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Multiplicities associated to graded families of ideals

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We prove that limits of multiplicities associated to graded families of ideals exist under very general conditions. Most of our results hold for analytically unramified equicharacteristic local rings with perfect residue fields. We give a number of applications, including a "volume = multiplicity" formula, generalizing the formula of Lazarsfeld and Mustaţă, and a proof that the epsilon multiplicity of Ulrich and Validashti exists as a limit for ideals in rather general rings, including analytic local domains. We prove a generalization of this to generalized symbolic powers of ideals proposed by Herzog, Puthenpurakal and Verma. We also prove an asymptotic "additivity formula" for limits of multiplicities and a formula on limiting growth of valuations, which answers a question posed by the author, Kia Dalili and Olga Kashcheyeva. Our proofs are inspired by a philosophy of Okounkov for computing limits of multiplicities as the volume of a slice of an appropriate cone generated by a semigroup determined by an appropriate filtration on a family of algebraic objects.

1. Introduction

In a series of papers, Okounkov interprets the asymptotic multiplicity of graded families of algebraic objects in terms of the volume of a slice of a corresponding cone (the Okounkov body). Okounkov's method employs an asymptotic version of a result of Khovanskii [1992] for finitely generated semigroups. One of his realizations of this philosophy [Okounkov 1996; 2003] gives a construction that computes the volume of a family of graded linear systems. This method was systematically developed by Lazarsfeld and Mustaţă [2009], who give many interesting consequences, including a new proof of Fujita approximation (see [Fujita 1994] for the original proof) and the fact that the volume of a big divisor on an irreducible projective variety over an algebraically closed field is a limit, which was earlier proven in [Lazarsfeld 2004] using Fujita approximation. More recently, Fulger [2011] has extended this result to compute local volumes of divisors on a log resolution of a normal variety over

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an algebraically closed field. Kaveh and Khovanskii [2012] have recently greatly generalized the theory of Newton–Okounkov bodies and applied this to general graded families of linear systems.

The method used in these papers is to choose a nonsingular closed point β on the d-dimensional variety X and then using a flag, a sequence of subvarieties

$$\{\beta\} = X_0 \subset X_1 \subset \cdots \subset X_{d-1} \subset X$$

that are nonsingular at β, to determine a rank-d valuation of the function field $k(X)$ that dominates the regular local ring $\mathcal{O}_{X,\beta}$. This valuation gives a very simple filtration of $\mathcal{O}_{X,\beta}$ represented by monomials in a regular system of parameters of $\mathcal{O}_{X,p}$, which are local equations of the flag. Since the residue field is algebraically closed, this allows us to associate a set of points in $\mathbb{Z}^d$ to a linear system on X (by means of a k-subspace of $k(X)$ giving the linear system) so that the number of these points is equal to the dimension of the linear system. In this way, a semigroup in $\mathbb{Z}^{d+1}$ is associated to a graded family of linear systems.

One of their applications is to prove a formula of equality of volume and multiplicity for a graded family $\{I_i\}_{i\in\mathbb{N}}$ of m_R-primary ideals in a local ring (R, m_R) such that R is a local domain that is essentially of finite type over an algebraically closed field k with $R/m_R = k$ [Lazarsfeld and Mustaţă 2009, Theorem 3.8]. These assumptions on R are all necessary for their proof. The proof involves interpreting the problem in terms of graded families of linear systems on a projective variety X on which R is the local ring of a closed point α. Then a valuation as above is constructed that is centered at a nonsingular point $\beta \in X$, and the cone methods are used to prove the limit. The formula "volume = multiplicity" for graded families of ideals was first proven by Ein, Lazarsfeld and Smith [Ein et al. 2003] for valuation ideals associated to an Abhyankar valuation in a regular local ring that is essentially of finite type over a field. Mustaţă [2002] proved the formula for regular local rings containing a field. In all of these cases, the volume $\mathrm{vol}(I_*)$ of the family, which is defined as a lim sup, is shown to be a limit.

Let $\{I_i\}$ be a graded family of ideals in a d-dimensional (Noetherian) local ring (R, m_R); that is, the family is indexed by the natural numbers with $I_0 = R$ and $I_i I_j \subset I_{i+j}$ for all i and j. Suppose that the ideals are m_R-primary (for $i > 0$). Let $\ell_R(N)$ denote the length of an R-module N. We find very general conditions on R under which the "volume"

$$\mathrm{vol}(I_*) = \limsup \frac{\ell_R(R/I_n)}{n^d/d!}$$

is actually a limit. For instance, we show that this limit exists if R is analytically unramified and equicharacteristic with perfect residue field (Theorem 5.8) or if R is regular (Theorem 4.6).

We thank the referee for pointing out that our basic result Theorem 4.2 is valid without our original assumption of excellence.

Our proof involves reducing to the case of a complete domain and then finding a suitable valuation that dominates R to construct an Okounkov body. The valuation that we use is of rank 1 and rational rank d. There are two issues that require special care in the proof. The first issue is to reduce to the case of an analytically irreducible domain. Analytic irreducibility is necessary to handle the boundedness restriction on the corresponding cone (condition (2)). The proof of boundedness is accomplished by using the linear Zariski subspace theorem of Hübl [2001] (which is valid if R is assumed excellent) or, as was pointed out by the referee, by an application of the version of Rees [1989] of Izumi's theorem, for which excellence is not required. The second issue is to handle the case of a nonclosed residue field. Our method for converting the problem into a problem of cones requires that the residue field of the valuation ring be equal to the residue field of R. Care needs to be taken when the base field is not algebraically closed. The perfect condition in Theorem 5.8 on the residue field is to prevent the introduction of nilpotents upon base change.

