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For $a/q \in \mathbb{Q}$ the Estermann function is defined as $D(s, a/q) := \sum_{n\geq 1} d(n)n^{-s}\,\mathrm{e}\big(n\frac{a}{q}\big)$ if $\Re(s) > 1$ and by meromorphic continuation otherwise. For q prime, we compute the moments of $D(s, a/q)$ at the central point $s = 1/2$, when averaging over $1 \leq a < q$.

As a consequence we deduce the asymptotic for the iterated moment of Dirichlet L-functions $\sum_{\chi_1,\dots,\chi_k \pmod q} \big|L\big(\frac{1}{2},\chi_1\big)\big|^2 \cdots \big|L\big(\frac{1}{2},\chi_k\big)\big|^2 \big|L\big(\frac{1}{2},\chi_1\cdots\chi_k\big)\big|^2$, obtaining a power saving error term.

Also, we compute the moments of certain functions defined in terms of continued fractions. For example, writing $f_\pm(a/q) := \sum_{j=0}^{r}(\pm 1)^j b_j$ where $[0; b_0, \dots, b_r]$ is the continued fraction expansion of a/q we prove that for $k \geq 2$ and q primes one has $\sum_{a=1}^{q-1} f_\pm(a/q)^k \sim 2(\zeta(k)^2/\zeta(2k))q^k$ as $q \to \infty$.

1. Introduction

Since the pioneering work of Hardy and Littlewood [1916], the study of moments of families of L-functions has gained a central role in number theory. This is mostly due their numerous applications on, e.g., nonvanishing (see [Iwaniec and Sarnak 2000; Soundararajan 2000]) and subconvexity estimates (see [Conrey and Iwaniec 2000]). Moreover, moments are also important as they highlight clearly the symmetry of each family.

In this paper we consider the moments of the Estermann function at the central point and, as a consequence, we obtain new results for moments of Dirichlet L-functions. We will describe the Estermann function in Section 1.1.2, we now focus on the family of Dirichlet L-functions. For this family only the second and fourth moments have been computed. The asymptotic for the second moment was obtained by Paley [1931], whereas Heath-Brown [1981] considered the fourth moment and showed

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \pmod q} \big|L\big(\tfrac{1}{2},\chi\big)\big|^4 \sim \frac{1}{2\pi^2}\prod_{p\mid q}\frac{(1-1/p)^3}{1+1/p}(\log q)^4, \tag{1-1}$$

provided that q doesn't have "too many prime divisors", a restriction that was later removed by Soundararajan [2007]. As usual, $\sum^*$ indicates that the sum is restricted to primitive characters and $\varphi^*(q)$ denotes the number of such characters. The problem of computing the full asymptotic expansion for the fourth moment was later solved by Young [2011a] in the case when q is prime. He proved

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \pmod q} \big|L\big(\tfrac{1}{2},\chi\big)\big|^4 = \sum_{i=0}^{4} c_i(\log q)^i + O(q^{-\frac{5}{512}+\varepsilon}) \tag{1-2}$$

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for some absolute constants c_i with $c_4 = (2\pi^2)^{-1}$. Recently, Blomer, Fouvry, Kowalski, Michel and Milićević [Blomer et al. 2017] introduced several improvements in Young's work improving the error term in (1-2) to $O(q^{-\frac{1}{32}+\varepsilon})$.

In this paper, we consider a variation of this problem and compute the asymptotic of

$$M_k(q) = \frac{1}{\varphi^*(q)^{k-1}} \sideset{}{^*}\sum_{\chi_1,\dots,\chi_{k-1} \ (\mathrm{mod}\ q)} \bigl|L\bigl(\tfrac{1}{2},\chi_1\bigr)\bigr|^2 \cdots \bigl|L\bigl(\tfrac{1}{2},\chi_{k-1}\bigr)\bigr|^2 \bigl|L\bigl(\tfrac{1}{2},\chi_1\cdots\chi_{k-1}\bigr)\bigr|^2, \tag{1-3}$$

where the sum has the extra restriction that $\chi_1\cdots\chi_{k-1}$ is primitive. If $k=2$, this coincides with the usual fourth moment of Dirichlet L-functions as computed by Young, whereas if $k>2$ then $M_k(q)$ should be thought of as an iterated fourth moment, since each character appears four times in the above expression. We shall prove the following theorem.

Theorem 1. *Let $k \ge 3$ and let q be prime. Then, there exists an absolute constant $A>0$ such that*

$$M_k(q) = \sum_{n=1}^{\infty} \frac{2^{\nu(n)}}{n^{k/2}} \Bigl(\bigl(\log \tfrac{q}{8n\pi}\bigr)^k + \bigl(-\tfrac{\pi}{2}\bigr)^k\Bigr) + O_\varepsilon(k^{Ak} q^{-\delta_k+\varepsilon}),$$

where $\nu(n)$ is the number of different prime factors of n, $\delta_k := (k-2-3\vartheta)/(2k+5)$ with $\vartheta = \frac{7}{64}$ being the best bound towards Selberg's eigenvalue conjecture. Also, the implicit constant depends on ε only.

Remark. Notice that δ_k is a increasing sequence such that $\delta_k \to \frac{1}{2}$ as $k \to \infty$. For $\vartheta = \frac{7}{64}$ the first few values of δ_k are $\delta_3 = \frac{43}{704}$, $\delta_4 = \frac{107}{832}$, $\delta_5 = \frac{57}{320}$.

Theorem 1 yields an asymptotic formula for $M_k(q)$ for $k < \eta(\log q)/(\log\log q)$ with $\eta > 0$ sufficiently small. Larger values of k are easier to deal with and one obtains the following corollary.

Corollary 2. *Let q be prime. Then as $q \to \infty$ we have*

$$M_k(q) \sim \frac{\zeta(k/2)^2}{\zeta(k)} (\log(q/(8\pi)) + \gamma)^k, \tag{1-4}$$

uniformly in $3 \le k = o(q^{\frac{1}{2}} \log q)$, where γ is the Euler–Mascheroni constant. Moreover this range is optimal, meaning that (1-4) is false if $k \gg q^{\frac{1}{2}} \log q$.

Remark. Notice that the main terms in (1-4) and Theorem 1 have a double pole at $k=2$. This is consistent with the fact that the main term for $M_2(q)$ has size $(\log q)^4$ rather than $(\log q)^2$. In principle one could treat the case $k=2$ together with the case $k \ge 3$. However, in order to do so one would need to include in (2-2) an extra main-term of size $q^{o(1)}$ coming from the diagonal term. For $k \ge 3$ this term is absorbed in the error term and so it is more convenient to simply exclude the case $k=2$.

A moment somewhat similar to (1-3) was previously considered by Chinta [2005] who used a multiple Dirichlet series approach to compute the asymptotic of the first moment of (roughly)

$$L\bigl(\tfrac{1}{2},\chi_{d_1}\bigr) L\bigl(\tfrac{1}{2},\chi_{d_2}\bigr) L\bigl(\tfrac{1}{2},\chi_{d_1}\chi_{d_2}\bigr), \tag{1-5}$$

where χ_d denotes the quadratic character associated to the extension $\mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$. We remark that there is a big difference between (1-3) and this case. Indeed, if χ_1, χ_2 are characters modulo q then so is $\chi_1\chi_2$, whereas if $d_1, d_2 \approx X$ then $\chi_{d_1}\chi_{d_2}$ is typically a character with conductor $\approx X^2$. This means that (1-3) roughly correspond to an iterated fourth moment, whereas the second moment of (1-5) roughly correspond to an iterated sixth moment of quadratic Dirichlet L-functions, and thus it doesn't seem to be attackable with the current technology. (As a comparison, the first moment computed by Chinta roughly correspond to an iterated third moment).

1.1. ***Twisted moments, the Estermann function, and continued fractions.*** A nice feature of Theorem 1 is that it can be essentially rephrased in terms of high moments of other functions appearing naturally in number theory. Indeed, the same computations give also the asymptotic for moments of twisted moments of Dirichlet L-functions, of the Estermann function, and of certain functions defined in terms of continued fractions. We now briefly describe each of these objects and give the corresponding version of Theorem 1.

1.1.1. *Moments of twisted moments.* Several classical methods to investigate the central values of Dirichlet L-functions pass through the study of the second moment of $L(s, \chi)$ times a Dirichlet polynomial $P_\vartheta(s, \chi) := \sum_{n \le q^\vartheta} a_n \cdot \chi(n) n^{-s}$:

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q)} \left|L\left(\tfrac{1}{2}, \chi\right) P_\vartheta\left(\tfrac{1}{2}, \chi\right)\right|^2. \tag{1-6}$$

For example, Iwaniec and Sarnak proved that $\frac{1}{3}$ of the Dirichlet L-functions do not vanish at the central point via proving the asymptotic for such average for $\vartheta < \frac{1}{2}$ (and choosing P_ϑ to be a mollifier). Moreover, it is easy to see that if one could extend such asymptotic to all polynomials of length $\vartheta < 1$, then the Lindelöf hypothesis would follow.

Expanding the square, using the multiplicativity of Dirichlet characters, and renormalizing, one immediately sees that (1-6) can be reduced to an average of twisted moments of the form

$$M(a, q) := \frac{q^{\frac{1}{2}}}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q)} \left|L\left(\tfrac{1}{2}, \chi\right)^2\right| \chi(a),$$

for $(a, q) = 1$. By the orthogonality of Dirichlet characters one can immediately rewrite Theorem 1 (and (1-1)) in terms of $M(a, q)$. In particular, one has

$$\sum_{a=1}^{q} M(a, q)^k = \frac{\varphi(q)}{\varphi^*(q)} q^{k/2} M_k(q) \sim \begin{cases} \dfrac{\zeta(k/2)^2}{\zeta(k)} (\log(q/(8\pi)) + \gamma)^k & \text{if } 3 \le k = o(q^{\frac{1}{2}} \log q), \\ \dfrac{1}{2\pi^2} (\log q)^4 & \text{if } k = 2, \end{cases} \tag{1-7}$$

as $q \to \infty$ with q prime, where φ is Euler's φ-function.

1.1.2. *Moments of the Estermann function.* For

$$(a, q) = 1, \quad q > 0, \quad \alpha, \beta \in \mathbb{C} \quad \text{and} \quad \Re(s) > 1 - \min(\Re(\alpha), \Re(\beta)),$$

the Estermann function is defined as

$$D_{\alpha,\beta}(s,a/q) := \sum_{n=1}^{\infty} \mathrm{e}(na/q)\frac{\tau_{\alpha,\beta}(n)}{n^s} = D_{\cos;\alpha,\beta}(s,a/q) + i\,D_{\sin;\alpha,\beta}(s,a/q), \tag{1-8}$$

where $D_{\cos}$ and $D_{\sin}$ have the same definition as D, but with $\mathrm{e}(na/q)$ replaced by $\cos(2\pi na/q)$ and $\sin(2\pi na/q)$ respectively. As usual, $\mathrm{e}(x) := e^{2\pi ix}$ and $\tau_{\alpha,\beta}(n) := \sum_{d_1d_2=n} d_1^{-\alpha}d_2^{-\beta}$.

$D_{\alpha,\beta}(s,a/q)$ was first introduced (with $\alpha=\beta=0$) by Estermann who proved that it extends to a meromorphic function on $\mathbb{C}$ satisfying a functional equation

$$\text{relating } D_{\alpha,\beta}(s,a/q) \quad \text{with } D_{-\alpha,-\beta}(1-s,\pm\bar{a}/q),$$

where $\bar{a}$ denotes the multiplicative inverse of a modulo q (and similarly for $D_{\sin}$ and $D_{\cos}$ which satisfy a more symmetric functional equation given by (3-2) below).

Since the work of Estermann [1930; 1932] on the number of representations of an integer as a sum of two or more products, the Estermann function has proved itself as a valuable tool when studying additive problems of similar flavor (see, e.g., [Motohashi 1980; 1994]) and in problems related to moments of L-functions (see, e.g., [Heath-Brown 1979; Young 2011a; Conrey et al. 1986]). These applications mainly use the functional equation for D as it encodes Voronoi's summation in an analytic fashion, allowing for a simpler computation of the main terms. However, the Estermann function is an interesting object by its own right, due to its surprising symmetries (see [Bettin 2016]) and to the connections with some interesting objects in analytic number theory. For example, by the work of Ishibashi [1995] (see also [Bettin and Conrey 2013a]) one has

$$D_{\sin;1,0}(0,a/q) = \pi\,\mathrm{s}(a,q), \qquad D_{\sin;0,0}(0,a/q) = \tfrac{1}{2}\,\mathrm{c}_0(a/q),$$

where $\mathrm{s}(a,q)$ is the classical Dedekind sum and $\mathrm{c}_0(a/q)$ is a cotangent sum, related to the Nyman–Beurling criterion for the Riemann hypothesis, which has been an object of intensive studies in recent years (see, for example, [Bettin and Conrey 2013b; Maier and Rassias 2016; Bettin 2015]). Ishibashi obtained similar identities also for other values of α, β, and in particular if α is a positive odd integer one obtains that $D_{\sin;\alpha,0}(0,a/q)$ is related to certain Dedekind cotangent sums studied by Beck [2003]. All these functions satisfy certain reciprocity relations and provide examples of "quantum modular forms" (see [Zagier 2010]).

Moreover, one can also obtain formulae relating the Estermann function to twisted moments of Dirichlet L-function (see [Bettin 2016; Conrey and Ghosh 2006]) and in particular for q prime and $(a,q)=1$, one has

$$D_{\cos;0,0}\big(\tfrac{1}{2},a/q\big) + D_{\sin;0,0}\big(\tfrac{1}{2},a/q\big) = M(a,q) + \frac{2(q^{\frac{1}{2}}-1)}{\varphi(q)}\zeta\big(\tfrac{1}{2}\big)^2. \tag{1-9}$$

By this formula and (1-7), it is clear that Theorem 1 gives an asymptotic formula for the high moments of $D_{\cos;0,0}\big(\tfrac{1}{2},a/q\big) + D_{\sin;0,0}\big(\tfrac{1}{2},a/q\big)$. The method however allows one to obtain the asymptotic for the joint moments of $D_{\cos;0,0}\big(\tfrac{1}{2},a/q\big)$ and $D_{\sin;0,0}\big(\tfrac{1}{2},a/q\big)$. We shall state this in Theorem 5 below where shifts are also included (all our results will be derived from this theorem). Here we content ourselves with giving the asymptotic for the high moments of the Estermann function:

Theorem 3. *Let q be prime. Then,*

$$\frac{1}{\varphi(q)}\sum_{a=1}^{q-1} D_{0,0}\big(\tfrac{1}{2},\tfrac{a}{q}\big)^k \sim q^{k/2-1}2^{1-k/2}\frac{\zeta(k/2)^2}{\zeta(k)}\Re\Big(\big(e^{\pi i/4}\big(\log\tfrac{q}{8\pi}+\gamma\big)-e^{-\pi i/4}\tfrac{\pi}{2}\big)^k\Big)$$

as $q\to\infty$, uniformly in $3\le k=o(q^{\frac{1}{2}}\log q)$. In particular, if $3\le k\ll 1$ then

$$\frac{1}{\varphi(q)}\sum_{a=1}^{q-1} D_{0,0}\big(\tfrac{1}{2},\tfrac{a}{q}\big)^k \sim q^{k/2-1}2^{1-k/2}\frac{\zeta(k/2)^2}{\zeta(k)}\big(\cos\big(\tfrac{k\pi}{4}\big)(\log q)^k-\tfrac{\pi}{2}\sin\big(\tfrac{k\pi}{4}\big)(\log q)^{k-1}\big)$$

as $q\to\infty$.

1.1.3. *Moments of certain functions defined in terms of continued fractions.* Finally, we discuss the relation with continued fractions. In [Bettin 2016] (see also [Young 2011b]), it was observed that $M(a,q)$, and more generally, $D_{\cos}$ and $D_{\sin}$, can be written in terms of the continued fraction expansion of a/q. Indeed, if $a,q\in\mathbb{Z}_{>0}$ and $[b_0;b_1,\dots,b_\kappa,1]$ is the continued fraction expansion of a/q, then for q prime one has

$$M(a,q)=\sum_{\substack{j=1\\ j\text{ odd}}}^{\kappa} b_j^{\frac{1}{2}}\big(\log\tfrac{b_j}{8\pi}+\gamma\big)-\tfrac{\pi}{2}\sum_{\substack{j=1\\ j\text{ even}}}^{\kappa} b_j^{\frac{1}{2}}+O(\log q). \tag{1-10}$$

It is therefore not surprising that Theorem 1 has an incarnation also in terms of moments for functions of the rationals defined as

$$f_{r,\pm}(a/q):=\sum_{j=1}^{\kappa}(\pm 1)^j b_j^{r/2},$$

where $r\in\mathbb{Z}_{\ge 1}$.

Theorem 4. *Let q be prime and let $k,r\in\mathbb{Z}_{\ge1}$ with $3\le kr=o((\log q)/(\log\log q))$. Then*

$$\sum_{a=1}^{q} f_{r,\pm}(a/q)^k\sim 2\frac{\zeta(kr/2)^2}{\zeta(kr)}q^{kr/2}$$

as $q\to\infty$.

Starting with the work of Heilbronn [1969], who considered the average value of $f_{0,+}$, there have been a very large number of papers computing the mean values of functions defined in terms of the continued fraction expansion. In particular, we cite the works [Porter 1975; Tonkov 1974] on $f_{0,+}$ and [Yao and Knuth 1975] where the asymptotic for the first moment of $f_{2,+}$ was given. However, to the knowledge of the author, Theorem 4 is the first result giving asymptotic formulae for k-th moments with $k\ge 3$ without exploiting an extra average over q (as in [Hensley 1994; Baladi and Vallée 2005]). For $k=2$ the only cases previously known where obtained by Bykovskiĭ [2005] (considering the second moment of $f_{0,+}$) and by the author [Bettin 2016] (considering the second moment of a variation of $f_{2,+}$). By combining the techniques employed in [Bettin 2016] and in this paper it seems possible to extend Theorem 4 to more general functions of similar shape.

1.2. ***Brief outline of the proof of Theorem 1.*** The approximate functional equation allows one to express $M_k(q)$ roughly in the form

$$\sum_{\substack{\pm n_1 \pm n_2 \cdots \pm n_k \equiv 0 \pmod q,\\ n_1 \cdots n_k \ll q^k}} \frac{d(n_1)\cdots d(n_k)}{n_1^{\frac{1}{2}} \cdots n_k^{\frac{1}{2}}}, \tag{1-11}$$

so that the problem of estimating $M_k(q)$ reduces to that of computing the asymptotic for this quadratic divisor problem. The diagonal terms (i.e., the terms with $\pm n_1 \pm n_2 \cdots \pm n_k = 0$) are a bit easier to study and give a main term; the main difficulties then lie in obtaining an asymptotic for the off-diagonal terms and in assembling the various main terms. In his proof of (1-2), which corresponds to (1-11) with $k = 2$, Young used a combination of several techniques each effective for some range of the variables n_1, n_2. In particular, when $n_1 \approx n_2$ (in the logarithmic scale) he followed an approach à la Motohashi [1997] using Kuznetsov formula, whereas when one variable is much larger than the other one, he used (new) estimates for the average value of the divisor function in arithmetic progressions.

Our approach is similar to that of Young, however there are several substantial differences which we will now discuss in some detail. First, the larger number of variables gives us the advantage of having to deal with more "flexible" sums enlarging the ranges where the various estimates are effective. For this reason, we can afford to use slightly weaker bounds employing the spectral theory only indirectly, through the bounds of Deshouilliers and Iwaniec [1982] (together with Kim and Sarnak's bound for the exceptional eigenvalues [Kim 2003]). It seems likely that one could use spectral methods in a more direct and efficient way, however the generalization of the methods in [Young 2011a] (or [Blomer et al. 2017]) to the $k \geq 3$ case is not straightforward and so we choose a simpler route as this is still sufficient for our purposes.

The larger number of variables also has a cost. Indeed, it introduces several new complications in the extraction and in the combination of the main terms, a process that requires a rather careful analysis and constitutes the central part of this paper. One of the causes of the complicated shape of the main terms (see (6-1)-(6-2)) is that with more than two variables the dichotomy "either one variable is much bigger than the other or the variables have the same size" doesn't hold for $k > 2$ and one has to (implicitly) deal also with cases such as $n_1 \approx \cdots \approx n_{k-1} \approx q^{1+1/k}$ and $n_k \approx 1$.

Another difference with Young's work arises when studying the diagonal terms. If $k = 2$, then one can handle these terms easily thanks to Ramanujan's formula $\sum_{n \geq 1} d(n)^2/n^s = \zeta(s)^4/\zeta(2s)$. If $k \geq 3$, we don't have such a nice exact formula, and we are left with the problem of showing that the series

$$\sum_{\pm n_1 \pm \cdots \pm n_k = 0} \frac{d(n_1)\cdots d(n_k)}{(n_1 \cdots n_k)^s}$$

can be meromorphically continued past the line $\Re(s) = 1 - 1/k$ which is the boundary of convergence. We shall leave this problem to a different paper, [Bettin 2017], where with similar (but a bit simpler) techniques we prove that this series admits meromorphic continuation to the region $\Re(s) > 1 - 2/(k+1)$.

Last, we mention a more technical problem. One of the steps in Young's proof requires separating n_1, n_2 in expressions of the form $(n_1 \pm n_2)^{-z}$ when $\Re(z) \approx 0$. This can be easily obtained by using some classical Mellin formulae; however, whereas the Mellin integral corresponding to $(1+x)^{-z}$ converges absolutely, the Mellin integral corresponding to $(1-x)^{-z}$ converges only conditionally so that the terms containing $(n_1 - n_2)^{-z}$ demand some caution. In our case this problem becomes rather more subtle as we need to apply these formulae iteratively in order to handle expressions such as $(n_1 \pm \cdots \pm n_k)^{-s}$. We overcome this difficulties by using a modification of the resulting "iterated" Mellin formula allowing us to write such expressions in terms of absolutely convergent integrals (see Section 10 for the details).

1.3. ***The structure of the paper.*** The paper is organized as follow. In Section 2 we state Theorem 5, a more general version of Theorem 3 providing the asymptotic for the mixed moments of $D_{\cos}$ and $D_{\sin}$ (as well as allowing for some small shifts). We then use this result to deduce Theorems 1, 3 and 4. In Section 3 we give some lemmas on the Estermann function which we shall need later on. It is in these lemmas that the spectral theory comes (indirectly) into play. The proof of Theorem 5 is carried out in Sections 5–9, after introducing some notation in Section 4, and constitutes the main body of the paper. Finally, in Section 10 we will prove the Mellin formula mentioned at the end of the previous section as well as some technical Lemmas needed in order to use this formula effectively.

2. Mixed moments of $D_{\cos}$ and $D_{\sin}$ and the deduction of the main theorems

Let $k \geq 1$, q be a prime and let $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \mathbb{C}$. Then, for any subset $\Upsilon \subseteq \{1, \ldots, k\}$ let $M_{\Upsilon,k}$ be the mixed shifted moment

$$M_{\Upsilon,k} := \frac{1}{\varphi(q)} \sum_{a=1}^{q-1} \prod_{i=1}^{k} D_{i;\alpha_i,\beta_i}\left(\tfrac{1}{2}, \tfrac{a}{q}\right),$$

where $D_{i;\alpha_i,\beta_i} := D_{\sin;\alpha_i,\beta_i}$ if $i \in \Upsilon$ and $D_i := D_{\cos;\alpha_i,\beta_i}$ otherwise. Also, let

$$\Gamma_i(s) := \begin{cases} \Gamma\left(\frac{1}{2}+s\right) & \text{if } i \in \Upsilon, \\ \Gamma(s) & \text{otherwise.} \end{cases} \tag{2-1}$$

Since $D_{\sin;\alpha_i,\beta_i}(s, -a/q) = -D_{\sin;\alpha_i,\beta_i}(s, a/q)$, then $M_{\Upsilon,k}$ is identically zero if $|\Upsilon|$ is odd. If $|\Upsilon|$ is even the asymptotic for $M_{\Upsilon,k}$ is given by the following theorem, provided that $k \geq 3$ (the corresponding theorem for $k = 2$ is essentially implicit in [Young 2011a], whereas the case $k = 1$ is trivial).

Theorem 5. *Let $\Upsilon \subseteq \{1, \ldots, k\}$ with $|\Upsilon|$ even. Let $k \geq 3$ and let q be a prime. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$, $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_k) \in \mathbb{C}^k$ with $|\alpha_i|, |\beta_i| \ll 1/\log q$ and $|\alpha_i|, |\beta_i| \leq \frac{1}{10}$ for all $i = 1, \ldots, k$. Then, there exists an absolute constant $A > 0$ such that for any $\varepsilon > 0$ we have*

$$M_{\Upsilon,k} = \sum_{\{\alpha_i', \beta_i'\} = \{\alpha_i, \beta_i\}} \mathscr{M}_{\boldsymbol{\alpha}', \boldsymbol{\beta}'} + O_\varepsilon(k^{Ak} q^{k/2-1-\delta_k+\varepsilon}), \tag{2-2}$$

where $\delta_k := \dfrac{k-2-3\vartheta}{2k+5}$,

$$\mathscr{M}_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \frac{q^{k/2-1}}{2^{k-1}} \frac{\zeta\big(\frac{k}{2}-\sum_{i=1}^k \alpha_i\big)\zeta\big(\frac{k}{2}+\sum_{i=1}^k \beta_i\big)}{\zeta\big(k-\sum_{i=1}^k(\alpha_i-\beta_i)\big)} \prod_{i=1}^k \frac{\Gamma_i\big(\frac14-\frac{\alpha_i}{2}\big)}{\Gamma_i\big(\frac14+\frac{\alpha_i}{2}\big)}\Big(\frac{q}{\pi}\Big)^{-\alpha_i}\zeta(1-\alpha_i+\beta_i) \tag{2-3}$$

and where the implicit constant in the error term depends on ε *only.*

Remark. If $\alpha_i = \beta_i$ for some $i = 1, \dots, k$, then $\mathscr{M}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ has to be interpreted as the limit for $\alpha_i \to \beta_i$ (see (2-4) below).

As mentioned in Section 1.3, we will prove Theorem 5 in Sections 5–9. We will now deduce Theorems 1, 3, and 4 from Theorem 5.

