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Lovász–Saks–Schrijver ideals and coordinate sections of determinantal varieties

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Motivated by questions in algebra and combinatorics we study two ideals associated to a simple graph G:

- the Lovász-Saks-Schrijver ideal defining the d-dimensional orthogonal representations of the graph complementary to G, and
- the determinantal ideal of the $(d+1)$-minors of a generic symmetric matrix with 0 in positions prescribed by the graph G.

In characteristic 0 these two ideals turn out to be closely related and algebraic properties such as being radical, prime or a complete intersection transfer from the Lovász–Saks–Schrijver ideal to the determinantal ideal. For Lovász–Saks–Schrijver ideals we link these properties to combinatorial properties of G and show that they always hold for d large enough. For specific classes of graphs, such a forests, we can give a complete picture and classify the radical, prime and complete intersection Lovász–Saks–Schrijver ideals.

1. Introduction

Let $\Bbbk$ be a field, $n \geq 1$ be an integer and set $[n] = \{1, \dots, n\}$. For a simple graph $G = ([n], E)$ with vertex set $[n]$ and edge set E we study the following two classes of ideals associated to G.

- *Lovász–Saks–Schrijver ideals*: For an integer $d \geq 1$ we consider the polynomial ring

$$S = \Bbbk\big[y_{i\ell} : i \in [n],\ \ell \in [d]\big].$$

For every edge $e = \{i, j\} \in \binom{[n]}{2}$ we set

$$f_e^{(d)} = \sum_{\ell=1}^{d} y_{i\ell} y_{j\ell}.$$

The ideal

$$L_G^{\Bbbk}(d) = (\, f_e^{(d)} : e \in E\,) \subseteq S$$

is called the Lovász–Saks–Schrijver ideal, LSS-ideal for short, of G with respect to $\Bbbk$. The ideal $L_G^{\Bbbk}(d)$ defines the variety of orthogonal representations of the graph complementary to G. We refer the reader to [Lovász et al. 1989; Lovász 2009] for background on orthogonal representations and

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results on the geometry of the variety of orthogonal representations which provided intuition for some of our results.

- *Coordinate sections of generic (symmetric) determinantal ideals*: Consider the polynomial ring $S = \Bbbk[x_{ij} : 1 \leq i \leq j \leq n]$ and let X be the generic $n \times n$ symmetric matrix, that is, the (i, j)-th entry of X is x_{ij} if $i \leq j$ and x_{ji} if $i > j$. Let X_G^{sym} be the matrix obtained from X by replacing the entries in positions (i, j) and (j, i) for $\{i, j\} \in E$ with 0. For an integer d let $I_d^{\Bbbk}(X_G^{\mathrm{sym}}) \subseteq S$ be the ideal of d-minors of X_G^{sym}. The ideal $I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ defines a coordinate hyperplane section of the generic symmetric determinantal variety. Similarly, we consider ideals defining coordinate hyperplane sections of the generic determinantal varieties and the generic skew-symmetric Pfaffian varieties.

We observe in Section 7 that the ideal $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ and the ideal $L_G^{\Bbbk}(d)$ are closely related. Indeed, if $\Bbbk$ has characteristic 0, classical results from invariant theory can be employed to show that $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ is radical (resp. is prime, has the expected height) provided $L_G^{\Bbbk}(d)$ is radical (resp. is prime, is a complete intersection). We also exhibit similar relations between variants of $L_G^{\Bbbk}(d)$ and ideals defining coordinate sections of determinantal and Pfaffian ideals.

These facts turn the focus on algebraic properties of the LSS-ideals $L_G^{\Bbbk}(d)$. In particular, we analyze the questions: When is $L_G^{\Bbbk}(d)$ a radical ideal? When is it a complete intersection? When is it a prime ideal? Other properties of ideals such as defining a normal ring or a UFD are interesting as well but will not be treated here. In Section 4 we prove the following:

Theorem 1.1. *Let $G = ([n], E)$ be a graph. Then*:

(1) *If $L_G^{\Bbbk}(d)$ is prime then $L_G^{\Bbbk}(d)$ is a complete intersection.*

(2) *If $L_G^{\Bbbk}(d)$ is a complete intersection then $L_G^{\Bbbk}(d+1)$ is prime.*

As an immediate consequence we have:

Corollary 1.2. *Let $G = ([n], E)$ be a graph. Then*:

(1) *If $L_G^{\Bbbk}(d)$ is prime* (*resp. complete intersection*) *then $L_G^{\Bbbk}(d+1)$ is prime* (*resp. complete intersection*).

(2) *If $L_G^{\Bbbk}(d)$ is prime* (*resp. complete intersection*) *then $L_{G'}^{\Bbbk}(d)$ is prime* (*resp. complete intersection*) *for every subgraph G' of G.*

In Section 5 we use these results to show that for d large enough $L_G^{\Bbbk}(d)$ is prime and complete intersection. To this end, for a graph $G = ([n], E)$ we define a graph theoretic invariant $\mathrm{pmd}(G) \in \mathbb{N}$, called the positive matching decomposition number of G. We prove in Lemma 5.4 that $\mathrm{pmd}(G) \leq \min\{2n-3, |E|\}$ and that $\mathrm{pmd}(G) \leq \min\{n-1, |E|\}$ if G is bipartite. We show the following:

Theorem 1.3. *Let $G = ([n], E)$ be a graph. Then for $d \geq \mathrm{pmd}(G)$ the ideal $L_G^{\Bbbk}(d)$ is a radical complete intersection. In particular, $L_G^{\Bbbk}(d)$ is prime if $d \geq \mathrm{pmd}(G) + 1$.*

The fact that $L_G^{\Bbbk}(d)$ is a complete intersection for large d also follows from [Sam and Weyman 2015, Theorems 3.5 and 3.8] or using the theory of strength from results in [Ananyan and Hochster 2016]. To have an explicit bound in Theorem 1.3 is crucial in order to use this result and the connection between $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ and $L_G^{\Bbbk}(d)$. Indeed, for deducing meaningful results, we need to single out cases where we can say something about $L_G^{\Bbbk}(d)$ for $d \leq n-1$. The results described in the following paragraph can be seen as steps in this direction.

Already in Section 4 we give necessary conditions for $L_G^{\Bbbk}(d)$ to be prime in terms of subgraphs of G, see Proposition 4.4. In particular, we prove that if $L_G^{\Bbbk}(d)$ is prime then G does not contain a complete bipartite subgraph $K_{a,b}$ with $a+b=d+1$ (i.e., $\bar{G}$ is $(n-d)$-connected). Similar results are obtained for complete intersections. In general these conditions are only necessary but in Section 6 we show that for small values of d they can be used to characterize the properties. For $d=1$ the characterization is obvious and in [Herzog et al. 2015] it is proved that $L_G^{\Bbbk}(2)$ is prime if and only if G is a matching. We obtain the following:

Theorem 1.4. *Let G be a graph. Then*:

(1) $L_G^{\Bbbk}(3)$ *is prime if and only if G does not contain $K_{1,3}$ and does not contain $K_{2,2}$.*

(2) $L_G^{\Bbbk}(2)$ *is a complete intersection if and only if G does not contain $K_{1,3}$ and does not contain C_{2m} for some $m \geq 2$.*

Here C_n denotes the cycle with n vertices. Finally for forests (i.e., graphs without cycles) we can give a complete picture.

Theorem 1.5. *Let G be a forest and denote by $\Delta(G)$ the maximal degree of a vertex in G. Then*:

(1) $L_G^{\Bbbk}(d)$ *is radical for all d.*

(2) $L_G^{\Bbbk}(d)$ *is a complete intersection if and only if $d \geq \Delta(G)$.*

(3) $L_G^{\Bbbk}(d)$ *is prime if and only if $d \geq \Delta(G)+1$.*

In Section 7 we demonstrate in characteristic 0 the above mentioned connection between $L_G^{\Bbbk}(d)$ and $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$. Using the results from the preceding sections we deduce sufficient conditions for $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ to be radical, prime or of expected height. Similar results are obtained for coordinate hyperplane sections of the generic determinantal varieties and the generic skew-symmetric Pfaffian varieties. To our knowledge coordinate sections of determinantal varieties have been systematically studied only in the case of maximal minors, see for example the results in [Boocher 2012; Eisenbud 1988; Giusti and Merle 1982].

In Section 8 we use the results from Section 4 and Section 7 to formulate obstructions that prevent $L_G^{\Bbbk}(d)$ to be prime or a complete intersection. We also study the exact asymptotics in terms of the number of vertices of the least d such that $L_G^{\Bbbk}(d)$ is prime for G a complete and a complete bipartite graph. Finally, in Section 9 we pose open problems, formulate conjectures and exhibit a relation between hypergraph LSS-ideals and coordinate sections of bounded rank tensor varieties.

To complete the outline of the paper we mention that Section 2 sets up the graph theory and Gröbner theory. Section 3 recalls results from [Herzog et al. 2015] for the case $d = 2$ which in particular show that $L_G^{\Bbbk}(2)$ is always radical if char $\Bbbk \neq 2$. We then exhibit and discuss counterexamples which demonstrate that this is not the case for $d = 3$.

2. Notations and generalities

2A. ***Graph and hypergraph theory.*** In the following we introduce graph theory notation. We mostly follow the conventions from [Diestel 1997]. For us a graph $G = (V, E)$ is a simple graph on a finite vertex set V. In particular, E is a subset of the set of 2-element subsets $\binom{V}{2}$ of V. In most of the cases we assume that $V = [n] = \{1, \dots, n\}$. A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. Given two graphs G and G' we say that G contains G' if G has a subgraph isomorphic to G'.

More generally, a hypergraph $H = (V, E)$ is a pair consisting of a finite set of vertices V and a set E of subsets of V. We are only interested in the situation when the sets in E are inclusionwise incomparable. Such a set of subsets is called a clutter.

For $m, n > 0$ we will use the following notation:

- K_n denotes the complete graph on n vertices, i.e., $K_n = ([n], \{\{i, j\} : 1 \leq i < j \leq n\})$,
- $K_{m,n}$ denotes the complete bipartite graph $([m] \cup [\tilde{n}], \{\{i, \tilde{j}\} : i \in [m], \tilde{j} \in [\tilde{n}]\})$ with bipartition $[m]$ and $[\tilde{n}] = \{\tilde{1}, \dots, \tilde{n}\}$.
- B_n denotes the subgraph of $K_{n,n}$ obtained by removing the edges $\{i, \tilde{i}\}$ for $i = 1, \dots, n$.
- For $n > 2$ we denote by C_n the cycle with n vertices, i.e., the subgraph of K_n with edges $\{1, 2\}$, $\{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}$.
- For $n > 1$ we denote by P_n the path with n vertices, i.e., the subgraph of K_n with edges $\{1, 2\}$, $\{2, 3\}, \dots, \{n-1, n\}$.

We denote by $\bar{G} = (V, \bar{E})$ with $\bar{E} = \binom{V}{2} \setminus E$ the graph complementary to $G = (V, E)$. Let $W \subseteq V$. We write $G_W = (W, \{e \in E : e \subseteq W\})$ for the graph induced by G on vertex set W and $G - W$ for the subgraph induced by G on $V \setminus W$. In case $W = \{v\}$ for some $v \in V$ we simply write $G - v$ for $G - \{v\}$.

A graph $G = ([n], E)$ with $n \geq k + 1$ is called k-(vertex)connected if for every $W \subset V$ with $|W| = k - 1$ the graph $G - W$ is connected. The degree $\deg(v)$ of a vertex v of G is $|\{e \in E : v \in e\}|$ and $\Delta(G) = \max_{v \in V} \deg(v)$. Clearly, if $G = ([n], E)$ is k-connected then every vertex has degree at least k and $\Delta(\bar{G}) \leq n - k - 1$. We denote by $\omega(G)$ the clique number of G, i.e., the largest a such that G contains K_a. The following well known fact follows directly from the definitions.

Lemma 2.1. *Given a graph $G = ([n], E)$ and an integer $1 \leq d \leq n$ the following conditions are equivalent:*

(1) *$\bar{G}$ is $(n-d)$-connected.*

(2) *G does not contain $K_{a,b}$ with $a + b = d + 1$.*

2B. ***Basics on LSS-ideals and their generalization to hypergraphs.*** Let $H = ([n], E)$ be a hypergraph. For an integer $d \geq 1$ we consider the polynomial ring $S = \Bbbk[y_{i\ell} : i \in [n], \ell \in [d]]$. We define for $e \in E$

$$f_e^{(d)} = \sum_{\ell=1}^{d} \prod_{i \in e} y_{i\ell}.$$

If E is a clutter we call the ideal

$$L_H^{\Bbbk}(d) = (\, f_e^{(d)} : e \in E\,) \subseteq S$$

the LSS-ideal of the hypergraph H.

It will sometimes be useful to consider $L_H^{\Bbbk}(d)$ as a multigraded ideal. For that we equip S with the multigrading induced by $\deg(y_{i\ell}) = \mathfrak{e}_i$ for the i-th unit vector $\mathfrak{e}_i$ in $\mathbb{Z}^n$ and $(i, \ell) \in [n] \times [d]$. Clearly, for $e \in E$ the polynomial $f_e^{(d)}$ is multigraded of degree $\sum_{i \in e} \mathfrak{e}_i$. In particular, $L_H^{\Bbbk}(d)$ is $\mathbb{Z}^n$-multigraded. The following remark is an immediate consequence of the fact that if E is a clutter the two polynomials $f_e^{(d)}$ and $f_{e'}^{(d)}$ corresponding to distinct edges $e, e' \in E$ have incomparable multidegrees.

