ is the sum of two terms:

$$\begin{cases} 0 & \text{if } \mathcal{U}_\infty \equiv 0 \\ \left(\lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-(n/2)}\right)\mathcal{U}_\infty(x_i) & \text{if } \mathcal{U}_\infty \not\equiv 0 \end{cases}$$

and

$$\sum_{j \in \mathcal{A}_i} \Lambda_j \begin{cases} \left(n(n-2)\right)^{(n-2)/2} \lambda_{i,j}/|z - z_{i,j}|^{n-2} & \text{if } j \in \mathcal{B}_i \text{ and } i \in \mathcal{A}_j, \\ \lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2}/s_{i,j,\alpha}^{n-2} & \text{if } j \notin \mathcal{B}_i \text{ or } i \notin \mathcal{A}_j. \end{cases}$$

As a remark, if $j \in \mathcal{A}_i$ and $i \in \mathcal{A}_j$, then $\mu_{i,\alpha} \sim \mu_{j,\alpha}$. In particular, $z_{i,j} \neq 0$ since, if not the case, we would get from the inequality $r_{i,\alpha} \leq s_{i,j,\alpha}$ that $r_{i,\alpha} = o(\mu_{i,\alpha})$ and then that $d_g(x_{i,\alpha}, x_{j,\alpha}) = o(\mu_{i,\alpha})$, a contradiction with the first equation in (4-5) of Proposition 4.2. By (9-1) and (9-2), standard elliptic theory gives the lemma, up to the proof that $\mathcal{H}_i(0) \neq 0$. Assume first that there exists $j \in \mathcal{A}_i$ such that $s_{i,j,\alpha} = r_{i,\alpha}$. Then in the term involving this j in the above sum over $\mathcal{A}_i$ there is at least one line which is positive. Since all the other terms are nonnegative, this proves that $\mathcal{H}_i(0) \neq 0$. The other possibility

is that $\mathcal{U}_\infty \not\equiv 0$ and that $r_{i,\alpha}^2 = \mu_{i,\alpha}$ so the first term in the definition of $\mathcal{H}_i$ is nonzero. Indeed, by the maximum principle, since

$$\Delta_g |\mathcal{U}_\infty|_\Sigma + \Lambda |\mathcal{U}_\infty|_\Sigma \geq 0$$

for some $\Lambda > 0$, where $|\mathcal{U}_\infty|_\Sigma = \sum_i u_{i,\infty}$ is the sum of the components of $\mathcal{U}_\infty$, we get that $|\mathcal{U}_\infty|_\Sigma > 0$ in M if $\mathcal{U}_\infty \not\equiv 0$. Then, here again, $\mathcal{H}_i(0) \neq 0$. Noting that the above two possibilities are the only two possibilities since our assumption on $r_{i,\alpha}$ clearly implies that $r_{i,\alpha} \to 0$ as $\alpha \to +\infty$, this ends the proof of Lemma 9.1. $\qquad\square$

As it can be checked from the above proof, we have an explicit formula for the X_i's in Lemma 9.1. We get that

$$X_i = \left(\lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-(n/2)}\right) \mathcal{U}_\infty(x_i) + \sum_{j \in (\mathcal{A}_i \setminus \mathcal{B}_i) \cup \Theta_i} \left(\lim_{\alpha \to +\infty} \frac{r_{i,\alpha}}{s_{i,j,\alpha}}\right)^{n-2} \Lambda_j, \tag{9-3}$$

where we adopt the convention that the first term in the right-hand side of (9-3) is zero if $\mathcal{U}_\infty \equiv 0$, that the second term is zero if $(\mathcal{A}_i \setminus \mathcal{B}_i) \cup \Theta_i = \varnothing$, and where $\Theta_i = \{j \in \mathcal{A}_i \text{ s.t. } i \notin \mathcal{A}_j\}$. Now, at this point, we can state Proposition 9.2 which establishes sharp asymptotics connecting the range of influence $r_{i,\alpha}$ of the blow-up points $x_{i,\alpha}$ to the weights $\mu_{i,\alpha}$ of the bubbles in the decomposition of Proposition 4.2.

Proposition 9.2. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 4$, $p \geq 1$ be an integer, and $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$ satisfying (H). Let also $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of (1-1) such that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. Let $i \in \{1, \dots, N\}$ and assume that, up to a subsequence, $r_{i,\alpha} = o(\sqrt{\mu_{i,\alpha}/\mu_\alpha})$. Then*

$$\left((A(x_i) - \tfrac{1}{6} S_g(x_i) \, \mathrm{Id}_p) \Lambda_i + o(1)\right) r_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}} = 2\mathcal{H}_i(0) + o(1) \tag{9-4}$$

if $n = 4$, and

$$\left(\left(A(x_i) - \frac{n-2}{4(n-1)} S_g(x_i) \, \mathrm{Id}_p\right) \Lambda_i + o(1)\right) \mu_{i,\alpha}^{4-n} r_{i,\alpha}^{n-2}$$

$$= \frac{n^{n-2}(n-2)^{n-1} \omega_{n-1}}{\int_{\mathbb{R}^n} u_0^2 \, dx} \left(\mathcal{H}_i(0) + \frac{n-4}{2} \langle \Lambda_i, \mathcal{H}_i(0) \rangle_{\mathbb{R}^p} \Lambda_i\right) + o(1) \tag{9-5}$$

if $n \geq 5$, where $\mathcal{H}_i$ is as in Lemma 9.1, the $r_{i,\alpha}$'s are as in (8-3), and u_0 is given by (4-3). Moreover, $\langle \Lambda_i, \nabla \mathcal{H}_i(0) \rangle_{\mathbb{R}^p} \equiv 0$.