2.1. ***Proof of Theorem 1, 3 and 4 and of Corollary 2.*** We start by observing that if $|\Upsilon|$ is even then from Theorem 5 one has

$$\sum_{a=1}^{q-1}\prod_{i=1}^{k} D_{i;0,0}\big(\tfrac12,\tfrac{a}{q}\big) = \frac{q^{k/2}}{2^{k-1}}\sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^{k/2}}\prod_{i=1}^{k}\big(\log\tfrac{q}{8n\pi}+\gamma-a_i\pi\big) + O_\varepsilon(k^{Ak}q^{k/2-\delta_k+\varepsilon}), \tag{2-4}$$

where $a_i = -\frac12$ if $i \in \Upsilon$ and $a_i = \frac12$ otherwise. Indeed, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy the hypothesis of Theorem 5 and $\alpha_i \neq \beta_i$ for all i, then by contour integration the main term on the right hand side of (2-2) can be rewritten as

$$\sum_{\{\alpha_i',\beta_i'\}=\{\alpha_i,\beta_i\}} \mathscr{M}_{\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*} = \frac{q^{k/2-1}}{2^{k-1}}\frac{1}{(2\pi i)^k}\oint_{|s_1|=\frac14}\cdots\oint_{|s_k|=\frac14}\frac{\zeta\big(\frac{k}{2}+\sum_{i=1}^k(s_i-\alpha_i-\beta_i)\big)\zeta\big(\frac{k}{2}+\sum_{i=1}^k s_i\big)}{\zeta\big(k+\sum_{i=1}^k(2s_i-\alpha_i-\beta_i)\big)}$$
$$\times\prod_{i=1}^{k}\frac{\Gamma_i\big(\frac14+\frac{s-\alpha_i-\beta_i}{2}\big)}{\Gamma_i\big(\frac14-\frac{s-\alpha_i-\beta_i}{2}\big)}\Big(\frac{q}{\pi}\Big)^{s_i-\alpha_i-\beta_i}\zeta(1+s_i-\alpha_i)\zeta(1+s_i-\beta_i)\,ds_i, \tag{2-5}$$

where the circles are integrated counterclockwise. Thus, taking the limit for $\boldsymbol{\alpha},\boldsymbol{\beta}\to 0$ and expanding $\zeta(s)^2/\zeta(2s)$ as a Dirichlet series (see [Titchmarsh 1986, (1.2.8)]), we obtain

$$\sum_{\{\alpha_i',\beta_i'\}=\{\alpha_i,\beta_i\}} \mathscr{M}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} = \frac{q^{k/2-1}}{2^{k-1}}\sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^{k/2}}\prod_{i=1}^{k}\frac{1}{2\pi i}\oint_{|s_i|=\frac14}\frac{\Gamma_i\big(\frac14+\frac{s_i}{2}\big)}{\Gamma_i\big(\frac14-\frac{s_i}{2}\big)}\Big(\frac{q}{n\pi}\Big)^{s_i}\zeta(1+s_i)^2 d\,s_i$$
$$= \frac{q^{k/2-1}}{2^{k-1}}\sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^{k/2}}\prod_{i=1}^{k}\big(\log\tfrac{q}{8n\pi}+\gamma-a_i\pi\big),$$

by the residue theorem. We remind that $\nu(n)$ is the number of distinct prime factors of n and γ is the Euler–Mascheroni constant. Equation (2-4) then follows.

To prove Theorem 1 we observe that by (2-4) we have (remember that if $|\Upsilon|$ is odd then $M_{\Upsilon,k}=0$)

$$\begin{aligned}\sum_{a=1}^{q-1}\bigl(D_{\cos;0,0}\bigl(\tfrac12,\tfrac aq\bigr)+D_{\sin;0,0}\bigl(\tfrac12,\tfrac aq\bigr)\bigr)^k\\ &=\sum_{r=0}^{k}\binom{k}{r}\sum_{a=1}^{q-1}D_{\cos;0,0}\bigl(\tfrac12,\tfrac aq\bigr)^{k-r}D_{\sin;0,0}\bigl(\tfrac12,\tfrac aq\bigr)^r\\ &=\frac{q^{k/2}}{2^{k-1}}\sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^{k/2}}\sum_{\substack{r=0\\ r\text{ even}}}\binom{k}{r}\bigl(\log\tfrac{q}{8n\pi}+\gamma-\tfrac{\pi}{2}\bigr)^{k-r}\bigl(\log\tfrac{q}{8n\pi}+\gamma+\tfrac{\pi}{2}\bigr)^{r}+\mathcal{E}_2\\ &=\frac{q^{k/2}}{2^{k}}\sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^{k/2}}\bigl(\bigl(2\log\tfrac{q}{8n\pi}+2\gamma\bigr)^k+\bigl(-\tfrac{\pi}{2}\bigr)^k\bigr)+\mathcal{E}_2\end{aligned}$$

for some $\mathcal{E}_2\ll_\varepsilon k^{Ak}q^{k/2-\delta_k+\varepsilon}$, where in the last step we used that

$$\sum_{\substack{r=0\\ r\text{even}}}^{k}\binom{k}{r}x^{k-r}y^r=\frac{(x+y)^k+(x-y)^k}{2}\quad\text{for all } x,y\in\mathbb{R}.$$

Thus, using (1-9) one obtains Theorem 1. One easily verifies that as $q\to\infty$

$$\sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^{k/2}}\bigl(\bigl(\log\tfrac{q}{8n\pi}+\gamma\bigr)^k+\bigl(-\tfrac{\pi}{2}\bigr)^k\bigr)\sim\frac{\zeta(k/2)^2}{\zeta(k)}\bigl(\log\tfrac{q}{8\pi}+\gamma\bigr)^k$$

uniformly in $k\geq 3$. If $k<\eta(\log q)/(\log\log q)$ with $\eta>0$ sufficiently small (but fixed), then the error term $\mathcal{E}_2$ is smaller than the above main term and so Corollary 2 follows on this range.

Now assume $k\geq\eta(\log q)/(\log\log q)$. First, we observe that by (1-10) for $a\neq 1$ we have

$$|M(a,q)|\leq(q/\eta)^{\frac12}\log q \tag{2-6}$$

for any fixed $1<\eta<2$ and q sufficiently large. Indeed, this is obvious if $a=-1$, whereas if $a\neq\pm1$ then $\max_j b_j\leq(q-1)/2$ and so the above bound follows since $b_1\cdots b_\kappa\leq q$. Furthermore, from the second moment estimate $\sum_{a=1}^{q}|M(a,q)|^2\ll(\log q)^4$ it follows that for every $C>0$ there are at most $O(q(\log q)^4/C^2)$ values of a in $1<a\leq q$ such that $|M(a,q)|\geq C$. Thus, by (2-6) we have

$$\begin{aligned}\sum_{a=2}^{q}M(a,q)^k&\leq\sum_{\substack{2\leq a\leq q\\ |M(a,q)|<C}}M(a,q)^k+\sum_{\substack{2\leq a\leq q\\ |M(a,q)|\geq C}}M(a,q)^k\\ &\ll C^kq+\frac{q^{k/2+1}(\log q)^{k+4}}{\eta^{k/2}C^2}\ll\frac{q^{k/2+2/(k+2)}}{\eta^{k/2}}(\log q)^{k+2}\end{aligned}$$

for $C=\eta^{-\frac12}q^{\frac12-1/(k+2)}\log q$. Note that if $k\gg(\log q)/(\log\log q)$, then

$$\text{error term}\ll q^{k/2}\eta^{-k/4}(\log q)^k=o(q^{k/2}(\log(q/(8\pi))+\gamma)^k)\quad\text{as } q\to\infty,\text{ uniformly in } k.$$

Finally, we have (see [Heath-Brown 1981])

$$M(1,q) = q^{\frac12}(\log(q/(8\pi))+\gamma) + 2\zeta\left(\tfrac12\right)^2 + O(q^{-\frac12})$$

so that

$$M(1,q)^k = q^{k/2}(\log(q/(8\pi))+\gamma)^k \exp\left(2\zeta\left(\tfrac12\right)^2 \frac{k}{q^{\frac12}\log q}(1+O(1/\log q))\right)$$

for q large enough. Thus, if $(\log q)/(\log\log q) \ll k = o(q^{\frac12}\log q)$ we have

$$M_k(q) = q^{-k/2}\sum_{a=1}^{q} M(a,q)^k \sim (\log(q/(8\pi))+\gamma)^k \sim \frac{\zeta(k/2)^2}{\zeta(k)}(\log(q/(8\pi))+\gamma)^k$$

as $q\to\infty$, whereas this asymptotic is false if $k \gg q^{\frac12}\log q$. This concludes the proof of Corollary 2.

The proof of Theorem 3 is analogous to those of Theorem 1 and Corollary 2, with the difference that in this case we use (1-8) rather than (1-9). Indeed for some $\mathcal{E}_1 \ll_\varepsilon k^{Ak} q^{k/2-\delta_k+\varepsilon}$ we have

$$\begin{aligned}
\sum_{a=1}^{q-1} & D_{0,0}\left(\tfrac12,\tfrac{a}{q}\right)^k \\
&= \sum_{r=0}^{k}\binom{k}{r}\sum_{a=1}^{q-1} D_{\cos;0,0}\left(\tfrac12,\tfrac{a}{q}\right)^{k-r} i^r D_{\sin;0,0}\left(\tfrac12,\tfrac{a}{q}\right)^r \\
&= \frac{q^{k/2}}{2^{k-1}}\sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^{k/2}}\sum_{\substack{r=0\\ r\text{ even}}}^{k}\binom{k}{r}\left(\log\tfrac{q}{8n\pi}+\gamma-\tfrac{\pi}{2}\right)^{k-r} i^r\left(\log\tfrac{q}{8n\pi}+\gamma+\tfrac{\pi}{2}\right)^r + \mathcal{E}_1 \\
&= \frac{q^{k/2}}{2^{k}}\sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^{k/2}}\left(\left((1+i)\left(\log\tfrac{q}{8n\pi}+\gamma\right)-(1-i)\tfrac{\pi}{2}\right)^k + \left((1-i)\left(\log\tfrac{q}{8n\pi}+\gamma\right)-(1+i)\tfrac{\pi}{2}\right)^k\right) + \mathcal{E}_1 \\
&= (q/2)^{k/2} 2\,\Re\left(\sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^{k/2}}\left(e^{\pi i/4}\left(\log\tfrac{q}{8n\pi}+\gamma\right)-e^{-\pi i/4}\tfrac{\pi}{2}\right)^k\right) + \mathcal{E}_1, \\
&\sim q^{k/2}2^{1-k/2}\frac{\zeta(k/2)^2}{\zeta(k)}\Re\left(\left(e^{\pi i/4}\left(\log\tfrac{q}{8\pi}+\gamma\right)-e^{-\pi i/4}\tfrac{\pi}{2}\right)^k\right)
\end{aligned}$$

as $q\to\infty$ with $3\le k = o((\log q)/(\log\log q))$. One then obtains Theorem 3 on the range $3 \le k = o(q^{\frac12}\log q)$ by proceeding as in the proof of Corollary 2.

2.2. ***Proof of Theorem 4.*** We compute the moments of $f_{r,+}$ only, the case of $f_{r,-}$ being analogous (using (2-8) instead of (2-7)).

We start by noticing that Corollary 11 of [Bettin 2016] gives

$$D_{\cos;0,0}\left(\tfrac12,\tfrac{a}{q}\right) = \frac12\sum_{j=1}^{\kappa} b_j^{\frac12}\left(\log\tfrac{b_j}{8\pi}+\gamma-\tfrac{\pi}{2}\right) + O(\log q), \tag{2-7}$$

$$D_{\sin;0,0}\big(\tfrac12,\tfrac aq\big)=\frac12\sum_{j=1}^{\kappa}(-1)^j b_j^{\frac12}\big(\log\tfrac{b_j}{8\pi}+\gamma+\tfrac{\pi}{2}\big)+O(\log q), \tag{2-8}$$

where $[0; b_1,\dots b_\kappa, 1]$ is the continued fraction expansion of a/q. Moreover, since $b_1\cdots b_j\asymp q$, then if one among $b_1,\dots,b_j$, say b_{j^*}, satisfies $b_{j^*}>q/(\log q)^{100}$, and thus in particular

$$\log b_{j*}=\log q+O(\log\log q),$$

then $b_j\ll(\log q)^{100}$ for $j\neq j^*$. In particular, if $\max_j b_j>q/(\log q)^{100}$ and $1\le r=o(\log q/\log\log q)$, then

$$\begin{aligned}
f_{r,+}\big(\tfrac aq\big)=\sum_{j=1}^{\kappa}b_j^r/2&=\max_{j=1,\dots\kappa}b_j^{r/2}+O((\log q)^{50r+1})=\frac{1}{(\log q)^r}\bigg(\max_{j=1,\dots\kappa}b_j^{\frac12}\log q\bigg)^r+O((\log q)^{50r+1})\\
&=\frac{1}{(\log q)^r}\bigg(\bigg(\max_{j=1,\dots\kappa}b_j^{\frac12}\big(\log\tfrac{b_j}{8\pi}+\gamma-\tfrac{\pi}{2}\big)\bigg)\big(1+O(\log\log q/\log q)\big)\bigg)^r+O((\log q)^{50r+1})\\
&=\frac{2^r}{(\log q)^r}D_{\cos}\big(\tfrac12,\tfrac aq\big)^r\big(1+O(r\log\log q/\log q)\big).
\end{aligned}\tag{2-9}$$

Moreover, from (2-7) it follows easily that

$$\sum_{j=1}^{\kappa}b_j^{\frac12}\le D_{\cos;0,0}\big(\tfrac12,\tfrac aq\big)+B\log q$$

for all a/q and some $B>0$. In particular, if $\max_j b_j\le q/(\log q)^{100}$ and q is large enough, then

$$\begin{aligned}
f_{r,+}\big(\tfrac aq\big)^k&\le\frac{q^{(k/2)(r-1)}}{(\log q)^{50k(r-1)}}\bigg(\sum_{j=1}^{\kappa}b_j^{\frac12}\bigg)^k\le\frac{q^{(k/2)(r-1)}}{(\log q)^{50k(r-1)}}\big(D_{\cos;0,0}\big(\tfrac12,\tfrac aq\big)+B\log q\big)^k\\
&\ll\frac{q^{kr/2-1}}{(\log q)^{50k(r-1)+47(k-2)}}\big(D_{\cos;0,0}\big(\tfrac12,\tfrac aq\big)+B\log q\big)^2
\end{aligned}\tag{2-10}$$

for $k\ge2$, since $\max_j b_j\le q/(\log q)^{100}$ implies $\big|D_{\cos}\big(\tfrac12,\tfrac aq\big)\big|+B\log q\le q^{\frac12}/(\log q)^{48}$ for q large enough. Now, we have

$$\sum_{a=1}^{q}f_{r,+}\big(\tfrac aq\big)^k=\sum_{\substack{1\le a<q\\ \max_j b_j>q/(\log q)^{100}}}f_{r,+}\big(\tfrac aq\big)^k+\sum_{\substack{1\le a<q\\ \max_j b_j\le q/(\log q)^{100}}}f_{r,+}\big(\tfrac aq\big)^k. \tag{2-11}$$

By (2-10) the second summand is bounded by

$$\begin{aligned}
\sum_{\substack{1\le a<q,\\ \max_j b_j\le q/(\log q)^{100}}}f_{r,+}\big(\tfrac aq\big)^k&\ll\frac{q^{kr/2-1}}{(\log q)^{50k(r-1)+48(k-2)}}\sum_{1\le a<q}\big(D_{\cos}\big(\tfrac12,\tfrac aq\big)+B\log q\big)^2\\
&\ll\frac{q^{kr/2}}{(\log q)^{50k(r-1)+48(k-2)-4}}\ll\frac{q^{kr/2}}{\log q}
\end{aligned}$$

for $kr \geq 3$ (if $k = 1$ one needs to modify slightly the argument, but the final bound still holds). By (2-9) the first summand of (2-11) can be written as

$$\sum_{\substack{1\leq a<q \\ \max_j b_j > q/(\log q)^{100}}}^{q} \frac{\left(2D_{\cos;0,0}\left(\frac{1}{2},\frac{a}{q}\right)\right)^{kr}}{(\log q)^{kr}}\left(1+O(kr\log\log q/\log q)\right)$$
$$= \sum_{1\leq a<q} \frac{\left(2D_{\cos;0,0}\left(\frac{1}{2},\frac{a}{q}\right)\right)^{kr}}{(\log q)^{kr}}\left(1+O(kr\log\log q/\log q)\right)+O(q^{kr/2}/\log q)$$
$$= 2\frac{\zeta(kr)^2}{\zeta(kr/2)}q^{kr/2}\left(1+O(kr\log\log q/\log q)\right)$$

by (2-4) for $3 \leq rk = o((\log q)/(\log\log q))$ and where one can complete the sum by proceeding as in the previous computation. Theorem 4 then follows.

3. The Estermann function and bounds for sums of Kloosterman sums

In this Section we give some results for the Estermann function and for the periodic zeta-function which will be needed in the proof of Theorem 5. In particular, in Section 3.1 we give the functional equation for both these functions, whereas in Section 3.2 we give a version of the approximate functional equation for the Estermann function. Finally, in Section 3.3 we give some estimates for products of the Estermann function and the periodic zeta-function, using the bounds of [Deshouillers and Iwaniec 1982] for sums of Kloosterman sums.

3.1. ***The functional equations.*** We start by giving the functional equation for the Estermann function.

Lemma 6. *For $(a,q)=1$, $q>0$ and $\alpha\in\mathbb{C}$, $D_{\alpha,\beta}(s,a/q)-q^{1-\alpha-\beta-2s}\zeta(s+\alpha)\zeta(s+\beta)$ can be extended to an entire function of s. Moreover, $D_{\alpha,\beta}(s,a/q)$ satisfies the functional equation*

$$D_{\alpha,\beta}\left(s,\tfrac{a}{q}\right)=\frac{2}{q}\left(\frac{q}{2\pi}\right)^{2-2s-\alpha-\beta}\Gamma(1-s-\alpha)\Gamma(1-s-\beta)$$
$$\times\left(\cos(\pi(\alpha-\beta)/2)D_{\alpha,\beta}\left(1-s,\tfrac{\bar{a}}{q}\right)-\cos\left(\tfrac{\pi}{2}(2s+\alpha+\beta)\right)D_{-\alpha,-\beta}\left(1-s,-\tfrac{\bar{a}}{q}\right)\right), \quad (3\text{-}1)$$

where, here and in the following, $\bar{a}$ denotes the multiplicative inverse of a modulo the denominator q.

Proof. This is Lemma 4 of [Conrey 1989]. □

Corollary 7. *Let*

$$\Lambda_{\cos;\alpha,\beta}\left(s,\tfrac{a}{q}\right):=\Gamma\left(\tfrac{s+\alpha}{2}\right)\Gamma\left(\tfrac{s+\beta}{2}\right)\left(\frac{q}{\pi}\right)^{s+(\alpha+\beta)/2}D_{\cos;\alpha,\beta}\left(s,\tfrac{a}{q}\right),$$
$$\Lambda_{\sin;\alpha,\beta}\left(s,\tfrac{a}{q}\right):=\Gamma\left(\tfrac{1+s+\alpha}{2}\right)\Gamma\left(\tfrac{1+s+\beta}{2}\right)\left(\frac{q}{\pi}\right)^{s+(\alpha+\beta)/2}D_{\sin;\alpha,\beta}\left(s,\tfrac{a}{q}\right).$$

Then, we have the functional equations

$$\Lambda_{\cos;\alpha,\beta}\left(s,\tfrac{a}{q}\right)=\Lambda_{\cos;-\alpha,-\beta}\left(1-s,\tfrac{\bar{a}}{q}\right),\quad \Lambda_{\sin;\alpha,\beta}\left(s,\tfrac{a}{q}\right)=\Lambda_{\sin;-\alpha,-\beta}\left(1-s,\tfrac{\bar{a}}{q}\right). \quad (3\text{-}2)$$

Proof. These functional equations follow from (3-1), using the reflection and the duplication formulas for the Γ-function. $\square$

We also need the basic properties of the periodic zeta-function which, for $x \in \mathbb{R}$ and $\Re(s) > 1$, is defined as

$$F(s, x) := \sum_{n=1}^{\infty} \frac{\mathrm{e}(nx)}{n^s}. \tag{3-3}$$

Notice that if $x \in \mathbb{Z}$, then $F(s, x) = \zeta(s)$.

Lemma 8. *Let $h, l \in \mathbb{Z}$ with $(h, \ell) = 1$ and $\ell > 0$, then $F(s, h/\ell)$ extends to an entire function of s with the exception of a simple pole at $s = 1$ if $\ell = 1$. Moreover, $F(s, x)$ satisfies the functional equation*

$$F(1-s, h/\ell) = \ell^{s-1} \sum_{b=1}^{\ell} \mathrm{e}(hb/\ell) \frac{\Gamma(s)}{(2\pi)^s} \big(e^{-\pi i s/2} F(s, b/\ell) + e^{\pi i s/2} F(s, -b/\ell)\big). \tag{3-4}$$

Finally, for $\ell \nmid h$ we have

$$F(0, h/\ell) = -\tfrac{1}{2} + \tfrac{i}{2} \cot(\pi h/\ell). \tag{3-5}$$

Proof. For (3-5) and the analytic continuation of F see [Apostol 1951, pp. 161, 164]. For (3-4), one divides the series for F into congruence classes modulo ℓ writing $F(s, h/\ell)$ as a sum of Hurwitz zeta-functions $\zeta(s, b/\ell)$; applying the functional equation [Apostol 1976, Theorem 12.6] for $\zeta(s, x)$ then gives (3-4). $\square$

3.2. ***The approximate functional equation.*** Next, we give an approximate functional equation allowing us to express a product of k Estermann functions as a sum of total length about $q^{k/2}$.

Lemma 9. *Let $k \geq 1$ and $\Upsilon \subseteq \{1, \dots, k\}$. Let $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ be an entire function satisfying $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(-s) = G_{-\boldsymbol{\alpha},-\boldsymbol{\beta}}(s)$, $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(0) = 1$ and $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}\big(\tfrac{1}{2} - \alpha_i\big) = G_{\boldsymbol{\alpha},\boldsymbol{\beta}}\big(\tfrac{1}{2} - \beta_i\big) = 0$ for $i = 1, \dots, k$ and decaying faster than any power of s on vertical strips. Let*

$$g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) := \pi^{-ks} \prod_{j=1}^{k} \frac{\Gamma_i\big(\big(\tfrac{1}{2} + s + \alpha_i\big)/2\big)\Gamma_i\big(\big(\tfrac{1}{2} + s + \beta_i\big)/2\big)}{\Gamma_i\big(\big(\tfrac{1}{2} + \alpha_i\big)/2\big)\Gamma_i\big(\big(\tfrac{1}{2} + \beta_i\big)/2\big)}, \tag{3-6}$$

$$X_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \prod_{j=1}^{k} \frac{\Gamma_i\big(\big(\tfrac{1}{2} - \alpha_i\big)/2\big)\Gamma_i\big(\big(\tfrac{1}{2} - \beta_i\big)/2\big)}{\Gamma_i\big(\big(\tfrac{1}{2} + \alpha_i\big)/2\big)\Gamma_i\big(\big(\tfrac{1}{2} + \beta_i\big)/2\big)} \Big(\frac{q}{\pi}\Big)^{-\alpha_i - \beta_i}$$

and for any $c_s > 0$ let

$$V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x) := \frac{1}{2\pi i} \int_{(c_s)} G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) x^{-s} \frac{ds}{s},$$

where, as usual, $\int_{(c)} \cdot \, ds$ indicates that the integral is taken along the vertical line from $c - i\infty$ to $c + i\infty$. Then for $a, q \in \mathbb{Z}$, with $q > 1$ and $(a, q) = 1$ we have

$$\prod_{i=1}^{k} D_{i;\alpha_i,\beta_i}\big(\tfrac{1}{2}, \tfrac{a}{q}\big) = S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, q) + X_{\boldsymbol{\alpha},\boldsymbol{\beta}} S_{-\boldsymbol{\alpha},-\boldsymbol{\beta}}(\bar{a}, q), \tag{3-7}$$

where $\bar{a}$ is the inverse of a modulo q and

$$S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,q) := \frac{i^{-|\Upsilon|}}{2^k} \sum_{\varepsilon=(\pm_1 1,\dots,\pm_k 1)\in\{\pm 1\}^k} \sum_{n_1,\dots,n_k\geq 1} \rho_\Upsilon(\varepsilon) \frac{\tau_{\alpha_1,\beta_1}(n_1)\cdots\tau_{\alpha_k,\beta_k}(n_k)}{(n_1\cdots n_k)^{\frac{1}{2}}} \times \mathrm{e}\Big(\frac{a(\pm_1 n_1 \pm_2 \cdots \pm_k n_k)}{q}\Big) V_{\boldsymbol{\alpha},\boldsymbol{\beta}}\Big(\frac{n_1\cdots n_k}{q^k}\Big),$$

with $\rho_\Upsilon(\varepsilon) := \prod_{i\in\Upsilon}(\pm_i 1)$.

Proof. By contour integration and the functional equation, we have

$$\begin{aligned}
&\prod_{i=1}^k \Lambda_{i;\alpha_i,\beta_i}\big(\tfrac12,\tfrac{a}{q}\big)\\
&\quad= \frac{1}{2\pi i}\Big(\int_{(2)} - \int_{(-2)}\Big) \prod_{i=1}^k \Lambda_{i;\alpha_i,\beta_i}\big(\tfrac12+s,\tfrac{a}{q}\big)\cdot G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)\,\frac{ds}{s}\\
&\quad= \frac{1}{2\pi i}\int_{(2)} \prod_{i=1}^k \Lambda_{i;\alpha_i,\beta_i}\big(\tfrac12+s,\tfrac{a}{q}\big)\cdot G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)\,\frac{ds}{s} + \frac{1}{2\pi i}\int_{(2)} \prod_{i=1}^k \Lambda_{i;-\alpha_i,-\beta_i}\big(\tfrac12+s,\tfrac{\bar a}{q}\big)\cdot G_{-\boldsymbol{\alpha},-\boldsymbol{\beta}}(s)\,\frac{ds}{s}.
\end{aligned}$$

Now, expanding the Estermann functions into their Dirichlet series, we see that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(2)} \prod_{i=1}^k \frac{\Lambda_{i;\alpha_i,\beta_i}\big(\tfrac12+s,\tfrac{a}{q}\big)}{\Gamma_i\big((\tfrac12+\alpha_i)/2\big)\Gamma_i\big((\tfrac12+\beta_i)/2\big)\big(\tfrac{q}{\pi}\big)^{\frac12+(\alpha_i+\beta_i)/2}}\cdot G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)\,\frac{ds}{s}\\
&\qquad= \frac{i^{-|\Upsilon|}}{2^k}\sum_{n_1,\dots,n_k\in\mathbb{Z}\setminus\{0\}} \mathrm{sgn}\Big(\prod_{i\in\Upsilon} n_i\Big)\frac{\tau_{\alpha_1,\beta_1}(|n_1|)\cdots\tau_{\alpha_k,\beta_k}(|n_k|)}{|n_1\cdots n_k|^{\frac12}}\,\mathrm{e}\Big(\frac{a(n_1+\cdots+n_k)}{q}\Big)\\
&\qquad\qquad\times\frac{1}{2\pi i}\int_{(2)} G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)\Big(\frac{|n_1\cdots n_k|}{q^k}\Big)^{-s}\frac{ds}{s}
\end{aligned}$$

and the lemma follows. □

3.3. *Estimates for the Estermann function.* In this section we give two bounds for certain averages of products the Estermann function and the periodic zeta-function. Both bounds depend on estimates for Kloosterman sums, more specifically on Weil's bound and on (a minor modification of) a bound by Deshouilliers and Iwaniec [1982]. We recall that the classical Kloosterman sum is defined as

$$S(m,n;\ell) := \sideset{}{^*}\sum_{c \pmod{\ell}} \mathrm{e}\Big(\frac{mc+n\bar{c}}{\ell}\Big)$$

for any $c,m,n\in\mathbb{Z}$, $c\geq 1$, where $\sum^*$ indicates that the sum is over $c \pmod{\ell}$ such that $(c,\ell)=1$. Also, we recall that Weil's bound gives $S(m,n;\ell) \ll d(\ell)(m,n,\ell)^{\frac12}\ell^{\frac12}$. Using this bound we obtain the following lemma.

Lemma 10. *Let $r>0$, $0<\delta<1$, $C\geq 2$, $\eta_0\neq 0$ and $(\eta_1,\dots,\eta_r)\in\{\pm 1\}^r$. Let $|a|\leq 2C\delta$, $|b|\leq C\delta$ and $|a_j|,|b_j|<\delta$ for $j=1,\dots,r$. Then, for some $A>0$ we have*

$$\sum_{\ell\geq 1}\frac{1}{\ell^{C+a}}\sideset{}{^*}\sum_{h \pmod{\ell}} F\big(1+C\big(s-\tfrac12\big)+b,\tfrac{\eta_0 h}{\ell}\big)\prod_{j=1}^{r}D_{a_j,b_j}\big(\tfrac12,\tfrac{\eta_j h}{\ell}\big)\ll_\delta (AC/\delta)^{A(r+C)}(1+|s|)^{A(r+C)} \tag{3-8}$$

in the strip

$$-\frac12+\frac{r+\frac32}{C+r-\frac12}+8\delta<\Re(s)<\frac12-2\delta,$$

where F is the periodic zeta-function defined in (3-3). Moreover, the left hand side of (3-8) is meromorphic in the half plane $\Re(s)>-\frac12+(r+\frac32)/(C+r-\frac12)+8\delta$ with poles at $s=\frac12-a_j$ and $s=\frac12-b_j$ for $j=1,\dots,r$ and $s=\frac12-b/C$ and these poles are simple if $a_1,\dots,a_r,b_1,\dots,b_r$ and b/C are all distinct.