Remark 2.2. Let $H = ([n], E)$ be a hypergraph such that E is clutter. The generators $f_e^{(d)}$, $e \in E$, of $L_H^{\Bbbk}(d)$ form a minimal system of generators. In particular, $L_H^{\Bbbk}(d)$ is a complete intersection if and only if the polynomials $f_e^{(d)}$, $e \in E$, form a regular sequence.

The following alternative description of $L_G^{\Bbbk}(d)$ for a graph G turns out to be helpful in some places.

Remark 2.3. Let $G = ([n], E)$ be a graph. Consider the $n \times d$ matrix $Y = (y_{i\ell})$. Then $L_G^{\Bbbk}(d)$ is the ideal generated by the entries of the matrix YY^T in positions (i, j) with $\{i, j\} \in E$. Here Y^T denotes the transpose of Y.

Similarly, for a bipartite graph G, say a subgraph of $K_{m,n}$, one considers two sets of variables y_{ij} with $(i, j) \in [m] \times [d]$, z_{ij} with $(i, j) \in [d] \times [n]$ and the matrices $Y = (y_{ij})$ and $Z = (z_{ij})$. Then $L_G^{\Bbbk}(d)$ coincides (after renaming the variables in the obvious way) with the ideal generated by the entries of the product matrix YZ in positions (i, j) for $\{i, \tilde{j}\} \in E$.

2C. ***Gröbner bases.*** We use the following notations and facts from Gröbner bases theory, see for example [Bruns and Conca 2003]. Consider the polynomial ring $S = \Bbbk[x_1, \ldots, x_m]$. For a vector

$$\mathfrak{w} = (w_i : i \in [m]) \in \mathbb{R}^m$$

and a nonzero polynomial

$$f = \sum_{\alpha \in \mathbb{N}^{[m]}} a_\alpha x^\alpha$$

we set $m_{\mathfrak{w}}(f) = \max_{a_\alpha \neq 0}\{\alpha \cdot \mathfrak{w}\}$ and

$$\operatorname{in}_{\mathfrak{w}}(f) = \sum_{\alpha \cdot \mathfrak{w} = m_{\mathfrak{w}}(f)} a_\alpha x^\alpha.$$

The latter is called the initial form of f with respect to $\mathfrak{w}$. For an ideal I we denote by $\operatorname{in}_{\mathfrak{w}}(I)$ the ideal generated by $\operatorname{in}_{\mathfrak{w}}(f)$ with $f \in I \setminus \{0\}$. For a term order $\prec$ we denote similarly by $\operatorname{in}_{\prec}(f)$ the largest term of f and by $\mathrm{in}_{\prec}(I)$ the ideal generated by $\mathrm{in}_{\prec}(f)$ with $f \in I \setminus \{0\}$. The following will allow us to deduce properties of ideals from properties of their initial ideals.

Proposition 2.4. *Let I be a homogeneous ideal in the polynomial ring S and let τ be either a term order $\prec$ or a vector $\mathfrak{w} \in \mathbb{R}^m$. If $\mathrm{in}_\tau(I)$ is radical or a complete intersection or prime then so is I. Moreover, if $I = (f_1, \dots, f_r)$ and the elements $\mathrm{in}_\tau(f_1), \dots, \mathrm{in}_\tau(f_r)$ form a regular sequence then $f_1, \dots, f_r$ form a regular sequence and $\mathrm{in}_\tau(I) = (\mathrm{in}_\tau(f_1), \dots, \mathrm{in}_\tau(f_r))$.*

3. Known results and counterexamples for Lovász–Saks–Schrijver ideals

We recall results from [Herzog et al. 2015] and present examples showing that $L_G^{\Bbbk}(3)$ is not radical in general. First observe that, for obvious reasons, $L_G^{\Bbbk}(1)$ is radical, it is a complete intersection if and only if G is a matching and it is prime if and only if G has no edges. For $d = 2$ the following result from [Herzog et al. 2015] gives a complete answer for two of the three properties under discussion.

Theorem 3.1 [Herzog et al. 2015, Theorems 1.1, 1.2 and Corollary 5.3]. *Let $G = ([n], E)$ be a graph. If* $\mathrm{char}\,\Bbbk \neq 2$ *then the ideal $L_G^{\Bbbk}(2)$ is radical. If* $\mathrm{char}\,\Bbbk = 2$ *then $L_G^{\Bbbk}(2)$ is radical if and only if G is bipartite. Furthermore, $L_G^{\Bbbk}(2)$ is prime if and only if G is a matching.*

Indeed, in [Herzog et al. 2015] the characterization of the graphs G for which $L_G^{\Bbbk}(2)$ is prime is given under the assumption that $\mathrm{char}\,\Bbbk \neq 1, 2 \bmod (4)$ but it turns out that the statement holds as well in arbitrary characteristic (see Proposition 4.4 for the missing details).

The next examples show that $L_G^{\Bbbk}(3)$ need not be radical. In the examples we assume that $\Bbbk$ has characteristic 0 but we consider it very likely that the ideals are not radical over any field.

A quick criterion implying that an ideal J in a ring S is not radical is to identify an element $g \in S$ such that $J : g \neq J : g^2$. We call such a g a witness (of the fact that J is not radical). Of course the potential witnesses must be sought among the elements that are "closely related" to J. Alternatively, one can try to compute the radical of J or even its primary decomposition directly and read off whether J is radical. But these direct computations are extremely time consuming for LSS-ideals and did not terminate on our computers in the examples below. Nevertheless, in all examples we have quickly identified witnesses.

Example 3.2. We present three examples of graphs G such that $L_G^{\Bbbk}(3)$ is not radical over any field $\Bbbk$ of characteristic 0. The first example has 6 vertices and 9 edges and it is the smallest example we have found (both in terms of edges and vertices). The second example has 7 vertices and 10 edges and it is a complete intersection. This shows that $L_G^{\Bbbk}(3)$ can be a complete intersection without being radical. The third example is bipartite, a subgraph of $K_{5,4}$, with 12 edges, and is the smallest bipartite example we have found. In all cases, since the LSS-ideal $L_G^{\Bbbk}(3)$ has integral coefficients, we may assume that $\Bbbk = \mathbb{Q}$ and exhibit a witness g, i.e., a polynomial g such that $L_G^{\Bbbk}(3) : g \neq L_G^{\Bbbk}(3) : g^2$. The latter inequality can be checked with the help of CoCoA [Abbott et al. 2018] or Macaulay 2 [Grayson and Stillman 1993].

(1) Let G be the graph with 6 vertices and 9 edges depicted in Figure 1, left, i.e., with edges

$$E = \big\{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,6\}, \{3,5\}, \{4,6\}\big\}.$$

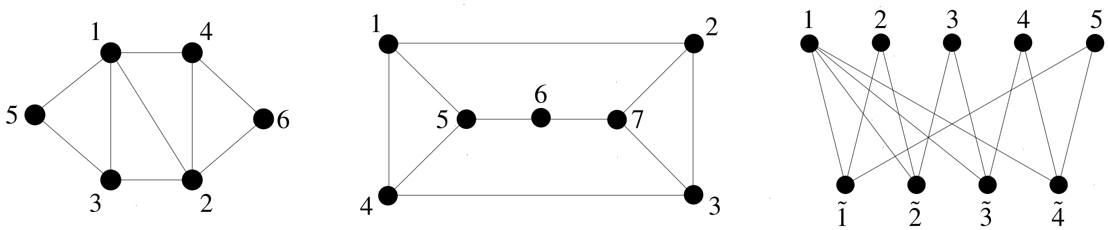

Figure 1. Graphs G with nonradical $L_G^{\Bbbk}(3)$.

Here the witness can be chosen as follows. Denote by $Y = (y_{ij})$ a generic 6×3 matrix. As discussed in Remark 2.3 the ideal $L_G^{\mathbb{Q}}(3)$ is generated by the entries of YY^T corresponding to the positions in E. Now g can be taken as the 3-minor of Y with row indices $1, 5, 6$.

(2) Let G be the graph with 7 vertices and 10 edges depicted in Figure 1, middle, i.e., with edges

$$E = \big\{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 7\}, \{3, 4\}, \{3, 7\}, \{4, 5\}, \{5, 6\}, \{6, 7\}\big\}.$$

Here the witness can be chosen as follows. Denote by $Y = (y_{ij})$ a generic 7×3 matrix. Again as discussed in Remark 2.3 the ideal $L_G^{\mathbb{Q}}(3)$ is generated by the entries of YY^T corresponding to the positions in E. Now g can be taken as the 3-minor of Y with row indices $1, 2, 4$. The fact that $L_G^{\mathbb{Q}}(3)$ is a complete intersection can be checked quickly with CoCoA [Abbott et al. 2018] or Macaulay 2 [Grayson and Stillman 1993].

(3) Let G be the subgraph of the complete bipartite graph $K_{5,4}$ depicted in Figure 1, right, i.e., with edges

$$E = \big\{\{1, \tilde{1}\}, \{1, \tilde{2}\}, \{1, \tilde{3}\}, \{1, \tilde{4}\}, \{2, \tilde{1}\}, \{2, \tilde{2}\}, \{3, \tilde{2}\}, \{3, \tilde{3}\}, \{4, \tilde{3}\}, \{4, \tilde{4}\}, \{5, \tilde{1}\}, \{5, \tilde{4}\}\big\}.$$

Denote by $X = (x_{ij})$ a generic 5×3 matrix and by $Y = (y_{ij})$ a generic 3×4 matrix. As explained in Remark 2.3 the ideal $L_G^{\mathbb{Q}}(3)$ is generated by the entries of XY corresponding to the positions in E. Now the witness g can be taken to be the 3-minor of X corresponding to the column indices $1, 2, 4$.

4. Stabilization of algebraic properties of $L_G^{\Bbbk}(d)$

In this section we prove Theorem 1.1 and state some of its consequences. We recall first some facts on the symmetric algebra of a module stating the results in the way that suit our needs best.

Recall that, given a ring R and an R-module M presented as the cokernel of an R-linear map

$$f : R^m \to R^n$$

the symmetric algebra $\mathrm{Sym}_R(M)$ of M is (isomorphic to) the quotient of $\mathrm{Sym}_R(R^n) = R[x_1, \dots, x_n]$ by the ideal J generated by the entries of $A\,(x_1, \dots, x_n)^T$, where A is the $m \times n$ matrix representing f. Vice versa every quotient of $R[x_1, \dots, x_n]$ by an ideal J generated by homogeneous elements of degree 1 in the x_i's is the symmetric algebra of an R-module.

Part (1) of the following is a special case of [Avramov 1981, Proposition 3] and part (2) a special case of [Huneke 1981, Theorem 1.1]. Here and in the rest of the paper for a matrix A with entries in a ring R and a number t we denote by $I_t(A)$ the ideal of R generated by the t-minors of A.

Theorem 4.1. *Let R be a complete intersection. Then*:

(1) $\mathrm{Sym}_R(M)$ *is a complete intersection if and only if* height $I_t(A) \geq m-t+1$ *for all* $t = 1, \dots, m$.

(2) $\mathrm{Sym}_R(M)$ *is a domain and* $I_m(A) \neq 0$ *if and only if R is a domain, and* height $I_t(A) \geq m-t+2$ *for all* $t = 1, \dots, m$.

The equivalent conditions of (2) *imply those of* (1).

Remark 4.2. Let $G = ([n], E)$ be a graph. The ideal $L_G^{\Bbbk}(d) \subseteq S = \Bbbk[y_{i,j} : i \in [n],\ j \in [d]]$ is generated by elements that have degree at most one in each block of variables. Hence $L_G^{\Bbbk}(d)$ can be seen as an ideal defining a symmetric algebra in various ways.

For example, set $G_1 = G - n$, $U = \{i \in [n-1] : \{i, n\} \in E\}$, $u = |U|$, $S' = \Bbbk[y_{i,j} : i \in [n-1],\ j \in [d]]$ and $R = S'/L_{G_1}^{\Bbbk}(d)$. Then $S/L_G^{\Bbbk}(d)$ is the symmetric algebra of the cokernel of the R-linear map

$$R^u \to R^d$$

associated to the $u \times d$ matrix $A = (y_{ij})$ with $i \in U$ and $j = 1, \dots, d$.

Remark 4.3. In order to apply Theorem 4.1 to the case described in Remark 4.2 it is important to observe that for every G no minors of the matrix $(y_{ij})_{(i,j)\in[n]\times[d]}$ vanish modulo $L_G^{\Bbbk}(d)$. This is because $L_G^{\Bbbk}(d)$ is contained in the ideal J generated by the monomials $y_{ik}y_{jk}$ and the terms in the minors of (y_{ij}) do not belong to J for obvious reasons.