We prove Proposition 9.2 by reverse induction on i. We let $i \in \{1, \dots, N\}$ be such that $\sqrt{\mu_\alpha} r_{i,\alpha} = o(\sqrt{\mu_{i,\alpha}})$ and, in case $i < N$, we assume that

for any $j = i + 1, \dots, N$, (9-4) and (9-5) hold for j if $\sqrt{\mu_\alpha} r_{j,\alpha} = o(\sqrt{\mu_{j,\alpha}})$. $\qquad$ (H$_i$)

If $i = N$ we do not assume anything. Then we aim to prove that (9-4) and (9-5) hold true for i. As a remark it should be noted that we always have

$$\mathcal{H}_i(0) + \frac{n-4}{2} \langle \Lambda_i, \mathcal{H}_i(0) \rangle_{\mathbb{R}^p} \Lambda_i \neq 0. \tag{9-6}$$

Let $i \in \{1, \dots, N\}$, $i < N$, be arbitrary. Assuming (H_i), we get that for any $j \in \mathcal{D}_i$,

$$s_{j,i,\alpha}^{2-n} = \begin{cases} O(-\ln \mu_{j,\alpha}) & \text{if } n = 4, \\ O(\mu_{j,\alpha}^{4-n}) & \text{if } n \geq 5. \end{cases} \tag{9-7}$$

Indeed, if $j \in \mathcal{D}_i$, then $j > i$. Moreover, for any $j \in \mathcal{D}_i$, we have $i \in \mathcal{A}_j$, so $s_{j,i,\alpha} \geq r_{j,\alpha}$, and clearly $s_{j,i,\alpha}^2 = o(\mu_{j,\alpha} \mu_{i,\alpha}^{-1}) = o(\mu_{j,\alpha} \mu_\alpha^{-1})$. In particular, $\sqrt{\mu_\alpha} r_{j,\alpha} = o(\sqrt{\mu_{j,\alpha}})$, and (9-7) is a direct consequence of (H_i), thanks to (9-6). Now we prove Proposition 9.2 in several steps. In the sequel we let $R_t(\alpha)$ represent any quantity such that

$$R_t(\alpha) = \begin{cases} o(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}) & \text{if } n = 4, \\ o(\mu_{i,\alpha}^2) & \text{if } n \geq 5. \end{cases} \tag{9-8}$$

The first step in the proof of Proposition 9.2 is as follows.

Step 1. *Let $i \in \{1, \dots, N\}$ be arbitrary. In case $i < N$, assume that (H_i) holds true. Let*

$$\mathcal{F}_\alpha = \left(64\omega_3\left(\langle A(x_i)\Lambda_i, \Lambda_i \rangle_{\mathbb{R}^p} - \tfrac{1}{6}S_g(x_i)\right) + o(1)\right)\mu_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}} + o\left(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}\right)$$

if $n = 4$, and

$$\mathcal{F}_\alpha = \left(\left(\langle A(x_i)\Lambda_i, \Lambda_i \rangle_{\mathbb{R}^p} - \frac{n-2}{4(n-1)}S_g(x_i)\right)\int_{\mathbb{R}^n} u_0^2 dx + o(1)\right)\mu_{i,\alpha}^2$$

if $n \geq 5$. Then we have, up to passing to a subsequence,

$$\mathcal{F}_\alpha = \left(\tfrac{1}{2}n^{n-2}(n-2)^n \omega_{n-1}\langle \Lambda_i, \mathcal{H}_i(0) \rangle_{\mathbb{R}^p} + o(1)\right)\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}$$

if $\sqrt{\mu_\alpha} r_{i,\alpha} = o(\sqrt{\mu_{i,\alpha}})$, and $\mathcal{F}_\alpha = O(\mu_{i,\alpha}^{(n-2)/2}\mu_\alpha^{(n-2)/2})$ otherwise, where $\mathcal{H}_i$ is as in Lemma 9.1, the $r_{i,\alpha}$'s are as in (8-3), and u_0 is as in (4-3).

Proof of Step 1. We apply the Pohozaev identity (7-2) of Proposition 7.1 in Section 7 to $\mathcal{U}_\alpha$ in $\Omega_{i,\alpha}$ with $X = X^\alpha$ given by

$$X^\alpha(x) = \left(1 - \frac{1}{6(n-1)}Rc_g^\sharp(x)\left(\nabla f_\alpha(x), \nabla f_\alpha(x)\right)\right)\nabla f_\alpha(x), \tag{9-9}$$

where $f_\alpha(x) = \tfrac{1}{2}d_g(x_{i,\alpha}, x)^2$, and $Rc_g^\sharp$ is the $(0, 2)$-tensor field we get from the $(2, 0)$-Ricci tensor Rc_g due to the musical isomorphism. We obtain

$$\int_{\Omega_{i,\alpha}} \langle A_\alpha \mathcal{U}_\alpha, X^\alpha(\nabla \mathcal{U}_\alpha) \rangle_{\mathbb{R}^p} dv_g$$

$$+ \frac{n-2}{4n}\int_{\Omega_{i,\alpha}}(\Delta_g(\text{div}_g X^\alpha))|\mathcal{U}_\alpha|^2 dv_g + \frac{n-2}{2n}\int_{\Omega_{i,\alpha}}(\text{div}_g X^\alpha)\langle A_\alpha \mathcal{U}_\alpha, \mathcal{U}_\alpha \rangle_{\mathbb{R}^p} dv_g$$

$$= Q_\alpha - \sum_{j \in \mathcal{D}_i} Q_\alpha^j + R_{1,\alpha} + R_{2,\alpha} - \sum_{j \in \mathcal{D}_i} R_{2,\alpha}^j, \quad (9\text{-}10)$$

where, if $\nu = \nu_\alpha$ stands for the unit outer normal to $\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})$, the Q_α's are given by

$$Q_\alpha = \frac{n-2}{2n} \int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} (\mathrm{div}_g X^\alpha)\langle \partial_\nu \mathcal{U}_\alpha, \mathcal{U}_\alpha \rangle_{\mathbb{R}^p} d\sigma_g$$

$$- \int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} \left(\tfrac{1}{2} X^\alpha(\nu)|\nabla \mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla \mathcal{U}_\alpha), \partial_\nu \mathcal{U}_\alpha \rangle_{\mathbb{R}^p} \right) d\sigma_g, \quad (9\text{-}11)$$

the Q_α^j's are given by

$$Q_\alpha^j = \frac{n-2}{2n} \int_{\partial \Omega_{i,j,\alpha}} (\mathrm{div}_g X^\alpha)\langle \partial_\nu \mathcal{U}_\alpha, \mathcal{U}_\alpha \rangle_{\mathbb{R}^p} d\sigma_g$$

$$- \int_{\partial \Omega_{i,j,\alpha}} \left(\tfrac{1}{2} X^\alpha(\nu)|\nabla \mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla \mathcal{U}_\alpha), \partial_\nu \mathcal{U}_\alpha \rangle_{\mathbb{R}^p} \right) d\sigma_g, \quad (9\text{-}12)$$