Proof. For $L\geq 1$, let

$$H_L(s):=\sum_{L<\ell\leq 2L}\frac{1}{\ell^{C+a}}\sideset{}{^*}\sum_{h \pmod{\ell}} F\big(1+C\big(s-\tfrac12\big)+b,\tfrac{\eta_0 h}{\ell}\big)\prod_{j=1}^{r}D_{a_j,b_j}\big(\tfrac12+s,\tfrac{\eta_j h}{\ell}\big),$$

$$K(s):=\prod_{j=1}^{r}\big(s-\tfrac12+a_j\big)\big(s-\tfrac12+b_j\big).$$

Notice that if $\ell\neq 1$ and $(h,\ell)=1$ then $F(x,h/\ell)$ is entire and thus so is $H_L(s)K(s)$ for all $L\geq 1$. Now, if $\Re(s)=\frac12+2\delta$, then a trivial bound gives

$$H_L(s)K(s)\ll(1+|s|^{2r})(A/\delta)^{2r+1}L^{-C+2+2\delta C}, \tag{3-9}$$

where, here and in the following, A denotes a sufficiently large positive constant, which might change from line to line.

Next, take $\Re(s)=-\frac12-2\delta$. Then, applying the functional equations (3-1) and (3-4) to D and F, expanding D and F into their Dirichlet series, and using Stirling's formula in the crude form

$$\Gamma(\sigma+it)\ll c^{-1}(1+A|\sigma|)^{|\sigma|}(1+|t|)^{\sigma-\frac12}e^{-(\pi/2)|t|},\quad \sigma\geq c>0, \tag{3-10}$$

we see that

$$\begin{aligned}H_L(s)K(s)\ll A^rC^{AC}(1+|s|)^{A(r+C)}L^{r-1+(5C+6r)\delta}&\sum_{L<\ell\leq 2L}\sum_{u=1}^{\ell}\big|F\big(C\big(\tfrac12-s\big)-b,u/\ell\big)\big|\\ &\times\sum_{n_1,\dots,n_r\in\mathbb{Z}_{\neq 0}}\frac{|\tau_{-a_1,-b_1}(|n_1|)\cdots\tau_{-a_r,-b_r}(|n_r|)|}{|n_1^{1+2\delta}\cdots n_r^{1+2\delta}|}\big|S(\eta_0u,n_1+\cdots+n_k;\ell)\big|,\end{aligned}$$

and thus by Weil's bound we obtain

$$H_L(s)K(s)\ll(A/\delta)^{2r+5}C^{AC}(1+|s|)^{A(r+C)}L^{r+\frac32+6(r+C)\delta}, \tag{3-11}$$

when $\Re(s)=-\frac{1}{2}-2\delta$. Thus, by (3-9), (3-11) and the Phragmén–Lindelöf principle, if $-\frac{1}{2}-2\delta\le\Re(s)\le\frac{1}{2}+2\delta$ we have

$$H_L(s)K(s)\ll(A/\delta)^{2r+5}C^{AC}(1+|s|)^{A(r+C)}L^{r+\frac{3}{2}-(C+r-\frac{1}{2})(\Re(s)+\frac{1}{2})+5\delta(r+C)}.$$

Moreover, if $\left|s-\frac{1}{2}\right|>2\delta$ then $K(s)\gg\delta^{2r}$ and thus, if $-\frac{1}{2}-2\delta\le\Re(s)\le\frac{1}{2}-2\delta$, we have

$$H_L(s)\ll(A/\delta)^{4r+5}C^{AC}(1+|s|)^{A(r+C)}L^{r+\frac{3}{2}-(C+r-\frac{1}{2})(\Re(s)+\frac{1}{2})+5\delta(r+C)}.$$

It follows that if

$$-\frac{1}{2}+\frac{r+\frac{3}{2}+6\delta(r+C)}{C+r-\frac{1}{2}}\le\Re(s)\le\frac{1}{2}-2\delta \tag{3-12}$$

then

$$\sum_{\ell>1}\frac{1}{\ell^{C+a}}\sideset{}{^*}\sum_{h\ (\mathrm{mod}\ \ell)}F\big(1+C\big(s-\tfrac{1}{2}\big)+b,\tfrac{\eta_0 h}{\ell}\big)\prod_{j=1}^{r}D_{a_j,b_j}\big(\tfrac{1}{2}+s,\tfrac{\eta_j h}{\ell}\big)\ll_\delta(A/\delta)^{4r+6}C^{AC}(1+|s|)^{A(r+C)}.$$

Finally, the contribution of the $\ell=1$ term to the left hand side of (3-8) is

$$\zeta\big(1+C\big(s-\tfrac{1}{2}\big)+b\big)\prod_{j=1}^{r}\zeta\big(\tfrac{1}{2}+s+a_j\big)\zeta\big(\tfrac{1}{2}+s+b_j\big)\ll(A/\delta)^{2r+1}C^{AC}(1+|s|)^{A(r+C)}$$

when s satisfies (3-12) and thus (3-8) follows. We conclude by remarking that the above computations also give the meromorphicity of the left hand side of (3-8) on $\Re(s)\ge-\frac{1}{2}+\big(r+\frac{3}{2}+5\delta(r+C)\big)/\big(C+r-\frac{1}{2}\big)$. □

We now states a variation of a bound by Deshouilliers–Iwaniec for sums of Kloosterman sums (see Theorem 9 and (1.52) of [Deshouillers and Iwaniec 1982]), which is also essentially implicit in the more general bounds given in [Blomer et al. 2007; Harman et al. 2004] (see [Watt 2005, Theorem 1.4]).

Lemma 11. *Let W be a smooth function supported in $[1,2]$ and satisfying $W^{(i)}(x)\ll C^i$ for $i=0,1,2$ and some $C>1$. Let a_m, $b_n\ll1$ be sequences of complex numbers supported in $[M,2M]$ and $[N,2N]$ respectively. Then, for $q\ge1$ and $\eta\in\{\pm1\}$ we have*

$$\sum_{m,n,\ell\ge1}W(\ell/L)a_mb_nS(qm,\eta n;\ell)\ll_\varepsilon q^{\vartheta+\varepsilon}C^{\frac{9}{2}+\varepsilon}(L^{1+\varepsilon}+q^{\frac{1}{2}})MN, \tag{3-13}$$

where $\vartheta=\frac{7}{64}$.

Proof. First we observe that we can assume that a_m is supported on integers which are coprime with q. Indeed, if (3-13) holds in the coprime case, then since $\vartheta<\frac{1}{2}$ we have

$$\begin{aligned}\sum_{m,n,\ell\ge1}W(\ell/L)a_mb_nS(qm,\eta n;\ell)&=\sum_{d\,|\,q^\infty}\sum_{\substack{m,n,\ell\ge1\\(m,q)=1}}W(\ell/L)a_{dm}b_nS(qdm,\eta n;\ell)\\&\ll\sum_{d\,|\,q^\infty}\frac{q^{\vartheta+\varepsilon}}{d^{1-\vartheta-\varepsilon}}C^{\frac{9}{2}+\varepsilon}(L^{1+\varepsilon}+(dq)^{\frac{1}{2}})MN\\&\ll q^{\vartheta+\varepsilon}C^{\frac{9}{2}+\varepsilon}(L^{1+\varepsilon}+q^{\frac{1}{2}})MN,\end{aligned}$$

as claimed. To prove (3-13) in the coprime case, one proceeds as in the proof of Theorem 9 of [Deshouillers and Iwaniec 1982] applying Kuznetsov's formula. Then one uses the multiplicativity of Hecke-eigenvalues to separate q and m and applies the Kim–Sarnak bound [Kim 2003] for Hecke eigenvalues to deal with the contribution of the q-coefficient. The rest of the proof carries on as in [Deshouillers and Iwaniec 1982] essentially unchanged other than for the parameter X which is now multiplied by $q^{\frac{1}{2}}$. We remark that the multiplicativity of Hecke eigenvalues holds since we are in the case of level 1 for which there are only new-forms.

The above argument was carried out in detail in [Blomer et al. 2007, Theorem 4], where the authors deal with the more general case of arbitrary level which introduces several difficulties especially when dealing with the contribution of the Eisenstein spectrum. In some ranges [Blomer et al. 2007, Theorem 4] gives a weaker bound than (3-13), but one can easily modify their proof to obtain (3-13). Indeed, for $D=1$ the bound on the last display of [Blomer et al. 2007, p. 75] can be modified to give (in the same notation as in [Blomer et al. 2007])

$$\ll_{\varepsilon,p_1,p_2} ((1+X)Zq)^{4\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MNq^{2\vartheta}\,\frac{Z^{\frac{3}{2}}+ZX+X^2+M/q}{1+X/Z}\|a_2\|_2^2. \tag{3-14}$$

If $Z^{1+\varepsilon}\geq X$ then this is obvious since this bound is weaker than the bound in [Blomer et al. 2007], aside from the fact that we removed the factors $(1+C/\sqrt{MN})^{2\vartheta}$ and $Z^{2\vartheta}$ since Selberg's eigenvalue conjecture holds when the level is $D=1$. If $X>Z^{1+\varepsilon}$, then $Z^{1+\varepsilon}<T_0=16X$ and so only the summands with $|t_j|\leq 1$ and $1\leq |t_j|\leq T_0=16X$ give a nonnegligible contribution. The terms with $|t_j|\leq 1$ then are bounded as in the first display of [Blomer et al. 2007, p. 75] without ignoring the extra saving $(1+X/Z)^{-1}$ as done there, whereas for the terms with $1\leq |t_j|\leq T_0$ we use the bound in the first line of the second display of [Blomer et al. 2007, p. 75] using $T_0=16X$.

In the case $D=1$ the contribution of both the holomorphic and the continuous spectrum can be treated in the same way without extra difficulties, obtaining that also their contribution is bounded by (3-14). Using these bounds we then obtain that the left hand side of (3-13) is

$$\begin{aligned}&\ll_\varepsilon q^{\vartheta+\varepsilon}C^{2+\varepsilon}\frac{(C^{\frac{3}{2}}+C\sqrt{qMN}/L+qMN/L^2+M)^{\frac{1}{2}}(C^{\frac{3}{2}}+C\sqrt{qMN}/L+qMN/L^2+N)^{\frac{1}{2}}}{1+\sqrt{qMN}/(LC)}\\ &\hspace{28em}\times L^{1+\varepsilon}\sqrt{MN}\\ &\ll q^{\vartheta+\varepsilon}C^{\frac{9}{2}+\varepsilon}(L^{1+\varepsilon}+q^{\frac{1}{2}})MN. \qquad\square\end{aligned}$$

Remark. Using the variation of the spectral large sieve given by Blomer and Milićević [2015, Theorem 8], one obtains a bound which improves upon (3-13) when the parameters are in certain ranges. It is likely that the use of such a bound in combination with (3-13) would lead to a better bound for the error term in Theorem 5. However for simplicity we choose to use (3-13) in all ranges, since this is sufficient for our purposes.

Using Lemma 11 we obtain the following result.

Lemma 12. *For $r \geq 1$, let $t_0, \dots, t_r \in \mathbb{R}$, $(\eta_1, \dots, \eta_r) \in \{\pm 1\}^r$, and let $\eta_0 \neq 0$. Furthermore, let $|a_j|, |b_j| < \delta$ for $j = 1, \dots, r$ and some $0 < \delta < 1$. Finally, let $L > 0$ and let $W(x)$ be a smooth function supported on $[1, 2]$ with $W^{(i)}(x) \ll 1$ for $i = 0, 1, 2$. Then, if $w \in \mathbb{C}$ and $\sigma \geq 2\delta$, we have that*

$$\mathfrak{S} := \sum_{\ell \geq 1} \frac{W(\ell/L)}{\ell^{1+w}} \sideset{}{^*}\sum_{h \pmod{\ell}} F\big(1+\sigma+it_0, \tfrac{\eta_0 h}{\ell}\big) \prod_{j=1}^{r} D_{a_j,b_j}\big(-\sigma+it_j, \tfrac{\eta_j h}{\ell}\big)$$

is bounded by

$$\mathfrak{S} \ll_\delta L^{r(3\sigma+1)-\Re(w)} \frac{A^{r(\sigma+1)}}{\delta^{2r}} K_r(\sigma, w, t_1, \dots, t_j) \times \begin{cases} |\eta_0|^{\vartheta+\delta} & \text{if } L \geq |\eta_0|^{\frac{1}{2}}, \\ |\eta_0|^{\frac{1}{6}+\frac{\vartheta}{3}+\delta} & \text{always}, \end{cases} \tag{3-15}$$

for some absolute $A > 0$ and where

$$K_r(s, w, t_1, \dots, t_j) := (1+\sigma)^{2r(2\sigma+1)}(1+|w|)^4 \prod_{j=1}^{r} (1+|s|+|t_j|)^{1+4\sigma}$$

Proof. Applying the functional equation (3-1), expanding D and F into their Dirichlet series, and using (3-10) we obtain

$$\mathfrak{S} \ll_\delta A^r(1+A\sigma)^{2r(\sigma+\delta+1)} \bigg(\prod_{j=1}^{r} (1+|t_j|)^{2(\sigma+\delta)+1}\bigg) \bigg|\sum_{\ell \geq 1} \sum_{m \geq 1} \sum_{n \in \mathbb{Z}} \frac{W_0(\ell) f_n}{m^{1+\sigma+it_0}} S(\eta_0 m, n; \ell)\bigg|,$$

where

$$W_0(x) := W(x/L) x^{r(2\sigma+1)-1-w-\sum_{j=1}^{r}(t_j+a_j+b_j)} \ll (2L)^{r(3\sigma+1)-1-\Re(w)},$$

$$f_n := \sum_{\substack{n_1,\dots,n_r \in \mathbb{Z}_{\neq 0} \\ n_1+\dots+n_r = n}} \frac{\tau_{-a_1,-b_1}(|n_1|)}{n_1^{1+s-it_1}} \cdots \frac{\tau_{-a_r,-b_r}(|n_r|)}{n_r^{1+s-it_r}} \ll_\delta (A/\delta)^{2r} \frac{1}{|n|^{1+\delta/2}}.$$

Splitting the sums over n and m into dyadic blocks and applying (3-13) one easily gets the bound

$$\mathfrak{S} \ll_\delta L^{r(3\sigma+1)-\Re(w)} \frac{A^{r(\sigma+1)}}{\delta^{2r}} K_r(\sigma, w, t_1, \dots, t_j) |\eta_0|^{\vartheta+\delta} (1+|\eta_0|^{\frac{1}{2}}/L), \tag{3-16}$$

which gives (3-15) in the case $L \geq |\eta_0|^{\frac{1}{2}}$. Applying Weil's bound rather than (3-13), one obtains

$$\mathfrak{S} \ll_\delta L^{r(3\sigma+1)-\Re(w)} \frac{A^{r(\sigma+1)}}{\delta^{2r}} K_r(\sigma, w, t_1, \dots, t_j) L^{\frac{1}{2}}, \tag{3-17}$$

and taking the minimum between (3-16) and (3-17) one gets (3-15) also in the case $L < |\eta_0|^{\frac{1}{2}}$. □

4. Some assumptions

In this section we set up some notation and make some simplifying assumptions, which we will use throughout the rest of the paper.

First, q will always denote a prime, k an integer greater than 2, and Υ a subset of $\{1, \dots, k\}$ of even cardinality. Moreover we shall use the convention that A and ε denote respectively a sufficiently large and an arbitrarily small positive constant on which the implicit bounds are allowed to depend and whose value might change from line to line.

Also, we assume $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k) \in \mathbb{A}^k_{2C}$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k) \in \mathbb{A}^k_{C/2}$ for some constant $C > 0$ $\big($with $4C/\log q \le \frac{1}{10}\big)$, where $\mathbb{A}_r$ denotes the annulus $\{s \in \mathbb{C} \mid r/\log q \le |s| \le 2r/\log q\}$. This assumption can then be removed in the proof of Theorem 5 by analytic continuation and the maximum modulus principle, since both the left hand side and, by (2-5), the main term on the right hand side of (2-2) are analytic functions of the shifts in $|\alpha_i|, |\beta_i| \le 4C/\log q$. We remark in particular, that with the above assumption, we have $|\alpha_i|, |\beta_i|, |\alpha_i - \beta_i| \asymp 1/\log q$.

Moreover, for the rest of the paper we fix an entire function $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ as follow:

$$G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) := \frac{Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)}{Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(0)} \frac{\xi\big(\frac{1}{2}+s\big)}{\xi\big(\frac{1}{2}\big)}, \tag{4-1}$$

where $\xi(s) := \frac{1}{2}s(s-1)\pi^{-\pi/2}\Gamma\big(\frac{1}{2}s\big)\zeta(s)$ is the Riemann ξ-function and

$$Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) := \prod_{i=1}^{k}\Big(\big(s^2 - (\alpha_i-\beta_i)^2\big)\big(\tfrac{1}{4} - (s+\alpha_i)^2\big)\big(\tfrac{1}{4} - (s+\beta_i)^2\big)\Big).$$

By the functional equation for the Riemann zeta-function we have $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(-s) = G_{-\boldsymbol{\alpha},-\boldsymbol{\beta}}(s)$ and so $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ satisfies the hypotheses of Lemma 9. Moreover, using Stirling's formula (3-10) we also obtain

$$G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) \ll (A\log q)^{2k} e^{-C_1|t|}(1+|\sigma|)^{A(|\sigma|+k)}, \tag{4-2}$$

for all $s = \sigma + it \in \mathbb{C}$ and some $C_1 > 0$.

Finally, we notice that from the functional equations (3-2), for $i = 1, \dots, k$ we have the convexity bound

$$D_{i,\alpha_i,\beta_i}\big(\tfrac{1}{2}, \tfrac{a}{q}\big) \ll q^{\frac{1}{2}}(\log q)^2$$

and so trivially $M_{\Upsilon,k} \ll (Aq^{\frac{1}{2}}(\log q)^2)^k$. Also, from (2-5) it is easy to see that one also has

$$\sum_{\{\alpha_i',\beta_i'\}=\{\alpha_i,\beta_i\}} \mathscr{M}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} \ll q^{k/2-1}(A\log q)^k.$$

It follows that Theorem 5 is trivial if $k \gg \log q/\log\log q$ since in this case $(Ak)^{Ak} \gg q^{A/2}$. Thus, we will assume $k = o(\log q/\log\log q)$. In particular, for q large enough we have $|\alpha_i|, |\beta_i| \le 4C/\log q < 1/(k\log\log q) < \frac{\varepsilon}{2k}$ and a fortiori

$$|\alpha_1| + \cdots + |\alpha_k| + |\beta_1| + \cdots + |\beta_k| < \varepsilon.$$

Moreover, notice that under these assumptions we also have the inequality $(k/\varepsilon)^{Ak} \ll (\log q)^{Ak} \ll q^{\varepsilon}$, which we shall often use.

5. Dividing into diagonal and off-diagonal terms and structure of the proof

By the approximate functional Equation (3-7) and the orthogonality of additive characters, we can decompose $M_{\Upsilon,k}$ into diagonal and off-diagonal terms:

$$M_{\Upsilon,k} := \frac{1}{\varphi(q)} \sum_{a=1}^{q-1} \prod_{i=1}^{k} D_{i;\alpha_i,\beta_i}\left(\tfrac{1}{2}, \tfrac{a}{q}\right) = \mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}} + X_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathcal{D}_{-\boldsymbol{\alpha},-\boldsymbol{\beta}} + \mathcal{O}_{\boldsymbol{\alpha},\boldsymbol{\beta}} + X_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathcal{O}_{-\boldsymbol{\alpha},-\boldsymbol{\beta}},$$

where

$$\mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \frac{i^{|\Upsilon|}}{2^k} \sum_{\varepsilon \in \{\pm 1\}^k} \rho_\Upsilon(\varepsilon) \sum_{\pm_1 n_1 \pm_2 \cdots \pm_k n_k = 0} \frac{\tau_{\alpha_1,\beta_1}(n_1) \cdots \tau_{\alpha_k,\beta_k}(n_k)}{(n_1 \cdots n_k)^{\frac{1}{2}}} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\frac{n_1 \cdots n_k}{q^k}\right),$$

$$\mathcal{O}_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \frac{i^{|\Upsilon|}}{2^k} \sum_{\varepsilon \in \{\pm 1\}^k} \rho_\Upsilon(\varepsilon) \mathcal{O}'_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}},$$

$$\mathcal{O}'_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}} := \sum_{d \mid q} d \frac{\mu(q/d)}{\varphi(q)} \sum_{\substack{d \mid (\pm_1 n_1 \pm_2 \cdots \pm_k n_k) \\ \pm_1 n_1 \pm_2 \cdots \pm_k n_k \neq 0}} \frac{\tau_{\alpha_1,\beta_1}(n_1) \cdots \tau_{\alpha_k,\beta_k}(n_k)}{(n_1 \cdots n_k)^{\frac{1}{2}}} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\frac{n_1 \cdots n_k}{q^k}\right),$$

and the sum over ε is a sum over $\varepsilon = \{\pm_1 1, \dots, \pm_k 1\} \in \{1, -1\}^k$.

The diagonal term $\mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ will be treated in Section 6, using the results of [Bettin 2017]. The terms with $d = 1$ in $\mathcal{O}'_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}}$ could be easily dealt with in a simple way, however it is more convenient to keep them together with the other off-diagonal terms.

Lemma 13. *We have*

$$\mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathscr{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}} + O(q^{k/2-2k/(k+1)+\varepsilon}), \tag{5-1}$$

where $\mathscr{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is as defined in (6-1).

For the off-diagonal terms we introduce partitions of unity. We need a function $P : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, satisfying

$$\sum_{N}^{\dagger} P(x/N) = 1, \quad \forall x > 0,$$

where by $\sum^\dagger$ we mean that the index runs through the elements of a certain (fixed) set of positive real numbers such that $\sum^\dagger_{X^{-1} \leq N \leq X} 1 \ll \log X$. Also, we require that $P(x)$ is supported on $1 \leq x \leq 2$ and $P^{(j)}(x) \ll j^{Aj}$ for some $A > 0$. It is not difficult to construct such a partition.[1] Notice that under these conditions, the Mellin transform of $P(x)$,

$$\widetilde{P}(s) := \int_0^\infty P(x) x^{s-1}\, dx,$$

is entire and satisfies

$$\widetilde{P}(\sigma + it) \ll (1 + j + |\sigma|)^{Aj} A^{|\sigma|} (1 + |t|)^{-j}, \quad \forall j \geq 0. \tag{5-2}$$

[1] For example take the set of indexes in $\sum^\dagger$ to be $\left\{\left(\frac{3}{2}\right)^n \mid n \in \mathbb{Z}\right\}$ and $P(x) = \int_1^{\frac{3}{2}} \eta(xy) \frac{dy}{y}$, where $\eta(x) = Ce^{-1/(1-(4x-7)^2)}$ for $\left|x - \frac{7}{4}\right| < \frac{1}{4}$ and $\eta(x) = 0$ otherwise, and where C is such that $\int_{\mathbb{R}} \eta(y) \frac{dy}{y} = 1$.

Using partitions of unity we can decompose $\mathcal{O}'_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ into

$$\mathcal{O}'_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}} := \sum_{N_1,\dots,N_k}^{\dagger} \mathcal{O}''_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}}(N_1,\dots,N_k),$$

where $\mathcal{O}''_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}}(N_1,\dots,N_k)$ is defined as $\mathcal{O}'_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}}$, with the only difference that the summands are multiplied by $P(n_1/N_1)\cdots P(n_k/N_k)$. In the following we will often omit to indicate the dependencies from $N_1,\dots N_k$ for ease of notation.

The following two Lemmas summarize our results on the off-diagonal terms. The first Lemma, which is effective when $N_1,\dots,N_k$ are close together, uses the spectral theory of automorphic forms (via the bounds proven in Section 3.3) and is proven in Section 7. The second lemma, which is effective when one of the N_i is considerably larger than the others, uses the bounds for sums of Kloosterman sums proven by Young [2011a] and is proven in Section 9.

Lemma 14. *Let $N_{\max}$ be the maximum among $N_1,\dots,N_k$. Then*

$$\sum_{\varepsilon\in\{\pm1\}^k} \rho_\Upsilon(\varepsilon)\mathcal{O}''_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}}(N_1,\dots,N_k) = \mathscr{M}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(N_1,\dots,N_k) + \mathcal{E}_{1;\boldsymbol{\alpha},\boldsymbol{\beta}}(N_1,\dots,N_k), \tag{5-3}$$

where

$$\mathcal{E}_{1;\boldsymbol{\alpha},\boldsymbol{\beta}} \ll \frac{N_{\max}^{\varepsilon}}{q^{1-\varepsilon}}\left(\frac{q^{\vartheta}N_{\max}^{k/2+\frac12}}{(N_1\cdots N_k)^{\frac12}} + \frac{q^{k/2-\frac13+\vartheta/3}N_{\max}^{\frac12}}{(N_1\cdots N_k)^{\frac12}} + \frac{q^{\frac16+\vartheta/3}(N_1\cdots N_k)^{\frac12}}{N_{\max}} + \frac{(N_1\cdots N_k)^{\frac12}}{N_{\max}^{\frac12}}\right) \tag{5-4}$$

and $\mathscr{M}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(N_1,\dots,N_k)$ is defined in (7-38). Moreover,

$$\mathscr{M}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(N_1,\dots,N_k) \ll q^{\varepsilon}(N_1\cdots N_k)^{\frac12}N_{\max}^{-1+\varepsilon}. \tag{5-5}$$

Lemma 15. *Let $N_{\max}$ be the maximum among $N_1,\dots,N_k$. Then*

$$\mathcal{O}''_{\varepsilon,\alpha,\beta}(N_1,\dots,N_k) \ll q^{\varepsilon}\left(\left(\frac{N_1\cdots N_k}{N_{\max}}\right)^{\frac12-1/(2(k-1))}\left(\frac{q^{\frac34}}{N_{\max}^{\frac12}}+1\right) + \frac{(N_1\cdots N_k)^{\frac12}}{N_{\max}^{3/4}}\right).$$

Notice that in the crucial case $N_1\cdots N_k \asymp q^k$ Lemma 14 is nontrivial for $N_{\max} \ll q^{2-2(\vartheta+1)/(k+1)-\delta}$ for any fixed $\delta>0$, whereas Lemma 15 is nontrivial as long as $N_{\max} \gg_k q^{\frac43+\delta}$. In particular, in order to have a nontrivial bound for all ranges we need $\vartheta < \frac{k-2}{3}$ and so for $k=3$ we need $\vartheta < \frac13$.

The following lemma, which we shall prove in Section 8, allows us to combine the various main terms.

Lemma 16. *We have*

$$\frac{i^{|\Upsilon|}}{2^k} \sum_{\substack{N_1,\dots,N_k\\ N_1\cdots N_k \ll q^{k+\varepsilon}}}^{\dagger} \left(\mathscr{M}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(N_1,\dots,N_k) + X_{\boldsymbol{\beta},\boldsymbol{\alpha}}\mathscr{M}_{-\boldsymbol{\beta},-\boldsymbol{\alpha}}(N_1,\dots,N_k)\right)$$
$$= -(\mathscr{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}} + X_{\boldsymbol{\beta},\boldsymbol{\alpha}}\mathscr{D}_{-\boldsymbol{\beta},-\boldsymbol{\alpha}}) + \sum_{\{\alpha'_i,\beta'_i\}=\{\alpha_i,\beta_i\}} \mathscr{M}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} + O(q^{k/2-\frac32+\iota_k+\varepsilon}),$$

where $\mathscr{M}_{\alpha,\beta}$ is as defined in (2-3) and $\iota_k = \frac{3}{14}$ if $k=4$ and $\iota_k=0$ otherwise.