Proposition 4.4. *Let $G = ([n], E)$ be a graph. If $L_G^{\Bbbk}(d)$ is prime then G does not contain $K_{a,b}$ with $a + b > d$.*

Proof. Suppose by contradiction that $L_G^{\Bbbk}(d)$ is prime and G contains $K_{a,b}$ for some $a+b > d$. We may decrease either a or b or both and assume right away that $a + b = d + 1$ with $a, b \geq 1$. In particular $a, b \leq d$ and $a + b \leq n$. We may assume that $K_{a,b}$ is a subgraph of G with edges $\{i, a + j\}$ for $i \in [a]$ and $j \in [b]$. Set $R = S/L_G^{\Bbbk}(d)$ and $Y = (y_{i\,\ell}) \in R^{a\times d}$ and $Z = (z_{\ell,i}) \in R^{d\times b}$ with $z_{\ell,i} = y_{i+a,\ell}$. Since $K_{a,b}$ is a subgraph of G we have $YZ = 0$ in R. By assumption R is a domain and $YZ = 0$ can be seen as a matrix identity over the field of fractions of R. Hence

$$\mathrm{rank}(Y) + \mathrm{rank}(Z) \leq d.$$

From $a + b = d + 1$ it follows that $\mathrm{rank}(Y) < a$ or $\mathrm{rank}(Z) < b$. This implies that $I_a(Y) = 0$ or $I_b(Z) = 0$ as ideals of R. But by Remark 4.3 none of the minors of Y and Z are in $L_G^{\Bbbk}(d)$. This is a contradiction and hence $L_G^{\Bbbk}(d)$ is not prime. □

Lemma 4.5. *Let A be an $m \times n$ matrix with entries in a Noetherian ring R. Assume $m \leq n$. Let $S = R[x] = R[x_1, \dots, x_m]$ be a polynomial ring over R and let B be the $m \times (n+1)$ matrix with entries* *in S obtained by adding the column $(x_1, \dots, x_m)^T$ to A. Then we have* $\operatorname{height} I_1(B) = \operatorname{height} I_1(A) + m$ *and*

$$\operatorname{height} I_t(B) \geq \min\bigl\{\operatorname{height} I_{t-1}(A),\ \operatorname{height} I_t(A) + m - t + 1\bigr\}$$

for all $1 < t \leq m$.

Proof. Set $u = \min\bigl\{\operatorname{height} I_{t-1}(A),\ \operatorname{height} I_t(A) + m - t + 1\bigr\}$. Let P be a prime ideal of S containing $I_t(B)$. We have to prove that $\operatorname{height} P \geq u$. If $P \supseteq I_{t-1}(A)$ then $\operatorname{height} P \geq \operatorname{height} I_{t-1}(A) \geq u$. If $P \not\supseteq I_{t-1}(A)$ then we may assume that the $(t-1)$-minor F corresponding to the first $(t-1)$ rows and columns of A is not in P. Hence, $\operatorname{height} P = \operatorname{height} P R_F[x]$ and $P R_F[x]$ contains $I_t(A) R_F[x]$ and $\bigl(x_j - F^{-1} G_j : j = t, \dots, m\bigr)$ with $G_j \in R[x_1, \dots, x_{t-1}]$. Since the elements $x_j - F^{-1} G_j$ are algebraically independent over R_F we have

$$\operatorname{height} P R_F[x] \geq \operatorname{height} I_t(A) R_F + (m - t + 1) \geq \operatorname{height} I_t(A) + (m - t + 1). \qquad \square$$

Proof of Theorem 1.1. To prove (1) we argue by induction on n. The induction base $n \leq 2$ is obvious. Assume $n > 2$. We use the notation from Remark 4.2 and set

$$S = \Bbbk\bigl[y_{ij} : i \in [n],\ j \in [d]\bigr], \quad S' = \Bbbk\bigl[y_{i,j} : i \in [n-1],\ j \in [d]\bigr],$$
$$G_1 = G - n, \quad U = \{i \in [n-1] : \{i, n\} \in E\}, \quad u = |U|.$$

Note, that $S'/L^{\Bbbk}_{G_1}(d)$ is an algebra retract of $S/L^{\Bbbk}_G(d)$. Therefore $L^{\Bbbk}_{G_1}(d) = L^{\Bbbk}_G(d) \cap S'$ and so $L^{\Bbbk}_{G_1}(d)$ is prime. By induction, it follows that $L^{\Bbbk}_{G_1}(d)$ is a complete intersection. Since u is the degree of the vertex n in G we have that $K_{1,u} \subset G$. Since $L^{\Bbbk}_G(d)$ is prime Proposition 4.4 implies $1 + u < d + 1$, i.e., $u < d$. By virtue of Remark 4.3 we have that the minors of the matrix A are nonzero in $S'/L^{\Bbbk}_{G_1}(d)$. In particular, $I_u(A) \neq 0$ in $S'/L^{\Bbbk}_{G_1}(d)$ and hence (2) in Theorem 4.1 holds. Then (1) in Theorem 4.1 holds as well, i.e., $L^{\Bbbk}_G(d)$ is a complete intersection.

To prove (2) we again argue by induction on n. For $n \leq 2$ the assertion is obvious. Assume $n > 2$. We again use the notation $G_1 = G - n$, $U = \{i \in [n] : \{i, n\} \in E\}$, $u = |U|$. In addition we set $Y = (y_{ij})_{(i,j) \in U \times [d+1]}$, $S = \Bbbk[y_{ij} : i \in [n],\ j \in [d+1]]$, $S' = \Bbbk[y_{ij} : i \in [n-1],\ j \in [d+1]]$ and $R = S'/L^{\Bbbk}_{G_1}(d+1)$. By construction, $S/L^{\Bbbk}_G(d+1)$ is the symmetric algebra of the R-module presented as the cokernel of the map $R^u \to R^{d+1}$ associated to Y.

By assumption, $L^{\Bbbk}_G(d)$ is a complete intersection and hence $L^{\Bbbk}_{G_1}(d)$ is a complete intersection as well. It then follows by induction that $L^{\Bbbk}_{G_1}(d+1)$ is prime and hence R is a domain. Since the polynomials $f^{(d)}_{\{i,n\}}$ with $i \in U$ are a regular sequence contained in the ideal $(y_{nj} : 1 \leq j \leq d)$ we have $u \leq d$ and by Remark 4.3 $I_u(Y) \neq 0$ in R. Therefore, by Theorem 4.1(2) we have

$$L^{\Bbbk}_G(d+1) \text{ is prime} \quad \Longleftrightarrow \quad \operatorname{height} I_t(Y) \geq u - t + 2 \text{ in } R \quad \text{for every } t = 1, \dots, u.$$

Equivalently, we have to prove that

$$\operatorname{height}\bigl(I_t(Y) + L^{\Bbbk}_{G_1}(d+1)\bigr) \geq u - t + 2 + g \text{ in } S' \quad \text{for every } t = 1, \dots, u,$$

where $g = \operatorname{height} L^{\Bbbk}_{G_1}(d+1) = |E| - u$.

Consider the weight vector $\mathfrak{w} \in \mathbb{R}^{n\times(d+1)}$ defined by $\mathfrak{w}_{ij} = 1$ and $\mathfrak{w}_{i\,d+1} = 0$ for all $j \in [d]$ and $i \in [n]$. By construction the initial forms of the standard generators of $\operatorname{in}_{\mathfrak{w}}(L^{\Bbbk}_{G_1}(d+1))$ are the standard generators of $L^{\Bbbk}_{G_1}(d)$. Since the standard generators of $I_t(Y)$ coincide with their initial forms with respect to $\mathfrak{w}$ it follows that $\operatorname{in}_{\mathfrak{w}}(I_t(Y)) \supseteq I_t(Y)$ (indeed equality holds but we do not need this fact).

Therefore, $\operatorname{in}_{\mathfrak{w}}(I_t(Y) + L^{\Bbbk}_{G_1}(d+1)) \supseteq I_t(Y) + L^{\Bbbk}_{G_1}(d)$ and it is enough to prove that

$$\operatorname{height}\big(I_t(Y) + L^{\Bbbk}_{G_1}(d)\big) \geq u - t + 2 + g \text{ in } S' \quad \text{for every } t = 1, \dots, u$$

or, equivalently,

$$\operatorname{height} I_t(Y) \geq u - t + 2 \text{ in } R' \quad \text{for every } t = 1, \dots, u,$$

where $R' = S'/L^{\Bbbk}_{G_1}(d)$.

The variables $y_{1\,d+1}, \dots, y_{n-1\,d+1}$ do not appear in the generators of $L^{\Bbbk}_{G_1}(d)$. Hence

$$R' = R''[y_{1\,d+1}, \dots, y_{n-1\,d+1}] \quad \text{with } R'' = \Bbbk\big[y_{ij} : (i,j) \in [n-1] \times [d]\big]/L^{\Bbbk}_{G_1}(d).$$

Let Y' be the matrix Y with the $(d+1)$-st column removed. Then $S/L^{\Bbbk}_{G}(d)$ can be regarded as the symmetric algebra of the R''-module presented as the cokernel of the map

$$(R'')^u \xrightarrow{Y'} (R'')^d. \tag{1}$$

By assumption $S/L^{\Bbbk}_{G}(d)$ is a complete intersection. Hence by Theorem 4.1(1) we know

$$\operatorname{height} I_t(Y') \geq u - t + 1 \text{ in } R'' \quad \text{for every } t = 1, \dots, u$$

Since Y is obtained from Y' by adding a column of variables over R'' by Lemma 4.5 we have

$$\operatorname{height} I_t(Y) \geq \min\big\{\operatorname{height} I_{t-1}(Y'),\ \operatorname{height} I_t(Y') + u - t + 1\big\} \geq u - t + 2$$

in R' and for all $t = 1, \dots, u$. □

Proof of Corollary 1.2. Assertion (1) in Corollary 1.2 is a formal consequence of Theorem 1.1. Assertion (2) is obvious for complete intersections. Finally assume that $L^{\Bbbk}_{G}(d)$ is prime. Then by Theorem 1.1 $L^{\Bbbk}_{G}(d)$ is a complete intersection. The statement now follows from a general fact: if a regular sequence generates a prime ideal in a standard graded algebra or in a local ring then so does every subset of the sequence. □

5. Positive matching decompositions

In this section we introduce positive matching decompositions and prove Theorem 1.3.

Definition 5.1. Given a hypergraph $H = (V, E)$ a positive matching of H is a subset $M \subset E$ of pairwise disjoint sets (i.e., a matching) such that there exists a weight function $w : V \to \mathbb{R}$ satisfying:

$$\sum_{i \in A} w(i) > 0 \quad \text{if } A \in M, \qquad \sum_{i \in A} w(i) < 0 \quad \text{if } A \in E \setminus M. \tag{2}$$

The next lemma summarizes some elementary properties of positive matchings.

Lemma 5.2. *Let $H = (V, E)$ be a hypergraph such that E is a clutter, $M \subseteq E$ and $V_M = \bigcup_{A\in M} A$.*

(1) *M is a positive matching for H if and only if M is a positive matching for the induced hypergraph $(V_M, \{A \in E : A \subseteq V_M\})$.*

(2) *Assume M is a positive matching on H and $A \in E$ is such that $M_1 = M \cup \{A\}$ is a matching. Assume also there is a vertex $a \in A$ such that*

$$\big\{B \in E : B \subset V_{M_1} \text{ and } a \in B\big\} = \{A\}.$$

Then $M \cup \{A\}$ is a positive matching of H.

(3) *If H is a bipartite graph with bipartition $V = V_1 \cup V_2$ then M is a positive matching if and only if M is a matching and directing the edges $e \in E$ from V_1 to V_2 if $e \in M$ and from V_2 to V_1 if $e \in E \setminus M$ yields an acyclic orientation.*

Proof. (1) Set $H_1 = (V_M, \{A \in E : A \subseteq V_M\})$. Clearly a weight function on V for which M is a positive matching restricts to V_M making M a positive matching of H_1. Conversely, assume we are given a weight function w on V_M that makes M a positive matching. Then we extend w to V by assigning to the vertices in $V \setminus V_M$ a weight sufficiently negative to induce a negative weight on the elements of E which contain at least one element from $V \setminus V_M$. For example, one can set $w(i) = -|V| \max\{w(j) : j \in V_M\}$ for every $i \in V \setminus V_M$. Such an extension makes M a positive matching for H.

(2) Let w be a weight that makes M a positive matching of H. In view of (1), it is enough to prove that there is a weight v defined on V_{M_1} making M_1 a positive matching for the restriction of H to V_{M_1}. We set $v(i) = w(i)$ if $i \in V_{M_1}$ and $i \neq a$ and we give $v(a)$ a high enough value to have $v(A) > 0$, i.e., $v(a) > -\sum_{i\in A\ i\neq a} w(i)$. Since there are no elements in E other than A that are contained in V_{M_1} and contain a the resulting weight v has the desired properties.

(3) We change the coordinates $w(i)$ to $-w(i)$ for $i \in V_2$ in the inequalities defining a positive matchings. As a simple reformulation of (2) we get that in these coordinates a matching M is positive if and only if there is a weight function such that for $\{i, j\} \in E,\ i \in V_1,\ j \in V_2$ we have

$$w(i) > w(j) \quad \text{if } \{i, j\} \in M, \qquad w(i) < w(j) \quad \text{if } \{i.j\} \in E \setminus M. \tag{3}$$

This is equivalent to the existence of a region in the arrangement of hyperplanes $w(i) = w(j)$ for $\{i, j\} \in E$ in $\mathbb{R}^V$ satisfying (3). But it is well known that the regions in this arrangement are in one to one correspondence with the acyclic orientations of G (see [Greene and Zaslavsky 1983, Lemma 7.1]). □

Now we are in position to introduce the key concept of this section.

Definition 5.3. Let $H = (V, E)$ be a hypergraph for which E is a clutter. A positive matching decomposition (or pm-decomposition) of G is a partition $E = \bigcup_{i=1}^{p} E_i$ into pairwise disjoint subsets such that E_i is a positive matching on $\big(V, E \setminus \bigcup_{j=1}^{i-1} E_j\big)$ for $i = 1, \dots, p$. The E_i are called the parts of the pm-decomposition. The smallest p for which G admits a pm-decomposition with p parts will be denoted by $\operatorname{pmd}(H)$.