where $\Omega_{i,j,\alpha}$ is as in (8-15), the $R_{1,\alpha}$'s are given by

$$R_{1,\alpha} = - \int_{\Omega_{i,\alpha}} (T_\nabla X^\alpha)_{\mathcal{U}_\alpha} dv_g, \quad (9\text{-}13)$$

where $(T_\nabla X)_{\mathcal{U}} = \sum_{i=1}^p S_X^\sharp(\nabla u_i, \nabla u_i)$ and S_X is as in (7-1), the $R_{2,\alpha}$'s are given by

$$R_{2,\alpha} = \frac{n-2}{2n} \int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} X^\alpha(\nu)|\mathcal{U}_\alpha|^{2^\star} d\sigma_g - \frac{n-2}{4n} \int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} (\partial_\nu(\mathrm{div}_g X^\alpha))|\mathcal{U}_\alpha|^2 d\sigma_g, \quad (9\text{-}14)$$

and the $R_{2,\alpha}^j$'s are given by

$$R_{2,\alpha}^j = \frac{n-2}{2n} \int_{\partial \Omega_{i,j,\alpha}} X^\alpha(\nu)|\mathcal{U}_\alpha|^{2^\star} d\sigma_g - \frac{n-2}{4n} \int_{\partial \Omega_{i,j,\alpha}} (\partial_\nu(\mathrm{div}_g X^\alpha))|\mathcal{U}_\alpha|^2 d\sigma_g. \quad (9\text{-}15)$$

Note that $\mathcal{D}_i = \varnothing$ if $i = N$. Thanks to the expression of the X^α's in (9-9) we have the estimates

$$|X^\alpha(x)| = O(d_g(x_{i,\alpha}, x)),$$

$$\mathrm{div}_g X^\alpha(x) - n = O(d_g(x_{i,\alpha}, x)^2),$$

$$|\nabla(\mathrm{div}_g X^\alpha)(x)| = O(d_g(x_{i,\alpha}, x)),$$

$$\Delta_g(\mathrm{div}_g X^\alpha)(x) = \frac{n}{n-1} S_g(x_{i,\alpha}) + O(d_g(x_{i,\alpha}, x)). \quad (9\text{-}16)$$

In what follows we estimate the different terms involved in (9-10). We start with estimates on the Q_α^j's and $R_{2,\alpha}^j$'s in (9-12) and (9-15). Since

$$d_g(x_{i,\alpha}, x) \leq d_g(x_{i,\alpha}, x_{j,\alpha}) + R_{i,j} s_{j,i,\alpha} = O\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_{j,\alpha}}} s_{j,i,\alpha} \right)$$

on $\partial \Omega_{i,j,\alpha}$, we obtain from Lemma 8.2, (9-7) and (9-16) that

$$Q_\alpha^j + R_{2,\alpha}^j = O\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_{j,\alpha}}} \mu_{j,\alpha}^{n-2} s_{j,i,\alpha}^{2-n} \right) = R_t(\alpha), \quad (9\text{-}17)$$

where $R_t(\alpha)$ is as in (9-8). Now we estimate the $R_{2,\alpha}$'s in (9-14). Still from Lemma 8.2, we obtain by direct computations, using (8-6) and (9-16), that

$$R_{2,\alpha} = O(\mu_{i,\alpha}^{n-2}) + o(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}). \quad (9\text{-}18)$$

Concerning the right-hand side of (9-10) it remains to estimate the Q_α's in (9-11) and the $R_{1,\alpha}$'s in (9-13). We start with estimates for the $R_{1,\alpha}$'s. We remark that $S^\sharp_{X^\alpha} = O(d_g(x_{i,\alpha}, x)^2)$ and that

$$(T_\nabla X^\alpha)_{\mathscr{B}^i_\alpha} = O\big(d_g(x_{i,\alpha}, x)^3 |\nabla B_{i,\alpha}|^2\big).$$

In particular, we can write

$$R_{1,\alpha} = O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^3 |\nabla B_{i,\alpha}|^2 dv_g\bigg) + O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\nabla B_{i,\alpha}| |\nabla (\mathcal{U}^\alpha - B_{i,\alpha}\Lambda_i)| dv_g\bigg)$$
$$+ O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\nabla (\mathcal{U}^\alpha - B_{i,\alpha}\Lambda_i)|^2 dv_g\bigg).$$

Direct computations lead to

$$\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^{2+\kappa} |\nabla B_{i,\alpha}|^2 dv_g = \begin{cases} O_\kappa\big(-\mu^2_{i,\alpha} \ln \mu_{i,\alpha}\big) & \text{if } n=4, \\ O_\kappa(\mu^2_{i,\alpha}) & \text{if } n \geq 5, \end{cases}$$

where $O_\kappa = O$ if $\kappa = 0$, and $O_\kappa = o$ if $\kappa = 1$. Integrating by parts and using Lemma 8.1, Lemma 8.2, and (9-7), we can write

$$\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\nabla(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)|^2 dv_g$$
$$= O\bigg(\int_{\partial\Omega_{i,\alpha}} |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i| d_g(x_{i,\alpha}, x)^2 |\nabla(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)| d\sigma_g\bigg) + O\bigg(\int_{\partial\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x) |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i|^2 d\sigma_g\bigg)$$
$$+ O\bigg(\int_{\Omega_{i,\alpha}} |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i|^2 dv_g\bigg) + \int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 \langle \mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i, \Delta_g(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)\rangle_{\mathbb{R}^p} dv_g,$$

and then

$$\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\nabla(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)|^2 dv_g$$
$$= \int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 \langle \mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i, \Delta_g(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)\rangle_{\mathbb{R}^p} dv_g + o\big(\mu^{n-2}_{i,\alpha} r^{2-n}_{i,\alpha}\big) + R_t(\alpha),$$

where $R_t(\alpha)$ is as in (9-8). It remains to remark that thanks to the equations satisfied by the $\mathcal{U}_\alpha$'s, and the expression of Δ_g in geodesic polar coordinates, we have