The limit $\lim_{n\to\infty} \ell_R(R/I^n)/(n^d/d!)$ is just the Hilbert–Samuel multiplicity $e(I)$, which is a positive integer, in the case when $I_n = I^n$ with I an m_R-primary ideal. In general, when working with the kind of generality allowed by a graded family of m_R-primary ideals, the limit will be irrational. For instance, given $\lambda \in \mathbb{R}_+$, the graded family of m_R-primary ideals I_n generated by the monomials $x^i y^j$ such that $(1/2\lambda)i + j \geq n$ in the power-series ring $R = k[\![x, y]\!]$ in two variables will give us the limit $\lim_{n\to\infty} \ell_R(R/I_n)/n^2 = \lambda$.

We also obtain irrational limits for more classical families of ideals. Suppose that R is an excellent d-dimensional local domain with perfect residue field and ν is a discrete valuation dominating R (the value group is $\mathbb{Z}$). Then the valuation ideals $I_n = \{f \in R \mid \nu(f) \geq n\}$ form a graded family of m_R-primary ideals, so Theorem 5.8 tells us that the limit $\lim_{n\to\infty} \ell_R(R/I_n)/n^d$ exists. This limit will however in general not be rational. [Cutkosky and Srinivas 1993, Example 6] gives such an example in a three-dimensional normal local ring.

We give a number of applications of this formula and these techniques to the computation of limits in commutative algebra.

We prove the formula "$\mathrm{vol}(I_*) = \mathrm{multiplicity}(I_*)$" for local rings R and graded families of m_R-primary ideals such that either R is regular or R is analytically unramified and equicharacteristic with perfect residue field in Theorem 6.5. In our proof, we use a critical result on volumes of cones, which is derived in [Lazarsfeld and Mustață 2009]. We generalize this result to obtain an asymptotic additivity formula for multiplicities of an arbitrary graded family of ideals (not required to be m_R-primary) in Theorem 6.10.

Another application is to show that the epsilon multiplicity of Ulrich and Validashti [2011], defined as a lim sup, is actually a limit in some new situations. We prove that this limit exists for graded families of ideals in a local ring R such that one of the following holds: R is regular, R is analytically irreducible and excellent with algebraically closed residue field or R is normal, excellent and equicharacteristic with perfect residue field. As an immediate consequence, we obtain the existence of the limit for graded families of ideals in an analytic local domain, which is of interest in singularity theory. In [Cutkosky et al. 2005], an example is given showing that this limit is in general not rational. Previously, the limit was shown to exist in some cases in [Cutkosky et al. 2010b], and the existence of the limit was proven (for more general modules) in some cases in [Kleiman 2010] and over a domain R that is essentially of finite type over a perfect field in [Cutkosky 2011]. The proof in the latter paper used Fujita approximation on a projective variety on which the ring R was the local ring of a closed point.

We prove in Corollary 6.4 a formula on asymptotic multiplicity of generalized symbolic powers proposed by Herzog, Puthenpurakal and Verma [Herzog et al. 2008, beginning of Introduction].

We also prove that a question raised in [Cutkosky et al. 2010a] about the growth of the semigroup of a valuation semigroup has a positive answer for very general valuations and domains. We prove in Theorem 7.1 that if R is a d-dimensional regular local ring or an analytically unramified local domain with algebraically closed residue field and ω is a zero-dimensional rank-1 valuation dominating R with value group contained in $\mathbb{R}$ and if $\varphi(n)$ is the number of elements in the semigroup of values attained on R that are less than n, then

$$\lim_{n\to\infty} \frac{\varphi(n)}{n^d}$$

exists. This formula was established if R is a regular local ring of dimension 2 with algebraically closed residue field in [Cutkosky et al. 2010a] and if R is an arbitrary regular local ring of dimension 2 in [Cutkosky and Vinh 2011] using a detailed analysis of a generating sequence associated to the valuation. Our proof of this result in general dimension follows, as an application of the existence of limits for graded families of m_R-primary ideals, from the fact that $\varphi(n) = \ell_R(R/I_n)$, where $I_n = \{f \in m_R \mid \nu(f) \geq n\}$ [Cutkosky et al. 2010a; Cutkosky and Teissier 2010]. It is shown in [Cutkosky et al. 2010a] that the limits $\lim_{n\to\infty} \varphi(n)/n^2$ that can be attained on a regular local ring of dimension 2 are the real numbers β with $0 \leq \beta < \frac{1}{2}$.

2. Notation

Let m_R denote the maximal ideal of a local ring R. $Q(R)$ will denote the quotient field of a domain R and $\ell_R(N)$ the length of an R-module N. $\mathbb{Z}_+$ denotes the positive integers and $\mathbb{N}$ the nonnegative integers. Suppose that $x \in \mathbb{R}$. Then $\lceil x \rceil$ is the smallest integer n such that $x \le n$ and $\lfloor x \rfloor$ the largest integer n such that $n \le x$.