Note that one has $\operatorname{pmd}(H) \le |E|$ because of the obvious pm-decomposition $\bigcup_{A\in E}\{A\}$. On the other hand $\operatorname{pmd}(G)$ is smaller than $|E|$ for most clutters. For graphs we have:

Lemma 5.4. *Let $G = ([n], E)$ be a graph. Then*:

(1) $\operatorname{pmd}(G) \le \min(2n-3, |E|)$.

(2) *If G is bipartite then* $\operatorname{pmd}(G) \le \min(n-1, |E|)$.

(3) $\operatorname{pmd}(G) \ge \Delta(G)$ *with equality if G is a forest.*

Proof. (1) Since we have already argued that $\operatorname{pmd}(G) \le |E|$ to prove the first statement we have to show that $\operatorname{pmd}(G) \le 2n-3$. To this end we may assume that G is the complete graph K_n because any pm-decomposition of K_n induces a pm-decomposition on its subgraphs. For $\ell = 1, \dots, 2n-3$ we set $E_\ell = \{\{i, j\} : i + j = \ell + 2\}$. Clearly one has $E = \bigcup_{\ell=1}^{2n-3} E_\ell$. So to prove that this is a pm-decomposition of K_n we have to prove that E_t is a positive matching on $G_t = \big([n], \bigcup_{\ell=t}^{2n-3} E_\ell\big)$. To this end we build E_t by inserting the edges one by one starting from those that involve vertices with smaller indices and repeatedly use Lemma 5.2(2) to prove that we actually get a positive matching. For example for $n = 8$, to prove that E_7 is a positive matching on G_7 we order the elements in E_7 as follows $\{4, 5\}, \{3, 6\}, \{2, 7\}, \{1, 9\}$. We assume we know already that $\{\{4, 5\}, \{3, 6\}\}$ is a positive matching and use Lemma 5.2(2) with $A = \{2, 7\}$ and $a = 2$ to prove that $\{\{4, 5\}, \{3, 6\}, \{2, 7\}\}$ is a positive matching as well.

(2) In this case it is enough to prove that $\operatorname{pmd}(K_{m,n}) \le n + m - 1$. For $\ell = 1, \dots, m + n - 1$ we set $E_\ell = \{\{i, \tilde{j}\} : i + j = \ell + 1\}$. Clearly one has $E = \bigcup_{\ell=1}^{m+n-1} E_\ell$. So to prove that this is a positive matching decomposition of $K_{m,n}$ we have to prove that E_ℓ is a positive matching on $E \setminus \bigcup_{k=1}^{\ell-1} E_k$ for $\ell = 1, \dots, m + n - 1$.

For $\ell = 1$ the assertion is obvious since E_1 contains a single edge. Now assume $\ell \ge 2$. By Lemma 5.2(3) it suffices to show that directing the edges in E_ℓ from $[m]$ to $[\tilde{n}]$ and the edges in $E \setminus \bigcup_{k=1}^{\ell} E_k$ in the other direction yields an acyclic orientation. Assume the resulting directed graph has a directed cycle. Let $\{i, \tilde{j}\} \in E_\ell$ be the edge from E_ℓ in this directed cycle for which j is minimal. The directed edge following the edge $i \to \tilde{j}$ in the directed cycle is of the form $\tilde{j} \to i'$ for some i' with $i' + j > \ell + 1$. This implies $i' > i$. Now let $i' \to \tilde{j}'$ be the edge following $\tilde{j} \to i'$ in the directed cycle. Then $\{i', \tilde{j}'\} \in E_\ell$ and $i' + j' = \ell + 1$. But this yields $j' < j$ which contradicts the minimality of j. Hence there is no directed cycle and E_ℓ is a positive matching on $E \setminus \bigcup_{k=1}^{\ell-1} E_k$.

(3) The inequality $\Delta(G) \le \operatorname{pmd}(G)$ is obvious. To prove that equality holds if G is a forest we argue by induction on the number of vertices. We may assume $\{n-1, n\} \in E$ and that n is a leaf of G. Hence $G_1 = G - n$ is a forest on $n - 1$ vertices and by induction there exists a positive matching decomposition $E_1, \dots, E_p$ of G_1 with $p = \Delta(G_1)$. If $\Delta(G_1) < \Delta(G)$ we may simply set $E_{p+1} = \{\{n-1, n\}\}$ and note that, by virtue of Lemma 5.2(1), $E_1, \dots, E_{p+1}$ is a positive matching decomposition of G. If instead $\Delta(G_1) = \Delta(G)$ then there exists i such that $n - 1 \notin V_{E_i}$ and hence $E_i' = E_i \cup \{\{n-1, n\}\}$ is a matching. Using (1) and (2) of Lemma 5.2 one easily checks that the resulting decomposition $E_1, \dots, E_{i-1}, E_i', E_{i+1} \dots, E_p$ is a positive matching decomposition of G. $\square$

Next we connect positive matching decompositions to algebraic properties of LSS-ideals.

Lemma 5.5. *Let $H=(V,E)$ be a hypergraph such that E is a clutter, $d \geq p = \operatorname{pmd}(H)$ and $E=\bigcup_{\ell=1}^{p} E_\ell$ a positive matching decomposition. Then there exists a term order $<$ on S such that for every ℓ and every $A \in E_\ell$ we have*

$$\operatorname{in}_<(f_A^{(d)}) = \prod_{i \in A} y_{i\ell}. \tag{4}$$

Proof. To define $<$ we first define weight vectors $\mathfrak{w}_1, \dots, \mathfrak{w}_p \in \mathbb{R}^{V \times [d]}$. For that purpose we use the weight functions $w_\ell : V \to \mathbb{R}$, associated to each matching E_ℓ, $\ell = 1, \dots, p$. The weight vector $\mathfrak{w}_\ell$ is defined as follows:

- $\mathfrak{w}_\ell(y_{ik}) = 0$ if $k \neq \ell$, and
- $\mathfrak{w}_\ell(y_{i\ell}) = w_\ell(i)$.

By construction it follows that:

$$\operatorname{in}_{\mathfrak{w}_1}(f_A^{(d)}) = \begin{cases} \prod_{i\in A} y_{i1} & \text{if } A \in E_1, \\ \sum_{k=2}^{d} \prod_{i \in A} y_{ik} & \text{if } A \in E \setminus \{E_1\}. \end{cases} \tag{5}$$

We define the term order $<$ as follows: $y^\alpha < y^\beta$ if

(1) $|\alpha| < |\beta|$, or

(2) $|\alpha| = |\beta|$ and $\mathfrak{w}_\ell(y^\alpha) < \mathfrak{w}_\ell(y^\beta)$ for the smallest ℓ such that $\mathfrak{w}_\ell(y^\alpha) \neq \mathfrak{w}_\ell(y^\beta)$, or

(3) $|\alpha| = |\beta|$ and $\mathfrak{w}_\ell(y^\alpha) = \mathfrak{w}_\ell(y^\beta)$ for all ℓ and $y^\alpha <_0 y^\beta$ for an arbitrary but fixed term order $<_0$.

Now a simple induction shows that for all ℓ and for all $A \in E_\ell$ we have $\operatorname{in}_<(f_A^{(d)}) = \prod_{i\in A} y_{i\ell}$. □

Proof of Theorem 1.3. Let $d \geq p = \operatorname{pmd}(G)$ and $E = \bigcup_{\ell=1}^{p} E_\ell$ a pm-decomposition of G. By Lemma 5.5 there is a term order $<$ satisfying (4). Since each E_ℓ, $\ell = 1, \dots, p$, is a matching (4) implies that the initial monomials of the generators $f_A^{(d)}$ of $L_H^{\Bbbk}(d)$ are pairwise coprime and square free. Then the assertion follows from Proposition 2.4. The rest follows from Theorem 1.1. □

The following is an immediate consequence of Theorem 1.3 and Lemma 5.4:

Corollary 5.6. *Let $G=([n],E)$ be a graph. Then $L_G^{\Bbbk}(d)$ is a radical complete intersection for $d \geq \min\{2n-3, |E|\}$ and prime for $d \geq \min\{2n-3, |E|\}+1$. If G is bipartite then $L_G^{\Bbbk}(d)$ is a radical complete intersection for $d \geq \min\{n-1, |E|\}$ and prime for $d \geq \min\{n-1,|E|\}+1$.*

6. Proofs of Theorem 1.4 and Theorem 1.5

Proof of Theorem 1.4. (1) By Proposition 4.4 if $L_G^{\Bbbk}(3)$ is prime then G does not contain $K_{1,3}$ and $K_{2,2}$. Now assume G does not contain $K_{1,3}$ and $K_{2,2}$. In addition, we may assume that $\Bbbk$ is algebraically closed. Since the tensor product over $\Bbbk$ of $\Bbbk$-algebras that are domains is a domain (see the Corollary to Proposition 1 in Bourbaki's Algebra [Bourbaki 1990, Chapter V, 17]) we may also assume that the graph is connected. A connected graph not containing $K_{1,3}$ and $K_{2,2}$ is either an isolated vertex or a path

P_n on $n>1$ vertices or a cycle C_n with n vertices for $n=3$ or $n\geq 5$. For an isolated vertex we have $L_G^{\Bbbk}(3)=(0)$. Hence we have to prove that $L_G^{\Bbbk}(3)$ is prime when $G=P_n$ for $n\geq 2$ or $G=C_n$ for $n=3$ or $n\geq 5$. If $G=P_n$ then by Lemma 5.4 $\operatorname{pmd}(P_n)=\Delta(P_n)\leq 2$. Hence by Theorem 1.3 it follows that $L_{P_n}^{\Bbbk}(3)$ is prime.

Now let $G=C_n$ for $n=3$ or $n\geq 5$ and set $m=n-1$. To prove that $L_{C_n}^{\Bbbk}(3)$ is prime we use the symmetric algebra perspective. Observe that C_n-n is $P_m=P_{n-1}$. Set $J=L_{P_m}^{\Bbbk}(3)$, $S=\Bbbk[y_{ij}:i\in[m],\ j\in[3]]$ and $R=S/J$. We have already proved that J is a prime complete intersection of height $m-1$. We have to prove that the symmetric algebra of the cokernel of the R-linear map

$$R^2\xrightarrow{Y}R^3\quad\text{with } Y=\begin{pmatrix}y_{11}&y_{12}&y_{13}\\ y_{m1}&y_{m2}&y_{m3}\end{pmatrix}$$

is a domain. Since by Remark 4.3 $I_2(Y)\neq 0$ in R, taking into consideration Remark 4.2 we may apply Theorem 4.1. Therefore, it is enough to prove that

$$\text{height } I_1(Y)\geq 3\quad\text{and}\quad\text{height } I_2(Y)\geq 2\qquad\text{in } R.$$

Equivalently, it is enough to prove that in S

$$\text{height } I_1(Y)+J\geq m+2, \tag{6}$$

$$\text{height } I_2(Y)+J\geq m+1. \tag{7}$$

First we prove (6). Since height $I_1(Y)=6$ in S then (6) is obvious for $m\leq 4$. For $m>4$ observe that $I_1(Y)+J$ can be written as $I_1(Y)+H$, where H is the LSS-ideal of the path with vertices $2,3,\dots,m-1$. Because $I_1(Y)$ and H use disjoint set of variables, we have

$$\text{height } I_1(Y)+H=6+m-3=m+3$$

and this proves (6). Now we note that the condition height $I_2(Y)\geq 1$ holds in R because R is a domain and $I_2(Y)\neq 0$. Hence we deduce from Theorem 4.1(1) that $L_{C_n}^{\Bbbk}(3)$ is a complete intersection for all $n\geq 3$.

It remains to prove (7). Since $I_2(Y)$ is a prime ideal of S of height 2 and $J\not\subset I_2(Y)$ the ideal $I_2(Y)+J$ has height at least 3. Hence the assertion (7) is obvious for $m=2$, i.e., $n=3$. Therefore, we may assume $m\geq 4$ (here we use $n\neq 4$). Let P be a prime ideal of S containing $I_2(Y)+J$. We have to prove that height $P\geq m+1$. If P contains $I_1(Y)$ then height $P\geq m+2$ by (6). So we may assume that P does not contain $I_1(Y)$, say $y_{11}\notin P$, and prove that height $PS_x\geq m+1$, where $x=y_{11}$. Since $I_2(Y)S_x=(y_{m2}-x^{-1}y_{m1}y_{12},\,y_{m3}-x^{-1}y_{m1}y_{13})$ we have

$$\begin{aligned} f_{m-1,m}^{(3)}&=y_{m-1,1}y_{m1}+y_{m-1,2}y_{m2}+y_{m-1,3}y_{m3}\\ &=y_{m-1,1}y_{m1}+y_{m-1,2}x^{-1}y_{m1}y_{12}+y_{m-1,3}x^{-1}y_{m1}y_{13}\\ &=x^{-1}y_{m1}f_{1,m-1}^{(3)}\mod I_2(Y)S_x. \end{aligned}$$

From $f^{(3)}_{m-1,m} \in J$ it follows that $y_{m1} f^{(3)}_{1,m-1} \in PS_x$. This implies that either $y_{m1} \in PS_x$ or $f^{(3)}_{1,m-1} \in PS_x$. In the first case PS_x contains y_{m1}, y_{m2}, y_{m3} and the LSS-ideal associated to the path with vertices $1, \dots, m-1$. Hence height $PS_x \geq 3 + m - 2 = m + 1$ as desired. Finally, if $f^{(3)}_{1,m-1} \in PS_x$ we have that PS_x contains the ideal $L^{\Bbbk}_{C_{m-1}}(3)$ associated to the cycle with vertices $1, \dots, m-1$ and we have already observed that this ideal is a complete intersection. Since $y_{m2} - x^{-1} y_{m1} y_{12}$, $y_{m3} - x^{-1} y_{m1} y_{13}$ are in PS_x as well it follows that height $PS_x \geq 2 + m - 1 = m + 1$.