$$\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 \langle \mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i, \Delta_g(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)\rangle_{\mathbb{R}^p} dv_g$$
$$= O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i| \big(|\mathcal{U}_\alpha|^{2^\star - 1} + B^{2^\star - 1}_{i,\alpha}\big) dv_g\bigg)$$
$$+ O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i| |\mathcal{U}_\alpha| dv_g\bigg) + O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^3 |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i| |\nabla B_{i,\alpha}| dv_g\bigg),$$

so that, by Lemma 8.1, using Hölder's inequalities,

$$R_{1,\alpha} = o\big(\mu^{n-2}_{i,\alpha} r^{2-n}_{i,\alpha}\big) + R_t(\alpha), \tag{9-19}$$

where $R_t(\alpha)$ is as in (9-8). Still concerning the right-hand side of (9-10) it remains to estimate the Q_α's in (9-11). Thanks to Lemma 8.2 and Lemma 9.1, we get by simple computations that

$$Q_\alpha = \left(-\tfrac{1}{2}n^{n-2}(n-2)^n\omega_{n-1}\langle\Lambda_i, \mathcal{H}_i(0)\rangle_{\mathbb{R}^p} + o(1)\right)\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n} + O\left(\mu_{i,\alpha}^{n-2}\right), \qquad (9\text{-}20)$$

if $r_{i,\alpha} = o\left(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\right)$, and $Q_\alpha = O\left(\mu_{i,\alpha}^{(n-2)/2}\mu_\alpha^{(n-2)/2}\right)$ otherwise. Now we concentrate on the left-hand side of (9-10). Writing $A_\alpha(x) = A_\alpha(x_{i,\alpha}) + O(d_g(x_{i,\alpha}, x))$, we get

$$\int_{\Omega_{i,\alpha}}\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p}dv_g = \sum_{j,k=1}^p A_{jk}^\alpha(x_{i,\alpha})\int_{\Omega_{i,\alpha}}u_{j,\alpha}X^\alpha(\nabla\mathcal{U}_\alpha)_k dv_g + O\left(\int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)^2|\mathcal{U}_\alpha||\nabla\mathcal{U}_\alpha|dv_g\right).$$

Using the Cauchy–Schwarz inequality, we can write

$$\int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)^2|\mathcal{U}_\alpha||\nabla\mathcal{U}_\alpha|dv_g \le \Pi_{j=0}^1\left(\int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)^{3-2j}|\nabla^{1-j}\mathcal{U}_\alpha|^2 dv_g\right)^{1/2}.$$

Using Lemma 8.1 it is easily checked that

$$\int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)|\mathcal{U}_\alpha|^2 dv_g = R_t(\alpha), \qquad (9\text{-}21)$$

where $R_t(\alpha)$ is as in (9-8). We integrate by parts and use the equations satisfied by the $\mathcal{U}_\alpha$'s, together with Lemma 8.1, Lemma 8.2, and (9-7), to obtain

$$\int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)^3|\nabla\mathcal{U}_\alpha|^2 dv_g = O\left(\int_{\partial\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)^3|\nabla\mathcal{U}_\alpha||\mathcal{U}_\alpha|d\sigma_g\right)$$
$$+ O\left(\int_{\partial\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)^2|\mathcal{U}_\alpha|^2 d\sigma_g\right) + O\left(\int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)^3|\mathcal{U}_\alpha|^{2^*}dv_g\right)$$
$$+ O\left(\int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)|\mathcal{U}_\alpha|^2 dv_g\right),$$

and then that

$$\int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)^3|\nabla\mathcal{U}_\alpha|^2 dv_g = o\left(\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n}\right) + R_t(\alpha),$$

where $R_t(\alpha)$ is as in (9-8). Thus we get that

$$\int_{\Omega_{i,\alpha}}\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p}dv_g = \sum_{j,k=1}^p A_{jk}^\alpha(x_{i,\alpha})\int_{\Omega_{i,\alpha}}u_{j,\alpha}X^\alpha(\nabla\mathcal{U}_\alpha)_k dv_g + o\left(\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n}\right) + R_t(\alpha).$$

Integrating by parts again, and estimating the different terms as above, it is easily checked that we actually have

$$\int_{\Omega_{i,\alpha}}\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p}dv_g = -\frac{n}{2}\sum_{j,k=1}^p A_{jk}^\alpha(x_{i,\alpha})\int_{\Omega_{i,\alpha}}u_{j,\alpha}u_{k,\alpha}dv_g + o\left(\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n}\right) + R_t(\alpha),$$

where $R_t(\alpha)$ is as in (9-8). Proceeding as above, thanks to (9-16), one finally gets that

$$\int_{\Omega_{i,\alpha}} \langle A_\alpha \mathcal{U}_\alpha, X^\alpha(\nabla \mathcal{U}_\alpha)\rangle_{\mathbb{R}^p} dv_g + \frac{n-2}{4n} \int_{\Omega_{i,\alpha}} (\Delta_g(\operatorname{div}_g X^\alpha))|\mathcal{U}_\alpha|^2 dv_g$$

$$+ \frac{n-2}{2n} \int_{\Omega_{i,\alpha}} (\operatorname{div}_g X^\alpha)\langle A_\alpha \mathcal{U}_\alpha, \mathcal{U}_\alpha\rangle_{\mathbb{R}^p} dv_g$$

$$= -\sum_{j,k=1}^p A_{jk}^\alpha(x_{i,\alpha}) \int_{\Omega_{i,\alpha}} u_{j,\alpha} u_{k,\alpha} dv_g + \frac{n-2}{4(n-1)} S_g(x_{i,\alpha}) \int_{\Omega_{i,\alpha}} |\mathcal{U}_\alpha|^2 dv_g + o\big(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}\big) + R_t(\alpha),$$

(9-22)

where $R_t(\alpha)$ is as in (9-8). We have

$$\int_{\Omega_{i,j,\alpha}} B_{i,\alpha}^2 dv_g = R_t(\alpha) \quad \text{for all } j \in \mathcal{D}_i.$$

(9-23)