We conclude the section with the deduction of Theorem 5 from the above lemmas.

Proof of Theorem 5. Lemma 14 gives us the asymptotic for $\sum_\varepsilon \rho_\Upsilon(\varepsilon)\mathcal{O}''_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}}$ in the range where the variables are close together. If one variable is much larger than the others then (5-5) and Lemma 15 give us that both $\mathcal{O}''_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}}$ and the main term are small and so we obtain a second formula for $\sum_\varepsilon \rho_\Upsilon(\varepsilon)\mathcal{O}''_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}}$,

$$\sum_{\varepsilon\in\{\pm1\}^k} \rho_\Upsilon(\varepsilon)\mathcal{O}''_{\varepsilon,\alpha,\beta}(N_1,\dots,N_k) = \mathscr{M}_{\alpha,\beta}(N_1,\dots,N_k)+\mathcal{E}_{2;\alpha,\beta}(N_1,\dots,N_k), \tag{5-6}$$

where

$$\mathcal{E}_{2;\alpha,\beta} \ll q^\varepsilon\biggl(\biggl(\frac{N_1\cdots N_k}{N_{\max}}\biggr)^{\frac12-1/(2(k-1))}\biggl(\frac{q^{\frac34}}{N_{\max}^{\frac12}}+1\biggr)+\frac{(N_1\cdots N_k)^{\frac12}}{N_{\max}^{\frac34}}\biggr).$$

Finally, in the range where $N_1\cdots N_k$ is much smaller than q^k one can improve upon (5-3) and (5-6) by simply bounding trivially $O''_{\varepsilon,\alpha,\beta}$ and $\mathscr{M}_{\alpha,\beta}$ by $q^{-1+\varepsilon}(N_1\cdots N_k)^{\frac12}$. We then record here the following third formula for $\sum_\varepsilon \rho_\Upsilon(\varepsilon)\mathcal{O}''_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}}$:

$$\sum_{\varepsilon\in\{\pm1\}^k} \rho_\Upsilon(\varepsilon)\mathcal{O}''_{\varepsilon,\alpha,\beta}(N_1,\dots,N_k) = \mathscr{M}_{\alpha,\beta}(N_1,\dots,N_k)+\mathcal{E}_{3;\alpha,\beta}(N_1,\dots,N_k), \tag{5-7}$$

with $\mathcal{E}_{3;\alpha,\beta}(N_1,\dots,N_k)\ll q^{-1+\varepsilon}(N_1\cdots N_k)^{\frac12}$.

Combining (5-3), (5-6) and (5-7), and adding the condition $N_1\cdots N_k\ll q^{k+\varepsilon}$ at a negligible cost, Lemma 16 gives

$$\begin{aligned}
&\mathcal{O}_{\boldsymbol{\alpha},\boldsymbol{\beta}}+X_{\boldsymbol{\beta},\boldsymbol{\alpha}}\mathcal{O}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\\
&\quad=\frac{i^{|\Upsilon|}}{2^k}\sum_{\varepsilon\in\{\pm1\}^k}\rho_\Upsilon(\varepsilon)\sideset{}{^\dagger}\sum_{\substack{N_1,\dots,N_k\\ N_1\cdots N_k\ll q^{k+\varepsilon}}}\bigl(\mathcal{O}'_{\alpha,\beta}(N_1,\dots,N_k)+X_{\boldsymbol{\beta},\boldsymbol{\alpha}}\mathcal{O}'_{-\boldsymbol{\beta},-\boldsymbol{\alpha}}(N_1,\dots,N_k)\bigr)+O(1)\\
&\quad=-(\mathscr{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}}+X_{\boldsymbol{\beta},\boldsymbol{\alpha}}\mathscr{D}_{-\boldsymbol{\beta},-\boldsymbol{\alpha}})+\sum_{\{\alpha_i',\beta_i'\}=\{\alpha_i,\beta_i\}}\mathscr{M}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}+\mathscr{E}_{\boldsymbol{\alpha},\boldsymbol{\beta}}+O(q^{k/2-\frac32+\iota_k+\varepsilon}),
\end{aligned}$$

where

$$\mathscr{E}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\ll\max_{\substack{N_1,\dots,N_k\\ N_1\cdots N_k\ll q^{k+\varepsilon}}}\bigl(\min(\mathscr{E}_1,\mathscr{E}_2,\mathscr{E}_3)\bigr).$$

Thus, since the term $-(\mathscr{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}}+X_{\boldsymbol{\beta},\boldsymbol{\alpha}}\mathscr{D}_{-\boldsymbol{\beta},-\boldsymbol{\alpha}})$ cancels out with the main term of the diagonal term given by (5-1), to conclude the proof of Theorem 5 we just need to show that $\mathscr{E}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\ll q^{k/2-1-\delta_k+\varepsilon}$. Writing $N_{\max}=q^a$ and $N_1\cdots N_k=q^b$ (and considering only the contribution from the first summand in (5-4), since it easy to see the other terms produce a contribution which is $O(q^{k/2-\frac32+\varepsilon})$), we have that it is sufficient to show that

$$\max_{i=1,2,3}\max_{\substack{a\le b\le k\\ ka\ge b}}\min\Bigl(\frac{k+1}{2}a-\frac b2-1+\vartheta,\,L_i(a,b),\,\frac b2-1\Bigr)=\frac k2-\frac32+\frac{3(3+2\vartheta)}{2(2k+5)}=\frac k2-1-\delta_k,$$

for $k\ge3$, where

$$L_1(a,b):=\frac34-\frac a2+(b-a)\Bigl(\frac12-\frac{1}{2(k-1)}\Bigr),\quad L_2(a,b):=(b-a)\Bigl(\frac12-\frac{1}{2(k-1)}\Bigr),\quad L_3(a,b):=\frac b2-\frac34a.$$

If the maximum is attained at the interior of $\{a \le b \le k,\, ka \ge b\}$, then it must occur when $\frac{k+1}{2}a - \frac{b}{2} - 1 = \frac{b}{2} - 1 = L_i(a,b)$ for $i = 1, 2$, or 3 and so it would be $\frac{7k}{20} - \frac{13}{20}$, $\frac{k}{3} - \frac{2}{3}$ and $\frac{k}{3} - \frac{2}{3}$ respectively. Along the lines $a = b$, $ka = b$ and $b = k$ we have

$$
\begin{aligned}
&\max_{i=1,2,3}\max_{0\le a\le k}\min\Big(\frac{k}{2}a - 1 + \vartheta,\, L_i(a,a),\, \frac{a}{2} - 1\Big) = \max_{0\le a\le k}\min\Big(L_i(a,a),\, \frac{a}{2} - 1\Big) = 0,\\
&\max_{i=1,2,3}\max_{0\le a\le 1}\min\Big(\frac{a}{2} - 1 + \vartheta,\, L_i(a,ka),\, \frac{ka}{2} - 1\Big) \le -\frac{1}{2} + \vartheta \le 0,\\
&\max_{i=1,2,3}\max_{1\le a\le k}\min\Big(\frac{k+1}{2}a - \frac{k}{2} - 1 + \vartheta,\, L_i(a,k),\, \frac{k}{2} - 1\Big)\\
&\qquad = \max\Big(\frac{k}{2} - \frac{7}{4} + \frac{8k(2+\vartheta) - 19 - 12\vartheta}{4(k^2+2k-4)},\, \frac{k}{2} - \frac{3}{2} + \frac{2(k+\vartheta) - 5 - 4\vartheta}{2(k^2+k-3)},\, \frac{k}{2} - \frac{3}{2} + \frac{3(3+2\vartheta)}{2(2k+5)}\Big)\\
&\qquad = \frac{k}{2} - \frac{3}{2} + \frac{3(3+2\vartheta)}{2(2k+5)},
\end{aligned}
$$

for $k \ge 3$ and $\vartheta \le \frac{1}{3}$. Theorem 5 then follows. □

6. The diagonal terms

In this section we prove Lemma 13 deducing it from the following Lemma in [Bettin 2017]. We recall that in Section 4 we assumed $|\alpha_i|, |\beta_i| < \frac{\varepsilon}{2k}$ for all $i = 1, \dots, k$.

Lemma 17. *For $\Re(s) > 1 - \frac{1}{k} - \frac{1}{k}\sum_{i=1}^{k}\min(\Re(\alpha_i), \Re(\beta_i))$, let*

$$
\mathcal{W}_{\alpha,\beta}(s) := \frac{i^{|\Upsilon|}}{2^k}\sum_{\varepsilon\in\{\pm1\}^k}\rho_\Upsilon(\varepsilon)\sum_{\pm_1 n_1 \pm_2 \cdots \pm_k n_k = 0}\frac{\tau_{\alpha_1,\beta_1}(n_1)\cdots\tau_{\alpha_k,\beta_k}(n_k)}{(n_1\cdots n_k)^s}.
$$

Also, let

$$
\begin{aligned}
\mathcal{W}^{\dagger}_{\alpha,\beta}(s) := \sum_{(\mathcal{I},\boldsymbol{\alpha}',\boldsymbol{\beta}')\in S_{\boldsymbol{\alpha},\boldsymbol{\beta}}}\frac{2^{|\mathcal{J}|+1}\pi^{|\mathcal{I}|/2-1}}{|\mathcal{I}|(s-1) + s_{\mathcal{I};\boldsymbol{\alpha}'} + 1}\Bigg(\prod_{i\in\mathcal{I}}\zeta(1 - \alpha_i' + \beta_i')\frac{\Gamma_i\big(-\alpha_i'/2 + (1 + s_{\mathcal{I};\boldsymbol{\alpha}'})/(2|\mathcal{I}|)\big)}{\Gamma_i\big(\frac{1}{2} + \alpha_i'/2 - (1 + s_{\mathcal{I};\boldsymbol{\alpha}'})/(2|\mathcal{I}|)\big)}\Bigg)\\
\times\sum_{\ell\ge1}\sideset{}{^*}\sum_{h \ (\mathrm{mod}\ \ell)}\frac{1}{\ell^{|\mathcal{I}'| - \sum_{i\in\mathcal{I}}(\alpha_i' - \beta_i')}}\prod_{i\notin\mathcal{I}}D_i\big(1 + \alpha_i' - (1 + s_{\mathcal{I};\boldsymbol{\alpha}'})/|\mathcal{I}|,\, \alpha_i' - \beta_i',\, h/\ell\big).
\end{aligned}
$$

where $s_{\mathcal{I};\boldsymbol{\alpha}'} := \sum_{i\in\mathcal{I}}\alpha_i'$ and

$$
S_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \left\{(\mathcal{I},\boldsymbol{\alpha}',\boldsymbol{\beta}')\ \middle|\ \begin{array}{l}\mathcal{I}\subseteq\{1,\dots k\},\ |\mathcal{I}| > |\mathcal{J}| + 1,\ |\mathcal{I}\cap\Upsilon|\ \text{even},\\ \{\alpha_i', \beta_i'\} = \{\alpha_i, \beta_i\}\ \forall i\in\mathcal{I},\ (\alpha_i', \beta_i') = (\alpha_i, \beta_i)\ \forall i\notin\mathcal{I}\end{array}\right\}.
$$

Then for any $\varepsilon > 0$, $\mathcal{W}_{\alpha,\beta}(s) - \mathcal{W}^{\dagger}_{\alpha,\beta}(s)$ extends to a holomorphic function on $\Re(s) \ge 1 - \frac{2-4\varepsilon}{k+1}$ and in such a half plane it satisfies $\mathcal{W}_{\alpha,\beta}(s) - \mathcal{W}^{\dagger}_{\alpha,\beta}(s) \ll \big(\frac{k}{\varepsilon}(1 + |s|)\big)^{Ak}$.

Proof. Theorem 3 of [Bettin 2017] gives the meromorphic continuation and the bound for each ε. Thus, one obtains the lemma by summing over ε (for the simplification of the polar term one proceeds as in Lemma 23; see also Remark 2 of [Bettin 2017]). □

Proof of Lemma 13. Writing $V_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ in terms of it's Mellin transform we have

$$\mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \frac{1}{2\pi i}\int_{(2)} G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) \mathcal{W}_{\alpha,\beta}\big(\tfrac12+s\big) q^{ks}\,\frac{ds}{s}.$$

We write $\mathcal{W}_{\alpha,\beta}\big(\frac12+s\big)$ as $\mathcal{W}^{\dagger}_{\alpha,\beta}\big(\frac12+s\big)+\big(\mathcal{W}_{\alpha,\beta}\big(\frac12+s\big)-\mathcal{W}^{\dagger}_{\alpha,\beta}\big(\frac12+s\big)\big)$. For the second term we move the line of integration to $\Re(s)=\frac12-(2-4\varepsilon)/(k+1)$ and bound trivially using (4-2) obtaining an error of size $O(k^{Ak}q^{k/2-2k/(k+1)+\varepsilon}) = O(q^{k/2-2k/(k+1)+\varepsilon})$. For the first term we move the line of integration to $\Re(s)=-\frac12$ picking up the residues from the poles. We obtain (5-1) with

$$\mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \sum_{\substack{\mathcal{I}\cup\mathcal{J}=\{1,\dots,k\},\ \mathcal{I}\cap\mathcal{J}=\varnothing\\ |\mathcal{I}|>|\mathcal{J}|+1,\ |\mathcal{I}\cap\Upsilon|\text{even}}}\ \sum_{\substack{\{\alpha'_i,\beta'_i\}=\{\alpha_i,\beta_i\}\ \forall i\in\mathcal{I}\\ (\alpha'_j,\beta'_j)=(\alpha_j,\beta_j)\ \forall j\in\mathcal{J}}} \mathscr{D}_{\mathcal{I};\boldsymbol{\alpha}',\boldsymbol{\beta}'} \tag{6-1}$$

where

$$\mathscr{D}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}} := 2^k\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s_{\mathcal{I};\boldsymbol{\alpha}})}{s_{\mathcal{I};\boldsymbol{\alpha}}\pi|\mathcal{I}|} g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s_{\mathcal{I};\boldsymbol{\alpha}}) q^{ks_{\mathcal{I};\boldsymbol{\alpha}}}\Bigg(\prod_{i\in\mathcal{I}}\frac{\pi^{\frac12}\zeta(1-\alpha_i+\beta_i)}{2^{\frac12+\alpha_i+s_{\mathcal{I},\boldsymbol{\alpha}}}}\frac{\Gamma_i\big(\frac14-(\alpha_i+s_{\mathcal{I},\boldsymbol{\alpha}})/2\big)}{\Gamma_i\big(\frac14+(\alpha_i+s_{\mathcal{I},\boldsymbol{\alpha}})/2\big)}\Bigg)$$
$$\times\sum_{\ell}\ \sideset{}{^*}\sum_{h \pmod{\ell}}\frac{1}{\ell^{|\mathcal{I}|-\sum_{i\in\mathcal{I}}(\alpha_i-\beta_i)}}\Bigg(\prod_{j\in\mathcal{J}} D_j\big(\tfrac12+\alpha_j+s_{\mathcal{I},\boldsymbol{\alpha}},\ \alpha_j-\beta_j,\ \pm_j\tfrac{h}{\ell}\big)\Bigg) \tag{6-2}$$

and $s_{\mathcal{I};\boldsymbol{\alpha}} := \sum_{i\in\mathcal{I}}\alpha_i$. □

7. The terms close to the diagonal

In this section we prove Lemma 14. First, we assume that N_1 is the maximum of $N_1,\dots,N_k$, as we can do since both the main term and the error terms in Lemma 14 are symmetric in the indexes. Moreover, since we assumed that $|\Upsilon|$ is even, then we have

$$\sum_{\varepsilon\in\{\pm1\}^k}\rho_\Upsilon(\varepsilon)\mathcal{O}''_{\varepsilon,\alpha,\beta} = 2\sum_{\substack{\varepsilon\in\{\pm1\}^k\\ \pm_1 1=-1}}\rho_\Upsilon(\varepsilon)\mathcal{O}''_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}},$$

where here and in the following $\varepsilon=(\pm_1 1,\pm_2 1,\dots,\pm_k 1)$. We split $\mathcal{O}''_{\varepsilon,\alpha,\beta}$ further, depending on the sign and the size of $\pm_* f := -n_1\pm_2 n_2\pm_3\cdots\pm_k n_k$ (with $f>0$), introducing another partition of unity controlling the size of f:

$$\sum_{\varepsilon\in\{\pm1\}^k}\rho_\Upsilon(\varepsilon)\mathcal{O}''_{\varepsilon,\boldsymbol{\alpha},\boldsymbol{\beta}} = 2\sideset{}{^\dagger}\sum_{N_*\ll kN_1q^{\varepsilon/k}}\ \sum_{\substack{\varepsilon\in\{\pm1\}^k\\ \pm_1=-1}}\rho_\Upsilon(\varepsilon)\sum_{\pm_* 1\in\{\pm1\}}\mathcal{K}_{\varepsilon,\pm_*,\boldsymbol{\alpha},\boldsymbol{\beta}}, \tag{7-1}$$

where

$$\mathcal{K}_{\varepsilon,\pm_*;\boldsymbol{\alpha},\boldsymbol{\beta}} := \sum_{d\mid q} d\frac{\mu(q/d)}{\varphi(q)}\sum_{\substack{f\ge1,\\ f\equiv0 \pmod d}}\ \sum_{\substack{n_1,\dots,n_k\ge1\\ n_1=\pm_2n_2\pm_3\cdots\pm_kn_k\pm_*f}}\frac{\tau_{\alpha_1,\beta_1}(n_1)\cdots\tau_{\alpha_k,\beta_k}(n_k)}{(n_1\cdots n_k)^{\frac12}}$$
$$\times V_{\boldsymbol{\alpha},\boldsymbol{\beta}}\bigg(\frac{n_1\cdots n_k}{q^k}\bigg)P\bigg(\frac{n_1}{N_1}\bigg)\cdots P\bigg(\frac{n_k}{N_k}\bigg)P\bigg(\frac{f}{N_*}\bigg).$$

Notice that in (7-1) we truncated the sum over N_* at $N_* \ll kN_1q^{\varepsilon/k}$, as we clearly could.

7.1. ***Separating the variables arithmetically.*** We wish to separate the variables in

$$\tau_{\alpha_1,\beta_1}(n_1) = \tau_{\alpha_1,\beta_1}(\pm_2 n_2 \pm_3 \cdots \pm_k n_k \pm_* f).$$

One can achieve this goal by using Ramanujan's identity

$$\tau_{a,b}(n) = n^{-a}\tau_{0,b-a}(n) = n^{-a}\zeta(1-a+b)\sum_{\ell=1}^{\infty}\frac{c_\ell(n)}{\ell^{1-a+b}}, \tag{7-2}$$

which holds for $n \neq 0$ and $\Re(a-b) < 0$. The coefficient $c_\ell(n)$ denotes the Ramanujan sum

$$c_\ell(n) := \sideset{}{^*}\sum_{h \pmod{\ell}} \mathrm{e}\Big(\frac{nh}{\ell}\Big).$$

However, since (7-2) doesn't hold in a neighborhood of $a = b = 0$, it is more convenient to follow Young's approach and use the following lemma, which rephrases (7-2) as an approximate functional equation for $\tau_{a,b}(n)$.

Lemma 18. *Let $n \in \mathbb{Z}_{>0}$ and let $a, b \in \mathbb{C}$. Then,*

$$\tau_{a,b}(n) = n^{-a}\sum_{\ell}\frac{c_\ell(n)}{\ell^{1-a+b}}\upsilon_{a-b}\Big(\frac{\ell^2}{n}\Big) + n^{-b}\sum_{\ell}\frac{c_\ell(n)}{\ell^{1+a-b}}\upsilon_{b-a}\Big(\frac{\ell^2}{n}\Big) \tag{7-3}$$

where

$$\upsilon_a(x) = \int_{(c_w)} x^{-w/2}\zeta(1-a+w)\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(w)}{w}\,dw,$$

where $c_w > |\Re(a-b)|$ and $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(w)$ is as defined in (4-1).

Proof. See Lemma 5.4 of Young [2011a]. □

Applying (7-3) and splitting the resulting sum over ℓ using another partition of unity (and adding the restriction $L \geq \frac{1}{2}$ as we can do since P is supported on $[1, 2]$), we rewrite $\mathcal{K}_{\varepsilon,\pm_*;\boldsymbol{\alpha},\boldsymbol{\beta}}$ as

$$\mathcal{K}_{\varepsilon,\pm_*;\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\substack{\{\alpha_1',\beta_1'\}=\{\alpha_1,\beta_1\} \\ (\alpha_j',\beta_j')=(\alpha_j,\beta_j)\ \forall j\neq 1}} \ \sideset{}{^\dagger}\sum_{L\geq\frac{1}{2}} \mathcal{L}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}, \tag{7-4}$$

where $\boldsymbol{\alpha}' := (\alpha_1', \dots, \alpha_k')$, $\boldsymbol{\beta}' := (\beta_1', \dots, \beta_k')$ and

$$\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \sum_{\substack{n_1,\dots,n_k,\ f\geq 1 \\ n_1=\pm_2 n_2\pm_3\cdots\pm_k n_k\pm_* f}} \ \sum_{\substack{d\,|\,q \\ d\,|\,f}} \frac{\mu(q/d)d}{\varphi(q)} \sum_{\ell} \ \sideset{}{^*}\sum_{h \pmod{\ell}} \frac{c_\ell(\pm_2 n_2 \pm_3 \cdots \pm_k n_k \pm_* f)}{\ell^{1-\alpha+\beta}}\upsilon_{\alpha_1-\beta_1}\Big(\frac{\ell^2}{n_1}\Big)$$
$$\times \frac{\tau_{\alpha_2,\beta_2}(n_2)\cdots\tau_{\alpha_k,\beta_k}(n_k)}{n_1^{\frac{1}{2}}\cdots n_k^{\frac{1}{2}}} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}\Big(\frac{n_1\cdots n_k}{q^k}\Big) P\Big(\frac{n_1}{N_1}\Big)\cdots P\Big(\frac{n_k}{N_k}\Big) P\Big(\frac{f}{N_*}\Big) P\Big(\frac{\ell}{L}\Big).$$

Notice that we have omitted to indicate the dependency of $\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ from ε and $\pm_*$ in order to save notation.

Expressing P, $\upsilon_{\alpha_1-\beta_1}$ and V in terms of their Mellin transform and making the change of variables $u_i \to u_i - s$, for $i = 1, \dots, k$, we see that $\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ can be written as

$$\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{d\,|\,q} \frac{\mu(q/d)d}{\varphi(q)} \sum_{\substack{n_2,\dots,n_k,\, f\geq 1,\, d\,|\,f \\ \pm_2 n_2 \pm_3 \cdots \pm_k n_k \pm_* f>0}} \sum_{\ell} \sideset{}{^*}\sum_{h \pmod{\ell}} \frac{1}{(2\pi i)^{k+3}}$$
$$\times \int_{(c_s,c_w,\boldsymbol{c_u},c_{u_*})} \frac{N_*^{u_*}}{f^{u_*}} P\Big(\frac{\ell}{L}\Big) \frac{N_1^{u_1-s}\cdots N_k^{u_k-s}}{\ell^{1-\alpha_1+\beta_1+w}} \frac{\tau_{\alpha_2,\beta_2}(n_2)\cdots\tau_{\alpha_k,\beta_k}(n_k) c_\ell(\pm_2 n_2 \pm_3 \cdots \pm_k n_k \pm_* f)}{(\pm_2 n_2 \pm_3 \cdots \pm_k n_k \pm_* f)^{\frac12+\alpha_1+u_1-w/2} n_2^{\frac12+u_2}\cdots n_k^{\frac12+u_k}}$$
$$\times \widetilde{P}(u_*)\widetilde{P}(u_1-s)\cdots\widetilde{P}(u_k-s) q^{ks} \frac{H_{\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s)}{ws}\, dw\, ds\, \boldsymbol{du}\, du_*, \qquad \text{(7-5)}$$

where $\boldsymbol{du} := du_1 \cdots du_k$, $\boldsymbol{c_u}$ denotes the lines of integration $c_{u_1}, \dots, c_{u_k}$ and

$$H_{\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s) := \zeta(1+w-\alpha_1+\beta_1) G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(w) g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s).$$

Notice that, by the definitions (4-1) and (3-6) of $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ and $g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$, $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s)$ is entire and decays rapidly in both variables w and s:

$$H_{\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s) \ll e^{-C_2(|\Im(s)|+|\Im(w)|)} (1+|\Re(s)|+|\Re(w)|)^{A(|\Re(s)|+|\Re(w)|+k)}, \qquad \text{(7-6)}$$

for some $C_2 > 0$. As lines of integration, we take

$$c_s := \varepsilon/k, \quad c_{u_1} = -3k - \tfrac12 - \alpha_1 + 7\varepsilon, \quad c_{u_*} = c_{u_2} = \cdots = c_{u_k} = 4k, \quad c_w = 10\varepsilon.$$

The real parts of the lines are chosen to be large enough so that the various sums are absolutely convergent.

7.2. ***Separating the variables analytically.*** To complete the separation of the variables, we need also to deal with the factor $(\pm_2 n_2 \pm_3 \cdots \pm_k n_k \pm_* f)^{\frac12+\alpha_1+u_1-w/2}$ in (7-5). In order to do so, we use Lemma 27, in Section 10. We apply the lemma with $\kappa := k+1$, $B := 3k$ and $\upsilon_1 = \frac12 - \alpha_1 - u_1 + \frac{w}{2}$, so that $\Re(\upsilon_1) = B + 1 - 2\varepsilon$. We get

$$\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\nu} \frac{B!}{\nu_2!\cdots\nu_k!\nu_*!} (\mathcal{N}_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} + \mathcal{N}'_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}), \qquad \text{(7-7)}$$

where the sum is over $\nu = (\nu_2, \dots, \nu_k, \nu_*) \in \mathbb{Z}^k_{\geq 0}$ satisfying

$$\nu_2 + \cdots + \nu_k + \nu_* = B, \qquad \nu_i = 0 \quad \text{if } \pm_i 1 = -1, \qquad \nu_* = 0 \quad \text{if } \pm_* 1 = -1,$$

and $\mathcal{N}_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}$ is defined by

$$\mathcal{N}_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} := \sum_{d|q} \frac{\mu(q/d)d}{\varphi(q)} \sum_{\substack{n_1,\dots,n_k,\, \ell\geq 1 \\ f\geq 1,\, d\,|\,f}} \frac{P(\ell/L)}{(2\pi i)^{2k+3}} \int_{(c_s,c_w,\boldsymbol{c_u},c_{u_*},c_{\upsilon_*})} \frac{c_\ell(\pm_2 n_2 \pm_3 \cdots \pm_k n_k \pm_* f)}{f^{\upsilon_*+u_*-\nu_*}} N_*^{u_*}$$
$$\times \frac{q^{ks} N_1^{u_1-s}}{\ell^{1-\alpha_1+\beta_1+w}} \widetilde{P}(u_*)\widetilde{P}(u_1-s) \bigg(\prod_{i=2}^{k} \int_{(c_{\upsilon_i})} \frac{\tau_{\alpha_i,\beta_i}(n_i) N_i^{u_i-s}}{n_i^{\frac12+u_i+\upsilon_i-\nu_i}} \widetilde{P}(u_i-s) \bigg)$$
$$\times \Psi_{\varepsilon^*,B}\big(\tfrac12 - \alpha_1 - u_1 + w/2, \boldsymbol{\upsilon}, \upsilon_*\big) \frac{H_{\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s)}{ws}\, ds\, dw\, \boldsymbol{du}\, \boldsymbol{d\upsilon}\, du_*\, d\upsilon_*, \qquad \text{(7-8)}$$

with $c_{v_2}=\cdots=c_{v_k}=c_{v_*}=\varepsilon/k$, and $\mathcal{N}'_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}$ is defined in the same way with lines of integrations $c'_{v_2}=\cdots=c'_{v_k}=c'_{v_*}=\frac{1}{2}$ in place of $c_{v_2},\dots,c_{v_k},c_{v_*}$. Also, in (7-8) we used the notation $\boldsymbol{v}:=(v_2,\dots,v_k)$, $\boldsymbol{dv}:=dv_2\cdots dv_k$ and $\varepsilon^*:=(\pm_1 1,\dots,\pm_k 1,\pm_* 1)$ and $\Psi_{\varepsilon^*,B}$ is as in (10-5).