(2) For the "only if" part we note that if $L^{\Bbbk}_G(2)$ is a complete intersection then $L^{\Bbbk}_G(3)$ is prime by Theorem 1.1 and hence G does cannot contain $K_{1,3}$ by Proposition 4.4. Suppose, by contradiction, that G contains C_{2m} for some $m \geq 2$. Hence $L^{\Bbbk}_{C_{2m}}(2)$ is a complete intersection of height $2m$. But the generators of $L^{\Bbbk}_{C_{2m}}(2)$ are (up to sign) among the 2-minors of the matrix

$$\begin{pmatrix} y_{11} & -y_{22} & y_{31} & \cdots & y_{2m-1,1} & -y_{2m,2} \\ y_{12} & y_{21} & y_{32} & \cdots & y_{2m-1,2} & y_{2m,1} \end{pmatrix}$$

and the ideal of 2-minors of such a matrix has height $2m-1$, a contradiction.

For the converse implication, we may assume that $\Bbbk$ is algebraically closed. Since the tensor product over a perfect field $\Bbbk$ of reduced $\Bbbk$-algebras is reduced [Bourbaki 1990, Theorem 3, Chapter V, 15], we may also assume that G is connected. A connected graph satisfying the assumptions is either an isolated vertex, or a path or a cycle with an odd number of vertices. We have already observed that $\operatorname{pmd}(P_n) = \Delta(P_n) \leq 2$. By Theorem 1.3 it follows that $L^{\Bbbk}_{P_n}(2)$ is a complete intersection. It remains to prove that $L^{\Bbbk}_{C_{2m+1}}(2)$ is a complete intersection (of height $2m+1$). Note that $L^{\Bbbk}_{P_{2m+1}}(2) \subset L^{\Bbbk}_{C_{2m+1}}(2)$ and we know already that $L^{\Bbbk}_{P_{2m+1}}(2)$ is a complete intersection of height $2m$. Hence it remains to prove that $f^{(2)}_{1,2m+1}$ does not belong to any minimal prime of $L^{\Bbbk}_{P_{2n+1}}(2)$. The generators of $L^{\Bbbk}_{P_{2n+1}}(2)$ are (up to sign) the adjacent 2-minors of the matrix

$$\begin{pmatrix} y_{11} & -y_{22} & y_{31} & \cdots & y_{2m-1,1} & -y_{2m,2} & y_{2m+1,1} \\ y_{12} & y_{21} & y_{32} & \cdots & y_{2m-1,2} & y_{2m,1} & y_{2m+1,2} \end{pmatrix}.$$

The minimal primes of $L^{\Bbbk}_{P_{2n+1}}(2)$ are described in the proof of [Diaconis et al. 1998, Theorem 4.3], see also [Hoşten and Sullivant 2004; Herzog et al. 2010]. By the description given in [Diaconis et al. 1998] it is easy to see that all minimal primes of $L^{\Bbbk}_{P_{2n+1}}(2)$ with the exception of $I_2(Y)$ are contained in the ideal $Q = (y_{ij} : 2 < i < 2m+1,\ 1 \leq j \leq 2)$. Clearly, $f^{(2)}_{1,2m+1} \notin Q$. Finally, one has $f^{(2)}_{1,2m+1} \notin I_2(Y)$ since the monomial $y_{11} y_{2m+1,1}$ is divisible by no monomials in the support of the generators of $I_2(Y)$. □

We proceed with the proof of Theorem 1.5. We first formulate a more general statement. For this we need to introduce the concept of Cartwright–Sturmfels ideals. This concept was coined in [Conca et al. 2016] inspired by earlier work in [Conca et al. 2015; Cartwright and Sturmfels 2010]. It was further developed and applied to various classes of ideals in [Conca et al. 2017; 2018].

Consider for $d_1, \dots, d_n \geq 1$ the polynomial ring $S = \Bbbk[y_{ij} : i \in [n],\ j \in [d_i]]$ with multigrading $\deg y_{ij} = \mathfrak{e}_i \in \mathbb{Z}^n$. The group $G = \mathrm{GL}_{d_1}(\Bbbk) \times \cdots \times \mathrm{GL}_{d_n}(\Bbbk)$ acts naturally on S as the group of $\mathbb{Z}^n$-graded K-algebra automorphism. The Borel subgroup of G is $B = U_{d_1}(\Bbbk) \times \cdots \times U_{d_n}(\Bbbk)$, where $U_d(\Bbbk)$ denotes the subgroup of upper triangular matrices in $\mathrm{GL}_d(\Bbbk)$. A $\mathbb{Z}^n$-graded ideal J is Borel fixed if $g(J)=J$ for every $g \in B$. A $\mathbb{Z}^n$-graded ideal I of S is called a Cartwright–Sturmfels ideal if there exists a radical Borel fixed ideal J with the same multigraded Hilbert-series.

Theorem 6.1. *For $d_1,\dots,d_n \geq 1$ let $S=\Bbbk[y_{ij} : i \in [n],\ j \in [d_i]]$ be the polynomial ring with $\mathbb{Z}^n$ multigrading induced by* $\deg y_{ij}=\mathfrak{e}_i \in \mathbb{Z}^n$ *and $G=(V,E)$ be a forest. For each $e=\{i,j\} \in E$ let $f_e \in S$ be a $\mathbb{Z}^n$-graded polynomial of degree $\mathfrak{e}_i+\mathfrak{e}_j$. Then $I=(f_e : e \in E)$ is a Cartwright–Sturmfels ideal. In particular, I and all its initial ideals are radical.*

Proof. First, we observe that we may assume that the generators f_e of I form a regular sequence. To this end we introduce new variables and for each $e=\{i,j\} \in E$ we add to f_e a monomial m_e in the new variables of degree e_i+e_j so that m_e and $m_{e'}$ are coprime if $e \neq e'$. The new polynomials f_e+m_e with $e \in E$ form a regular sequence by Proposition 2.4 since their initial terms with respect to an appropriate term order are the pairwise coprime monomials m_e. The ideal I arises as a multigraded linear section of the ideal $(f_e+m_e : e \in E)$ by setting all new variables to 0. By [Conca et al. 2015, Theorem 1.16(5)] the family of Cartwright–Sturmfels ideals is closed under any multigraded linear section. Hence it is enough to prove the statement for the ideal $(f_e+m_e : e \in E)$. Equivalently we may assume right away that the generators f_e of I form a regular sequences.

The multigraded Hilbert series of a multigraded S-module M can by written as

$$\frac{K_M(z_1,\dots,z_n)}{\prod_{i=1}^n (1-z_i)^{d_i}}.$$

The numerator $K_M(z_1,\dots,z_n)$ is a Laurent polynomial with integral coefficients called the K-polynomial of M. Since the f_e's form a regular sequence the K-polynomial of S/I is the polynomial

$$F(z)=F(z_1,\dots,z_n)=\prod_{\{i,j\}\in E}(1-z_iz_j) \in \mathbb{Q}[z_1,\dots,z_n].$$

To prove that I is Cartwright–Sturmfels we have to prove that there is a Borel-fixed radical ideal J such that the K-polynomial of S/J is $F(z)$. Taking into consideration the duality between Cartwright–Sturmfels ideals and Cartwright–Sturmfels* ideals discussed in [Conca et al. 2016], it is enough to exhibit a monomial ideal J whose generators are in the polynomial ring $S'=\Bbbk[y_1,y_2,\dots,y_n]$ equipped with the (fine) $\mathbb{Z}^n$-grading $\deg y_i=\mathfrak{e}_i \in \mathbb{Z}^n$ such that the K-polynomial of J regarded as an S'-module is $F(1-z_1,\dots,1-z_n)$, that is,

$$\prod_{\{i,j\}\in E}(z_i+z_j-z_iz_j).$$

We claim that, under the assumption that $([n],E)$ is a forest, the ideal

$$J=\prod_{\{i,j\}\in E}(y_i,y_j)$$

has the desired property. In other words, we have to prove that the tensor product

$$T_E = \bigotimes_{\{i,j\}\in E} T_{\{i,j\}}$$

of the truncated Koszul complexes

$$T_{\{i,j\}} : 0 \to S'(-\mathfrak{e}_i - \mathfrak{e}_j) \to S'(-\mathfrak{e}_i) \oplus S'(-\mathfrak{e}_j) \to 0$$

associated to y_i, y_j resolves the ideal J. Consider a leaf $\{a, b\}$ of E. Set $E' = E \setminus \{\{a, b\}\}$,

$$J' = \prod_{\{i,j\}\in E'} (y_i, y_j)$$

and $J'' = (y_a, y_b)$. Then by induction on the number of edges we have that $T_{E'}$ resolves the ideal J'. Then the homology of T_E is $\mathrm{Tor}_*^{S'}(J', J'')$. Since $\{a, b\}$ is a leaf, one of the two variables y_a, y_b does not appear at all in the generators of J'. Hence y_a, y_b forms a regular J'-sequence. Then $\mathrm{Tor}_{\geq 1}^{S'}(J', J'') = 0$ and hence T_E resolves $J' \otimes J''$. Finally, $J' \otimes J'' = J'J''$ since $\mathrm{Tor}_1^{S'}(J', S/J'') = 0$. This concludes the proof that the ideal I is a Cartwright–Sturmfels ideal. Every initial ideal of a Cartwright–Sturmfels ideal is a Cartwright–Sturmfels ideal as well because this property just depends on the Hilbert series. In particular, every initial ideal of a Cartwright–Sturmfels ideal is radical. □

Proof of Theorem 1.5. Setting $d_1 = \cdots = d_n = d$ and $f_e = f_e^{(d)}$ in Theorem 6.1 we have that $L_G^{\Bbbk}(d)$ is a Cartwright–Sturmfels ideal and hence radical. Assertions (2) and (3) follow from Lemma 5.4, Theorem 1.3, Proposition 4.4 and Theorem 1.1. □

7. Invariant theory, determinantal ideals of matrices with 0's and their relation to LSS-ideals

The first goal of this section is to recall some classical results from invariant theory, see for example the paper by De Concini and Procesi [1976]. In particular, we recall how determinantal/Pfaffian rings arise as invariant rings of group actions. We assume throughout this section that the base field $\Bbbk$ is of characteristic 0. After the recap of invariant theory we will establish the connection to LSS-ideals.

7A. ***Generic determinantal rings as rings of invariants* (gen).** We take an $m \times n$ matrix of variables $X_{m,n}^{\mathrm{gen}} = (x_{ij})$ and consider the ideal $I_{d+1}^{\Bbbk}(X_{m,n}^{\mathrm{gen}})$ of $S^{\mathrm{gen}} = \Bbbk[x_{ij} : (i, j) \in [m] \times [n]]$ generated by the $(d+1)$-minors of $X_{m,n}^{\mathrm{gen}}$. Consider two matrices of variables Y and Z of size $m \times d$ and $d \times n$ and the following action of $\mathfrak{G} = \mathrm{GL}_d(\Bbbk)$ on the polynomial ring $\Bbbk[Y, Z]$: The matrix $A \in \mathfrak{G}$ acts by the $\Bbbk$-algebra automorphism of $\Bbbk[Y, Z]$ that sends $Y \to YA$ and $Z \to A^{-1}Z$. The entries of the product matrix YZ are clearly invariant under this action. Hence the ring of invariants $\Bbbk[Y, Z]^{\mathfrak{G}}$ contains the subalgebra $\Bbbk[YZ]$ generated by the entries of the product YZ. The *first main theorem of invariant theory* for this action says that $\Bbbk[Y, Z]^{\mathfrak{G}} = \Bbbk[YZ]$. We have a surjective $\Bbbk$-algebra map

$$\phi : S^{\mathrm{gen}} \to \Bbbk[Y, Z]^{\mathfrak{G}} = \Bbbk[YZ]$$

sending X to YZ. Clearly the product matrix YZ has rank d and hence we have $I^{\Bbbk}_{d+1}(X^{\mathrm{gen}}_{m,n}) \subseteq \operatorname{Ker}\phi$. The *second main theorem of invariant theory* says that $I^{\Bbbk}_{d+1}(X^{\mathrm{gen}}_{m,n}) = \operatorname{Ker}\phi$. Hence

$$S/I^{\Bbbk}_{d+1}(X^{\mathrm{gen}}_{m,n}) \simeq \Bbbk[YZ]. \tag{8}$$

7B. ***Generic symmetric determinantal rings as rings of invariants (*****sym*****).*** We take an $n \times n$ symmetric matrix of variables $X^{\mathrm{sym}}_n = (x_{ij})$ and consider the ideal $I^{\Bbbk}_{d+1}(X^{\mathrm{sym}}_n)$ in $S^{\mathrm{sym}} = \Bbbk[x_{ij} : 1 \le i \le j \le n]$ generated by the $(d+1)$-minors of X^{sym}_n. Consider a matrix of variables Y of size $n \times d$ and the following action of the orthogonal group $\mathfrak{G} = \mathcal{O}_d(\Bbbk) = \{A \in \mathrm{GL}_d(\Bbbk) : A^{-1} = A^T\}$ on the polynomial ring $\Bbbk[Y]$: any $A \in \mathfrak{G}$ acts by the $\Bbbk$-algebra automorphism of $\Bbbk[Y]$ that sends Y to YA. The entries of the product matrix YY^T are invariant under this action and hence the ring of invariants contains the subalgebra $\Bbbk[YY^T]$ generated by the entries of YY^T. The *first main theorem of invariant theory* for this action asserts that $\Bbbk[Y]^G = \Bbbk[YY^T]$. Then we have a surjective presentation

$$\phi : S^{\mathrm{sym}} \to \Bbbk[YY^T]$$

sending X to YY^T. Since the product matrix YY^T has rank d we have $I_{d+1}(X) \subseteq \operatorname{Ker}\phi$. The *second main theorem of invariant theory* then says that $I_{d+1}(X) = \operatorname{Ker}\phi$. Hence

$$S^{\mathrm{sym}}/I^{\Bbbk}_{d+1}(X^{\mathrm{sym}}_n) \simeq \Bbbk[YY^T]. \tag{9}$$