Indeed, if $d_g(x_{i,\alpha}, x_{j,\alpha})/s_{j,i,\alpha} \to +\infty$ as $\alpha \to +\infty$, then

$$\int_{B_{x_{j,\alpha}}(R_{i,j} s_{j,i,\alpha})} B_{i,\alpha}^2 dv_g = O\big(s_{j,i,\alpha}^n B_{i,\alpha}(x_{j,\alpha})^2\big) = O\big(\mu_{j,\alpha}^{n-2} s_{j,i,\alpha}^{4-n}\big) = R_t(\alpha),$$

thanks to (9-7), and if $d_g(x_{i,\alpha}, x_{j,\alpha}) = O(s_{j,i,\alpha})$, then $s_{j,i,\alpha} = o(\mu_{i,\alpha})$ and

$$\int_{B_{x_{j,\alpha}}(R_{i,j} s_{j,i,\alpha})} B_{i,\alpha}^2 dv_g = O\big(\mu_{i,\alpha}^{2-n} s_{j,i,\alpha}^n\big) = o(\mu_{i,\alpha}^2).$$

Clearly, (9-23) follows from these two equations. Plugging (9-23) into (9-22), we get from Lemma 8.1 that

$$\int_{\Omega_{i,\alpha}} \langle A_\alpha \mathcal{U}_\alpha, X^\alpha(\nabla \mathcal{U}_\alpha)\rangle_{\mathbb{R}^p} dv_g + \frac{n-2}{4n} \int_{\Omega_{i,\alpha}} (\Delta_g(\operatorname{div}_g X^\alpha))|\mathcal{U}_\alpha|^2 dv_g + \frac{n-2}{2n} \int_{\Omega_{i,\alpha}} (\operatorname{div}_g X^\alpha)\langle A_\alpha \mathcal{U}_\alpha, \mathcal{U}_\alpha\rangle_{\mathbb{R}^p} dv_g$$

$$= -\Big(\langle A(x_i)\Lambda_i, \Lambda_i\rangle_{\mathbb{R}^p} - \frac{n-2}{4(n-1)} S_g(x_i) + o(1)\Big) \int_{B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} B_{i,\alpha}^2 dv_g + o\big(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}\big) + R_t(\alpha). \quad (9\text{-}24)$$

We have

$$\int_{B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} B_{i,\alpha}^2 dv_g = \begin{cases} 64\omega_3 \mu_{i,\alpha}^2 \ln(r_{i,\alpha}/\mu_{i,\alpha}) + o\big(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}\big) & \text{if } n = 4, \\ \big(\int_{\mathbb{R}^n} u_0^2 dx\big) \mu_{i,\alpha}^2 + o(\mu_{i,\alpha}^2) & \text{if } n \geq 5, \end{cases}$$

(9-25)

where u_0 is given by (4-3). Combining (9-10), (9-17)–(9-20), (9-24), and (9-25) yields the proof of Step 1. □

Step 2. *Let $i \in \{1, \ldots, N\}$ be arbitrary. In case $i < N$, assume that* (H$_i$) *holds. Let $\mathcal{K}_\alpha$ be given by*

$$\mathcal{K}_\alpha = \Big(64\omega_3\big(A(x_i)\Lambda_i - \tfrac{1}{6} S_g(x_i)\Lambda_i\big) + o(1)\Big)\mu_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}} + o\big(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}\big)$$

in case $n = 4$, and

$$\mathcal{K}_\alpha = \Big(\big(A(x_i)\Lambda_i - \frac{n-2}{4(n-1)} S_g(x_i)\Lambda_i\big) \int_{\mathbb{R}^n} u_0^2 dx + o(1)\Big)\mu_{i,\alpha}^2$$

in case $n \geq 5$. Then, up to passing to a subsequence, we have

$$\mathcal{K}_\alpha = \left(n^{n-2}(n-2)^{n-1}\omega_{n-1}\left(\mathcal{H}_i(0) + \frac{n-4}{2}\langle \mathcal{H}_i(0), \Lambda_i\rangle\Lambda_i \right) + o(1) \right)\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n},$$

if $\sqrt{\mu_\alpha}r_{i,\alpha} = o(\sqrt{\mu_{i,\alpha}})$, and $\mathcal{K}_\alpha = O(\mu_{i,\alpha}^{(n-2)/2}\mu_\alpha^{(n-2)/2})$ otherwise, where $\mathcal{H}_i$ is as in Lemma 9.1, the $r_{i,\alpha}$'s are as in (8-3), and u_0 is as in (4-3).

Proof of Step 2. We multiply the line k of the system (1-1) by $u_{l,\alpha}$ and integrate over $\Omega_{i,\alpha}$. This leads to

$$\int_{\Omega_{i,\alpha}} u_{l,\alpha}\Delta_g u_{k,\alpha}dv_g + \sum_{m=1}^{p}\int_{\Omega_{i,\alpha}} A_{km}^\alpha u_{l,\alpha}u_{m,\alpha}dv_g = \int_{\Omega_{i,\alpha}} |\mathcal{U}_\alpha|^{2^\star-2}u_{k_\alpha}u_{l,\alpha}dv_g. \tag{9-26}$$

Let the $\Lambda_{i,k}$'s, $k = 1, \ldots, p$, be the components of Λ_i, and the $\mathcal{H}_{i,k}$'s be the components of $\mathcal{H}_i$. We define $S_{k,l}^\alpha$ by

$$S_{k,l}^\alpha = \left(n^{n-2}(n-2)^{n-1}\omega_{n-1}\left(\Lambda_{i,k}\mathcal{H}_{i,l}(0) - \Lambda_{i,l}\mathcal{H}_{i,k}(0) \right) + o(1) \right)\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n},$$

if $r_{i,\alpha} = o(\sqrt{\mu_{i,\alpha}/\mu_\alpha})$, and $S_{k,l}^\alpha = O\left(\mu_{i,\alpha}^{(n-2)/2}\mu_\alpha^{(n-2)/2} \right)$ otherwise. We also define $T_{k,l}^\alpha$ by

$$T_{k,l}^\alpha = \begin{cases} \left(64\omega_3 W_{k,l} + o(1)\right)\mu_{i,\alpha}^2 \ln(r_{i,\alpha}/\mu_{i,\alpha}) + o\left(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}\right) & \text{if } n = 4, \\ \left(W_{k,l}\int_{\mathbb{R}^n} u_0^2 dx + o(1)\right)\mu_{i,\alpha}^2 & \text{if } n \geq 5, \end{cases}$$

where

$$W_{k,l} = \sum_{m=1}^{p}\left(A(x_i)_{lm}\Lambda_{i,k}\Lambda_{i,m} - A(x_i)_{km}\Lambda_{i,l}\Lambda_{i,m} \right),$$