We recall some notation on multiplicity from [Zariski and Samuel 1960, Chapter VIII, §10; Serre 1965, p. V-2; Bruns and Herzog 1993, § 4.6]. Suppose that (R, m_R) is a (Noetherian) local ring, N is a finitely generated R-module with $r = \dim N$ and a is an ideal of definition of R. Then

$$e_a(N) = \lim_{k\to\infty} \frac{\ell_R(N/a^k N)}{k^r/r!}.$$

We write $e(a) = e_a(R)$.

If $s \ge r = \dim N$, then we define

$$e_s(a, N) = \begin{cases} e_a(N) & \text{if } \dim N = s, \\ 0 & \text{if } \dim N < s. \end{cases}$$

A local ring is analytically unramified if its completion is reduced. In particular, a reduced excellent local ring is analytically unramified.

3. Semigroups and cones

Suppose that $\Gamma \subset \mathbb{N}^{d+1}$ is a semigroup. Set

$$\Sigma = \Sigma(\Gamma) = \text{closed convex cone}(\Gamma) \subset \mathbb{R}^{d+1},$$
$$\Delta = \Delta(\Gamma) = \Sigma \cap (\mathbb{R}^d \times \{1\}).$$

For $m \in \mathbb{N}$, put

$$\Gamma_m = \Gamma \cap (\mathbb{N}^d \times \{m\}),$$

which can be viewed as a subset of $\mathbb{N}^d$. Consider the following three conditions on Γ:

(1) $\Gamma_0 = \{0\}$.

(2) There exist finitely many vectors $(v_i, 1)$ spanning a semigroup $B \subset \mathbb{N}^{d+1}$ such that $\Gamma \subset B$.

Let $G(\Gamma)$ is the subgroup of $\mathbb{Z}^{d+1}$ generated by Γ.

(3) $G(\Gamma) = \mathbb{Z}^{d+1}$.

We will use the convention that $\{e_i\}$ is the standard basis of $\mathbb{Z}^{d+1}$.

Assuming the boundedness condition (2), condition (1) simply states that 0 is in the semigroup Γ.

Theorem 3.1 [Okounkov 2003, §3; Lazarsfeld and Mustață 2009, Proposition 2.1]. *Suppose that* Γ *satisfies (1)–(3). Then*

$$\lim_{m\to\infty} \frac{\#\Gamma_m}{m^d} = \mathrm{vol}(\Delta(\Gamma)).$$

Recently, it has been shown that limits exist under much weaker conditions by Kaveh and Khovanskii [2012].

Theorem 3.2 [Lazarsfeld and Mustaţă 2009, Proposition 3.1]. *Suppose that* Γ *satisfies (1)–(3). Fix* $\varepsilon > 0$. *Then there is an integer* $p_0 = p_0(\varepsilon)$ *such that if* $p \geq p_0$, *then the limit*

$$\lim_{k\to\infty} \frac{\#(k\Gamma_p)}{k^d p^d} \geq \mathrm{vol}(\Delta(\Gamma)) - \varepsilon$$

exists, where

$$k\Gamma_p = \{x_1 + \cdots + x_k \mid x_1, \ldots, x_k \in \Gamma_p\}.$$

4. An asymptotic theorem on lengths

Definition 4.1. A graded family of ideals $\{I_i\}$ in a ring R is a family of ideals indexed by the natural numbers such that $I_0 = R$ and $I_i I_j \subset I_{i+j}$ for all i and j. If R is a local ring and I_i is m_R-primary for $i > 0$, then we will say that $\{I_i\}$ is a graded family of m_R-primary ideals.

In this section, we prove the following theorem:

Theorem 4.2. *Suppose that* R *is an analytically irreducible local domain of dimension* $d > 0$ *and* $\{I_n\}$ *is a graded family of ideals in* R *such that*

$$\text{there exists } c \in \mathbb{Z}_+ \text{ such that } m_R^c \subset I_1. \tag{4}$$

Suppose that there exists a regular local ring S *such that* S *is essentially of finite type and birational over* R (R *and* S *have the same quotient field*) *and the residue field map* $R/m_R \to S/m_S$ *is an isomorphism. Then*

$$\lim_{i\to\infty} \frac{\ell_R(R/I_i)}{i^d}$$

exists.

We remark that the assumption $m_R^c \subset I_1$ implies that either I_n is m_R-primary for all positive n or there exists $n_0 > 1$ such that $I_{n_0} = R$. In this case, $m_R^{cn_0} \subset I_n$ for all $n \geq n_0$, so $\ell_R(R/I_i)$ is actually bounded.

Let assumptions be as in Theorem 4.2. Let $y_1, \ldots, y_d$ be a regular system of parameters in S. Let $\lambda_1, \ldots, \lambda_d$ be rationally independent real numbers such that

$$\lambda_i \geq 1 \quad \text{for all } i. \tag{5}$$

We define a valuation ν on $Q(R)$ that dominates S by prescribing

$$\nu(y_1^{a_1} \cdots y_d^{a_d}) = a_1\lambda_1 + \cdots + a_d\lambda_d$$

for $a_1, \ldots, a_d \in \mathbb{Z}_+$ and $\nu(\gamma) = 0$ if $\gamma \in S$ has nonzero residue in S/m_S.