The contribution of $\mathcal{N}'_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}$ can be bounded by moving the lines of integration c_{u_i} to $c_{u_i}=2\varepsilon+\nu_i$ for $i=2,\dots,k$ and c_{u_*} to $c_{u_*}=\frac{1}{2}+\nu_*+\varepsilon$ and bounding trivially. We obtain

$$\mathcal{N}'_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}\ll q^{-1+\varepsilon}N_1^{-B-\frac{1}{2}+A\varepsilon}N_*^{\frac{1}{2}+\nu_*+2\varepsilon}N_2^{\nu_2+2\varepsilon}\cdots N_k^{\nu_k+2\varepsilon}L^{-\varepsilon}$$

and thus

$$\sum_{\nu}\frac{B!}{\nu_2!\cdots\nu_k!\nu_*!}\mathcal{N}'_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}\ll q^{-1+A\varepsilon}N_1^{A\varepsilon}L^{-\varepsilon},$$

since N_1 is the maximum among $N_1,\dots,N_k$ and $N_*\ll kN_1q^{\varepsilon/k}$.

Next, we open the Ramanujan sum in (7-8) and we execute the sums over $n_2,\dots,n_k,f$ as we can do since the integrals and sums are absolutely convergent since $\nu_i,\nu_*\le B=3k$ and $c_{u_i}=c_{u_*}=4k$ for all i. We obtain

$$\begin{aligned}\mathcal{N}_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}=\sum_{d|q}\frac{\mu(q/d)}{\varphi(q)}\sum_{\ell}\frac{P(\ell/L)}{(2\pi i)^{2k+3}}\int_{(c_s,c_w,\boldsymbol{c_u},c_{u_*},c_{v_*})}\ \sideset{}{^*}\sum_{h \pmod{\ell}}\frac{d^{1-v_*-u_*+\nu_*}}{\ell^{1-\alpha_1+\beta_1+w}}F\big(v_*+u_*-\nu_*,\pm_*\tfrac{dh}{\ell}\big)q^{ks}\\ \times N_*^{u_*}\widetilde{P}(u_*)\bigg(\prod_{i=2}^{k}\int_{(c_{v_i})}D_{\alpha_i,\beta_i}\big(\tfrac{1}{2}+u_i+v_i-\nu_i,\tfrac{\pm_i h}{\ell}\big)\widetilde{P}(u_i-s)N_i^{u_i-s}\bigg)\\ \times N_1^{u_1-s}\widetilde{P}(u_1-s)\Psi_{\varepsilon^*,B}\big(\tfrac{1}{2}-\alpha_1-u_1+\tfrac{w}{2},\boldsymbol{v},v_*\big)\frac{H_{\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s)}{ws}\,ds\,dw\,\boldsymbol{du}\,\boldsymbol{dv}\,du_*\,dv_*,\end{aligned}$$

where, after moving the lines of integration $c_{u_2},\dots,c_{u_k},c_{u_*}$, we have

$$\begin{aligned}c_s:=\varepsilon/k,\quad c_{u_1}=-B-\tfrac{1}{2}-\Re(\alpha_1)+7\varepsilon,\quad c_w=10\varepsilon,\\ c_{v_2}=\cdots=c_{v_k}=c_{v_*}=\varepsilon/k,\quad c_{u_*}=1+\nu_*+2\varepsilon-\varepsilon/k,\end{aligned}\tag{7-9}$$

and $c_{u_i}=\frac{1}{2}+\nu_i+\varepsilon/k$, for $i=2,\dots,k$.

Remarks. (1), Thanks to (7-6) and to Lemma 28 in Section 10, the integrals in $\mathcal{N}_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}$ are all absolutely convergent when the line of integration are chosen so that $\Re(v_2)=\cdots=\Re(v_k)=\Re(v_*)=\varepsilon/k$ and $\Re(v_1):=\Re\big(\frac{1}{2}-\alpha_1-u_1+\frac{w}{2}\big)=B+1-2\varepsilon$ (and even if an extra factor of $\prod_{i=2}^{k}(1+|u_i|+|v_i|)^{1+4\varepsilon}$ is introduced inside the integrals, as will be relevant later on in the argument). In the following computations, until Lemma 20, we will (almost) always arrange the lines of integration in a way such that $\Re(v_1)$ is kept equal to $B+1-2\varepsilon$.[2] This ensures the absolute convergence of the integrals in all the bounds we give.

(2) We also observe that, by the definition (10-5), the poles of $\Psi_{\varepsilon^*,B}(v_1,\boldsymbol{v},v_*)$ are contained in the set

$$\big\{(v_1,\boldsymbol{v},v_*)\in\mathbb{C}^{k+1}\mid v_i\in\mathbb{Z}_{\le 0}\text{ for some }i\in\{1,\dots,k\}\text{ or }v_1+\cdots+v_k+v_*=B+1\big\}.$$

[2]The only exception is in the proof of (7-19), where we need to take $\Re(v_1)=B-2\varepsilon$. One can easily verify however that the integrals are all absolutely convergent also in that case.

(3) One should morally think of having $B = \nu_* = \nu_2 = \cdots = \nu_k = 0$, as their presence is just an artificial effect of forcing the various integrals over v_i to be absolutely convergent. Also, we chose and shall keep c_{u_*} in a way so that we stay just to the right of the pole of F. Aside from this, in the following computations our goal will typically be that of moving $c_{u_2}, \dots, c_{u_k}$ to the left thus obtaining savings in $N_2, \dots, N_k$. Since $D\bigl(s, \frac{h}{\ell}\bigr)$ grows roughly like $\ell^{1-\Re(s)}$ when $0 < \Re(s) < 1$, we then need to move w to the right to insure the convergence of the sum over ℓ. This in turn forces us to move c_{u_1} to the right since we need $\Re\bigl(\frac{1}{2} - \alpha_1 - u_1 + \frac{w}{2} + v_2 + \cdots + v_k + v_*\bigr) < B + 1$ to avoid a pole of $\Psi_{\varepsilon^*,B}$. Doing so we lose a power of N_1; however, since in the first argument of $\Psi_{\varepsilon^*,B}$ u_1 appears with a coefficient which is (negative the) double of that of w, we have that the gain in the exponents of $N_2, \dots, N_k$ is superior to the loss in the exponent of N_1. This will then produce a saving when the variables are close to the diagonal, that is when N_1 is not much larger than $(N_2 \cdots N_k)^{1/(k-1)}$.

7.3. ***Picking up the residues of the Estermann function.*** For each $i = 2, \dots, k$ we move the line of integration c_{u_i} to $c_{u_i} = -\frac{1}{2} + \nu_i - 2\varepsilon$, passing through the poles of the Estermann function at

$$u_i = \tfrac{1}{2} - \alpha_i - v_i + \nu_i \quad \text{and} \quad u_i = \tfrac{1}{2} - \beta_i - v_i + \nu_i .$$

By Lemma 6 and the residue theorem, we obtain

$$\mathcal{N}_{\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\substack{I \cup J = \{2,\dots,k\} \\ I \cap J = \varnothing}} \; \sum_{\substack{\{\alpha_i', \beta_i'\} = \{\alpha_i, \beta_i\} \ \forall i \in I \\ (\alpha_j', \beta_j') = (\alpha_j, \beta_j) \ \forall j \in J \cup \{1\}}} \mathcal{P}_{I;\nu;\boldsymbol{\alpha}',\boldsymbol{\beta}'}, \tag{7-10}$$

where, for $I \cup J = \{2, \dots, k\}$, $I \cap J = \varnothing$,

$$\begin{aligned} \mathcal{P}_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} := \sum_{d \mid q} \frac{\mu(q/d)}{\varphi(q)} \sum_{\ell} \frac{P(\ell/L)}{(2\pi i)^5} \int_{(c_s, c_w, c_{u_1}, c_{u_*}, c_{v_*})} \frac{q^{ks} d^{1-v_*-u_*+\nu_*}}{\ell^{\sum_{i \in I \cup \{1\}} (1-\alpha_i+\beta_i) + w}} \sideset{}{^*}\sum_{h \ (\mathrm{mod}\ \ell)} F\bigl(v_* + u_* - \nu_*, \pm_* \tfrac{dh}{\ell}\bigr) \\ \times \Biggl(\prod_{j \in J} \frac{1}{(2\pi i)^2} \int_{(c_{u_j}, c_{v_j})} D_{\alpha_j,\beta_j}\bigl(\tfrac{1}{2} + u_j + v_j - \nu_j, \tfrac{\pm_j h}{\ell}\bigr) \widetilde{P}(u_j - s) N_j^{u_j - s} \, du_j \Biggr) \\ \times \Biggl(\prod_{i \in I} \frac{1}{2\pi i} \int_{(c_{v_i})} \widetilde{P}\bigl(\tfrac{1}{2} - \alpha_i - v_i + \nu_i - s\bigr) N_i^{\frac{1}{2} - \alpha_i - v_i + \nu_i - s} \Biggr) N_*^{u_*} \widetilde{P}(u_*) N_1^{u_1 - s} \widetilde{P}(u_1 - s) \\ \times \Psi_{\varepsilon^*,B}\bigl(\tfrac{1}{2} - \alpha_1 - u_1 + \tfrac{w}{2}, \boldsymbol{v}, v_*\bigr) \frac{H'_{I;\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s)}{ws} \, ds \, dw \, du_1 \, \boldsymbol{dv} \, du_* \, dv_* \end{aligned}$$

and

$$H'_{I;\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s) := G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(w) g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) \zeta(1 + w - \alpha_1 + \beta_1) \prod_{i \in I} \zeta(1 - \alpha_i + \beta_i), \tag{7-11}$$

so that

$$H'_{I;\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s) \ll (A \log q)^{|I|} e^{-C_2(|\Im(s)| + |\Im(w)|)} (1 + |\Re(s)| + |\Re(w)|)^{A(|\Re(s)| + |\Re(w)| + k)}. \tag{7-12}$$

We remind also that the lines of integrations are given by (7-9) and

$$c_{u_j} = -\tfrac{1}{2} + \nu_j - 2\varepsilon, \quad \text{for all } j \in J. \tag{7-13}$$

7.4. ***Applying the bounds on sums of Kloosterman sums.*** In this section, we apply Lemma 12 to give a bound for $\mathcal{P}_{I,\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}$ under certain conditions.

Lemma 19. *Let $I \subseteq \{2, \dots, k\}$ and let $J := \{2, \dots, k\} \setminus I$. Then, if $|I| \le |J|$ we have*

$$\mathcal{P}_{I,\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} \ll q^{-1+A\varepsilon} N_1^{A\varepsilon} \big(q^{\vartheta} N_1^{(k+1)/2} + q^{k/2-\frac{1}{3}+\vartheta/3} N_1^{\frac{1}{2}}\big)(N_1 \cdots N_k)^{-\frac{1}{2}} L^{-\varepsilon}, \tag{7-14}$$

whereas if $|I| > |J|$ and $\nu_j > 0$ for some $j \in J$, then

$$\mathcal{P}_{I,\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} \ll q^{-\frac{5}{6}+\vartheta/3+A\varepsilon} N_1^{-1+A\varepsilon} (N_1 \cdots N_k)^{\frac{1}{2}} L^{-\varepsilon}. \tag{7-15}$$

Proof. First, we bound the sums over h and ℓ by Lemma 12 and we bound trivially the integrals which are all convergent by (5-2), (7-12) and (10-6) when the lines of integrations are given by (7-9) and (7-13). Doing so, we obtain

$$\begin{aligned}
\mathcal{P}_{I,\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} &\ll q^{-\frac{5}{6}+\vartheta/3+A\varepsilon} N_1^{-B-\frac{1}{2}+A\varepsilon} N_*^{1+\nu_*+\varepsilon} \Big(\prod_{i\in I} N_i^{\frac{1}{2}+\nu_i}\Big)\Big(\prod_{j\in J} N_j^{-\frac{1}{2}+\nu_j}\Big) L^{|J|-|I|-\varepsilon} \\
&\qquad \times \int_{(c_s,c_w,c_{u_1},c_{u_*},c_{v_*})} \Big(\prod_{j\in J} \int_{(c_{u_j})} (1+|v_j|+|u_j|)^{1+4\varepsilon} |\widetilde{P}(u_j - s)||du_j|\Big) \\
&\qquad \times \Big(\prod_{i\in I} \int_{(c_{v_i})} \big|\widetilde{P}\big(\tfrac{1}{2} - \alpha_i - v_i + \nu_i - s\big)\big|\Big) |\widetilde{P}(u_*)||\widetilde{P}(u_1 - s)| \\
&\qquad \times \big|\Psi_{\varepsilon^*,B}\big(\tfrac{1}{2} - \alpha_1 - u_1 + \tfrac{w}{2}, \boldsymbol{v}, v_*\big)\big| \frac{|H'_{I;\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s)|}{|ws|} |\, ds\, dw\, du_1\, \boldsymbol{dv}\, du_*\, dv_*| \\
&\ll q^{-\frac{5}{6}+\frac{\vartheta}{3}+A\varepsilon} N_1^{-B-\frac{1}{2}+A\varepsilon} N_*^{1+\nu_*} \Big(\prod_{i\in I} N_i^{\frac{1}{2}+\nu_i}\Big)\Big(\prod_{j\in J} N_j^{-\frac{1}{2}+\nu_j}\Big) L^{|J|-|I|-\varepsilon}.
\end{aligned} \tag{7-16}$$

If $|I| - |J| > 0$ and at least one of the ν_j is greater than zero, then this is bounded by

$$\begin{aligned}
\mathcal{P}_{I,\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} &\ll q^{-\frac{5}{6}+\vartheta/3+A\varepsilon} N_1^{A\varepsilon} \frac{N_*^{\nu_*+1}}{N_1^{\nu_*+1}} (N_1 \cdots N_k)^{\frac{1}{2}} \Big(\prod_{i\in I} \frac{N_i^{\nu_i}}{N_1^{\nu_i}}\Big)\Big(\prod_{j\in J} \frac{N_j^{\nu_j-1}}{N_1^{\nu_j}}\Big) L^{-\varepsilon} \\
&\ll q^{-\frac{5}{6}+\vartheta/3+A\varepsilon} N_1^{-1+A\varepsilon} (N_1 \cdots N_k)^{\frac{1}{2}} L^{-\varepsilon},
\end{aligned}$$

since $B = \nu_1 + \cdots + \nu_k$ and $N_2, \dots, N_k \le N_1$, $N_* \ll k N_1 q^{\varepsilon/k}$.

Now assume $|I|-|J|\le 0$ and let $L\le q^{\frac12}$. In this case (7-16) gives

$$\begin{aligned}\mathcal{P}_{I,\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} &\ll q^{-\frac56+\vartheta/3+A\varepsilon}N_1^{\frac12+|I|+A\varepsilon}\frac{N_*^{\nu_*+1}}{N_1^{\nu_*+1}}(N_1\cdots N_k)^{-\frac12}\biggl(\prod_{i\in I}\frac{N_i^{1+\nu_i}}{N_1^{1+\nu_i}}\biggr)\biggl(\prod_{j\in J}\frac{N_j^{\nu_j}}{N_1^{\nu_j}}\biggr)q^{\frac12(|J|-|I|)-\varepsilon}\\ &\ll q^{-\frac56+\vartheta/3+\frac12(|J|-|I|)+A\varepsilon}N_1^{\frac12+|I|+A\varepsilon}(N_1\cdots N_k)^{-\frac12}\\ &\ll q^{-\frac13+\vartheta/3+A\varepsilon}(N_1^{k/2+A\varepsilon}q^{-\frac12}+q^{k/2-1}N_1^{\frac12})(N_1\cdots N_k)^{-\frac12},\end{aligned}\tag{7-17}$$

since $|I|=k-1-|J|$ and $\frac{k-1}{2}\le|J|\le k-1$.

Finally, if $|I|-|J|\le 0$ and $L>q^{\frac12}$, then we move the lines of integration c_w and c_{u_1} to

$$c_w=|J|-|I|+10\varepsilon=k-1-2|I|+10\varepsilon,\quad c_{u_1}=-1-B+\tfrac{k}{2}-\Re(\alpha_1)-|I|+7\varepsilon.$$

Then, we use Lemma 12 and bound trivially the integrals (using (5-2), (7-12) and (10-6)) and we obtain

$$\begin{aligned}\mathcal{P}_{I,\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} &\ll q^{-1+\vartheta+A\varepsilon}N_1^{-1-B+k/2-|I|+A\varepsilon}N_*^{1+\nu_*}\biggl(\prod_{i\in I}N_i^{\frac12+\nu_i}\biggr)\biggl(\prod_{j\in J}N_j^{-\frac12+\nu_j}\biggr)L^{-\varepsilon}\\ &\ll\frac{q^{-1+\vartheta+A\varepsilon}}{L^{\varepsilon}}N_1^{\frac{k}{2}+A\varepsilon}\frac{N_*^{\nu_*+1}}{N_1^{\nu_*+1}}\biggl(\prod_{i\in I}\frac{N_i}{N_1}\biggr)\biggl(\prod_{i=2}^{k}\frac{N_i^{-\frac12+\nu_i}}{N_1^{\nu_i}}\biggr)\ll\frac{q^{-1+\vartheta+A\varepsilon}N_1^{k/2+\frac12+A\varepsilon}}{(N_1\cdots N_k)^{\frac12}L^{\varepsilon}}.\end{aligned}\tag{7-18}$$

Thus, since $N_1^{k/2}q^{-\frac12}\ll q^{k/2-1}N_1^{\frac12}+q^{-1}N_1^{k/2+\frac12}$, we have that (7-17) and (7-18) imply (7-14). □

7.5. ***Reassembling the sum over ν and further manipulations.*** By the previous section, we only need to consider the $\mathcal{P}_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}$ with $|I|>|J|$ and $\nu_j=0$ for all $j\in J$ (and lines of integration given in (7-9) and (7-13)). For each $j\in J$, we move c_{u_j} to $\frac12+\nu_j-2\varepsilon$ and simultaneously c_{v_j} to $c_{v_j}=-1+\varepsilon/k$, passing through the pole of $\Psi_{\varepsilon^*,B}$ at $v_j=0$. The contribution of the integral on the new line of integration can be bounded by

$$\ll q^{-\frac56+\vartheta/3+A\varepsilon}N_1^{-1+A\varepsilon}(N_1\cdots N_k)^{\frac12}L^{-\varepsilon},\tag{7-19}$$

as can be see by moving c_{u_1} to $c_{u_1}=-B-\frac32-\alpha_1+7\varepsilon$ and bounding the sums and integrals as in the proof of (7-15). Thus we only need to consider the residue at $v_j=0$ for all $j\in J$.

In the same way, we move the line of integration c_{v_*} to $c_{v_*}=1+\varepsilon/k$ and c_{u_*} to $c_{u_*}=\nu_*+2\varepsilon-\varepsilon/k$, passing through the pole of $\Psi_{\varepsilon^*,B}$ at $v_*=B+1-\bigl(\frac12-\alpha_1-u_1+\frac{w}{2}\bigr)-\sum_{i\in I}v_i$. The contribution of the new line of integration can be bounded by (7-19) in a similar way, so again we only need to consider the contribution of the residue. Thus, summarizing (and recalling (7-7) and (7-10)), we arrive at

$$\begin{aligned}\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}=&\sum_{\substack{I\cup J=\{2,\dots,k\}\\ I\cap J=\varnothing,\,|I|>|J|}}\ \sum_{\substack{\{\alpha_i',\beta_i'\}=\{\alpha_i,\beta_i\}\ \forall i\in I\\ (\alpha_j',\beta_j')=(\alpha_j,\beta_j)\ \forall j\in J\cup\{1\}}}\ \sum_{\nu}\frac{B!}{\nu_2!\cdots\nu_k!\nu_*!}\mathcal{Q}_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}\\ &+O\biggl(\frac{q^{-1+A\varepsilon}N_1^{A\varepsilon}}{(N_1\cdots N_k)^{\frac12}}\bigl(q^{\vartheta}N_1^{(k+1)/2}+q^{k/2-\frac13+\vartheta/3}N_1^{\frac12}+q^{\frac16+\vartheta/3}N_2\cdots N_k\bigr)L^{-\varepsilon}\biggr),\end{aligned}\tag{7-20}$$

where the sum over ν is now over $\nu = (\nu_2, \dots, \nu_k, \nu_*) \in \mathbb{Z}_{\geq 0}^k$ satisfying

$$\nu_2 + \cdots + \nu_k + \nu_* = B, \qquad \nu_i = 0 \quad \text{if } \pm_i 1 = -1 \text{ or } i \in J, \qquad \nu_* = 0 \quad \text{if } \pm_* 1 = -1,$$

and where

$$\begin{aligned}\mathcal{Q}_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} := &\sum_{d \mid q} \frac{\mu(q/d)}{\varphi(q)} \sum_{\ell} \sideset{}{^*}\sum_{h \pmod{\ell}} \frac{P(\ell/L)}{(2\pi i)^{4+|I|}} \int_{(c_s, c_w, c_{u_1}, c_{u_*},\, c_{v_i} \forall i \in I)} \frac{d^{\frac{1}{2}-B-\alpha_1-u_1+w/2+\sum_{i\in I} v_i - u_* + \nu_*}}{\ell^{\sum_{i\in I\cup\{1\}}(1-\alpha_i+\beta_i)+w}} \\ &\times q^{ks} F\big(B + \tfrac{1}{2} + \alpha_1 + u_1 - \tfrac{w}{2} - \textstyle\sum_{i\in I} v_i + u_* - \nu_*, \pm_* \tfrac{dh}{\ell}\big) \frac{H'_{I;\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s)}{ws} \\ &\times \Bigg(\prod_{j\in J} \frac{1}{2\pi i} \int_{(c_{u_j})} D_{\alpha_j,\beta_j}\big(\tfrac{1}{2} + u_j, \tfrac{\pm_j h}{\ell}\big) \widetilde{P}(u_j - s) N_j^{u_j - s}\, du_j \Bigg) \\ &\times N_*^{u_*} \widetilde{P}(u_*) N_1^{u_1 - s} \widetilde{P}(u_1 - s) \Psi'_{I;\varepsilon_I^*,B}\big(\tfrac{1}{2} - \alpha_1 - u_1 + \tfrac{w}{2}, \boldsymbol{v}_I\big) \\ &\times \Bigg(\prod_{i\in I} \widetilde{P}(\tfrac{1}{2} - \alpha_i - v_i + \nu_i - s) N_i^{\frac{1}{2}-\alpha_i - v_i + \nu_i - s} d^{v_i}\, dv_i \Bigg) dw\, du_1\, ds\, du_*, \end{aligned} \tag{7-21}$$

for $\boldsymbol{v}_I := (v_i)_{i\in I}$, $\varepsilon_I^* = (\pm_i 1)_{i \in I\cup\{*\}}$ and

$$\begin{aligned} \Psi'_{I;\varepsilon_I^*,B}(v_1, \boldsymbol{v}_I) &:= \frac{\Gamma\big(B + 1 - v_1 - \sum_{i\in I} v_i\big) \prod_{i\in I\cup\{1\}} \Gamma(v_i)}{\Gamma(V_{\mp_*;\varepsilon_I}(v_1, \boldsymbol{v}_I))\Gamma(B + 1 - V_{\mp_*;\varepsilon_I}(v_1, \boldsymbol{v}_I))}, \\ V_{\pm,\varepsilon_I}(v_1, \boldsymbol{v}_I) &:= \sum_{\substack{i \in I\cup\{1\} \\ \pm_i 1 = \pm 1}} v_i. \end{aligned} \tag{7-22}$$

We also remind that the line of integrations are

$$\begin{aligned} &c_s := \varepsilon/k, \quad c_w = 10\varepsilon, \quad c_{u_*} = 1 + \nu_* + 2\varepsilon - \varepsilon/k, \\ &c_{u_j} = -\tfrac{1}{2} + \nu_j - 2\varepsilon \quad \forall j \in J, \qquad c_{v_i} = \frac{\varepsilon}{k} \quad \forall i \in I \end{aligned} \tag{7-23}$$

and $c_{u_1} = -B - \frac{1}{2} - \Re(\alpha_1) + 7\varepsilon$.

Remark. Notice that the integrand in (7-21) decays rapidly along a vertical strip in each of the variables of integration. In particular, in the following computations we will always be able to bound the integrals trivially.

At this point, we wish to execute the sum over the partitions of unity N_*. However, first we need to remove the truncation $N_* \ll kN_1 q^{\varepsilon/k}$. This can be done at a negligible cost, as shown in the following lemma.

Lemma 20. *We have*

$$\sideset{}{^\dagger}\sum_{N_* \ll kN_1 q^{\varepsilon/k}} \mathcal{Q}_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} = \sideset{}{^\dagger}\sum_{N_*} \mathcal{Q}_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} + O\big(q^{-2+A\varepsilon} N_1^{A\varepsilon} (N_1 \cdots N_k)^{\frac{1}{2}} L^{-\varepsilon}\big). \tag{7-24}$$

Proof. Assume $N_* \gg kN_1 q^{\varepsilon/k}$. Given a large positive integer Δ, we move c_{v_i} to $c_{v_i} = -\Delta k + \frac{1}{2}$ for all $i \in I$ in (7-21). Doing so, we pass through the poles of

$$\Psi'_{I;\varepsilon_I^*,B}\big(\tfrac{1}{2} - \alpha_1 - u_1 + \tfrac{w}{2}, \boldsymbol{v}_I\big) \quad \text{at } v_i \in S_\Delta := \{-r \mid r \in \mathbb{Z}, 0 \le r < \Delta k\},$$

so that we have to deal with a sum of $(\Delta k + 1)^{|I|}$ terms coming from the contribution of the residues and of the integrals on the new lines of integration.[3] Then, for each of these terms, we move the line of integration c_{u_1} to $c_{u_1} = \Delta k$. This can be done without crossing any pole of $\Psi'_{I;\varepsilon_I^*,B}$ if the term was coming from picking up a residue in each of the variables v_i for all $i \in I$, since in this case the Γ factor in the denominator of $\Psi'_{I;\varepsilon_I^*,B}$ cancel the poles of $\Gamma(v_1) = \Gamma\big(\frac{1}{2} - \alpha_1 - u_1 + \frac{w}{2}\big)$. Otherwise, we also have to consider the residues of $\Psi'_{I;\varepsilon_I^*,B}$ at $\frac{1}{2} - \alpha_1 - u_1 + \frac{w}{2} \in S_\Delta$. In all cases, however, all the terms will still have at least one integral left (besides the w and u_* integrals) with line of integration $c_{v_i} = -\Delta k + \frac{1}{2}$ or $c_{u_1} = \Delta k$. Finally, for each of these terms we place the line of integration c_{u_*} so that the real part of the argument of the function F in (7-21) is still $1 + (4 - |I|/k)\varepsilon$ (we can do this without crossing poles).