7C. ***Generic Pfaffian rings as rings of invariants (*****skew*****).*** We take an $n \times n$ skew-symmetric matrix of variables $X^{\mathrm{skew}}_n = (x_{ij})$ and consider the ideal $\mathrm{Pf}^{\Bbbk}_{2d+2}(X)$ generated by the Pfaffians of size $(2d+2)$ of X^{skew}_n in $S^{\mathrm{skew}} = \Bbbk[x_{ij} : 1 \le i < j \le n]$. Consider a matrix of variables Y of size $n \times 2d$ and let J be the $2d \times 2d$ block matrix with d blocks

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

on the diagonal and 0 in the other positions. The symplectic group $\mathfrak{G} = \mathrm{Sp}_{2d}(\Bbbk) = \{A \in \mathrm{GL}_{2t}(\Bbbk) : AJA^T = J\}$ acts on the polynomial ring $\Bbbk[Y]$ as follows: an $A \in \mathfrak{G}$ acts on $\Bbbk[Y]$ by the automorphism that sends $Y \to YA$. The entries of the product matrix YJY^T are invariant under this action and hence the ring of invariants contains the subalgebra $\Bbbk[YJY^T]$ generated by the entries of YJY^T. The *first main theorem of invariant theory* for the current action says that $\Bbbk[Y]^G = \Bbbk[YJY^T]$. Then we have a surjective presentation: $\phi : S^{\mathrm{skew}} \to \Bbbk[YY^T]$ sending X to YJY^T. The product matrix YJY^T has rank $2d$ and hence we have $\mathrm{Pf}^{\Bbbk}_{2d+2}(X) \subseteq \operatorname{Ker}\phi$. The *second main theorem of invariant theory* for this action says that $\mathrm{Pf}^{\Bbbk}_{2d+2}(X) = \operatorname{Ker}\phi$. Hence

$$S^{\mathrm{skew}}/\,\mathrm{Pf}^{\Bbbk}_{2d+2}(X^{\mathrm{skew}}_n) \simeq \Bbbk[YJY^T]. \tag{10}$$

7D. ***Determinantal ideals of matrices with* 0*'s and their relation to LSS-ideals.*** The classical invariant theory point of view shows that the generic determinantal and Pfaffian ideals are prime as they are kernels of ring maps whose codomains are integral domains. Their height is also well known (see for example [Bruns and Vetter 1988]):

(gen) The height of the ideal $I_d^{\Bbbk}(X_{m,n}^{\mathrm{gen}})$ of d-minors of a $m \times n$ matrix of variables is $(n+1-d)(m+1-d)$.

(sym) The height of the ideal $I_d^{\Bbbk}(X_n^{\mathrm{sym}})$ of d-minors of a symmetric $n \times n$ matrix of variables is $\binom{n-d+2}{2}$.

(skew) The height of the ideal of Pfaffians $\mathrm{Pf}_{2d}^{\Bbbk}(X_n^{\mathrm{skew}})$ of size $2d$ (and degree d) of an $n \times n$ skew-symmetric matrix of variables is $\binom{n-2d+2}{2}$.

If one replaces the entries of the matrices with general linear forms in, say, u variables, then Bertini's theorem in combination with the fact that the generic determinantal/Pfaffian rings are Cohen–Macaulay implies that the determinantal/Pfaffian ideals remain prime as long as $u \geq 2 + \text{height}$ and radical if $u \geq 1 + \text{height}$.

But what about the case of special linear sections of determinantal ideals of matrices? And what about the case of coordinate sections? Are the corresponding ideals prime or radical? To describe coordinate sections we employ the following notation.

(gen) In the generic case we take a bipartite graph $G = ([m] \cup [\tilde{n}], E)$ and denote by X_G^{gen} the matrix obtained from the $m \times n$ matrix of variables by replacing the entries in position (i, j) with 0 for all $\{i, \tilde{j}\} \in E$.

(sym) In the generic symmetric case we take a subgraph $G = ([n], E)$ of K_n and denote by X_G^{sym} the matrix obtained from the $n \times n$ symmetric matrix of variables by replacing with 0 the entries in position (i, j) and (j, i) for all $\{i, j\} \in E$.

(skew) In the generic skew-symmetric case we take a subgraph $G = ([n], E)$ of K_n and denote by X_G^{skew} the matrix obtained from the skew-symmetric matrix of variables by replacing with 0 the entries in position (i, j) and (j, i) for all $\{i, j\} \in E$.

In this terminology $I_d^{\Bbbk}(X_G^{\mathrm{gen}})$ is the ideal of d-minors of X_G^{gen} in S^{gen} and similarly in the symmetric case. We write $\mathrm{Pf}_{2d}^{\Bbbk}(X_G^{\mathrm{skew}})$ for the ideal of Pfaffians of size $2d$ of X_G^{skew} in S^{skew}. We ask for conditions on G that imply that $I_d^{\Bbbk}(X_G^{\mathrm{gen}})$, $I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ or $\mathrm{Pf}_{2d}^{\Bbbk}(X_G^{\mathrm{skew}})$ is radical or prime or has the expected height.

Clearly, special linear sections of generic determinantal ideals can give nonprime and nonradical ideals. On the positive side, for maximal minors, we have the following results:

Remark 7.1. (1) Eisenbud [1988] proved that the ideal of maximal minors of a 1-generic $m \times n$ matrix of linear forms is prime and remains prime even after modding out any set of $\leq m-2$ linear forms. In particular, the ideal of maximal minors of an $m \times n$ matrix of linear forms is prime provided the ideal generated by the entries of the matrix has at least $m(n-1)+2$ generators.

(2) Giusti and Merle [1982] studied the ideal of maximal minors of coordinate sections in the generic case. One of their main results, [Giusti and Merle 1982, Theorem 1.6.1] characterizes, in combinatorial terms, the subgraphs G of $K_{m,n}$, $m \leq n$, such that the variety associated to $I_m^{\Bbbk}(X_G^{\mathrm{gen}})$ is irreducible, i.e., the radical of $I_m^{\Bbbk}(X_G^{\mathrm{gen}})$ is prime.

(3) Boocher [2012] proved that for any subgraph G of $K_{m,n}$, $m \le n$, the ideal $I_m^{\Bbbk}(X_G^{\mathrm{gen}})$ is radical. Combining his result with the result of Giusti and Merle, one obtains a characterization of the graphs G such that $I_m^{\Bbbk}(X_G^{\mathrm{gen}})$ is prime.

(4) Generalizing the result of Boocher, in [Conca et al. 2015; 2016] it is proved that ideals of maximal minors of a matrix of linear forms that is either row or column multigraded is radical.

In the generic case every nonzero minor of a matrix of type X_G^{gen} has no multiple factors because its multidegree is square-free. This explains, at least partially, why the determinantal ideals of X_G^{gen} have the tendency to be radical. However, the following example shows that they are not radical in general.

Example 7.2. Let X_G^{gen} be the 6×6 matrix associated to the graph from Example 3.2(3). That is, in the 6×6 generic matrix we set to 0 the entries in positions

$$(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (3,2), (3,3), (4,3), (4,4), (5,1), (5,4).$$

Then $I_4^{\Bbbk}(X_G^{\mathrm{gen}})$ is not radical over a field of characteristic 0 and very likely over any field. Here the "witness" is $g = x_{1,5}$, i.e., $I_4^{\Bbbk}(X_G^{\mathrm{gen}}) : g \neq I_4^{\Bbbk}(X_G^{\mathrm{gen}}) : g^2$. Since G is contained in $K_{5,4}$ one can consider as well $I_4^{\Bbbk}(X_G^{\mathrm{gen}})$ in the 5×5 matrix but that ideal turns out to be radical.

Similarly for symmetric matrices we have:

Example 7.3. Let X_G^{sym} be the 7×7 generic symmetric matrix associated to the graph from Example 3.2(1). That is, in the 7×7 generic symmetric matrix we set to 0 the entries in positions

$$\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,6\}, \{3,5\}, \{4,6\}$$

as well as in the symmetric positions. Then $I_4^{\Bbbk}(X_G^{\mathrm{sym}})$ is not radical over a field of characteristic 0. The witness here is $g = x_{1,6}$. Since G is contained in K_6 one can consider as well $I_4^{\Bbbk}(X_G^{\mathrm{sym}})$ in the 6×6 matrix but that ideal turns out to be radical.

It turns out that Examples 3.2, 7.2 and 7.3 are indeed closely related as we now explain.

Let $G = ([m] \cup [\tilde{n}], E)$ be a subgraph of the complete bipartite graph $K_{m,n}$. In view of the isomorphism (8) we have that

$$S^{\mathrm{gen}} / \big(I_{d+1}^{\Bbbk}(X_{m,n}^{\mathrm{gen}}) + (x_{ij} : \{i, \tilde{j}\} \in E) \big) \simeq \Bbbk[YZ]/J_G(d),$$

where $Y = (y_{ij})$, $Z = (z_{ij})$ are respectively $m \times d$ and $d \times n$ matrices of variables and $J_G(d)$ is the ideal of $\Bbbk[YZ]$ generated by $(YZ)_{i,j}$ with $\{i, \tilde{j}\} \in E$. Furthermore

$$I_{d+1}^{\Bbbk}(X_{m,n}^{\mathrm{gen}}) + (x_{ij} : \{i, \tilde{j}\} \in E) = I_{d+1}^{\Bbbk}(X_G^{\mathrm{gen}}) + (x_{ij} : \{i, \tilde{j}\} \in E).$$

The LSS-ideal $L_G^{\Bbbk}(d) \subset \Bbbk[Y, Z]$ is indeed equal to $J_G(d)\Bbbk[Y, Z]$. Now it is a classical result in invariant theory (derived from the fact that linear groups are reductive in characteristic 0), that $\Bbbk[YZ]$ is a direct summand of $\Bbbk[Y, Z]$ in characteristic 0. This implies that

$$J_G(d) = L_G^{\Bbbk}(d) \cap \Bbbk[YZ].$$

The next proposition is an immediate consequence.

Proposition 7.4. *Let* $\Bbbk$ *be a field of characteristic* 0, $d \geq 1$ *and* $G = ([m] \cup [\tilde{n}], E)$ *be a subgraph of* $K_{m,n}$. *If* $L^{\Bbbk}_G(d)$ *is radical* (*resp. is a complete intersection*, *is prime*) *then* $I^{\Bbbk}_{d+1}(X^{\text{gen}}_G)$ *is radical* (*resp. has maximal height*, *is prime*).

Now we start from a subgraph G of K_n. For $d + 1 \leq n$ we may consider the coordinate section $I^{\Bbbk}_{d+1}(X^{\text{sym}}_G)$ of $I^{\Bbbk}_{d+1}(X^{\text{sym}}_n)$. Using the isomorphism (9) we obtain:

Proposition 7.5. *Let* $\Bbbk$ *be a field of characteristic* 0 *and* $G = ([n], E)$ *a graph. If* $L^{\Bbbk}_G(d)$ *is radical* (*resp. is a complete intersection*, *is prime*) *then* $I^{\Bbbk}_{d+1}(X^{\text{sym}}_G)$ *is radical* (*resp. has maximal height*, *is prime*).

For $2d+2 \leq n$ we may consider the coordinate section $\mathrm{Pf}^{\Bbbk}_{2d+2}(X^{\text{skew}}_G)$ of $\mathrm{Pf}^{\Bbbk}_{2d+2}(X^{\text{skew}}_n)$. We may as well consider the associated twisted LSS-ideal $\hat{L}^{\Bbbk}_G(d)$ defined as follows. For every $i \in [n]$ we consider $2d$ indeterminates $y_{i\,1}, \ldots, y_{i\,2d}$. For $e = \{i, j\}$, $1 \leq i < j \leq n$ we set $\hat{f}^{(d)}_e$ to be the entry of the matrix YJY^T in row i and column j, i.e.,

$$\hat{f}^{(d)}_e = \sum_{k=1}^{d} \big(y_{i\,2k-1} y_{j\,2k} - y_{i\,2k} y_{j\,2k-1}\big).$$

Then we define the twisted LSS-ideal associated to G as follows:

$$\hat{L}^{\Bbbk}_G(d) = (\hat{f}^{(d)}_e : e \in E).$$

For $d = 1$ the twisted LSS-ideal coincides with the so-called binomial edge ideal defined and studied in [Herzog et al. 2010; Kiani and Saeedi Madani 2016; Matsuda and Murai 2013; Ohtani 2011].