Let C be a coefficient set of S. Since S is a regular local ring, for $r \in \mathbb{Z}_+$ and $f \in S$, there is a unique expression

$$f = \sum s_{i_1,\dots,i_d} y_1^{i_1} \cdots y_d^{i_d} + g_r$$

with $g_r \in m_S^r$, $s_{i_1,\dots,i_d} \in S$ and $i_1 + \cdots + i_d < r$ for all $i_1, \dots, i_d$ appearing in the sum. Take r so large that $r > i_1\lambda_1 + \cdots + i_d\lambda_d$ for some term with $s_{i_1,\dots,i_d} \neq 0$. Then define

$$\nu(f) = \min\{i_1\lambda_1 + \cdots + i_d\lambda_d \mid s_{i_1,\dots,i_d} \neq 0\}. \tag{6}$$

This definition is well-defined, and we calculate that $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$ and $\nu(fg) = \nu(f) + \nu(g)$ (by the uniqueness of the expansion (6)) for all $0 \neq f, g \in S$. Thus, ν is a valuation. Let V_ν be the valuation ring of ν (in $Q(R)$). The value group of V_ν is the (nondiscrete) ordered subgroup $\mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_d$ of $\mathbb{R}$. Since there is a unique monomial giving the minimum in (6), we have that the residue field of V_ν is $S/m_S = R/m_R$.

For $\lambda \in \mathbb{R}$, define ideals K_λ and K_{λ^+} in V_ν by

$$K_\lambda = \{f \in Q(R) \mid \nu(f) \geq \lambda\},$$
$$K_{\lambda^+} = \{f \in Q(R) \mid \nu(f) > \lambda\}.$$

We follow the usual convention that $\nu(0) = \infty$ is larger than any element of $\mathbb{R}$.

Lemma 4.3. *There exists $\alpha \in \mathbb{Z}_+$ such that $K_{\alpha n} \cap R \subset m_R^n$ for all $n \in \mathbb{N}$.*

Proof. Let $\rho = \lceil \max\{\lambda_1, \dots, \lambda_d\} \rceil \in \mathbb{Z}_+$. Suppose that $\lambda \in \mathbb{R}_+$. K_λ is generated by the monomials $y_1^{i_1} \cdots y_d^{i_d}$ such that $i_1\lambda_1 + \cdots + i_d\lambda_d \geq \lambda$, which implies that

$$\frac{\lambda}{\rho} \leq i_1 + \cdots + i_d$$

so that

$$K_\lambda \cap S \subset m_S^{\lceil \lambda/\rho \rceil}. \tag{7}$$

We now establish the following equation: there exists $a \in \mathbb{Z}_+$ such that

$$m_S^{a\ell} \cap R \subset m_R^{\ell} \tag{8}$$

for all $\ell \in \mathbb{N}$.

In the case when R is excellent, this is immediate from the linear Zariski subspace theorem [Hübl 2001, Theorem 1].

We now give a proof of (8) that was provided by the referee, which is valid without assuming that R is excellent. Let ω be the m_S-adic valuation. Let ν_i be the Rees valuations of m_R. The ν_i extend uniquely to the Rees valuations of $m_{\hat{R}}$. By the version of Rees [1989] of Izumi's theorem, the topologies defined on R by ω

and the ν_i are linearly equivalent. Let $\bar{\nu}_{m_R}$ be the reduced order of m_R. By the Rees valuation theorem (recalled in [Rees 1989]),

$$\bar{\nu}_{m_R}(x) = \min_i \left\{ \frac{\nu_i(x)}{\nu_i(m_R)} \right\}$$

for all $x \in R$, so the topology defined by ω on R is linearly equivalent to the topology defined by $\bar{\nu}_{m_R}$. The $\bar{\nu}_{m_R}$ topology is linearly equivalent to the m_R-topology by [Rees 1956] since R is analytically unramified. Thus, (8) is established.

Let $\alpha = \rho a$, where ρ is the constant of (7) and a is the constant of (8).

$$K_{\alpha n} \cap S = K_{\rho a n} \cap S \subset m_S^{an}$$

by (7), and thus,

$$K_{\alpha n} \cap R \subset m_S^{an} \cap R \subset m_R^n$$

by (8). □

Remark 4.4. The conclusions of Lemma 4.3 fail if R is not analytically irreducible as can be seen from the example

$$R = (k[x, y]/y^2 - x^2(x+1))_{(x,y)} \to S = k[s]_{(s)},$$

where $s = y/x - 1$.

For $0 \neq f \in R$, define

$$\varphi(f) = (n_1, \dots, n_d) \in \mathbb{N}^d$$

if $\nu(f) = n_1\lambda_1 + \cdots + n_d\lambda_d$.