Thus, bounding these terms by using Lemma 12 and the estimates (5-2), (7-12) and (10-6), we obtain

$$\mathcal{Q}_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} \ll q^{-\frac{5}{6}+\vartheta/3+A\varepsilon} \sum_{\substack{r_1,\dots r_k \in (\{0,1,\dots,\Delta-1\} \cup \{\Delta k - \frac{1}{2}\}) \\ r_i = \Delta k - \frac{1}{2} \text{ for some } i \in (\{1\} \cup I) \\ r_i = 0 \text{ if } i \in J}} \frac{N_1^{\frac{1}{2}+r_1+A\varepsilon} N_2^{r_2+\frac{1}{2}+\nu_2} \cdots N_k^{r_k+\frac{1}{2}+\nu_k}}{N_*^{B+\sum_{i=1}^k r_i - \nu_* + (4-|I|/k)\varepsilon} L^\varepsilon} \prod_{j \in J} N_j^{-1}$$

$$\ll q^{-1+A\varepsilon} \frac{N_1^{A\varepsilon}(N_1 \cdots N_k)^{\frac{1}{2}}}{q^{\Delta\varepsilon} N_*^\varepsilon} \bigg(\prod_{j \in J} N_j^{-1}\bigg) L^{-\varepsilon} \ll q^{-2+A\varepsilon} N_1^{A\varepsilon}(N_1 \cdots N_k)^{\frac{1}{2}} (LN_*)^{-\varepsilon},$$

if Δ is large enough with respect to ε. Equation (7-24) then follows. □

We now move the line of integration c_{u_1} to $c_{u_1} = \frac{1}{2} + 3\varepsilon$ and then execute the sum over N_*, which we do by using the following lemma.

Lemma 21. *Let $K(s)$ be a function which is analytic and grows at most polynomially on a strip $|\Re(s)| < c$ for some $c > 0$. Then, for any $-c < c_u < c$ we have*

$$\sum_N{}^{\dagger} \frac{1}{2\pi i} \int_{(c_u)} K(u) \widetilde{P}(u) N^u \, du = K(0).$$

Proof. For $\varepsilon > 0$, let $\hat{g}_\varepsilon(x)$ be the Mellin transform of $g_\varepsilon(u) := K(u)e^{\varepsilon u^2}$. We have

$$\sum_N{}^{\dagger} \frac{1}{2\pi i} \int_{(c_u)} K(u) \widetilde{P}(u) N^u \, du = \lim_{\varepsilon \to 0} \sum_N{}^{\dagger} \frac{1}{2\pi i} \int_{(0)} K(u) e^{\varepsilon u^2} \widetilde{P}(u) N^u \, du$$

as can be seen by splitting the sum according to whether $N \ge 1$ or $N < 1$ and moving the line of integration accordingly to $-c/2$ or $c/2$. We then write $g_\varepsilon(u)$ in terms of its Mellin transform. Exchanging the order

[3] In total there are $(\Delta k + 1)^{|I|}$ terms because for each v_i we have the possibility of taking the residue at $v_i = -r_i \in S_\Delta$ or to take the integral at $c_{v_i} = -\Delta k + \frac{1}{2}$.

of the integrals, as allowed by the bound $\hat{g}_\varepsilon(x) \ll \min(x^{c/2}, x^{-c/2})$, and executing the integral over u we obtain

$$\lim_{\varepsilon\to 0} \sum_N^\dagger \int_0^\infty g_\varepsilon(x) P(1/Nx) \frac{dx}{x} = \lim_{\varepsilon\to 0} \int_0^\infty g_\varepsilon(x) \frac{dx}{x} = F(0). \qquad \square$$

We move the line of integration c_{u_1} to $c_{u_1} = \frac{1}{2} + 3\varepsilon$ without crossing poles and apply the above lemma obtaining

$$\begin{aligned}\sum_{N_*}^\dagger \mathcal{Q}_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{d\,|\,q} \frac{\mu(q/d)}{\varphi(q)} \sum_\ell \sum_{h \ (\mathrm{mod}\ \ell)}^* \frac{P(\ell/L)}{(2\pi i)^{3+|I|}} \int_{(c_s, c_w, c_{u_1},\, c_{v_i}\,\forall i\in I)} \frac{d^{\frac{1}{2}-B-\alpha_1-u_1+w/2+\sum_{i\in I} v_i+\nu_*}}{\ell^{\sum_{i\in I\cup\{1\}}(1-\alpha_i+\beta_i)+w}} \\ \times q^{ks} F\big(B+\tfrac{1}{2}+\alpha_1+u_1-\tfrac{w}{2}-\textstyle\sum_{i\in I} v_i-\nu_*,\ \pm_* \tfrac{h}{\ell}\big) \\ \times \bigg(\prod_{j\in J} \frac{1}{2\pi i}\int_{(c_{u_j})} D_{\alpha_j,\beta_j}\big(\tfrac{1}{2}+u_j, \tfrac{\pm_j h}{\ell}\big) \widetilde{P}(u_j-s) N_j^{u_j-s}\, du_j\bigg) \\ \times N_1^{u_1-s} \widetilde{P}(u_1-s) \Psi'_{I;\varepsilon_I^*,B}\big(\tfrac{1}{2}-\alpha_1-u_1+\tfrac{w}{2}, \boldsymbol{v}_I\big) \frac{H'_{I;\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s)}{ws} \\ \times \prod_{i\in I}\Big(\widetilde{P}(\tfrac{1}{2}-\alpha_i-v_i+\nu_i-s) N_i^{\frac{1}{2}-\alpha_i-v_i+\nu_i-s} d^{v_i}\, dv_i\Big)\, dw\, du_1\, ds,\end{aligned} \tag{7-25}$$

with lines of integrations that we can take to be given by (7-23) and $c_{u_1} = \frac{1}{2} + 3\varepsilon$.

We are finally ready to execute the sum over ν. We do this in the following lemma, which also summarizes the previous computations.

Lemma 22. *We have*

$$\begin{aligned}\sum_{\varepsilon\in\{\pm 1\}^k} \rho_\Upsilon(\varepsilon) \mathcal{O}''_{\varepsilon,\alpha,\beta} = 2\sum_L^\dagger \sum_{\substack{I\cup J=\{2,\dots,k\}\\ I\cap J=\varnothing,\ |I|>|J|}} \sum_{\substack{\{\alpha'_i,\beta'_i\}=\{\alpha_i,\beta_i\}\,\forall i\in I_1\\ (\alpha'_j,\beta'_j)=(\alpha_j,\beta_j)\,\forall j\in J}} \sum_{\substack{\varepsilon\in\{\pm 1\}^k\\ \pm_1 1=-1}} \rho_\Upsilon(\varepsilon) \sum_{\pm_* 1\in\{\pm 1\}} \mathcal{R}_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}} \\ + O\bigg(\frac{q^{-1+A\varepsilon} N_1^{A\varepsilon}}{(N_1\cdots N_k)^{\frac{1}{2}} L^\varepsilon}\Big(q^\vartheta N_1^{(k+1)/2} + q^{k/2-\frac{1}{3}+\vartheta/3} N_1^{\frac{1}{2}} + q^{\frac{1}{6}+\vartheta/3} N_2\cdots N_k\Big)\bigg),\end{aligned} \tag{7-26}$$

where $I_1 := I\cup\{1\}$ *and*

$$\begin{aligned}\mathcal{R}_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}} := \sum_{d\,|\,q} \frac{\mu(q/d)}{\varphi(q)} \sum_\ell \sum_{h \ (\mathrm{mod}\ \ell)}^* \frac{P(\ell/L)}{(2\pi i)^{3+|I|}} \int_{(c_s, c_w, c_{u_1},\, c_{v_i}\,\forall i\in I)} \frac{q^{ks} d^{\frac{1}{2}-\alpha_1-u_1+w/2+\sum_{i\in I} v_i} N_1^{u_1-s}}{\ell^{\sum_{i\in I_1}(1-\alpha_i+\beta_i)+w}} \\ \times \bigg(\prod_{j\in J} \frac{1}{2\pi i}\int_{(c_{u_j})} D_{\alpha_j,\beta_j}\big(\tfrac{1}{2}+u_j, \tfrac{\pm_j h}{\ell}\big) \widetilde{P}(u_j-s) N_j^{u_j-s} du_j\bigg) \\ \times \widetilde{P}(u_1-s) \times F\big(\tfrac{1}{2}+\alpha_1+u_1-\tfrac{w}{2}-\textstyle\sum_{i\in I} v_i, \pm_* \tfrac{dh}{\ell}\big) \Psi'_{I;\varepsilon_I^*,0}\big(\tfrac{1}{2}-\alpha_1-u_1+\tfrac{w}{2}, \boldsymbol{v}_I\big) \\ \times \bigg(\prod_{i\in I} \widetilde{P}\big(\tfrac{1}{2}-\alpha_i-v_i-s\big) N_i^{\frac{1}{2}-\alpha_i-v_i-s} d^{v_i}\, dv_i\bigg) \frac{H'_{I;\boldsymbol{\alpha},\boldsymbol{\beta}}(w,s)}{ws}\, dw\, du_1\, ds\end{aligned}$$

and lines of integrations given by (7-23) *and* $c_{u_1} = \frac{1}{2} + 3\varepsilon$.

Proof. Using (7-1), (7-4), (7-20) and (7-24) we obtain (7-26), with $\mathcal{R}_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}}$ replaced by

$$\mathcal{R}'_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}} := \sum_{\nu} \frac{B!}{\nu_2! \cdots \nu_k! \nu_*!} \sum_{N_*}^{\dagger} \mathcal{Q}_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}$$

and $\sum^{\dagger} \mathcal{Q}_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}$ as in (7-25). Thus, the lemma reduces to showing that $\mathcal{R}_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathcal{R}'_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}}$. This is an immediate consequence of Lemma 29 below, which is applicable since the pole of F is canceled by the sum over d. □

7.6. ***Reassembling the sum over ε.*** Now, we can also get rid of the integral over w. To do this, first we move the lines of integration c_{u_1} and c_{u_j} for $j \in J$ (without passing through poles), so that we have

$$c_{u_1} = -\Re(\alpha_1) + 7\varepsilon, \qquad c_{u_j} = \tfrac{1}{2} - \Re(\alpha_j) - 2\varepsilon \quad \forall j \in J \qquad c_s := \varepsilon/k, \qquad c_{v_i} = \frac{\varepsilon}{k} \quad \forall i \in I \tag{7-27}$$

and then we move c_w to $c_w = -1 + 10\varepsilon$ passing through a pole at $w = 0$. The contribution of the new line of integration is trivially bounded by

$$\ll q^{-1+A\varepsilon} N_1^{-\frac{1}{2}+A\varepsilon} (N_1 \cdots N_k)^{\frac{1}{2}} L^{-\varepsilon}. \tag{7-28}$$

since we have the convexity bound $D\big(1-2\varepsilon+it, \alpha_j-\beta_j, \frac{h}{\ell}\big) \ll \ell^{3\varepsilon}(1+|t|)^{3\varepsilon}$ and $|I| \geq 2$ (since $|I| > |J|$ and $k \geq 3$). Thus, we only need to consider the contribution from the residue at $w = 0$.

By the convexity bound

$$F\big(\tfrac{1}{2} + 7\varepsilon - |I|\varepsilon/k + it, \tfrac{h}{\ell}\big) \ll \ell^{\frac{1}{2}} (1+|t|)^{\frac{1}{2}},$$

the contribution of the $d = 1$ term is also bounded by (7-28). Thus, using also that $\varphi(q)^{-1} = q^{-1} + O(q^{-2})$ for q prime, we have

$$\mathcal{R}_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathcal{S}_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}} + O\big(q^{-1+A\varepsilon} N_1^{-\frac{1}{2}+A\varepsilon} (N_1 \cdots N_k)^{\frac{1}{2}} L^{-\varepsilon}\big) \tag{7-29}$$

with

$$\begin{aligned}
\mathcal{S}_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\ell} \sum_{h \pmod{\ell}}^{*} \frac{P(\ell/L)}{(2\pi i)^{2+|I|}} \int_{(c_s, c_{u_1}, c_{v_i} \forall i \in I)} & \frac{q^{ks-\frac{1}{2}-\alpha_1-u_1+\sum_{i\in I} v_i}}{\ell^{\sum_{i\in I\cup\{1\}}(1-\alpha_i+\beta_i)}} N_1^{u_1-s} \widetilde{P}(u_1-s) \\
&\times \Big(\prod_{j\in J} \frac{1}{2\pi i} \int_{(c_{u_j})} D_{\alpha_j,\beta_j}\big(\tfrac{1}{2}+u_j, \pm_* \tfrac{\pm_j h}{\ell}\big) \widetilde{P}(u_j-s) N_j^{u_j-s}\, du_j \Big) \\
&\times F\big(\tfrac{1}{2}+\alpha_1+u_1-\textstyle\sum_{i\in I} v_i, \tfrac{qh}{\ell}\big) \Psi'_{I;\varepsilon_I^*,0}\big(\tfrac{1}{2}-\alpha_1-u_1, \boldsymbol{v}_I\big) \\
&\times \Big(\prod_{i\in I} \widetilde{P}\big(\tfrac{1}{2}-\alpha_i-v_i-s\big) N_i^{\frac{1}{2}-\alpha_i-v_i-s} q^{v_i}\, dv_i \Big) \frac{H'_{I;\boldsymbol{\alpha},\boldsymbol{\beta}}(0,s)}{s}\, du_1\, ds
\end{aligned}$$

and lines of integration given by (7-27). Notice that we made the change of variable $h \to \pm_* h$.

We are ready to reassemble the sum over ε. To do this, first we split ε into ε_{I_1} and ε_J, where $\varepsilon_S := (\pm_i 1)_{i\in S}$; in particular, $\rho_\Upsilon(\varepsilon) = \rho_\Upsilon(\varepsilon_{I_1})\rho_\Upsilon(\varepsilon_J)$ where, with a slight abuse of notation, we write

$\rho_\Upsilon(\varepsilon_S) := \prod_{i\in\Upsilon\cap S}(\pm_i 1)$. Then, we observe that

$$\sum_{\varepsilon_J\in\{\pm1\}^{|J|}} \rho_\Upsilon(\varepsilon_J)\prod_{j\in J} D_{\alpha_j,\beta_j}\big(s_j, \pm_* \tfrac{\pm_j h}{\ell}\big) = 2^{|J|}(\pm_* i)^{|\Upsilon\cap J|}\prod_{j\in J} D_{j;\alpha_j,\beta_j}\big(s_j, \tfrac{h}{\ell}\big).$$

Thus,

$$\begin{aligned}
&\sum_{\substack{\varepsilon\in\{\pm1\}^k\\ \pm_1 1=-1}} \rho_\Upsilon(\varepsilon)\sum_{\pm_* 1\in\{\pm1\}} \mathcal{S}_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}}\\
&\quad= 2^{|J|} i^{|\Upsilon\cap J|}\sum_{\ell}\ \sideset{}{^*}\sum_{h \ (\mathrm{mod}\ \ell)} \frac{P(\ell/L)}{(2\pi i)^{2+|I|}} \frac{q^{ks-\frac12-\alpha_1-u_1+\sum_{i\in I} v_i}}{\ell^{\sum_{i\in I\cup\{1\}}(1-\alpha_i+\beta_i)}}\\
&\qquad\times\int_{(c_s,c_{u_1},\,c_{v_i}\forall i\in I)}\bigg(\prod_{j\in J}\frac{1}{2\pi i}\int_{(c_{u_j})} D_{j;\alpha_j,\beta_j}\big(\tfrac12+u_j, \tfrac{h}{\ell}\big)\widetilde{P}(u_j-s)N_j^{u_j-s}\,du_j\bigg)\\
&\qquad\quad\times N_1^{u_1-s}\widetilde{P}(u_1-s)F\big(\tfrac12+\alpha_1+u_1-\textstyle\sum_{i\in I} v_i, \tfrac{qh}{\ell}\big)\mathcal{X}_I\big(\tfrac12-u_1-\alpha_1, \boldsymbol{v}_I\big)\\
&\qquad\qquad\times\bigg(\prod_{i\in I}\widetilde{P}(\tfrac12-\alpha_i-v_i-s)N_i^{\frac12-\alpha_i-v_i-s}q^{v_i}dv_i\bigg)\frac{H'_{I;\boldsymbol{\alpha},\boldsymbol{\beta}}(0,s)}{s}\,du_1\,ds,
\end{aligned}\tag{7-30}$$

with

$$\begin{aligned}
\mathcal{X}_I(v_1,\boldsymbol{v}_I) &:= \sum_{\pm_* 1\in\{\pm1\}}(\pm_* 1)^{|\Upsilon\cap J|}\sum_{\substack{\varepsilon_{I_1}\in\{\pm1\}^{|I_1|}\\ \pm_1 1=-1}}\rho_\Upsilon(\varepsilon_{I_1})\Psi'_{I;\varepsilon_I^*,0}(v_1,\boldsymbol{v}_I),\\
&= \Gamma\bigg(1-\sum_{i\in I_1} v_i\bigg)\prod_{i\in I_1}\Gamma(v_i)\sum_{\pm_* 1\in\{\pm1\}}(\pm_* 1)^{|\Upsilon\cap I_1|}\\
&\qquad\times\sum_{\substack{\varepsilon_{I_1}\in\{\pm1\}^{|I_1|}\\ \pm_1 1=-1}}\rho_\Upsilon(\varepsilon_{I_1})\bigg(\Gamma\bigg(\sum_{\substack{i\in I_1\\ \pm_i 1=\mp_* 1}} v_i\bigg)\Gamma\bigg(1-\sum_{\substack{i\in I_1\\ \pm_i 1=\mp_* 1}} v_i\bigg)\bigg)^{-1},
\end{aligned}$$

by the definition (7-22) of $\Psi'_{I;\varepsilon_I^*,0}$ and since $(\pm_* 1)^{|\Upsilon\cap J|} = (\pm_* 1)^{|\Upsilon\cap(I\cup\{1\})|}$ for $|\Upsilon|$ even (as we have assumed). Also, we remind that we defined $\varepsilon_I^* := (\pm_i 1)_{i\in I\cup\{*\}}$ and $I_1 := I\cup\{1\}$.

We will now give a Γ-function identity, which we will use to give a symmetric expression for $\mathcal{X}_I(v_1,\boldsymbol{v}_I)$.

Lemma 23. *Let $r\ge 1$, $\Theta\subseteq\{1,\dots,r\}$ and $(s_1,\dots,s_r)\in\mathbb{C}^r$. For $\varepsilon_r = (\pm_1,\dots,\pm_r 1)\in\{\pm1\}^r$ let $\rho_\Theta(\varepsilon) := \prod_{i\in\Theta}(\pm_i 1)$. Then,*[4]

$$\begin{aligned}
&\prod_{i=1}^r\Gamma(s_i)\sum_{\pm_* 1\in\{\pm1\}}(\pm_* 1)^{|\Theta|}\sum_{\substack{\varepsilon\in\{\pm1\}^r\\ \pm_1 1=-1}}\rho_\Theta(\varepsilon)\bigg(\Gamma\bigg(\sum_{\pm_i 1=\mp_* 1} s_i\bigg)\Gamma\bigg(1-\sum_{\pm_i 1=\mp_* 1} s_i\bigg)\bigg)^{-1}\\
&\qquad= \frac{2^{s_1+\cdots+s_r}}{\pi^{1-r/2}}\bigg(\prod_{i\in\Theta}\frac{\Gamma\big(\frac12+\frac{s_i}{2}\big)}{\Gamma\big(1-\frac{s_i}{2}\big)}\bigg)\bigg(\prod_{i\notin\Theta}\frac{\Gamma\big(\frac{s_i}{2}\big)}{\Gamma\big(\frac12-\frac{s_i}{2}\big)}\bigg)\sin\Big(\frac{\pi}{2}(s_1+\cdots+s_r)-\frac{\pi}{2}|\Theta|\Big).
\end{aligned}$$

[4] The identity has to be interpreted as an identity between meromorphic functions.

Proof. First, we observe that, by analytic continuation, we can assume that $s_1, \dots, s_r \in \mathbb{R} \setminus \mathbb{Z}$. Thus, using the reflection formula for the Gamma function to have

$$\Bigl(\Gamma\Bigl(\sum_{\pm_i 1 = \mp_* 1} s_i\Bigr)\Gamma\Bigl(1 - \sum_{\pm_i 1 = \mp_* 1} s_i\Bigr)\Bigr)^{-1} = \pi^{-1}\sin\Bigl(\pi \sum_{\pm_i 1 = \mp_* 1} s_i\Bigr) = \pi^{-1}\Im\Bigl(\exp\Bigl(\pi i \sum_{\pm_i 1 = \mp_* 1} s_i\Bigr)\Bigr).$$

It follows that

$$\begin{aligned} S &:= \sum_{\pm_* 1 \in \{\pm 1\}} (\pm_* 1)^{|\Theta|} \sum_{\substack{\varepsilon \in \{\pm 1\}^r \\ \pm_1 1 = -1}} \rho_\Theta(\varepsilon)\Bigl(\Gamma\Bigl(\sum_{\pm_i 1 = \mp_* 1} s_i\Bigr)\Gamma\Bigl(1 - \sum_{\pm_i 1 = \mp_* 1} s_i\Bigr)\Bigr)^{-1} \\ &= \pi^{-1}\Im\bigl(e^{\pi i s_1} A_+ + (-1)^{|\Theta|} A_-\bigr), \end{aligned}$$

where

$$A_\pm := \sum_{\substack{\varepsilon \in \{\pm 1\}^r \\ \pm_1 1 = -1}} \rho_\Theta(\varepsilon) \exp\Bigl(\pi i \sum_{\substack{i=2 \\ \pm_i 1 = \mp 1}}^{r} s_i\Bigr).$$

Now, since $\rho_\Theta(\varepsilon) = \prod_{i \in \Theta}(\pm_i 1) = (-1)^{|\Theta \cap \{1\}|} \prod_{i \in \Theta \setminus \{1\}} (\pm_i 1)$, we have

$$\begin{aligned} A_\pm &= (-1)^{|\Theta \cap \{1\}|}\Bigl(\prod_{\substack{i=2 \\ i \in \Theta}}^{r} (\pm 1 \mp e^{\pi i s_i})\Bigr)\Bigl(\prod_{\substack{i=2 \\ i \notin \Theta}}^{r} (1 + e^{\pi i s_i})\Bigr) \\ &= (\pm 1)^{|\Theta \cap \{1\}|}(\mp 1)^{|\Theta|} i^{|\Theta \setminus \{1\}|} 2^{r-1} \exp\Bigl(\frac{\pi i}{2}\sum_{i=2}^{r} s_i\Bigr)\Bigl(\prod_{\substack{i=2 \\ i \in \Theta}}^{r} \sin\Bigl(\frac{\pi s_i}{2}\Bigr)\Bigr)\Bigl(\prod_{\substack{i=2 \\ i \notin \Theta}}^{r} \cos\Bigl(\frac{\pi s_i}{2}\Bigr)\Bigr) \end{aligned}$$

and thus

$$\begin{aligned} S &= \frac{2^{r-1}}{\pi}\Bigl(\prod_{\substack{i \in \Theta \\ i \neq 1}} \sin\Bigl(\frac{\pi s_i}{2}\Bigr)\Bigr)\Bigl(\prod_{\substack{i \notin \Theta \\ i \neq 1}} \cos\Bigl(\frac{\pi s_i}{2}\Bigr)\Bigr)(-1)^{|\Theta|}\Im\bigl((e^{\pi i s_1} + (-1)^{|\Theta \cap \{1\}|})e^{(\pi i/2)|\Theta \setminus \{1\}| + (\pi i/2)\sum_{i=2}^{r} s_i}\bigr) \\ &= \frac{2^r}{\pi}\Bigl(\prod_{i \in \Theta} \sin\Bigl(\frac{\pi s_i}{2}\Bigr)\Bigr)\Bigl(\prod_{i \notin \Theta} \cos\Bigl(\frac{\pi s_i}{2}\Bigr)\Bigr)(-1)^{|\Theta|}\Im\bigl(i^{|\Theta \cap \{1\}|} e^{(\pi i/2)|\Theta \setminus \{1\}| + (\pi i/2)\sum_{i=1}^{r} s_i}\bigr) \\ &= \frac{2^r}{\pi}\Bigl(\prod_{i \in \Theta} \sin\Bigl(\frac{\pi s_i}{2}\Bigr)\Bigr)\Bigl(\prod_{i \notin \Theta} \cos\Bigl(\frac{\pi s_i}{2}\Bigr)\Bigr)\sin\Bigl(-\frac{\pi}{2}|\Theta| + \frac{\pi}{2}\sum_{i=1}^{r} s_i\Bigr). \end{aligned}$$

By the duplication and the reflection formula for the Γ-function we have

$$\sin\Bigl(\frac{\pi s}{2}\Bigr)\Gamma(s) = \pi^{\frac{1}{2}} 2^{s-1}\frac{\Gamma\bigl(\frac{1}{2} + \frac{s}{2}\bigr)}{\Gamma\bigl(1 - \frac{s}{2}\bigr)}, \quad \cos\Bigl(\frac{\pi s}{2}\Bigr)\Gamma(s) = \pi^{\frac{1}{2}} 2^{s-1}\frac{\Gamma\bigl(\frac{s}{2}\bigr)}{\Gamma\bigl(\frac{1}{2} - \frac{s}{2}\bigr)}$$

and thus the lemma follows. □

Applying Lemma 23 with $r = |I_1|$ and using the definition (2-1) of Γ_i, we obtain

$$\mathcal{X}_I(v_1, \boldsymbol{v}_I) = \Gamma\Bigl(1 - \sum_{i \in I_1} v_i\Bigr)\frac{2^{\sum_{i \in I_1} v_i}}{\pi^{1 - |I_1|/2}}\Bigl(\prod_{i \in I_1} \frac{\Gamma_i\bigl(\frac{v_i}{2}\bigr)}{\Gamma_i\bigl(\frac{1}{2} - \frac{v_i}{2}\bigr)}\Bigr)\sin\Bigl(-\frac{\pi}{2}|I_1 \cap \Upsilon| + \frac{\pi}{2}\sum_{i \in I_1} v_i\Bigr).$$

Thus, plugging this expression into (7-30), making the change of variables

$$u_i = \tfrac{1}{2} - \alpha_i - v_i - s \quad \forall i \in I, \qquad u_j \to u_j + s \quad \forall j \in (J \cup \{1\}),$$

and moving slightly the lines of integration, we obtain

$$\sum_{\substack{\varepsilon\in\{\pm 1\}^k\\ \pm_1 1=-1}} \rho_\Upsilon(\varepsilon) \sum_{\pm_* 1\in\{\pm 1\}} \mathcal{S}_{I;\varepsilon^*;\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathcal{U}_{I_1;\boldsymbol{\alpha},\boldsymbol{\beta}}, \tag{7-31}$$

where for $\mathcal{I} \subseteq \{1,\dots,k\}$ (with $|\mathcal{I}| \geq 2$), $\mathcal{J} := \{1,\dots,k\} \setminus \mathcal{I}$, we define

$$\begin{aligned}\mathcal{U}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}} := -2^{|\mathcal{J}|} i^{|\Upsilon\cap\mathcal{J}|} \sum_{\ell} \sideset{}{^*}\sum_{h \pmod \ell} \frac{P(\ell/L)}{(2\pi i)^{1+k}} \int_{(c_s,\boldsymbol{c_u})} \frac{q^{ks-1}}{\pi \ell^{\sum_{i\in\mathcal{I}}(1-\alpha_i+\beta_i)}} \frac{H''_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}(s)}{s} \\ \times \Bigl(\prod_{j\in\mathcal{J}} D_{j;\alpha_j,\beta_j}\bigl(\tfrac{1}{2}+u_j+s, \tfrac{h}{\ell}\bigr) \widetilde{P}(u_j) N_j^{u_j} \Bigr) \Gamma\bigl(1+|\mathcal{I}|\bigl(s-\tfrac{1}{2}\bigr) + \textstyle\sum_{i\in\mathcal{I}}(u_i+\alpha_i)\bigr) \\ \times \sin\Bigl(\frac{\pi}{2}|\Upsilon\cap\mathcal{I}| + \frac{\pi}{2}|\mathcal{I}|\bigl(s-\tfrac{1}{2}\bigr) + \frac{\pi}{2}\sum_{i\in\mathcal{I}}(u_i+\alpha_i) \Bigr) \\ \times F\bigl(1+|\mathcal{I}|(s-\tfrac{1}{2}) + \textstyle\sum_{i\in\mathcal{I}}(u_i+\alpha_i), \tfrac{qh}{\ell}\bigr) \Bigl(\displaystyle\prod_{i\in\mathcal{I}} \frac{\pi^{\frac{1}{2}} \widetilde{P}(u_i) N_i^{u_i}}{(2q)^{u_i+\alpha_i+s-\frac{1}{2}}} \frac{\Gamma_i\bigl(\frac{1}{4}-\frac{u_i+\alpha_i+s}{2}\bigr)}{\Gamma_i\bigl(\frac{1}{4}+\frac{u_i+\alpha_i+s}{2}\bigr)} \Bigr) \, ds \, \boldsymbol{du},\end{aligned} \tag{7-32}$$

with lines of integration

$$c_{u_i} = \tfrac{1}{2} - \Re(\alpha_i) - 3\frac{\varepsilon}{k} - \varepsilon \quad \forall i \in \{1,\dots,k\}, \qquad c_s := \varepsilon/k, \tag{7-33}$$

and, recalling (7-11),

$$H''_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}(s) := G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) \prod_{i\in\mathcal{I}} \zeta(1-\alpha_i+\beta_i). \tag{7-34}$$

For future use we remark that if we move $c_{u'_i}$ to $c_{u'_i} = -\frac{1}{2} - \Re(\alpha_1) - 5\varepsilon$ for some $i' \in \mathcal{I}$ we get

$$\mathcal{U}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}} \ll N_{i'}^{-1+A\varepsilon} q^{A\varepsilon} (N_1 \cdots N_k)^{\frac{1}{2}} L^{-\varepsilon}. \tag{7-35}$$

Also, if $|\mathcal{I}| > |\mathcal{J}|$, then moving the line of integration to c_{u_j} to $c_{u_j} = -\frac{1}{2} + 5\frac{\varepsilon}{k} - \varepsilon$ for all $j \in \mathcal{J}$ (leaving the other lines of integration as in (7-33)), we obtain

$$\mathcal{U}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}} \ll q^{-1+A\varepsilon} (N_1 \cdots N_k)^{\frac{1}{2}} L^{-\varepsilon} \prod_{j\in\mathcal{J}} N_j^{-1+A\varepsilon}. \tag{7-36}$$

Finally, moving c_s to $c_s = \frac{1}{2} + B - 3\frac{\varepsilon}{k}$ and c_{u_i} to $c_{u_i} = -B$ for all $i = 1,\dots,k$ we obtain

$$\mathcal{U}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}} \ll_B (N_1 \cdots N_k/q^k)^{-B} q^{k/2-1+A\varepsilon}, \tag{7-37}$$

if $|\mathcal{I}| \geq 2$.