and u_0 is given by (4-3). Integrating by parts, thanks to Lemma 8.2 and Lemma 9.1, we have

$$\int_{\Omega_{i,\alpha}} u_{l,\alpha}\Delta_g u_{k,\alpha}dv_g = \int_{\Omega_{i,\alpha}} u_{k,\alpha}\Delta_g u_{l,\alpha}dv_g + \int_{\partial\Omega_{i,\alpha}}\left(u_{k,\alpha}\partial_\nu u_{l,\alpha} - u_{l,\alpha}\partial_\nu u_{k,\alpha} \right)d\sigma_g$$

$$= \int_{\Omega_{i,\alpha}} |\mathcal{U}_\alpha|^{2^\star-2}u_{k_\alpha}u_{l,\alpha}dv_g - \sum_{m=1}^{p}\int_{\Omega_{i,\alpha}} A_{lm}^\alpha u_{k,\alpha}u_{m,\alpha}dv_g + O\left(\sum_{j\in\mathcal{D}_i}\mu_{j,\alpha}^{n-2}s_{j,i,\alpha}^{2-n} \right) + S_{k,l}^\alpha.$$

Now we write $A_\alpha(x) = A_\alpha(x_{i,\alpha}) + O(d_g(x_{i,\alpha}, x))$. With similar estimates as in the proof of Step 1, thanks to (9-21), we get that

$$\sum_{m=1}^{p}\int_{\Omega_{i,\alpha}} A_{lm}^\alpha u_{k,\alpha}u_{m,\alpha}dv_g - \sum_{m=1}^{p}\int_{\Omega_{i,\alpha}} A_{km}^\alpha u_{l,\alpha}u_{m,\alpha}dv_g = o\left(\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n}\right) + T_{k,l}^\alpha.$$

Coming back to (9-26) with all these estimates, thanks to (9-7), we obtain that $S_{k,l}^\alpha = T_{k,l}^\alpha$. In particular, $\sum_k S_{k,l}^\alpha \Lambda_{i,k} = \sum_k T_{k,l}^\alpha \Lambda_{i,k}$ and Step 2 follows from Step 1. This ends the proof of Step 2. $\qquad\square$

Conclusion of the proof of Proposition 9.2. Equations (9-4) and (9-5) follow from Step 2. It remains to prove that $\langle\Lambda_i, \nabla\mathcal{H}_i(0)\rangle_{\mathbb{R}^p} \equiv 0$. We assume here that $\sqrt{\mu_\alpha}r_{i,\alpha} = o(\sqrt{\mu_{i,\alpha}})$. In particular, $r_{i,\alpha} \to 0$ as $\alpha \to +\infty$. Let Y be an arbitrary 1-form in $\mathbb{R}^n$. We apply once more the Pohozaev identity (7-2) to $\mathcal{U}_\alpha$ in $\Omega_{i,\alpha}$. However, here we choose $X = X^\alpha$ to be given in the exponential chart at $x_{i,\alpha}$ by

$$X_\kappa^\alpha = Y_\kappa - \frac{2}{3}R_{\kappa jkl}(x_{i,\alpha})x^j x^k Y^l,$$

where $Y^l = Y_l$ for all l and the $R_{\kappa jkl}$ are the components of the Riemann tensor Rm_g at $x_{i,\alpha}$ in the exponential chart. As is easily checked, still in geodesic normal coordinates at $x_{i,\alpha}$,

$$(\nabla X^\alpha)_{\kappa j} = -\mathscr{R}_{\kappa jkl}(x_{i,\alpha})x^k Y^l + O(|x|^2),$$

so that $\operatorname{div}_g(X^\alpha) = O(|x|^2)$. Then, thanks to the symmetries of the Riemann tensor, we obtain with the Pohozaev identity that

$$
\int_{\partial\Omega_{i,\alpha}} \left(\tfrac{1}{2} X^\alpha(\nu)|\nabla\mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla\mathcal{U}_\alpha), \partial_\nu\mathcal{U}_\alpha\rangle_{\mathbb{R}^p} \right) d\sigma_g + \int_{\Omega_{i,\alpha}} \langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p} dv_g
$$

$$
= O\left(\int_{\Omega_{i,\alpha}} |\mathcal{U}_\alpha|^2 dv_g \right) + O\left(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha},x)^2 |\nabla\mathcal{U}_\alpha|^2 dv_g \right) + O\left(\int_{\partial\Omega_{i,\alpha}} |\mathcal{U}_\alpha|^{2^*} d\sigma_g \right) \quad \text{(9-27)}
$$

$$
+ O\left(\int_{\partial\Omega_{i,\alpha}} |\mathcal{U}_\alpha|^2 d\sigma_g \right) + O\left(\int_{\partial\Omega_{i,\alpha}} |\partial_\nu\mathcal{U}_\alpha||\mathcal{U}_\alpha| d\sigma_g \right),
$$

Estimating the right-hand side of (9-27) via (9-7) and using Lemma 8.1 and Lemma 8.2, we get

$$
\int_{\partial\Omega_{i,\alpha}} \left(\tfrac{1}{2} X^\alpha(\nu)|\nabla\mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla\mathcal{U}_\alpha), \partial_\nu\mathcal{U}_\alpha\rangle_{\mathbb{R}^p} \right) d\sigma_g + \int_{\Omega_{i,\alpha}} \langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p} dv_g = \hat{R}_t(\alpha), \quad \text{(9-28)}
$$

where $\hat{R}_t(\alpha)$ is such that

$$
\hat{R}_t(\alpha) = \begin{cases} O\left(-\mu_{i,\alpha}^2 \ln\mu_{i,\alpha}\right) + O\left(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}\right) & \text{if } n = 4, \\ O(\mu_{i,\alpha}^2) + O\left(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}\right) & \text{if } n \geq 5. \end{cases} \quad \text{(9-29)}
$$