Lemma 4.5. *Suppose that $I \subset R$ is an ideal and $\lambda \in \mathbb{R}_+$. Then there are isomorphisms of R/m_R-modules*

$$K_\lambda \cap I / K_{\lambda^+} \cap I \cong \begin{cases} k & \text{if there exists } f \in I \text{ such that } \nu(f) = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $f, g \in K_\lambda \cap I$ are such that $\nu(f) = \nu(g) = \lambda$. Then $\nu(f/g) = 0$. Let $\bar{\alpha}$ be the class of f/g in $V_\nu/m_\nu \cong R/m_R$. Let $\alpha \in R$ be a lift of $\bar{\alpha}$ to R. Then $\nu(f - \alpha g) > \lambda$, and the class of f in $K_\lambda \cap I / K_{\lambda^+} \cap I$ is equal to $\bar{\alpha}$ times the class of g in $K_\lambda \cap I / K_{\lambda^+} \cap I$. □

Suppose that $I \subset R$ is an ideal and $K_\beta \cap R \subset I$ for some $\beta \in \mathbb{R}_+$. Then

$$\begin{aligned} \ell_R(R/I) &= \ell_R(R/K_\beta \cap R) - \ell_R(I/K_\beta \cap R) \\ &= \dim_k \Big(\bigoplus_{\lambda<\beta} K_\lambda \cap R / K_{\lambda^+} \cap R \Big) - \dim_k \Big(\bigoplus_{\lambda<\beta} K_\lambda \cap I / K_{\lambda^+} \cap R \Big) \\ &= \#\{(n_1, \dots, n_d) \in \varphi(R) \mid n_1\lambda_1 + \cdots + n_d\lambda_d < \beta\} \\ &\qquad - \#\{(n_1, \dots, n_d) \in \varphi(I) \mid n_1\lambda_1 + \cdots + n_d\lambda_d < \beta\}. \end{aligned} \tag{9}$$

Let $\beta=\alpha c\in\mathbb{Z}_+$, where c is the constant of (4) and α is the constant of Lemma 4.3 so that, for all $i\in\mathbb{Z}_+$,

$$K_{\beta i}\cap R=K_{\alpha ci}\cap R\subset m_R^{ic}\subset I_i. \tag{10}$$

We have from (9) that

$$\begin{aligned}\ell_R(R/I_i)=\#\{(n_1,\ldots,n_d)\in\varphi(R)\mid n_1\lambda_1+\cdots+n_d\lambda_d<\beta i\}\\ -\#\{(n_1,\ldots,n_d)\in\varphi(I_i)\mid n_1\lambda_1+\cdots+n_d\lambda_d<\beta i\}.\end{aligned} \tag{11}$$

Now $(n_1,\ldots,n_d)\in\varphi(R)$ and $n_1+\cdots+n_d\geq\beta i$ imply $n_1\lambda_1+\cdots+n_d\lambda_d\geq\beta i$ by (5) so that $(n_1,\ldots,n_d)\in\varphi(I_i)$ by (10). Thus,

$$\begin{aligned}\ell_R(R/I_i)=\#\{(n_1,\ldots,n_d)\in\varphi(R)\mid n_1+\cdots+n_d\leq\beta i\}\\ -\#\{(n_1,\ldots,n_d)\in\varphi(I_i)\mid n_1+\cdots+n_d\leq\beta i\}.\end{aligned} \tag{12}$$

Let $\Gamma\subset\mathbb{N}^{d+1}$ be the semigroup

$$\Gamma=\{(n_1,\ldots,n_d,i)\mid(n_1,\ldots,n_d)\in\varphi(I_i)\text{ and }n_1+\cdots+n_d\leq\beta i\}.$$

$I_0=R$ (and $\nu(1)=0$) implies (1) holds. The semigroup

$$B=\{(n_1,\ldots,n_d,i)\mid(n_1,\ldots,n_d)\in\mathbb{N}^d\text{ and }n_1+\cdots+n_d\leq\beta i\}$$

is generated by $B\cap(\mathbb{N}^d\times\{1\})$ and contains Γ, so (2) holds.

Write $y_i=f_i/g_i$ with $f_i,g_i\in R$ for $1\leq i\leq d$. Let $0\neq h\in I_1$. Then $hf_i,hg_i\in I_1$. There exists $c'\in\mathbb{Z}_+$ such that $c'\geq c$ and $hf_i,hg_i\notin m_R^{c'}$ for $1\leq i\leq d$. We may replace c with c' in (4). Then $\varphi(hf_i),\varphi(hg_i)\in\Gamma_1=\Gamma\cap(\mathbb{N}^d\times\{1\})$ for $1\leq i\leq d$ since hf_i and hg_i all have values $n_1\lambda_1+\cdots+n_d\lambda_d<\beta i$ so that $n_1+\cdots+n_d<\beta i$. We have that $\varphi(hf_i)-\varphi(hg_i)=e_i$ for $1\leq i\leq d$. Thus,

$$(e_i,0)=(\varphi(hf_i),1)-(\varphi(hg_i),1)\in G(\Gamma)$$

for $1\leq i\leq d$. Since $(\varphi(hf_i),1)\in G(\Gamma)$, we have that $(0,1)\in G(\Gamma)$, so $G(\Gamma)=\mathbb{Z}^{d+1}$ and (3) holds. By Theorem 3.1,

$$\lim_{i\to\infty}\frac{\#\Gamma_i}{i^d}=\mathrm{vol}(\Delta(\Gamma)). \tag{13}$$

Let $\Gamma'\subset\mathbb{N}^{d+1}$ be the semigroup

$$\Gamma'=\{(n_1,\ldots,n_d,i)\mid(n_1,\ldots,n_d)\in\varphi(R)\text{ and }n_1+\cdots+n_d\leq\beta i\}.$$

Our calculation for Γ shows that (1)–(3) hold for Γ'. By Theorem 3.1,

$$\lim_{i\to\infty}\frac{\#\Gamma'_i}{i^d}=\mathrm{vol}(\Delta(\Gamma')). \tag{14}$$

We obtain the conclusions of Theorem 4.2 from Equations (12), (13) and (14).