We are now ready to complete the proof of Lemma 14. Using (7-26), (7-29) and (7-31) we obtain

$$\begin{aligned}\sum_{\varepsilon\in\{\pm1\}^k}\rho_\Upsilon(\varepsilon)\mathcal{O}''_{\varepsilon,\alpha,\beta}&=\sideset{}{^\dagger}\sum_{L}\sum_{\substack{I\cup J=\{2,\dots,k\}\\ I\cap J=\varnothing,\ |I|>|J|}}\sum_{\substack{\{\alpha'_i,\beta'_i\}=\{\alpha_i,\beta_i\}\ \forall i\in I_1\\ (\alpha'_j,\beta'_j)=(\alpha_j,\beta_j)\ \forall j\in J}}2\mathcal{U}_{I_1;\boldsymbol{\alpha},\boldsymbol{\beta}}\\&\quad+O\biggl(\frac{N_1^{A\varepsilon}}{q^{1-A\varepsilon}}\biggl(\frac{q^{\vartheta}N_1^{k/2}+q^{k/2-\frac13+\vartheta/3}}{(N_2\cdots N_k)^{\frac12}}+(q^{\frac16+\vartheta/3}N_1^{-\frac12}+1)(N_2\cdots N_k)^{\frac12}\biggr)\biggr)\\&=\mathcal{M}_{\alpha,\beta}+O\biggl(\frac{N_1^{A\varepsilon}}{q^{1-A\varepsilon}}\biggl(\frac{q^{\vartheta}N_1^{k/2}+q^{k/2-\frac13+\vartheta/3}}{(N_2\cdots N_k)^{\frac12}}+(q^{\frac16+\vartheta/3}N_1^{-\frac12}+1)(N_2\cdots N_k)^{\frac12}\biggr)\biggr)\end{aligned}$$

where

$$\mathcal{M}_{\alpha,\beta}(N_1,\dots,N_k):=\sideset{}{^\dagger}\sum_{L}\sum_{\substack{\mathcal{I}\cup\mathcal{J}=\{1,\dots,k\}\\ \mathcal{I}\cap\mathcal{J}=\varnothing,\ |\mathcal{I}|>|\mathcal{J}|+1}}\sum_{\substack{\{\alpha'_i,\beta'_i\}=\{\alpha_i,\beta_i\}\ \forall i\in\mathcal{I}\\ (\alpha'_j,\beta'_j)=(\alpha_j,\beta_j)\ \forall j\in\mathcal{J}}}2\mathcal{U}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}. \tag{7-38}$$

and $\mathcal{U}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}$ as defined in (7-32). Noticed that in the last step we used (7-36) to extend the sum over the subsets of $\{1\ \dots,k\}$ to include also the sets $\mathcal{I}$ that do not contain 1. Moreover, by (7-35) and (7-36) we also have

$$\mathcal{M}_{\alpha,\beta}(N_1,\dots,N_k)\ll q^{A\varepsilon}(N_1\cdots N_k)^{\frac12}N_1^{-1+A\varepsilon}.$$

and thus the proof of Lemma 14 is complete. Also, by (7-37) for any $B>0$ we have

$$\mathcal{M}_{\alpha,\beta}(N_1,\dots,N_k)\ll_B q^{k/2-1+A\varepsilon}(N_1\cdots N_k/q^k)^{-B} \tag{7-39}$$

We remark that we reached a formula for $\mathcal{M}_{\alpha,\beta}$ which is completely symmetric in the $N_1,\dots,N_k$. This is important in order to remove the assumption that N_1 is the largest of $N_1,\dots,N_k$, so that we can sum over the partitions of unity.

8. Assembling the main terms

In this section we prove Lemma 16.

We start by moving c_{u_i} to $c_{u_i}=0$ for all $i\in\mathcal{I}$ and c_s to $\frac12-3\frac{\varepsilon}{k}$ (we can do this without passing through any pole nor having a problem of convergence). Then, after extending the sum over the partitions of unity $L,N_1,\dots,N_k$ using (7-39) and summing over them using Lemma 21 we obtain

$$\sideset{}{^\dagger}\sum_{\substack{N_1,\dots,N_k\\ N_1\cdots N_k\ll q^{k+\varepsilon}}}\mathcal{M}_{\alpha,\beta}(N_1,\dots,N_k)=\sum_{\substack{\mathcal{I}\cup\mathcal{J}=\{1,\dots,k\}\\ \mathcal{I}\cap\mathcal{J}=\varnothing,\ |\mathcal{I}|>|\mathcal{J}|+1}}\sum_{\substack{\{\alpha'_i,\beta'_i\}=\{\alpha_i,\beta_i\}\ \forall i\in\mathcal{I}\\ (\alpha'_j,\beta'_j)=(\alpha_j,\beta_j)\ \forall j\in\mathcal{J}}}2\mathcal{V}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}+O(1)$$

with

$$\mathscr{V}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}} := -2^{|\mathcal{J}|} i^{|\Upsilon\cap\mathcal{J}|} \frac{1}{2\pi i}\int_{(c_s)} \sum_{\ell} \sum_{h \pmod{\ell}}^{*} \frac{q^{\frac{1}{2}|\mathcal{I}|+|\mathcal{J}|s-1}}{\ell^{\sum_{i\in\mathcal{I}}(1-\alpha_i+\beta_i)}} \Gamma\big(1+|\mathcal{I}|\big(s-\tfrac{1}{2}\big)+\textstyle\sum_{i\in\mathcal{I}}\alpha_i\big)$$
$$\times \Bigg(\prod_{j\in\mathcal{J}} D_{j;\alpha_j,\beta_j}\big(\tfrac{1}{2}+s,\tfrac{h}{\ell}\big)\Bigg)\Bigg(\prod_{i\in\mathcal{I}} \frac{(2\pi)^{\frac{1}{2}}}{2^{\alpha_i+s}q^{\alpha_i}} \frac{\Gamma_i\big(\frac{1}{4}-\frac{\alpha_i+s}{2}\big)}{\Gamma_i\big(\frac{1}{4}+\frac{\alpha_i+s}{2}\big)}\Bigg) \frac{H''_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}(s)}{\pi s}$$
$$\times F\big(1+|\mathcal{I}|\big(s-\tfrac{1}{2}\big)+\textstyle\sum_{i\in\mathcal{I}}\alpha_i,\tfrac{h}{\ell}\big) \sin\Bigg(\frac{\pi}{2}|\Upsilon\cap\mathcal{I}|+\frac{\pi}{2}|\mathcal{I}|\big(s-\tfrac{1}{2}\big)+\frac{\pi}{2}\sum_{i\in\mathcal{I}}\alpha_i\Bigg)\,ds, \tag{8-1}$$

and line of integration $c_s = \frac{1}{2} - 3\frac{\varepsilon}{k}$.

Now, for each integral we move the line of integration c_s to

$$c_s = \max\Bigg(0, -\frac{1}{2}+\frac{|\mathcal{J}|+\frac{3}{2}}{|\mathcal{I}|+|\mathcal{J}|-\frac{1}{2}}\Bigg)+9\frac{\varepsilon}{k} = \begin{cases} \frac{3}{4k-2}+9\frac{\varepsilon}{k} & \text{if } |\mathcal{I}|=|\mathcal{J}|+2=\frac{k}{2}+1 \text{ with } k \text{ even},\\ 9\frac{\varepsilon}{k} & \text{if } |\mathcal{I}|>|\mathcal{J}|+2. \end{cases}$$

picking up the residue of the pole of the Γ-function at

$$s' = s'(\boldsymbol{\alpha}) = \frac{1}{2} - \frac{1+\sum_{i\in\mathcal{I}}\alpha_i}{|\mathcal{I}|}$$

(unless $k=4$, $|\mathcal{I}|=3$ in which case we stay on the right of such pole). Notice that Lemma 10 guarantees the convergence of the sum over ℓ on the new line of integration. Also, a quick computation shows that if $\mathcal{I}\neq I_k := \{1,\dots,k\}$ (and $|\mathcal{I}|>|\mathcal{J}|+1$) then

$$\frac{1}{2}|\mathcal{I}|+|\mathcal{J}|c_s - 1 \le \frac{k}{2}-\frac{3}{2}+\iota_k+9\frac{\varepsilon}{k}$$

where $\iota_k = \frac{3}{14}$ if $k=4$ and $\iota_k=0$ otherwise. In particular, if $\mathcal{I}\neq I_k$, then by (3-8) the contribution of the integral on the new line of integration is $O(q^{k/2-\frac{3}{2}+\iota_k+A\varepsilon})$ and we obtain

$$\mathscr{V}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathscr{X}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}} + O(q^{k/2-\frac{3}{2}+\iota_k+A\varepsilon}), \tag{8-2}$$

where

$$\mathscr{X}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}} := -\frac{2^{|\mathcal{J}|} i^{|\Upsilon\cap\mathcal{J}|}}{|\mathcal{I}|} \sum_{\ell} \sum_{h \pmod{\ell}}^{*} \frac{q^{\frac{1}{2}|\mathcal{I}|+|\mathcal{J}|s'-1}}{\ell^{\sum_{i\in\mathcal{I}}(1-\alpha_i+\beta_i)}} F\big(0,\tfrac{h}{\ell}\big)\sin\Big(\frac{\pi}{2}(|\Upsilon\cap\mathcal{I}|-1)\Big)$$
$$\times \Bigg(\prod_{j\in\mathcal{J}} D_{j;\alpha_j,\beta_j}\big(\tfrac{1}{2}+s',\tfrac{h}{\ell}\big)\Bigg)\Bigg(\prod_{i\in\mathcal{I}} \frac{(2\pi)^{\frac{1}{2}}}{2^{s'+\alpha_i}q^{\alpha_i}} \frac{\Gamma_i\big(\frac{1}{4}-\frac{\alpha_i+s'}{2}\big)}{\Gamma_i\big(\frac{1}{4}+\frac{\alpha_i+s'}{2}\big)}\Bigg) \frac{H''_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}(s')}{\pi s'}.$$

(Notice that (8-2) holds also in the case $k=4$, $|\mathcal{I}|=3$, since from (3-5) and a trivial bound it follows that $\mathscr{X}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}$ is convergent and $O(q^{\frac{2}{3}+A\varepsilon}) = O(q^{k/2-\frac{3}{2}+\iota_k+A\varepsilon})$).

If $\mathcal{I}=I_k$, then

$$\mathscr{V}_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathscr{X}_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}} + \mathscr{V}'_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}},$$

where $\mathscr{V}'_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}}$ is as in (8-1), but with the line of integration $c_s = 9\varepsilon/k$.

Now, notice that if $|\Upsilon\cap\mathcal{J}|$ is odd (and thus so is $|\Upsilon\cap\mathcal{I}|$ since $|\Upsilon|$ is even), then the sine in the expression defining $\mathscr{X}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}$ is equal to 0 and thus so is $\mathscr{X}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}$. If $|\Upsilon\cap\mathcal{J}|$ is even, then the product of the Estermann functions in the definition of $\mathscr{X}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}$ is invariant under the change $h\mapsto -h$; in particular, using the identity $F(0,h/\ell)+F(0,-h/\ell)=-1$ (which follows immediately from (3-5)), we obtain $\mathscr{X}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}=-\mathscr{X}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}+\mathscr{K}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}$ and so $\mathscr{X}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}=\frac{1}{2}\mathscr{K}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}$, where

$$\begin{aligned}\mathscr{K}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}:=&-\frac{2^{|\mathcal{J}|}}{|\mathcal{I}|}\sum_{\ell}\sideset{}{^*}\sum_{h \pmod{\ell}}\frac{q^{\frac12|\mathcal{I}|+|\mathcal{J}|s'-1}}{\ell^{\sum_{i\in\mathcal{I}}(1-\alpha_i+\beta_i)}}\Bigl(\prod_{j\in\mathcal{J}}D_{j;\alpha_j,\beta_j}\bigl(\tfrac12+s',\tfrac{h}{\ell}\bigr)\Bigr)\\&\qquad\times\Biggl(\prod_{i\in\mathcal{I}}\frac{(2\pi)^{\frac12}}{2^{s+\alpha_i}q^{\alpha_i}}\frac{\Gamma_i\bigl(\frac14-\frac{\alpha_i+s'}{2}\bigr)}{\Gamma_i\bigl(\frac14+\frac{\alpha_i+s'}{2}\bigr)}\Biggr)\frac{H''_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}(s')}{\pi s'}\\=&-\mathscr{D}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}},\end{aligned}$$

where $\mathscr{D}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}$ is as in (6-2), since $\frac12|\mathcal{I}|+|\mathcal{J}|s'-1=ks'+\bigl(\frac12-s'\bigr)|\mathcal{I}|-1=ks'+\sum_{i\in\mathcal{I}}\alpha_i$ and by the definition (7-34) of $H''_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}(s')$. It follows that

$$\sum_{\substack{\mathcal{I}\cup\mathcal{J}=\{1,\dots,k\}\\ \mathcal{I}\cap\mathcal{J}=\varnothing\\ \frac{k}{2}+\frac34<|\mathcal{I}|}}\ \sum_{\substack{\{\alpha_i',\beta_i'\}=\{\alpha_i,\beta_i\}\ \forall i\in\mathcal{I}\\ (\alpha_j',\beta_j')=(\alpha_j,\beta_j)\ \forall j\in\mathcal{J}}}2\mathscr{V}_{\mathcal{I};\boldsymbol{\alpha},\boldsymbol{\beta}}=\sum_{\{\alpha_i',\beta_i'\}=\{\alpha_i,\beta_i\}\ \forall i\in\mathcal{I}_k}2\mathscr{V}'_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}}-\mathscr{D}_{\boldsymbol{\alpha},\boldsymbol{\beta}}+O(q^{k/2-\frac12+\iota_k+A\varepsilon})$$

and thus to conclude the proof of Lemma 16, we just need to show $\mathscr{V}'_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}}+X_{\boldsymbol{\beta},\boldsymbol{\alpha}}\mathscr{V}'_{I_k;-\boldsymbol{\beta},-\boldsymbol{\alpha}}=\mathscr{M}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ with $\mathscr{M}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ as in (2-3). First, we need the following lemma.

Lemma 24. *For $\Re(s_1+s_1)>2$ and $\Re(s_2)>1$ we have*

$$\sum_{\ell}\frac{1}{\ell^{s_2}}\sideset{}{^*}\sum_{h \pmod{\ell}}F\bigl(s_1,\tfrac{h}{\ell}\bigr)=\frac{\zeta(s_1)\zeta(s_1+s_2-1)}{\zeta(s_2)}. \tag{8-3}$$

Proof. From the functional equation for $F(s,x)$ and the Phragmén–Lindelöf principle one sees that if $|s_1-1|>\varepsilon'>0$, then if

$$\sideset{}{^*}\sum_{h \pmod{\ell}}\bigl|F\bigl(s_1,\tfrac{h}{\ell}\bigr)\bigr|\ll_{s,\varepsilon,\varepsilon'}1+\ell^{1-\Re(s_1)+\varepsilon}$$

for all $\varepsilon>0$. It follows that the left hand side of (8-3) defines a meromorphic function in s_1,s_2 on the region $\Re(s_2)>1$, $\Re(s_1+s_2)>2$. Now, assume $\Re(s_1),\Re(s_2)>1$. Expanding F into its Dirichlet expansion (3-3), executing the sum over h, and using (7-2), we obtain

$$\sum_{\ell}\frac{1}{\ell^{s_2}}\sideset{}{^*}\sum_{h \pmod{\ell}}F\bigl(s_1,\tfrac{h}{\ell}\bigr)=\sum_{\ell}\frac{1}{\ell^{s_2}}\sum_{n}\frac{c_\ell(n)}{n^{s_1}}=\frac{1}{\zeta(s_2)}\sum_{n}\frac{\sigma_{1-s_2}(n)}{n^{s_1}}=\frac{\zeta(s_1)\zeta(s_1+s_2-1)}{\zeta(s_2)}.$$

The lemma then follows by analytic continuation. ∎

Applying this Lemma, we see that

$$\mathscr{V}'_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}} = -\frac{q^{k/2-1}}{2\pi i}\int_{(c_s)} \Gamma\Big(1+ks-\frac{k}{2}+\sum_{i=1}^{k}\alpha_i\Big)\frac{\zeta\big(1+ks-\frac{k}{2}+\sum_{i=1}^{k}\alpha_i\big)\zeta\big(\frac{k}{2}+ks+\sum_{i=1}^{k}\beta_i\big)}{\zeta\big(k-\sum_{i=1}^{k}(\alpha_i-\beta_i)\big)} \times \sin\Big(\frac{\pi}{2}\Big(|\Upsilon|-\frac{k}{2}+ks+\sum_{i=1}^{k}\alpha_i\Big)\Big)\Big(\prod_{i=1}^{k}\frac{\Gamma_i\big(\frac14-\frac{\alpha_i+s}{2}\big)}{\Gamma_i\big(\frac14+\frac{\alpha_i+s}{2}\big)}\frac{\pi^{\frac12}q^{-\alpha_i}}{2^{s-\frac12+\alpha_i}}\Big)\frac{H''_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}}(s)}{\pi s}\,ds,$$

so that by the functional equation (using that $|\Upsilon|$ is even) and the definition (7-34) of H'' we obtain

$$\mathscr{V}'_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}} = (-1)^{|\Upsilon|/2}\frac{q^{\frac{k}{2}-1}}{2\pi i}\int_{(c_s)}\frac{\zeta\big(\frac{k}{2}-ks-\sum_{i=1}^{k}\alpha_i\big)\zeta\big(\frac{k}{2}+ks+\sum_{i=1}^{k}\beta_i\big)}{\zeta\big(k-\sum_{i=1}^{k}(\alpha_i-\beta_i)\big)} \times G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)\Big(\prod_{i=1}^{k}\zeta(1-\alpha_i+\beta_i)\frac{\Gamma_i\big(\frac14-\frac{\alpha_i+s}{2}\big)\Gamma_i\big(\frac14+\frac{\beta_i+s}{2}\big)}{\Gamma_i\big((\frac12+\alpha_i)/2\big)\Gamma_i\big((\frac12+\beta_i)/2\big)}\Big(\frac{q}{\pi}\Big)^{-\alpha_i}\Big)\frac{ds}{s}.$$

Notice that changing s into $-s$ we obtain exactly $-X_{\boldsymbol{\beta},\boldsymbol{\alpha}}$ times the analogous term coming from $\mathscr{V}'_{I_k;-\boldsymbol{\beta},-\boldsymbol{\alpha}}$, but with line of integration $c_s=-9\frac{\varepsilon}{k}$. Thus, $\mathscr{V}'_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}}+X_{\boldsymbol{\beta},\boldsymbol{\alpha}}\mathscr{V}'_{I_k;-\boldsymbol{\beta},-\boldsymbol{\alpha}}$ coincides with the residue of the above integral at $s=0$, that is

$$\mathscr{V}'_{I_k;\boldsymbol{\alpha},\boldsymbol{\beta}}+X_{\boldsymbol{\beta},\boldsymbol{\alpha}}\mathscr{V}'_{I_k;-\boldsymbol{\beta},-\boldsymbol{\alpha}} = (-1)^{|\Upsilon|/2}q^{k/2-1}\frac{\zeta\big(\frac{k}{2}-\sum_{i=1}^{k}\alpha_i\big)\zeta\big(\frac{k}{2}+\sum_{i=1}^{k}\beta_i\big)}{\zeta\big(k-\sum_{i=1}^{k}(\alpha_i-\beta_i)\big)}\prod_{i=1}^{k}\zeta(1-\alpha_i+\beta_i)\frac{\Gamma_i\big(\frac14-\frac{\alpha_i}{2}\big)}{\Gamma_i\big(\frac14+\frac{\alpha_i}{2}\big)}\Big(\frac{q}{\pi}\Big)^{-\alpha_i}.$$

Thus, Lemma 16 follows.

9. The terms far from the diagonal

We will use the following result of Young to prove Lemma 15.

Lemma 25. *Let q be prime and let L, $K \ll q^{1+\varepsilon}$ and let W be a smooth function with compact support on $\mathbb{R}_{>0}$. Then,*

$$\sum_{0<\ell<L}\Big|\sum_{(k,q)=1}\mathrm{e}\Big(\frac{\ell\bar{k}}{q}\Big)W\Big(\frac{k}{K}\Big)\Big| \ll L^{\frac12}q^{\frac34+\varepsilon}+q^{\varepsilon}K^{\frac12}L.$$

Proof. This is Proposition 4.3 of [Young 2011a] with the extra condition requiring q to be prime which easily allows us to remove the condition $(q,\ell)=1$ from the first sum. ☐

Proof of Lemma 15. For simplicity we shall take $\boldsymbol{\alpha}=\boldsymbol{\beta}=\mathbf{0}:=(0,\dots,0)$, as the shifts don't play any role in this lemma and the same argument with obvious modifications works also when $\alpha_i,\beta_i \ll 1/\log q$.

By symmetry, we can assume that N_1 is the maximum of the N_i and that N_2 is the second largest. Also, we assume $N_1\cdots N_k \ll q^{k+\varepsilon}$ and $N_1 \gg q^{1+3\varepsilon}$, since otherwise the result is trivial.

Now, we start by observing that we can remove the condition $\pm_1 n_1 \pm_2 \cdots \pm_k n_k \neq 0$ in $\mathcal{O}''_\varepsilon := \mathcal{O}''_{\varepsilon,\mathbf{0},\mathbf{0}}$ at a cost of an admissible error:

$$\mathcal{O}''_\varepsilon = \sum_{d\,|\,q} d\frac{\mu(q/d)}{\varphi(q)} \sum_{d|(\pm_1 n_1 \pm_2 \cdots \pm_k n_k)} \frac{d(n_1)\cdots d(n_k)}{(n_1\cdots n_k)^{\frac12}} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}\Big(\frac{n_1\cdots n_k}{q^k}\Big) P\Big(\frac{n_1}{N_1}\Big)\cdots P\Big(\frac{n_k}{N_k}\Big) + O\big(q^{A\varepsilon}(N_1\cdots N_k)^{\frac12}/N_1\big).$$

Next, we decompose n_1 into $n_1 = f_1 g_1$ and attach to the new variables two partitions of unity so that $f_1 \asymp F_1$, $g_1 \asymp G_1$ with $F_1 \geq G_1$ and $F_1 G_1 \asymp N_1$. We shall also add the condition $(g_1, q) = 1$ at a cost of an error which is easily seen to be $O(q^{k/2-2+A\varepsilon})$. Writing V and $P(n_1/N_1)$ in terms of their Mellin transform, we obtain

$$\mathcal{O}''_\varepsilon = \sideset{}{^\dagger}\sum_{\substack{F_1G_1\asymp N_1\\ G_1\leq F_1}} \int'_{(c_s,c_{u_1})} q^{ks} N_1^{u_1} \sum_{0\leq |m|<M} w_m(s)\mathscr{A}_m(s+u_1) G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) g_{\boldsymbol{\alpha},\boldsymbol{\beta}} \widetilde{P}(u_1)\,\frac{ds}{s}\,du_1 + O\big(q^{A\varepsilon}(N_1\cdots N_k)^{\frac12}/N_1 + q^{k/2-2+A\varepsilon}\big), \quad (9\text{-}1)$$

where $M := \min(2kN_2, q)$, the $\int'$ indicates that the integrals are truncated at $|u_1|, |s| \leq q^\varepsilon$, the lines of integrations are $c_{u_1} = 0$, $c_s = \varepsilon/k$, and

$$\mathscr{A}_m(s) := \sum_{d\,|\,q} d\frac{\mu(q/d)}{\varphi(q)} \sum_{(g_1,q)=1} \sum_{f_1g_1\equiv m \pmod d} \frac{1}{(f_1g_1)^{\frac12+s}} P\Big(\frac{f_1}{F_1}\Big)P\Big(\frac{g_1}{G_1}\Big),$$
$$w_m(s) := \sum_{\pm_2 n_2 \pm_3 \cdots \pm_k n_k \equiv -m \pmod q} \frac{d(n_2)\cdots d(n_k)}{(n_2\cdots n_k)^{\frac12+s}} P\Big(\frac{n_2}{N_2}\Big)\cdots P\Big(\frac{n_k}{N_k}\Big).$$

Now, we apply Poisson's summation formula to the sum over f_1 and we see that for $\Re(s) = \varepsilon/k$

$$\mathscr{A}_m(s) = \sum_{d\,|\,q} \frac{\mu(q/d)}{\varphi(q)} \sum_{(g_1,q)=1} \frac{P(g_1/G_1)}{g_1^{\frac12+s}} \sum_{0\leq|\ell|\leq \frac{dq^{A\varepsilon}}{F_1}} \mathrm{e}\Big(\frac{\ell m \overline{g_1}}{d}\Big) \int_0^\infty \frac{P(x/F_1)}{x^{\frac12+s}}\, \mathrm{e}\Big(-\frac{\ell x}{d}\Big)\,dx + O(q^{-1})$$
$$= \mathscr{A}^*_m(s) + O(q^{-1}),$$

where

$$\mathscr{A}^*_m(s) = \frac{F_1^{\frac12-s}}{\varphi(q)} \sum_{(g_1,q)=1} \frac{P(g_1/G_1)}{g_1^{\frac12+s}} \sum_{0<|\ell|\leq L} \mathrm{e}\Big(\frac{\ell m\overline{g_1}}{q}\Big) \int_0^\infty \frac{P(x)}{x^{\frac12+s}}\,\mathrm{e}\Big(-\frac{\ell F_1 x}{q}\Big)\,dx$$

and $L = q^{1+A\varepsilon}/F_1$. Indeed, the sum over ℓ in the $d=1$ summands contains only the term $\ell = 0$ which cancel with the $\ell = 0$ term from $d = q$. Thus, (9-1) becomes

$$\mathcal{O}''_\varepsilon = \sideset{}{^\dagger}\sum_{\substack{F_1G_1\asymp N_1,\\ G_1\leq F_1}} \int'_{(c_s,c_{u_1})} q^{ks} N_1^{u_1} \sum_{0\neq |m|<M} w_m(s)\mathscr{A}^*_m(s+u_1) G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) g_{\boldsymbol{\alpha},\boldsymbol{\beta}} \widetilde{P}(u_1)\,\frac{ds}{s}\,du_1 + O(q^{k/2-\frac32+A\varepsilon} + q^{A\varepsilon}(N_1\cdots N_k)^{\frac12}/N_1),$$

since the contribution of the terms with $m=0$ can be bounded trivially by

$$\begin{aligned}&\ll \frac{L(F_1G_1)^{\frac12}}{q}(1+N_2/q)N_2^{-\frac12}(N_3\cdots N_k)^{\frac12}q^{A\varepsilon}\\&\ll q^{A\varepsilon}N_2^{-\frac12}(N_3\cdots N_k)^{\frac12}+q^{-1+A\varepsilon}(N_2\cdots N_k)^{\frac12}\ll q^{k/2-\frac32+A\varepsilon}.\end{aligned}$$

since $N_2^{-1}N_3\cdots N_k\ll q^{k-3+A\varepsilon}$ and $N_2\cdots N_k\ll q^{k-2+A\varepsilon}$. Also, we assume $N_1\le F_1^2\le q^{2+2A\varepsilon}$, since otherwise $\mathscr{A}_m^*(s)$ is identically zero.