Remark 7.6. Assume G is bipartite with bipartition $[n] = V_1 \cup V_2$ then the coordinate transformation (see [Bolognini et al. 2018, Corollary 6.2])

$$\begin{array}{llll} y_{i\,2k-1} \mapsto y_{i\,2k-1} & \text{and} & y_{i\,2k} \mapsto y_{i\,2k} & \text{for } i \in V_1, \\ y_{j\,2k} \mapsto y_{j\,2k-1} & \text{and} & y_{j\,2k-1} \mapsto -y_{j\,2k} & \text{for } j \in V_2, \end{array}$$

sends $\hat{L}^{\Bbbk}_G(d)$ to $L^{\Bbbk}_G(2d)$. In particular, for a bipartite graph G we have that $\hat{L}^{\Bbbk}_G(d)$ is radical (resp. prime) if and only if $L^{\Bbbk}_G(2d)$ is radical (resp. prime).

Using the isomorphism (10) we obtain:

Proposition 7.7. *Let* $\Bbbk$ *be a field of characteristic* 0 *and* $G = ([n], E)$ *a graph. If* $\hat{L}^{\Bbbk}_G(d)$ *is radical* (*resp. is a complete intersection*, *is prime*) *then* $\mathrm{Pf}^{\Bbbk}_{2d+2}(X^{\text{skew}}_G)$ *is radical* (*resp. has maximal height*, *is prime*).

Now, in characteristic 0, the results that we have established for LSS-ideals can be turned into statements concerning coordinate sections of determinantal ideals.

Theorem 7.8. *Let* $\Bbbk$ *be a field of characteristic* 0.

(1) *For every subgraph* G *of* $K_{m,n}$ *the ideals* $I^{\Bbbk}_2(X^{\text{gen}}_G)$ *and* $I^{\Bbbk}_3(X^{\text{gen}}_G)$ *are radical.*

(2) *For every subgraph* G *of* K_n *the ideals* $I^{\Bbbk}_2(X^{\text{sym}}_G)$ *and* $I^{\Bbbk}_3(X^{\text{sym}}_G)$ *are radical.*

(3) *For every subgraph G of K_n the ideal* $\mathrm{Pf}_4^{\Bbbk}(X_G^{\mathrm{skew}})$ *is radical.*

Furthermore if G is a forest then:

(4) $I_d^{\Bbbk}(X_G^{\mathrm{gen}})$, $I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ *and* $\mathrm{Pf}_{2d}^{\Bbbk}(X_G^{\mathrm{skew}})$ *are radical for all d.*

(5) $I_d^{\Bbbk}(X_G^{\mathrm{gen}})$ *and* $I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ *have maximal height if* $d \geq \Delta(G)+1$.

(6) $I_d^{\Bbbk}(X_G^{\mathrm{gen}})$ *and* $I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ *are prime if* $d \geq \Delta(G)+2$.

Proof. The statements for ideals of 2-minors follow from Propositions 7.4 and 7.5 using the fact that the edge ideal of a graph is radical. Indeed these results hold over a field of arbitrary characteristic as the corresponding ideals are "toric."

By [Herzog et al. 2015, Theorem 1.1] the ideal $L_G^{\Bbbk}(2)$ is radical for all graphs G. Using Propositions 7.4 and 7.5 this implies that $I_3^{\Bbbk}(X_G^{\mathrm{gen}})$ is radical for bipartite graphs G and $I_3^{\Bbbk}(X_G^{\mathrm{sym}})$ is radical for arbitrary graphs.

By [Herzog et al. 2010, Corollary 2.2] the ideal $\hat{L}_G^{\Bbbk}(1)$ is radical for all graphs G. Using Proposition 7.7 this implies that $\mathrm{Pf}_4^{\Bbbk}(X_G^{\mathrm{skew}})$ is radical for arbitrary graphs.

Finally, for a forest G the results in the case of minors are derived from Propositions 7.4, 7.5 and Theorem 1.5. In the Pfaffian case they follow using Theorem 6.1 and Proposition 7.7. $\square$

The following corollary is an immediate consequence of assertion (3) in Theorem 7.8.

Corollary 7.9. *Let $G(2,n)$ be the coordinate ring of the Grassmannian of 2-dimensional subspaces in $\Bbbk^n$ in its standard Plücker coordinates. Then any subset of the Plücker coordinates generates a radical ideal in $G(2,n)$.*

A statement analogous to Corollary 7.9 for higher order Grassmannians is not true. Indeed, the point is that a set of m-minors of a generic matrix $m \times n$ does not generate a radical ideal in general (as it does for $m = 2$). For example, in the Grassmannian $G(3,6)$ modulo [123], [124], [135], [236] the class of [125][346] is a nonzero nilpotent.

Next we look into necessary conditions for $I_d^{\Bbbk}(X_G^{\mathrm{gen}})$ and $I_d^{\Bbbk}(X_G^{\mathrm{sym}})$ to be prime.

Lemma 7.10. *Let $G = ([n], G)$ be a graph.*

(1) *If $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ is prime then G does not contain $K_{a,b}$ for $a+b > d$ (i.e., $\overline{G}$ is $(n-d)$-connected).*

(2) *If $G = B_d$ with $d \geq 4$ and X is the generic $(d+2)\times(d+2)$ matrix then $I_{d+1}^{\Bbbk}(X_G^{\mathrm{gen}})$ is not prime.*

Proof. (1) Assume by contradiction that G contains $K_{a,b}$ for $a+b = d+1$. We may assume that the corresponding set of vertices are $[a]$ and $\{a+j : j \in [b]\}$. But then the submatrix of X_G^{sym} of the first $d+1$ rows and columns is block-diagonal with (at least) two blocks. Hence its determinant is nonzero, is reducible and has degree $d+1$. Since all the generators of $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ have degree $d+1$ it follows that $I_{d+1}^{\Bbbk}(X_G^{\mathrm{sym}})$ cannot be prime.

(2) Set $Y_d = X^{\text{gen}}_{B_d}$, i.e.,

$$Y_d = \begin{pmatrix} x_{11} & 0 & \cdots & 0 & x_{1,d+1} & x_{1,d+2} \\ 0 & x_{22} & \cdots & 0 & x_{2,d+1} & x_{2,d+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & x_{dd} & \vdots & \vdots \\ x_{d+1,1} & x_{d+1,2} & \cdots & \cdots & x_{d+1,d+1} & x_{d+1,d+2} \\ x_{d+2,1} & x_{d+2,2} & \cdots & \cdots & x_{d+2,d+1} & x_{d+2,d+2} \end{pmatrix}.$$

and $J = I_{d+1}(Y_d)$ and let S be the polynomial ring whose indeterminates are the nonzero entries of Y_d. First, we prove that for every $d \geq 1$ the ideal J has the expected height, i.e., height $J = 4$. For $d = 1, 2, 3$ the ideal J is indeed prime of height 4: for $d = 1$ this is obvious because Y_1 is the generic 3×3 matrix; for $d = 2$ and $d = 3$ it follows from the fact that the corresponding LSS-ideal is prime by virtue of Proposition 7.4. For $d > 3$ let P be a prime containing J. If P contains $(x_{11}, x_{22}, x_{33}, x_{44})$ then height $P \geq 4$. If P does not contain $(x_{11}, x_{22}, x_{33}, x_{44})$ we may assume $x_{11} \notin P$. Inverting x_{11} and using the standard localization trick for determinantal ideals one sees that $PS_{x_{11}}$ contains, up to a change of variables, $I_d(Y_{d-1})$. Hence height $P =$ height $PS_x \geq 4$. Now that we know that J has height 4 to prove that J is not prime for $d \geq 4$ it is enough to observe that $J \subset (x_{11}, x_{22}, x_{33}, x_{44})$. The latter is straightforward since mod $(x_{11}, x_{22}, x_{33}, x_{44})$ the submatrix of Y consisting of the first 4-rows has rank 2. □

8. Obstructions to algebraic properties and asymptotic behavior

In this section we return to the study of LSS-ideals $L^{\Bbbk}_G(d)$. Using results from Section 4 and results about $I_{d+1}(X^{\text{gen}}_{B_d})$ from Section 7 we derive necessary conditions for $L^{\Bbbk}_G(d)$ to be a complete intersection or prime. In addition, we discuss the exact asymptotic behavior of these properties for complete and complete bipartite graphs. To this end it is convenient to introduce the following notation. Given an algebraic property $\mathcal{P}$ of ideals and a graph G we set

$$\text{asym}_{\Bbbk}(\mathcal{P}, G) = \inf\big\{d : L^{\Bbbk}_G(d') \text{ has property } \mathcal{P} \text{ for all } d' \geq d\big\}.$$

Here we are interested in the properties $\mathcal{P} \in \{\text{radical, c.i., prime}\}$. By Theorem 1.1, Corollary 1.2 and Theorem 1.3 we know that for every graph G we have

$$\begin{aligned} \text{asym}_{\Bbbk}(\text{c.i.}, G) &= \min\{d : L^{\Bbbk}_G(d) \text{ is c.i.}\} &&\leq \text{pmd}(G), \\ \text{asym}_{\Bbbk}(\text{prime}, G) &= \min\{d : L^{\Bbbk}_G(d) \text{ is prime}\} &&\leq \text{pmd}(G) + 1, \\ \text{asym}_{\Bbbk}(\text{c.i.}, G) &\leq \text{asym}_{\Bbbk}(\text{prime}, G) &&\leq \text{asym}_{\Bbbk}(\text{c.i.}, G) + 1. \end{aligned}$$

Furthermore there are graphs such that $\text{asym}_{\Bbbk}(\text{prime}, G) = \text{asym}_{\Bbbk}(\text{c.i.}, G) + 1$ (e.g., odd cycles or forests) and others such that $\text{asym}_{\Bbbk}(\text{prime}, G) = \text{asym}_{\Bbbk}(\text{c.i.}, G)$ (e.g., even cycles). We have the following obstructions:

Proposition 8.1. *Let* $G = ([n], E)$. *Then*:

(1) *If $L^{\Bbbk}_G(d)$ is prime then G does not contain $K_{a,b}$ with $a+b=d+1$. Furthermore, if $d>3$ and* $\operatorname{char}\Bbbk=0$ *then G does not contain B_d.*

(2) *If $L^{\Bbbk}_G(d)$ is a complete intersection then G does not contain $K_{a,b}$ with $a+b=d+2$. Furthermore, if $d>2$ and* $\operatorname{char}\Bbbk=0$ *then G does not contain B_{d+1}.*

Proof. (1) The first assertion has been already proved in Proposition 4.4. For the second let $\operatorname{char}\Bbbk=0$ and $d>3$. By contradiction, assume G contains B_d. Then by Corollary 1.2 we know that $L^{\Bbbk}_{B_d}(d)$ is prime because $L^{\Bbbk}_G(d)$ is prime. Then Proposition 7.4 implies that $I_{d+1}(X^{\mathrm{gen}}_{B_d})$ is prime for a generic matrix X of arbitrary size and this contradicts Lemma 7.10(2).

(2) Assertion (2) follows from (1) by using Theorem 1.1. □

Another obstruction is described in the following proposition.

Proposition 8.2. *Let $\Bbbk$ be a field of characteristic 0 and $n\in\mathbb{N}$. Let w_n be the largest positive integer such that $\binom{w_n}{2}\le n$. Then:*

(1) *$L^{\Bbbk}_{K_n}(d)$ is not prime for $d=n+\binom{w_n-2}{2}-1$.*

(2) *$L^{\Bbbk}_{K_n}(d)$ is not a complete intersection for $d=n+\binom{w_{n+1}-2}{2}-2$.*

Proof. (1) We set $h_n=\binom{w_n}{2}$ and $m_n=w_n+d-1$. The numbers are chosen so that, using the formulas for the height of determinantal ideals mentioned in Section 7, the ideal $I_{d+1}(X)$ of $(d+1)$-minors of a generic symmetric $m_n\times m_n$ matrix X has height h_n. Consider K_n as the graph $\left([m_n],\binom{[n]}{2}\right)$ on m_n vertices where the vertices $n+1,\dots,m_n$ do not appear in edges. Assume, by contradiction, that the ideal $L^{\Bbbk}_{K_n}(d)$ is prime. Then by Proposition 7.5 the ideal $I^{\Bbbk}_{d+1}(X^{\mathrm{sym}}_{K_n})$ is prime and of height h_n. But one has

$$I^{\Bbbk}_{d+1}(X^{\mathrm{sym}}_{K_n})\subset(x_{11},x_{22},\dots,x_{h_nh_n}) \tag{11}$$

which is a contradiction. To check (11) it is enough to prove that the rank of the matrix

$$X^{\mathrm{sym}}_{K_n}\mod(x_{11},x_{22},\dots,x_{h_nh_n})$$

is at most d. That is, we have to check that the rank of an $(m_n\times m_n)$-matrix with block decomposition

$$\begin{pmatrix}0 & A\\ B & C\end{pmatrix},$$

where 0 is the zero matrix of size $(h_n\times n)$, is at most d. Since $d=m_n-n+m_n-h_n$ the latter is obvious.

(2) We set $h_n=\binom{w_{n+1}}{2}$ and $m_n=w_{n+1}+d-1$. As above, the numbers are chosen so that the ideal $I_{d+1}(X)$ of $(d+1)$-minors of a generic symmetric $m_n\times m_n$ matrix X has height h_n.

Assume, by contradiction, that $L^{\Bbbk}_{K_n}(d)$ is a complete intersection. From Proposition 7.5 it follows that $I^{\Bbbk}_{d+1}(X^{\mathrm{sym}}_{K_n})$ has height h_n. But

$$I^{\Bbbk}_{d+1}(X^{\mathrm{sym}}_{K_n})\subset(x_{11},x_{22},\dots,x_{h_n-1,h_n-1}) \tag{12}$$

which is a contradiction. As above (12) boils down to an obvious statement about the rank of a matrix with a zero submatrix of a certain size. □

Using this result we can now analyze the asymptotic behavior of $\operatorname{asym}_{\Bbbk}(\text{c.i.}, K_n)$ and $\operatorname{asym}_{\Bbbk}(\text{prime}, K_n)$.