The following is an immediate consequence of Theorem 4.2, taking $S=R$:

Theorem 4.6. *Suppose that R is a regular local ring of dimension $d>0$ and $\{I_n\}$ is a graded family of m_R-primary ideals in R. Then the limit*

$$\lim_{n\to\infty}\frac{\ell_R(R/I_n)}{n^d}$$

exists.

5. A theorem on asymptotic lengths in more general rings

Lemma 5.1. *Suppose that R is a d-dimensional reduced local ring and $\{I_n\}$ is a graded family of m_R-primary ideals in R. Let $p_1,\ldots,p_s$ be the minimal primes of R, set $R_i=R/p_i$, and let S be the ring $S=\bigoplus_{i=1}^s R_i$. Then there exists $\alpha\in\mathbb{Z}_+$ such that for all $n\in\mathbb{Z}_+$,*

$$\left|\sum_{i=1}^s \ell_{R_i}(R_i/I_nR_i)-\ell_R(R/I_n)\right|\le \alpha n^{d-1}.$$

Proof. There exists $c\in\mathbb{Z}_+$ such that $m_R^c\subset I_1$. Since S is a finitely generated R-submodule of the total ring of fractions $T=\bigoplus_{i=1}^s Q(R_i)$ of R, there exists a nonzero divisor $x\in R$ such that $xS\subset R$.

The natural inclusion $R\to S$ induces exact sequences of R-modules

$$0\to R\cap I_nS/I_n\to R/I_n\to S/I_nS\to N_n\to 0. \tag{15}$$

We also have exact sequences of R-modules

$$0\to A_n\to R/I_n\xrightarrow{x} R/I_n\to M_n\to 0. \tag{16}$$

We have that $x(R\cap I_nS)\subset I_n$ and $A_n=I_n:x/I_n$ so that

$$\ell_R(R\cap I_nS/I_n)\le \ell_R(A_n). \tag{17}$$

Now $M_n\cong (R/x)/I_n(R/x)$, so

$$\ell_R(M_n)\le \ell_R((R/x)/m_R^{nc}(R/x))\le \beta(nc)^{d-1}$$

for some β, computed from the Hilbert–Samuel polynomial of R/x and the finitely many values of the Hilbert–Samuel function of R/x that do not agree with this polynomial. Thus,

$$\ell_R(A_n)=\ell(M_n)\le \beta c^{d-1}n^{d-1} \tag{18}$$

by (16).

Since $xS\subset R$, we have that

$$N_n\cong (S/R+I_nS)=S/(R+I_nS+xS).$$

Thus,

$$\ell_R(N_n) \le \ell_R((S/xS)/m_R^{nc}(S/xS)) \le \gamma(nc)^{d-1} \tag{19}$$

for some γ, computed from the Hilbert–Samuel polynomial of the semilocal ring S/x with respect to the ideal of definition $m_R(S/xS)$. Thus,

$$|\ell_R(R/I_n) - \ell_R(S/I_nS)| \le \max\{\beta, \gamma\} c^{d-1} n^{d-1}.$$

The lemma now follows since

$$\ell_R(S/I_nS) = \sum \ell_{R_i}(R_i/I_nR_i). \qquad \square$$

Theorem 5.2. *Suppose that R is an analytically unramified local ring with algebraically closed residue field. Let $d > 0$ be the dimension of R. Suppose that $\{I_n\}$ is a graded family of m_R-primary ideals in R. Then*

$$\lim_{i\to\infty} \frac{\ell_R(R/I_i)}{i^d}$$

exists.

Proof. Let $\hat{R}$ be the m_R-adic completion of R, which is reduced and excellent. Since the I_n are m_R-primary, we have that $R/I_n \cong \hat{R}/I_n\hat{R}$ and $\ell_R(R/I_n) = \ell_{\hat{R}}(\hat{R}/I_n\hat{R})$ for all n. Let $\{q_1, \dots, q_s\}$ be the minimal primes of $\hat{R}$. By Lemma 5.1, we reduce to proving the theorem for the families of ideals $\{I_n\hat{R}/q_i\}$ in $\hat{R}/q_i$ for $1 \le i \le s$. We may thus assume that R is a complete domain. Let $\pi : X \to \operatorname{Spec}(R)$ be the normalization of the blow-up of m_R. X is of finite type over R since R is excellent. Since $\pi^{-1}(m_R)$ has codimension 1 in X and X is normal, there exists a closed point $x \in X$ such that the local ring $\mathcal{O}_{X,x}$ is a regular local ring. Let S be this local ring. $S/m_S = R/m_R$ since S/m_S is finite over R/m_R, which is an algebraically closed field. $\square$

Lemma 5.3. *Suppose that R is a Noetherian local domain that contains a field k. Suppose that k' is a finite separable field extension of k such that $k \subset R/m_R \subset k'$. Let $S = R \otimes_k k'$. Then S is a reduced Noetherian semilocal ring. Let $p_1, \dots, p_r$ be the maximal ideals of S. Then $m_RS = p_1 \cap \cdots \cap p_r$.*

Proof. Let K be the quotient field of R. Then $K \otimes_k k'$ is reduced [Zariski and Samuel 1958, Theorem 39, p. 195]. Since k' is flat over k, we have an inclusion $R \otimes_k k' \subset K \otimes_k k'$, so $S = R \otimes_k k'$ is reduced. $S/m_RS \cong (R/m_R) \otimes_k k'$ is also reduced by [Zariski and Samuel 1958, Theorem 39]. Thus, $m_RS = p_1 \cap \cdots \cap p_r$. $\square$

Remark 5.4. In the case that R is a regular local ring, we have that $S = R \otimes_k k'$ is a regular ring.