Now we can write

$$
\int_{\Omega_{i,\alpha}} \langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p} dv_g = \sum_{k,l=1}^p A_{kl}^\alpha(x_{i,\alpha}) \int_{\Omega_{i,\alpha}} u_{k,\alpha} X^\alpha(\nabla\mathcal{U}_\alpha)_l dv_g + O(T_\alpha),
$$

where

$$
T_\alpha = \int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha},x)|\nabla\mathcal{U}_\alpha||\mathcal{U}_\alpha| dv_g,
$$

obtaining

$$
\int_{\Omega_{i,\alpha}} \langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p} dv_g
$$

$$
= \frac{1}{2} \sum_{k,l=1}^p A_{kl}^\alpha(x_{i,\alpha}) \int_{\partial\Omega_{i,\alpha}} u_{k,\alpha} u_{l,\alpha} X^\alpha(\nu) d\sigma_g - \frac{1}{2} \sum_{k,l=1}^p A_{kl}^\alpha(x_{i,\alpha}) \int_{\Omega_{i,\alpha}} u_{k,\alpha} u_{l,\alpha} \operatorname{div}_g\left(X^\alpha\right) dv_g + O(T_\alpha).
$$

As above, estimating the various terms in this equation, it follows that

$$
\int_{\Omega_{i,\alpha}} \langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p} dv_g = \hat{R}_t(\alpha), \quad \text{(9-30)}
$$

where $\hat{R}_t(\alpha)$ is as in (9-29). As a consequence, coming back to (9-28), thanks to (9-30), we get

$$
\int_{\partial\Omega_{i,\alpha}} \left(\tfrac{1}{2} X^\alpha(\nu)|\nabla\mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla\mathcal{U}_\alpha), \partial_\nu\mathcal{U}_\alpha\rangle_{\mathbb{R}^p} \right) d\sigma_g = \hat{R}_t(\alpha), \quad \text{(9-31)}
$$

where $\hat{R}_t(\alpha)$ is as in (9-29). By Lemmas 8.2 and 9.1, together with (9-7), we have

$$\int_{\partial\Omega_{i,\alpha}} \left(\tfrac{1}{2} X^\alpha(\nu)|\nabla\mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla\mathcal{U}_\alpha), \partial_\nu\mathcal{U}_\alpha\rangle_{\mathbb{R}^p} \right) d\sigma_g$$
$$= \left(n^{n-2}(n-2)^{n-1}\omega_{n-1}\langle\Lambda_i, (Y(\nabla\mathcal{H}_i))_0\rangle_{\mathbb{R}^p} + o(1) \right) \mu_{i,\alpha}^{n-2} r_{i,\alpha}^{1-n} + \hat{W}_\alpha, \quad (9\text{-}32)$$

where $(Y(\nabla\mathcal{H}_i))_0 \in \mathbb{R}^p$ is such that $(Y(\nabla\mathcal{H}_i))_0^l = \sum_{\kappa=1}^n Y_\kappa(\nabla_\kappa\mathcal{H}_{i,l})(0)$ for all $l = 1, \ldots, p$, and

$$\hat{W}_\alpha = \begin{cases} o\big(\mu_{i,\alpha}^2(-\ln\mu_{i,\alpha})^{3/2}\big) & \text{if } n = 4, \\ o(\mu_{i,\alpha}^2) & \text{if } n \geq 5. \end{cases}$$

As a consequence of Step 2 we have

$$r_{i,\alpha} = \begin{cases} O\big((-\ln\mu_{i,\alpha})^{-1/2}\big) & \text{if } n = 4, \\ O\big(\mu_{i,\alpha}^{(n-4)/(n-2)}\big) & \text{if } n \geq 5. \end{cases}$$

Coming back to (9-31)–(9-32), it follows that $\langle\Lambda_i, (Y(\nabla\mathcal{H}_i))_0\rangle_{\mathbb{R}^p} = 0$, and since Y is arbitrary, we get $\langle\Lambda_i, \nabla\mathcal{H}_i(0)\rangle_{\mathbb{R}^p} \equiv 0$. $\qquad\square$

10. Proof of Theorem 0.1

We prove Theorem 0.1 using Proposition 9.2. We let $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$ satisfying (H) and (H'). We also let $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of (1-1) and we assume by contradiction that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. We order the blow-up points of the $\mathcal{U}_\alpha$'s in such a way that

$$\mu_\alpha = \mu_{1,\alpha} \geq \cdots \geq \mu_{N,\alpha},$$

where the $\mu_{i,\alpha}$'s are the weights of the vector bubble $(\mathcal{B}_\alpha^i)_\alpha$ in Proposition 4.2, and we let $\mathcal{A}_i$ be as in (8-4). We consider $\mathcal{A}_1$. By (H'), $\operatorname{Ker} A_n(x) \cap \operatorname{Vect}_+(\mathbb{R}^p) = \{0\}$ for all $x \in M$, where A_n is as in (0-2). In particular, if the $r_{i,\alpha}$'s are as in (8-3), it follows from Step 2 in Section 9 that $r_{1,\alpha} \to 0$ as $\alpha \to +\infty$. As a direct consequence, $\mathcal{A}_1 \neq \varnothing$. Let $i \in \mathcal{A}_1$. Still by Step 2 in Section 9, we have $r_{i,\alpha} \to 0$ as $\alpha \to +\infty$. By Proposition 9.2, since $\operatorname{Ker} A_n(x) \cap \operatorname{Vect}_+(\mathbb{R}^p) = \{0\}$ for all $x \in M$, for any $i \in \mathcal{A}_1 \cup \{1\}$, there exists $C_i > 0$ such that

$$r_{i,\alpha}^2 \ln\frac{1}{\mu_{i,\alpha}} \to C_i \text{ if } n = 4 \quad \text{and} \quad r_{i,\alpha}^{n-2}\mu_{i,\alpha}^{4-n} \to C_i \text{ if } n \geq 5 \qquad (10\text{-}1)$$

as $\alpha \to +\infty$. By (10-1), $\mu_{i,\alpha} = o(r_{i,\alpha})$ for all $i \in \mathcal{A}_1 \cup \{1\}$. We also get from (10-1) that for any $i \in \mathcal{A}_1 \cup \{1\}$,

$$\mu_{i,\alpha} = o(r_{i,\alpha}^2) \text{ if } n = 4, 5 \quad \text{and} \quad r_{i,\alpha}^2 = o(\mu_{i,\alpha}) \text{ if } n \geq 7. \qquad (10\text{-}2)$$