Corollary 8.3. *Let* $\Bbbk$ *be a field of characteristic* 0*. Then*

$$\lim_{n\to\infty} \frac{\operatorname{asym}_{\Bbbk}(\text{c.i.}, K_n)}{n} = \lim_{n\to\infty} \frac{\operatorname{asym}_{\Bbbk}(\text{prime}, K_n)}{n} = 2. \tag{13}$$

Proof. By Corollary 5.6 we have $\operatorname{asym}_{\Bbbk}(\text{prime}, K_n) \le 2n-2$. By Proposition 8.2 we have

$$n + \binom{w_{n+1}-2}{2} - 1 \le \operatorname{asym}_{\Bbbk}(\text{c.i.}, K_n) \le \operatorname{asym}_{\Bbbk}(\text{prime}, K_n). \tag{14}$$

Hence the equalities in (13) follow from the fact that

$$\lim_{n\to\infty} \frac{\binom{w_{n+1}-2}{2}}{n} = 1. \qquad \square$$

Using Proposition 8.2 and Theorem 1.1 we obtain further obstructions.

Corollary 8.4. *Let* G *be a graph on* n *vertices and* $\Bbbk$ *a field of characteristic* 0 *and denote by* $\alpha = \omega(G)$ *the clique number of* G*. Then* $L_G^{\Bbbk}(d)$ *is not a complete intersection for* $d \le \alpha + \binom{w_{\alpha+1}-2}{2} - 2$ *and* $L_G^{\Bbbk}(d)$ *is not prime for* $d \le \alpha + \binom{w_\alpha - 2}{2} - 1$*, where* w_α *is defined as in Proposition 8.2.*

To give an actual feeling for the obstruction, we present one example:

Example 8.5. For $n = 15$ one has $w_n = 6$ and $L_{K_n}^{\Bbbk}(d)$ is not prime for $d = 15 + \binom{6-2}{2} - 1 = 20$. Therefore $L_G^{\Bbbk}(20)$ is not prime if G contains K_{15}, i.e., $\omega(G) \ge 15$.

For the case of complete bipartite graphs $K_{m,n}$ results of De Concini and Strickland [1981] or Musili and Seshadri [1983] on the varieties of complexes imply the following:

Theorem 8.6. *Let* $G = K_{m,n}$*. Then*:

(1) $L_G^{\Bbbk}(d)$ *is radical for every* d.

(2) $L_G^{\Bbbk}(d)$ *is a complete intersection if and only if* $d \ge m+n-1$.

(3) $L_G^{\Bbbk}(d)$ *is prime if and only if* $d \ge m+n$.

(4) $\operatorname{pmd}(G) = m+n-1$.

Proof. Taking into account Remark 2.3, the assertions (1), (2), and (3) follow form general results on the variety of complexes proved in [De Concini and Strickland 1981] and, with different techniques, in [Musili and Seshadri 1983]. It has been observed by Tchernev [2001] that the assertions in [De Concini and Strickland 1981] that refer to the so-called Hodge algebra structure of the variety of complexes are not correct. However, those assertions can be replaced with statements concerning Gröbner bases as it is done, for example, in a similar case in [Tchernev 2001]. Hence, (1), (2) and (3) can still be deduced from the arguments in [De Concini and Strickland 1981].

Alternative proofs of (2) and (3) are obtained by combining Proposition 8.1 and Corollary 5.6. Finally (4) is a consequence of Lemma 5.4 and Proposition 8.1. $\square$

9. Questions and open problems

We have seen that for the properties "complete intersection" and "prime" of $L_G^{\Bbbk}(d)$ there is persistence along the parameter d but Example 3.2 shows persistence does not occur in general for the property of being radical.

Question 9.1. What patterns can occur in the set $\{d : L_G^{\Bbbk}(d) \text{is radical}\}$ for a graph G?

Since the complete intersection property and prime property of $L_G^{\Bbbk}(d)$ for a given d are inherited by subgraphs, the properties can be characterized by means of forbidden subgraphs. We have explicitly identified the forbidden subgraphs in Theorem 1.4 for $d = 2$ and complete intersection and for $d = 3$ and prime. For $d = 3$ and complete intersection we do not even have a conjecture for the set of forbidden graphs. For $d = 4$ results from Lovász's book [2009, Chapter 9.4] suggest the following:

Question 9.2. Is it true that $L_G^{\Bbbk}(4)$ is prime if and only if G does not contain $K_{a,b}$ for $a+b=5$ and B_4?

Via the fact that primeness of $L_G^{\Bbbk}(d)$ implies primeness of $I_{d+1}^{\Bbbk}(X_G)$ a result by Giusti and Merle [1982, Theorem 1.6.1] guides the intuition behind the following question.

Question 9.3. Let G be a subgraph of $K_{m,n}$ graph and assume $m \le n$. Is it true that $L_G^{\Bbbk}(m-1)$ is prime if and only if G does not contain $K_{a,b}$ for $a+b \ge m$?

By Propositions 7.4 and 7.5 we know that if $L_G^{\Bbbk}(d)$ is radical or prime then so are $I_{d+1}^{\Bbbk}(X_G^{\text{gen}})$ and $I_{d+1}^{\Bbbk}(X_G^{\text{sym}})$ respectively. But our general bounds for $\text{asym}_{\Bbbk}(\text{radical}, G)$ and $\text{asym}_{\Bbbk}(\text{prime}, G)$ from Corollary 5.6 are not good enough to make use of this implication. Indeed, Corollary 8.3 shows that for the properties complete intersection and prime and n large enough there are graphs G for which Proposition 7.5 does not prove primality of an interesting ideal. On the other hand the use of Theorem 1.5 in Theorem 7.8 shows that one can take advantage of this connection in some cases. It would be interesting to exhibit classes different from forests where this is possible.

Question 9.4. Are there more interesting classes of graphs $G = ([n], E)$ for which $\text{asym}_{\Bbbk}(\text{c.i.}, G) \le n-1$ or $\text{asym}_{\Bbbk}(\text{prime}, G) \le n$?

Despite the fact that Proposition 8.2 destroys the hope for using Theorem 7.8 for general graphs, it would be interesting to replace the asymptotic result by an actual value. By Corollary 8.3 for n large we have $\text{asym}_{\Bbbk}(\text{prime}, K_n) = 2n - c_n$ for some numbers $c_n \in o(n)$ which using the notation of Proposition 8.2 satisfy $n - \binom{w_n-2}{2} + 1 \ge c_n \ge 2$. But we have no conjecture for an actual formula for c_n.

Question 9.5. What is the exact value of $\text{asym}_{\Bbbk}(\text{prime}, K_n)$?

For radicality we have a concrete conjecture in the case $G = K_n$.

Conjecture 9.6.

$$\text{asym}_{\Bbbk}(\text{radical}, K_n) = 1 \quad (\textit{at least if } \operatorname{char} \Bbbk = 0).$$

In other words, given a matrix of variables X of size $n \times d$ we conjecture the ideal of the off-diagonal entries of XX^T is radical for all n, d.

It would also be interesting to study the ideal generated by all the entries of XX^T. We note that the symplectic version of this problem has been investigated by De Concini [1979].

Next we turn to open problems about hypergraph LSS-ideals. We know from Theorem 1.3 that for a hypergraph $H = (V, E)$ for which E is a clutter the ideal $L_H^{\Bbbk}(d)$ is a radical complete intersection for $d \geq \mathrm{pmd}(G)$. But we prove in Theorem 1.3 that $L_H^{\Bbbk}(d)$ is prime for $d \geq \mathrm{pmd}(H) + 1$ only in the case that H is a graph.

Question 9.7. Is it true that for a hypergraph $H = (V, E)$, where E is a clutter, we have $L_H^{\Bbbk}(d)$ is prime for $d \geq \mathrm{pmd}(H) + 1$?

Similarly, the persistence results from Theorem 1.1 ask for generalizations.

Question 9.8. Let $H = (V, E)$ be a hypergraph, where E is a clutter. Is it true that if $L_H^{\Bbbk}(d)$ is a complete intersection (resp. prime) then so is $L_H^{\Bbbk}(d+1)$?

For a number $r \geq 1$ we call a hypergraph $H = (V, E)$ an r-uniform graph if every element of E has cardinality r. In particular, E is a clutter. We say that an r-uniform graph $H = (V, E)$ is r-partite if there is a partition $V = V_1 \cup \cdots \cup V_r$ such that $\#(A \cap V_i) = 1$ for all $i \in [r]$ and for all $A \in E$. Now we connect the study of ideal $L_H^{\Bbbk}(d)$ for r-uniform (r-partite) graphs with the study of coordinate sections of the variety of tensors with a given rank. We consider two mappings:

(ϕ) Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ be the standard basis vectors of $\Bbbk^n$. For vectors $v_i = (v_{i,j})_{j \in [d]} \in \Bbbk^d,\ i \in [r]$, consider the map ϕ that sends $(v_1, \ldots, v_r) \in (\Bbbk^d)^n$ to the tensor
$$\sum_{j=1}^{d} \sum_{\sigma \in S_r} v_{\sigma(1),j} \cdots v_{\sigma(r),j}\, \mathfrak{e}_{\sigma(1)} \otimes \cdots \otimes \mathfrak{e}_{\sigma(r)} \in \underbrace{\Bbbk^n \otimes \cdots \otimes \Bbbk^n}_{r}.$$
We take the sums over the different tensors arising from $\mathfrak{e}_{i_1} \otimes \cdots \otimes \mathfrak{e}_{i_r}$, for numbers $1 \leq i_1 \leq \cdots \leq i_r \leq n$, by permuting the positions as standard basis of the space of symmetric tensors.

(ψ) Let $n = n_1 + \cdots + n_r$ for natural numbers $n_1, \ldots, n_r \geq 1$. Let $\mathfrak{e}_i^{(j)} \in \Bbbk^{n_j}$ be the i-th standard basis vector of $\Bbbk^{n_j}$, $i \in [n_j]$ and $j \in [r]$. For vectors $v_i^{(j)} = (v_{i,j})_{j \in [d]} \in \Bbbk^d$ for $i \in [n_j]$ and $j \in [r]$ consider the map ψ that sends $(v_i^{(j)})_{(i,j) \in [n_j] \times [r]}$ to
$$\sum_{(i_1, \ldots, i_r) \in [n_1] \times \cdots \times [n_r]} v_{i_1}^{(1)} \cdots v_{i_r}^{(r)}\, e_{i_1}^{(1)} \otimes \cdots \otimes \mathfrak{e}_{i_r}^{(r)} \in \Bbbk^{n_1} \otimes \cdots \otimes \Bbbk^{n_r}.$$
We take the tensors $\mathfrak{e}_{i_1}^{(1)} \otimes \cdots \otimes \mathfrak{e}_{i_r}^{(r)}$ for numbers $i_j \in [n_j],\ j \in [r]$ as the standard basis of $\Bbbk^{n_1} \otimes \cdots \otimes \Bbbk^{n_r}$.

Recall that a (symmetric) tensor has (symmetric) rank $\leq d$ it can be written as a sum of $\leq d$ decomposable (symmetric) tensors. For more details on tensor rank and the geometry of bounded rank tensors we refer the reader to [Landsberg 2012]. Let $H = (V, E)$ be a hypergraph. We write $\mathcal{V}(L_H^{\Bbbk}(d))$ for the vanishing locus of $L_H^{\Bbbk}(d)$. The definition of the maps ϕ and ψ immediately implies the following proposition.

Proposition 9.9. *Let $H = ([n], E)$ be an r-uniform hypergraph and $\Bbbk$ an algebraically closed field.*

(1) *Then the restriction of the map ϕ to $\mathcal{V}(L_H^{\Bbbk}(d))$ is a parametrization of the variety of symmetric tensors in $\Bbbk^n \otimes \cdots \otimes \Bbbk^n$ (with r factors $\Bbbk^n$) of rank $\leq d$ which when expanded in the standard basis has coefficient zero for the basis elements indexed by $1 \leq i_1 < \cdots < i_r \leq n$ and $\{i_1, \ldots, i_r\} \in E$. In particular, the Zariski-closure of the image of the restriction is irreducible if $L_H^{\Bbbk}(d)$ is prime.*

(2) *If H is r-partite with respect to the partition $V = V_1 \cup \cdots \cup V_r$, where $|V_i| = n_i$, $i \in [r]$, then the restriction of the map ψ to $\mathcal{V}(L_H^{\Bbbk}(d))$ is a parametrization of the variety of tensors in $\Bbbk^{n_1} \otimes \cdots \otimes \Bbbk^{n_r}$ of rank $\leq d$ which when expanded in the standard basis have coefficient zero for the basis elements indexed by $i_1, \ldots, i_r$ where $\{i_1, \ldots, i_r\} \in E$. In particular, the Zariski-closure of the image of the restriction is irreducible if $L_H^{\Bbbk}(d)$ is prime.*

Proposition 9.9 gives further motivation to Question 9.7. Indeed, it suggests to strengthen Question 9.4.

Question 9.10. Let $\Bbbk$ be an algebraically closed field. Can one describe classes of r-uniform hypergraphs H for which $L_H^{\Bbbk}(d)$ is prime for some d bounded from above by the maximal symmetric rank of a symmetric tensor in $\Bbbk^n \otimes \cdots \otimes \Bbbk^n$ (with r factors $\Bbbk^n$)?

An analogous question can be asked for r-partite r-uniform hypergraphs and tensors of bounded rank.

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