Proof. Since R is a regular local ring, m_R is generated by $d = \dim R$ elements. For $1 \le i \le r$, we thus have that $p_iS_{p_i} = m_RS_{p_i}$ is generated by $d = \dim R = \dim S_{p_i}$ elements. Thus, S_{p_i} is a regular local ring. $\square$

Remark 5.5. If k' is Galois over k, then $S/p_i \cong k'$ for $1 \le i \le r$.

Proof. Let $\tilde{k} = R/m_R$. By our assumption, $\tilde{k}$ is a finite separable extension of k. Thus, $\tilde{k} = k[\alpha]$ for some $\alpha \in k'$. Let $f(x) \in k[x]$ be the minimal polynomial of α. Since k' is a normal extension of k containing α, $f(x)$ splits into linear factors in $k'[x]$. Thus,

$$\bigoplus_{i=1}^{r} R/p_i \cong S/m_R S \cong \tilde{k} \otimes_k k' \cong k'[x]/(f(x)) \cong (k')^r. \qquad \square$$

Remark 5.6. If R is complete in the m_R-adic topology, then $R \otimes_k k'$ is complete in the $m_R R \otimes_k k'$-adic topology [Zariski and Samuel 1960, Theorem 16, p. 277]. If $p_1, \dots, p_r$ are the maximal ideals of $R \otimes_k k'$, then $R \otimes_k k' \cong \bigoplus_{i=1}^{r} (R \otimes_k k')_{p_i}$ [Matsumura 1986, Theorem 8.15]. Thus, each $(R \otimes_k k')_{p_i}$ is a complete local ring.

Lemma 5.7. *Let assumptions and notation be as in Lemma 5.3, and suppose that I is an m_R-primary ideal in R. Then*

$$[k' : k]\ell_R(R/I) = \sum_{i=1}^{r} [S/p_i : R/m_R]\ell_{S_{p_i}}((S/IS)_{p_i}).$$

Proof. We have

$$\dim_k R/I = [R/m_R : k]\ell_R(R/I),$$
$$\dim_k S/IS = \dim_k (R/I) \otimes_k k' = [k' : k] \dim_k (R/I).$$

S/IS is an Artin local ring so that $S/IS \cong \bigoplus_{i=1}^{r} (S/IS)_{p_i}$. Thus,

$$\dim_k (S/IS) = \sum_{i=1}^{r} [S/p_i : k]\ell_{S_{p_i}}((S/IS)_{p_i}). \qquad \square$$

We will need the following definition. A commutative ring A containing a field k is said to be geometrically irreducible over k if $A \otimes_k k'$ has a unique minimal prime for all finite extensions k' of k.

Theorem 5.8. *Suppose that R is an analytically unramified equicharacteristic local ring with perfect residue field. Let $d > 0$ be the dimension of R. Suppose that $\{I_n\}$ is a graded family of m_R-primary ideals in R. Then*

$$\lim_{i \to \infty} \frac{\ell_R(R/I_i)}{i^d}$$

exists.

Proof. There exists $c \in \mathbb{Z}_+$ such that $m_R^c \subset I_1$. Let $\hat{R}$ be the m_R-adic completion of R. Since the I_n are m_R-primary, we have that $R/I_n \cong \hat{R}/I_n\hat{R}$ and $\ell_R(R/I_n) = \ell_{\hat{R}}(\hat{R}/I_n\hat{R})$ for all n. $\hat{R}$ is reduced since R is analytically unramified.

Let $\{q_1, \dots, q_s\}$ be the minimal primes of $\hat{R}$. By Lemma 5.1, we reduce to proving the theorem for the families of ideals $\{I_n\hat{R}/q_i\}$ in $\hat{R}/q_i$ for $1 \le i \le s$. In the case of a minimal prime q_i of R such that $\dim R/q_i < d$, the limits

$$\lim_{n\to\infty} \frac{\ell_R(R_i/I_nR_i)}{n^d}$$

are all zero since $\ell_R(R_i/I_nR_i) \le \ell_R(R_i/m_R^{nc}R_i)$ for all n.

We may thus assume that R is a complete domain. $\hat{R}$ contains a coefficient field $k \cong R/m_R$ by the Cohen structure theorem as R is complete and equicharacteristic. Let k' be the separable closure of k in $Q(R)$, and let $\overline{R}$ be the integral closure of R in $Q(R)$. We have that $k' \subset \overline{R}$. $\overline{R}$ is a finitely generated R-module since R is excellent. Let $n \subset \overline{R}$ be a maximal ideal lying over m_R. Then the residue field extension $R/m_R \to \overline{R}/n$ is finite. Since $k' \subset \overline{R}/n$, we have that k' is a finite extension of k. By [Grothendieck 1965, Corollary 4.5.11], there exists a finite extension L of k (which can be taken to be Galois over k) such that if $q_1, \dots, q_r$ are the minimal primes of $R \otimes_k L$, then each ring $R \otimes_k L/q_i$ is geometrically irreducible over L.