As a remark, it follows from (10-2) that $\mathcal{U}_\infty \equiv 0$ when $n = 4, 5$ since, if not the case, $r_{i,\alpha}^2 \leq \mu_{i,\alpha}$. It also follows from (10-2) that for any $i \in \mathcal{A}_1 \cup \{1\}$, $\mathcal{A}_i \cap \mathcal{B}_i \neq \varnothing$, where the $\mathcal{B}_i$'s are as in (8-8). By (9-3), we get with (10-2) that

$$\mathcal{H}_i(z) = \sum_{j \in \mathcal{A}_i \cap \mathcal{B}_i} \frac{\lambda_{i,j}\Lambda_j}{|z - z_{i,j}|^{n-2}}, \qquad (10\text{-}3)$$

where $\mathcal{H}_i$ is as in Lemma 9.1. In particular, the Λ_i's are the S^{p-1} projections of the bubbles $(\mathcal{B}_\alpha^i)_\alpha$. Let $\mathcal{E}_1 = (\mathcal{A}_1 \cap \mathcal{B}_1) \cup \{1\}$. For any $i \in \mathcal{A}_1 \cap \mathcal{B}_1$, we have $\mathcal{A}_i \cap \mathcal{B}_i = \mathcal{E}_1 \setminus \{i\}$. We pick up some $i \in \mathcal{E}_1$ such that $d_g(x_{1,\alpha}, x_{i,\alpha}) \geq d_g(x_{1,\alpha}, x_{j,\alpha})$ for all $j \in \mathcal{E}_1$. By Proposition 9.2 we have $\langle \Lambda_i, \nabla \mathcal{H}_i(0) \rangle_{\mathbb{R}^p} = 0$. Together with (10-3), this implies that $\langle \Lambda_i, \Lambda_j \rangle_{\mathbb{R}^p} = 0$ for all $j \in \mathcal{E}_1 \setminus \{i\}$. Repeating the operation with $\mathcal{E}_1 \setminus \{i\}$, and so on up to exhaust all the indices in $\mathcal{E}_1$, we obtain that $\langle \Lambda_i, \Lambda_j \rangle_{\mathbb{R}^p} = \delta_{ij}$ for all $i, j \in \mathcal{E}_1$. Moreover, it follows from (9-4) and (9-5) in Proposition 9.2 that $V = \text{Vect}\{\Lambda_i, \ i \in \mathcal{E}_1\}$ is a stable vector space of $A_n(x_1)$. Noting that $\langle \Lambda_i, \mathcal{H}_i(0) \rangle_{\mathbb{R}^p} = 0$ for all $i \in \mathcal{E}_1$, we also get with (9-4) and (9-5) in Proposition 9.2 that the Λ_i's are isotropic vectors for $A_n(x_1)$ for all $i \in \mathcal{E}_1$. In particular, we get a contradiction with (H'). This proves Theorem 0.1 when $n \neq 6$. When $n = 6$, thanks to Proposition 2.1, it remains to prove that our systems are weakly stable, and thus that we necessarily have $\mathcal{U}_\infty \not\equiv 0$ if we assume (H'). When $n = 6$, it follows from (10-1) that $r_{i,\alpha}^2 \sim \mu_{i,\alpha}$. Then, by (9-3),

$$\mathcal{H}_i(z) = \sum_{j \in \mathcal{A}_i \cap \mathcal{B}_i} \frac{\lambda_{i,j} \Lambda_j}{|z - z_{i,j}|^{n-2}} + C \mathcal{U}_\infty(x_1), \qquad (10\text{-}4)$$

where $r_{i,\alpha}^{-4} \mu_{i,\alpha}^2 \to C$ as $\alpha \to +\infty$. As above, $\langle \Lambda_i, \Lambda_j \rangle_{\mathbb{R}^p} = \delta_{ij}$ for all $i, j \in \mathcal{E}_1$, but we may have $\mathcal{E}_1 = \{1\}$. By Proposition 9.2, $V = \text{Vect}\{\Lambda_i, \ i \in \mathcal{E}_1\}$ is a stable vector space of $A_6(x_1)$ and the Λ_i's are isotropic vectors for $A_6(x_1)$ for all $i \in \mathcal{E}_1$ if $\mathcal{U}_\infty(x_1) = 0$. In particular, we do get a contradiction with (H') if $\mathcal{U}_\infty(x_1) = 0$. This proves Theorem 0.1 when $n = 6$.

As a remark, if $n = 6$ and $A_6 < 0$ in M in the sense of bilinear forms, where A_6 is as in (0-2), then we also get a contradiction by (9-5) in Proposition 9.2 since $r_{i,\alpha}^2 \sim \mu_{i,\alpha}$ and $\mathcal{H}_i(0), \langle \Lambda_i, \mathcal{H}_i(0) \rangle_{\mathbb{R}^p} \Lambda_i \in \text{Vect}_+(\mathbb{R}^p)$. In particular, we recover analytic stability for our systems if we assume that $A_6 < 0$ in M in the sense of bilinear forms. More precisely, letting (M, g) be a smooth compact six-dimensional Riemannian manifold, $p \geq 1$ be an integer, and $A : M \to M_p^s(\mathbb{R})$ be a C^1-map such that A satisfies (H), the system (0-1) associated with A is analytically stable if $A_6(x) < 0$ in the sense of bilinear forms for all x.

As another remark, it is easily seen from (9-4) and (9-5) in Proposition 9.2 that for any $n \geq 4$, and any $i \in \mathcal{A}_1 \cup \{1\}$, $A_n(x_1) \Lambda_i \in \text{Vect}_+(\mathbb{R}^p)$. In particular, we can replace (H') in Theorem 0.1 by the slightly more general condition that for any $x \in M$, and any $k \in \{1, \ldots, p\}$, there does not exist an orthonormal family $(e_1, \ldots, e_k)$ of vectors in $\text{Is}_{A_n(x)} \cap \text{Vect}_+(\mathbb{R}^p)$ such that $A_n(x)V \subset V$ and $A_n(x)e_i \in \text{Vect}_+(\mathbb{R}^p)$ for all i, where V is the k-dimensional subspace of $\mathbb{R}^p$ with basis $(e_1, \ldots, e_k)$.

As a final remark we mention that Theorem 0.1 still holds true, and can be proved with only slight modifications in the arguments of Section 9, if the C^1 convergence of the A_α's is replaced by a $C^{0,\theta}$-convergence of the A_α's with $\theta = 1$ when $n = 4$, and $\theta > 2/(n-2)$ when $n \geq 5$.

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