Changing the order of summation and integration and bounding trivially $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s)g_{\boldsymbol{\alpha},\boldsymbol{\beta}}\widetilde{P}(u_1)$, we see that

$$\mathscr{O}''_\varepsilon\ll \sum_{\substack{F_1G_1\asymp N_1\\ G_1\le F_1}}^{\dagger}\int'_{(c_s,c_{u_1})}q^{\varepsilon}|E(s,u_1)|\,|ds\,du_1|+q^{k/2-\frac32+A\varepsilon}+q^{A\varepsilon}\frac{(N_1\cdots N_k)^{\frac12}}{N_1}, \tag{9-2}$$

where

$$\begin{aligned}E(s,u_1)&:=\frac{F_1^{\frac12}}{\varphi(q)}\sum_{0<|\ell|\le L}\sum_{0<|m|<M}|w_m(s)|\Bigg|\sum_{(g_1,q)=1}\frac{P(g_1/G_1)}{g_1^{\frac12+s+u_1}}\,\mathrm{e}\,\frac{\ell m\overline{g_1}}{q}\Bigg|\\&\ll\frac{F_1^{\frac12}}{qG_1^{\frac12}}\max_{0<|r|\le R}c_r\sum_{0\ne|r|\le R}\Bigg|\sum_{(g_1,q)=1}P\Big(\frac{g_1}{G_1}\Big)\Big(\frac{G_1}{g_1}\Big)^{\frac12+s+u_1}\mathrm{e}\Big(\frac{r\overline{g_1}}{q}\Big)\Bigg|,\end{aligned}$$

with $R:=\min(2kLN_2,q)\le 2kq^{1+A\varepsilon}\min(N_2/F_1,1)$ and

$$\begin{aligned}c_r&:=\sum_{\substack{\ell m\equiv r\pmod q\\ 0<|m|\le M,\ 0<|\ell|\le L}}|w_m(s)|\ll\sum_{\substack{0<|\ell|\le L,\ n_2\asymp N_2,\dots,n_k\asymp N_k\\ (\pm_2n_2\pm_3\cdots\pm_kn_k)\ell\equiv -r\pmod q}}q^{A\varepsilon}d(n_2)\cdots d(n_2)(N_2\cdots N_k)^{-\frac12}\\&\ll\sum_{\substack{0<|\ell|\le L,\ |n|\ll kN_2\\ n\ell\equiv -r\pmod q}}q^{A\varepsilon}N_2^{-\frac12}(N_3\cdots N_k)^{\frac12}\ll\sum_{\substack{|a|\ll kLN_2\\ a\equiv -r\pmod q}}d(a)q^{A\varepsilon}N_2^{-\frac12}(N_3\cdots N_k)^{\frac12}\\&\ll q^{A\varepsilon}N_2^{-\frac12}(N_3\cdots N_k)^{\frac12}(1+LN_2/q).\end{aligned}$$

Thus, by Lemma 25, for $|s|,|u_1|\ll q^{A\varepsilon}$, $\Re(s)=\varepsilon/k$, $\Re(u_1)=0$ we have

$$\begin{aligned}E(s,u_1)&\ll q^{-1+A\varepsilon}F_1^{\frac12}(G_1N_2)^{-\frac12}(N_3\cdots N_k)^{\frac12}(1+LN_2/q)(R^{\frac12}q^{\frac34}+G_1^{\frac12}R)\\&\ll q^{A\varepsilon}(F_1/G_1N_2)^{\frac12}(N_3\cdots N_k)^{\frac12}(1+N_2/F_1)\min\big(F_1^{-\frac12}N_2^{\frac12}q^{\frac14}+G_1^{\frac12}N_2/F_1,\ q^{\frac14}+G_1^{\frac12}\big)\\&=q^{A\varepsilon}(N_3\cdots N_k)^{\frac12}(1+N_2/F_1)\min\Bigg(\frac{q^{\frac14}}{G_1^{\frac12}}+\frac{N_2^{\frac12}}{F_1^{\frac12}},\ \frac{F_1^{\frac12}}{N_2^{\frac12}}\frac{q^{\frac14}}{G_1^{\frac12}}+\frac{F_1^{\frac12}}{N_2^{\frac12}}\Bigg).\end{aligned}$$

For $x, y > 0$ we have $(1+x^2)\min(y+x, y/x+1/x) \le (x+1)(y+1)$, whence

$$
\begin{aligned}
E(s,u_1) &\ll q^{A\varepsilon}(N_3\cdots N_k)^{\frac{1}{2}}\big(N_2^{\frac{1}{2}}q^{\frac{1}{4}}N_1^{-\frac{1}{2}} + N_2^{\frac{1}{2}}F_1^{-\frac{1}{2}} + q^{\frac{1}{4}+\varepsilon}G_1^{-\frac{1}{2}} + 1\big)\\
&\ll q^{A\varepsilon}(N_3\cdots N_k)^{\frac{1}{2}}\big(N_2^{\frac{1}{2}}N_1^{-\frac{1}{4}} + q^{\frac{3}{4}}N_1^{-\frac{1}{2}} + 1\big)\\
&\ll q^{A\varepsilon}\big((N_2\cdots N_k)^{\frac{1}{2}}N_1^{-\frac{1}{4}} + (N_2\cdots N_k)^{\frac{1}{2}-1/(2(k-1))}(q^{\frac{3}{4}}N_1^{-\frac{1}{2}}+1)\big),
\end{aligned}
\tag{9-3}
$$

where in the second inequality we used that $N_1 \gg q$, $F_1^{-\frac{1}{2}} \le N_1^{-\frac{1}{4}}$ and $G_1^{-\frac{1}{2}} \asymp (F_1/N_1)^{\frac{1}{2}} \le N_1^{-\frac{1}{2}}q^{\frac{1}{2}+A\varepsilon}$, and in the third one that $N_3\cdots N_k \le N_2^{k-2}$ (so that $N_3\cdots N_k \le (N_2\cdots N_k)^{(k-2)/(k-1)}$). The lemma then follows by inserting (9-3) in (9-2). □

10. A Mellin formula

In this section we prove a formula to separate the variables in expressions of the form $(\pm_1 x_1 \pm_2 \cdots \pm_\kappa x_\kappa)^{-s}$ which generalizes the Mellin transforms given in the following lemma.

Lemma 26. *Let $x, y > 0$. Then*

$$
(x+y)^{-b} = \frac{1}{2\pi i}\int_{(c_v)} \frac{\Gamma(v)\Gamma(b-v)}{\Gamma(b)} x^{v-b}y^{-v}\,dv, \tag{10-1}
$$

for $0 < c_v < \Re(b)$. Moreover, for $\Re(b) < 0 < c_w$, we have

$$
(x-y)^{-b}\chi_{\mathbb{R}_{>0}}(x-y) = \frac{1}{2\pi i}\int_{(c_w)} \frac{\Gamma(w)\Gamma(1-b)}{\Gamma(1-b+w)} x^{w-b}y^{-w}\,dw, \tag{10-2}
$$

where $\chi_X(x)$ is the indicator function of the set X.

Equation (10-1) can be used repeatedly to give a formula for $(x_1+\cdots+x_\kappa)^{-s}$ valid for $\Re(s) > 0$. However, it is not straightforward to obtain a satisfactory formula valid in the case when there are some minus signs, as the integrals obtained by repeatedly applying (10-1) and (10-2) are not absolutely convergent. The following Lemma overcomes this problem by introducing an extra integration.

Lemma 27. *Let $\kappa \ge 2$ and $x_1, \dots x_\kappa > 0$. Let $\varepsilon = (\pm_1, \dots, \pm_\kappa 1) \in \{\pm 1\}^\kappa$, with $\pm_1 1 = -1$. Let $B \in \mathbb{Z}_{\ge 0}$ be such that $\frac{\kappa}{2} + \frac{1}{2} < \Re(v_1) < B+1$. Moreover, let $c_{v_2}, \dots, c_{v_\kappa}, c'_{v_2}, \dots, c'_{v_\kappa} > 0$ be such that*

$$
\Re(v_1) + c_{v_2} + \cdots + c_{v_\kappa} < B+1 < \Re(v_1) + c'_{v_2} + \cdots + c'_{v_\kappa}. \tag{10-3}
$$

Then

$$
\begin{aligned}
&(\pm_2 x_2 \pm_3 \cdots \pm_\kappa x_\kappa)^{v_1-1}\chi_{\mathbb{R}_{>0}}(\pm_2 x_2 \pm_3 \cdots \pm_\kappa x_\kappa)\\
&= \sum_{\substack{\nu=(\nu_2,\dots,\nu_\kappa)\in\mathbb{Z}_{\ge 0}^{\kappa-1}\\ \nu_2+\cdots+\nu_\kappa = B\\ \nu_i = 0 \text{ if } \pm_i = -1}} \frac{B!}{\nu_2!\cdots\nu_\kappa!}\frac{1}{(2\pi i)^{\kappa-1}}\bigg(\int_{(c_{v_2},\dots,c_{v_\kappa})} - \int_{(c'_{v_2},\dots,c'_{v_\kappa})}\bigg)\frac{\Psi_{\varepsilon,B}(v_1,\dots,v_\kappa)}{x_2^{v_2-\nu_2}\cdots x_\kappa^{v_\kappa-\nu_\kappa}}\,dv_2\cdots dv_\kappa,
\end{aligned}
\tag{10-4}
$$

where

$$\begin{aligned}\Psi_{\varepsilon,B}(s_1,\dots,s_\kappa) &:= \frac{\Gamma(s_1)\cdots\Gamma(s_\kappa)}{\Gamma(V_{+;\varepsilon}(s_1,\dots,s_\kappa))\Gamma(V_{-;\varepsilon}(s_1,\dots,s_\kappa))}\,\frac{G(B+1-s_1-\cdots-s_\kappa)}{B+1-s_1-\cdots-s_\kappa},\\ V_{\pm;\varepsilon}(v_1,\dots,v_\kappa) &:= \sum_{\substack{1\le i\le\kappa\\ \pm_i 1=\pm 1}} v_i\end{aligned}\tag{10-5}$$

and $G(s)$ is any entire function such that $G(0)=1$ and $G(\sigma+it)\ll e^{-C_1|t|}(1+|\sigma|)^{C_2|\sigma|}$ for some fixed $C_1, C_2>0$.

Remarks. (1) If $\varepsilon=(-1,\dots,-1)$, then Ψ_ε has to be interpreted as being identically zero.

(2) If $\xi(s)$ is the Riemann ξ-function, then $G(s)=\xi(s)/\xi(0)$ satisfies the hypothesis of the lemma.

Before giving a proof for Lemma 27, we give the following lemma which implies that the integrals in (10-4) are absolutely convergent.

Lemma 28. *Let $s_i=\sigma_i+it_i$ for $i=1,\dots,\kappa$. Then, for some $A>0$ we have*

$$\Psi_{\varepsilon,B}(s_1,\dots,s_\kappa)\ll\frac{1}{\delta^\kappa}\frac{(1+B+|\sigma_1|+\cdots+|\sigma_\kappa|)^{A(1+B+|\sigma_1|+\cdots+|\sigma_\kappa|)}}{(1+|t_1|)^{\frac12-\sigma_1}\cdots(1+|t_\kappa|)^{\frac12-\sigma_\kappa}(1+|t_1|+\cdots+|t_\kappa|)^{\sigma_1+\cdots+\sigma_\kappa-1}},\tag{10-6}$$

provided that the s_i are located at a distance greater than $\delta>0$ from the poles of Ψ_ε.

Proof. By Stirling's formula (and the reflection's formula for the Gamma function), if the distance of $s=\sigma+it$ from the poles of $\Gamma(s)$ is greater than δ, then we have

$$\Gamma(s)\ll\frac{1}{\delta}(1+A_1|\sigma|)^{|\sigma|}(1+|t|)^{\sigma-\frac12}e^{-(\pi/2)|t|},\qquad \Gamma(s)^{-1}\ll(1+A_1|\sigma|)^{|\sigma|}(1+|t|)^{-\sigma+\frac12}e^{(\pi/2)|t|},$$

for some $A_1>0$. It follows that

$$\begin{aligned}&\Psi_{\varepsilon,B}(s_1,\dots,s_\kappa)\\ &\ll\frac{1}{\delta^\kappa}\frac{(1+B+|\sigma_1|+\cdots+|\sigma_\kappa|)^{A_2(1+B+|\sigma_1|+\cdots+|\sigma_\kappa|)}}{(1+|t_1|)^{\frac12-\sigma_1}\cdots(1+|t_\kappa|)^{\frac12-\sigma_\kappa}}\\ &\qquad\times\frac{e^{-(\pi/2)(|t_1|+\cdots+|t_\kappa|-|V_{+;\varepsilon}(t_1,\dots,t_\kappa)|-|V_{-;\varepsilon}(t_1,\dots,t_\kappa)|)-C_1|t_1+\cdots+t_\kappa|}}{(1+|V_{+;\varepsilon}(t_1,\dots,t_\kappa)|)^{V_{+;\varepsilon}(\sigma_1,\dots,\sigma_\kappa)-\frac12}(1+|V_{-;\varepsilon}(t_1,\dots,t_\kappa)|)^{V_{-;\varepsilon}(\sigma_1,\dots,\sigma_\kappa)-\frac12}},\end{aligned}\tag{10-7}$$

for some $A_2>0$. Now, we have

$$\frac{e^{-C_1|x+y|}}{(1+|x|)^{\eta_1}(1+|y|)^{\eta_2}}\ll\frac{(1+|\eta_1|+|\eta_2|)^{A_3(|\eta_1|+|\eta_2|)}}{(1+|x|+|y|)^{\eta_1+\eta_2}},$$

for some $A_3>0$ (depending on C_1). Thus, the factor on the second line of (10-7) is

$$\begin{aligned}&\ll(1+|\sigma_1|+\cdots+|\sigma_\kappa|)^{A_4(|\sigma_1|+\cdots+|\sigma_\kappa|)}\frac{e^{-(\pi/2)(|t_1|+\cdots+|t_\kappa|-|V_{+;\varepsilon}(t_1,\dots,t_\kappa)|-|V_{-;\varepsilon}(t_1,\dots,t_\kappa)|)}}{\big(1+|V_{+;\varepsilon}(t_1,\dots,t_\kappa)|+|V_{-;\varepsilon}(t_1,\dots,t_\kappa)|\big)^{\sigma_1+\cdots+\sigma_\kappa-1}}\\ &\ll\frac{(1+|\sigma_1|+\cdots+|\sigma_\kappa|)^{A_5(|\sigma_1|+\cdots+|\sigma_\kappa|)}}{(1+|t_1|+\cdots+|t_\kappa|)^{\sigma_1+\cdots+\sigma_\kappa-1}},\end{aligned}$$

and (10-6) follows. □

Proof of Lemma 27. First, we remark that the estimate (10-6) implies the absolute convergence of the integrals on the right hand side of (10-4) and justifies the following computations.

we prove the lemma by induction. First we consider the case $\kappa = 2$. From (10-1) we have

$$\begin{aligned}(x_2+x_3)^{v_1-1} &= (x_2+x_3)^B(x_2+x_3)^{v_1-1-B}\\ &= \sum_{\substack{\nu_2,\nu_3\in\mathbb{Z}_{\geq 0}\\ \nu_2+\nu_3=B}} \frac{B!}{\nu_2!\nu_3!} x^{\nu_2} x_3^{\nu_3} \frac{1}{2\pi i}\int_{(c_{v_3})} \frac{\Gamma(v_3)\Gamma(1+B-v_1-v_3)}{\Gamma(1+B-v_1)x_2^{B+1-v_1-v_3}x_3^{v_3}}\,dv_3,\end{aligned} \tag{10-8}$$

for $0 < c_{v_3} < 1+B-\Re(v_1)$. Now, by contour integration,

$$\frac{\Gamma(1+B-v_1-v_3)}{\Gamma(1+B-v_1)} x_2^{v_1+v_3-B-1} = \frac{1}{2\pi i}\left(\int_{(c_{v_2})} - \int_{(c'_{v_2})}\right) \frac{\Gamma(v_2)x_2^{-v_2}}{\Gamma(v_2+v_3)} \frac{G(B+1-v_1-v_2-v_3)}{B+1-v_1-v_2-v_3}\,dv_2,$$

where $c_{v_2}, c'_{v_2} > 0$ and $c_{v_2} < -\Re(v_1+v_3)+B+1 < c'_{v_2}$. Inserting this into (10-8) we obtain (10-4) in the case $\varepsilon = (-1, 1, 1)$.

The case $\varepsilon = (-1, 1, -1)$ (and thus its permutation $\varepsilon = (-1, -1, 1)$) follows in the same way from (10-2).

Now, let $\varepsilon = (-1, \pm_2, \dots, \pm_{\kappa+1}) \in \{\pm 1\}^{\kappa+1}$ with $\kappa \geq 2$ and suppose (10-4) holds for all $\varepsilon' \in \{\pm 1\}^{\kappa}$ with $\pm'_1 1 = -1$. Since $\kappa+1 \geq 3$ there are two indexes $2 \leq i < j \leq \kappa+1$ such that $\pm_i 1 = \pm_j 1$ and without loss of generality we can assume $i = \kappa,\ j = \kappa+1$. Then, letting $\varepsilon' = (-1, \pm_2, \dots, \pm_\kappa)$, we have

$$\begin{aligned}&(\pm_2 x_2 \pm_3 \cdots \pm_{\kappa+1} x_{\kappa+1})^{v_1-1}\chi_{\mathbb{R}_{>0}}(\pm_2 x_2 \pm_3 \cdots \pm_{\kappa+1} x_{\kappa+1})\\ &\quad= \sum_{\substack{\nu=(\nu_2,\dots,\nu_\kappa)\in\mathbb{Z}^\kappa_{\geq 0}\\ \nu_2+\dots+\nu_{\kappa+1}=B\\ \nu_i=0 \text{ if } \pm_i=-1}} \frac{B!}{\nu_2!\cdots\nu_\kappa!}\frac{1}{(2\pi i)^{\kappa-1}}\\ &\qquad\times\left(\int_{(c_{v_2},\dots,c_{v_\kappa})} - \int_{(c'_{v_2},\dots,c'_{v_\kappa})}\right) \frac{\Psi_{\varepsilon',B}(v_1,\dots,v_\kappa)}{x_2^{v_2-\nu_2}\cdots x_{\kappa-1}^{v_{\kappa-1}-\nu_{\kappa-1}}}(x_\kappa+x_{\kappa+1})^{-v_\kappa+\nu_\kappa}\,dv_2\cdots dv_\kappa\end{aligned}$$

where $c_{v_2}, \dots, c_{v_\kappa}, c'_{v_2}, \dots, c'_{v_\kappa} > 0$ satisfy (10-3). Then, we expand the binomial $(x_\kappa + x_{\kappa+1})^{\nu_\kappa}$ and apply (10-1) to $(x_\kappa+x_{\kappa+1})^{-v_\kappa}$. We obtain

$$\begin{aligned}&(\pm_2 x_2 \pm_3 \cdots \pm_{\kappa+1} x_{\kappa+1})^{v_1-1}\chi_{\mathbb{R}_{>0}}(\pm_2 x_2 \pm_3 \cdots \pm_{\kappa+1} x_{\kappa+1})\\ &\quad= \sum_{\substack{\nu=(\nu_2,\dots,\nu_{\kappa+1})\in\mathbb{Z}^\kappa_{\geq 0}\\ \nu_2+\dots+\nu_{\kappa+1}=B\\ \nu_i=0 \text{ if } \pm_i=-1}} \frac{B!}{\nu_2!\cdots\nu_{\kappa+1}!}\frac{1}{(2\pi i)^{\kappa}}\left(\int_{(c_{v_2},\dots,c_{v_{\kappa+1}})} - \int_{(c'_{v_2},\dots,c'_{v_{\kappa+1}})}\right)\\ &\qquad\times \frac{\Psi_{\varepsilon',B}(v_1,\dots,v_\kappa)}{x_2^{v_2-\nu_2}\cdots x_{\kappa-1}^{v_{\kappa-1}-\nu_{\kappa-1}} x_\kappa^{v_\kappa-v_{\kappa+1}-\nu_\kappa} x_{\kappa+1}^{v_{\kappa+1}-\nu_{\kappa+1}}} \frac{\Gamma(v_{\kappa+1})\Gamma(v_\kappa-v_{\kappa+1})}{\Gamma(v_\kappa)}\,dv_2\cdots dv_\kappa,\end{aligned}$$

where $c_{v_{\kappa+1}}, c'_{v_{\kappa+1}} > 0$ are such that $0 < c_{v_{\kappa+1}} < c_{v_\kappa}$. We make the change of variables $v_\kappa \to v_\kappa + v_{\kappa+1}$ and the lemma follows. □

Lemma 29. *Let $\kappa \geq 2$ and let $\varepsilon^* = (\pm_1 1, \dots, \pm_\kappa 1, \pm_* 1) \in \{\pm 1\}^{\kappa+1}$, with $\pm_1 1 = -1$. For $B \geq 0$, let*

$$\Psi'_{\varepsilon^*,B}(v_1, \dots, v_\kappa) := \frac{\Gamma(B+1-v_1-\cdots-v_\kappa)\Gamma(v_1)\cdots\Gamma(v_\kappa)}{\Gamma(V_{\mp_*;\varepsilon}(v_1, \dots, v_\kappa))\Gamma(B+1-V_{\mp_*;\varepsilon}(v_1, \dots, v_\kappa))},$$

where $V_{\pm;\varepsilon}$ is defined in (10-5).

Let $\mathcal{F}(v_0, \dots, v_\kappa)$ be analytic on $\{(v_0, \dots, v_\kappa) \in \mathbb{C}^{\kappa+1} \mid 0 < \Re(v_0) < B+1\}$ and assume that for $0 < \Re(v_0) < B+1$ and any $A > 0$ one has that $\mathcal{F}$ satisfies

$$\mathcal{F}(v_0, \dots, v_\kappa) \ll \prod_{i=2}^{\kappa} (1+|v_i|)^{-A}$$

where the implicit constant may depend on A, v_1 and $\Re(v_0)$. Then for any $v_1 \in \mathbb{C}$ and $c_{v_2}, \dots, c_{v_\kappa} > 0$ satisfying $0 < \Re(v_1) + c_{v_2} + \cdots + c_{v_\kappa} < 1$ we have

$$\begin{aligned}\sum_{\nu} \frac{B!}{\nu_*!\nu_2!\cdots\nu_\kappa!} \int_{(c_{v_2},\dots,c_{v_\kappa})} & \Psi'_{\varepsilon,B}(v_1, \dots, v_\kappa) \\ & \times \mathcal{F}\big(B+1-\nu_*-v_1-\cdots-v_\kappa, v_1, v_2-\nu_2, \dots, v_\kappa-\nu_\kappa\big)\, dv_2\cdots dv_\kappa \\ & = \int_{(c_{v_2},\dots,c_{v_\kappa})} \Psi'_{\varepsilon,0}(v_1, \dots, v_\kappa)\mathcal{F}\big(1-v_1-\cdots-v_\kappa, v_1, \dots, v_\kappa\big)\, dv_2\cdots dv_\kappa, \end{aligned} \tag{10-9}$$

where the sum on the left is taken over $\nu = (\nu_2, \dots, \nu_\kappa, \nu_*) \in \mathbb{Z}^\kappa_{\geq 0}$ satisfying

$$\nu_2 + \cdots + \nu_\kappa + \nu_* = B, \qquad \nu_i = 0 \quad \text{if } \pm_i = -1 \text{ or } i \in J, \qquad \nu_* = 0 \quad \text{if } \pm_* = -1.$$

Proof. Making the change of variables $v_i \to v_i + \nu_i$, for $i = 2, \dots, \kappa$, moving back the lines of integration to c_{v_i} (as we can do without crossing any pole), and switching the order of summation and integration, we see that the left hand side of (10-9) is equal to

$$\int_{(c_{v_2},\dots,c_{v_\kappa})} \sum_{\nu} \frac{B!}{\nu_*!\nu_2!\cdots\nu_\kappa!} \Psi'_\varepsilon(v_1, v_2+\nu_2, \dots, v_\kappa+\nu_\kappa)\mathcal{F}(1-v_1-\cdots-v_\kappa, v_1, \dots, v_\kappa)\, dv_2\cdots dv_\kappa.$$

Now, the identity $\mathrm{B}(s_1+1, s_2) + \mathrm{B}(s_1, s_2+1) = \mathrm{B}(s_1, s_2)$, satisfied by the Beta function $\mathrm{B}(s_1, s_2) := \Gamma(s_1)\Gamma(s_2)\Gamma(s_1+s_2)^{-1}$, can be generalized to

$$\sum_{\substack{(r_1,\dots,r_m)\in\mathbb{Z}^m_{\geq 0} \\ r_1+\cdots+r_m=r}} \frac{r!}{r_1!\cdots r_m!} \frac{\Gamma(s_1+r_1)\cdots\Gamma(s_m+r_m)}{\Gamma(r+s_1+\cdots+s_m)} = \frac{\Gamma(s_1)\cdots\Gamma(s_m)}{\Gamma(s_1+\cdots+s_m)},$$

for $m, r \geq 1$, $s_1, \dots, s_m \in \mathbb{C}$. Thus, we have

$$\sum_{\substack{\nu=(\nu_2,\dots,\nu_\kappa,\nu_*)\in\mathbb{Z}^\kappa_{\geq 0} \\ \nu_2+\cdots+\nu_\kappa+\nu_*=B \\ \nu_i=0 \text{ if } \pm_i=-1 \\ \nu_*=0 \text{ if } \pm_*=-1}} \frac{B!}{\nu_*!\nu_2!\cdots\nu_\kappa!} \Psi_{\varepsilon,B}(v_1, v_2+\nu_2 \dots, v_\kappa+\nu_\kappa) = \Psi'_{\varepsilon,0}(v_1, \dots, v_\kappa)$$

and the lemma follows. □

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