$R \otimes_k L$ is a reduced semilocal ring by Lemma 5.3, and by Remark 5.5, the residue field of all maximal ideals of $R \otimes_k L$ is L, which is a perfect field. By Remark 5.6 and Lemmas 5.1 and 5.7, we reduce to the case where R is a complete local domain with perfect coefficient field k such that R is geometrically irreducible over k. Let $\pi : X \to \mathrm{Spec}(R)$ be the normalization of the blow-up of m_R. Since R is excellent, π is projective and birational. Since $m_R\mathcal{O}_X$ is locally principal, $\pi^{-1}(m_R)$ has codimension 1 in X. Since X is normal, it is regular in codimension 1, so there exists a closed point $q \in X$ such that $\pi(q) = m_R$ and $S = \mathcal{O}_{X,q}$ is a regular local ring. Let $k' = S/m_S$. Then k' is finite over k and is thus a separable extension of the perfect field k.

Let k'' be a finite Galois extension of k containing k'. Let $R' = R \otimes_k k''$. R' is a local domain with residue field k''. R' is complete by Remark 5.6. $S \otimes_k k''$ is regular and semilocal by Remark 5.4. Let $p \in S \otimes_k k''$ be a maximal ideal. Let $S' = (S \otimes_k k'')_p$. There exist $f_0, \dots, f_t \in Q(R)$ such that S is a localization of $R[f_1/f_0, \dots, f_t/f_0]$ at a maximal ideal that necessarily contracts to m_R. Thus, S' is essentially of finite type and birational over R' since we can regard $f_0, \dots, f_t \in R'$. Since S' is a regular local ring and $k'' = S'/m_{S'} = R'/m_{R'}$ by Remark 5.5, we have that Theorem 5.8 follows from Lemma 5.7 and Theorem 4.2. □

6. Some applications to asymptotic multiplicities

Theorem 6.1. *Suppose that R is a local ring of dimension $d > 0$ such that one of the following holds*:

(1) *R is regular or*

(2) *R is analytically irreducible with algebraically closed residue field or*

(3) *R is normal, excellent and equicharacteristic with perfect residue field.*

Suppose that $\{I_i\}$ and $\{J_i\}$ are graded families of nonzero ideals in R. Further suppose that $I_i \subset J_i$ for all i and there exists $c \in \mathbb{Z}_+$ such that

$$m_R^{ci} \cap I_i = m_R^{ci} \cap J_i \tag{20}$$

for all i. Then the limit

$$\lim_{i\to\infty} \frac{\ell_R(J_i/I_i)}{i^d}$$

exists.

Remark 6.2. An analytic local domain R satisfies the hypotheses of Theorem 6.1(2). The fact that R is analytically irreducible ($\hat{R}$ is a domain) follows from [Grothendieck 1965, Corollary 18.9.2].

Proof of Theorem 6.1. We will apply the method of Theorem 4.2. When R is regular, we take $S = R$, and in case (2), we construct S by the argument of the proof of Theorem 5.2. We will consider case (3) at the end of the proof.

Let ν be the valuation of $Q(R)$ constructed from S in the proof of Theorem 4.2 with associated valuation ideals K_λ in the valuation ring V_ν of ν.

Apply Lemma 4.3 if R is not regular to find $\alpha \in \mathbb{Z}_+$ such that

$$K_{\alpha n} \cap R \subset m_R^n$$

for all $n \in \mathbb{Z}_+$. When R is regular so that $R = S$, the existence of such an α follows directly from (7). We will use the function $\varphi : R \setminus \{0\} \to \mathbb{N}^{d+1}$ of the proof of Theorem 4.2. We have that

$$K_{\alpha c n} \cap I_n = K_{\alpha c n} \cap J_n$$

for all n. Thus,

$$\ell_R(J_n/I_n) = \ell_R(J_n/K_{\alpha c n} \cap J_n) - \ell_R(I_n/K_{\alpha c n} \cap I_n) \tag{21}$$

for all n. Let $\beta = \alpha c$ and

$$\Gamma(J_*) = \{(n_1, \dots, n_d, i) \mid (n_1, \dots, n_d) \in \varphi(J_i) \text{ and } n_1 + \cdots + n_d \le \beta i\},$$
$$\Gamma(I_*) = \{(n_1, \dots, n_d, i) \mid (n_1, \dots, n_d) \in \varphi(I_i) \text{ and } n_1 + \cdots + n_d \le \beta i\}.$$

We have that

$$\ell_R(J_n/I_n) = \#\Gamma(J_*)_n - \#\Gamma(I_*)_n \tag{22}$$

as explained in the proof of Theorem 4.2. As in the proof of Theorem 4.2, we have that $\Gamma(J_*)$ and $\Gamma(I_*)$ satisfy the conditions (1)–(3). Thus,

$$\lim_{n\to\infty} \frac{\#\Gamma(J_*)_n}{n^d} = \mathrm{vol}(\Delta(\Gamma(J_*))) \quad \text{and} \quad \lim_{n\to\infty} \frac{\#\Gamma(I_*)_n}{n^d} = \mathrm{vol}(\Delta(\Gamma(I_*)))$$

by Theorem 3.1. The theorem (in cases (1) or (2)) now follows from (22).

Now suppose that R satisfies the assumptions of case (3). Then the m_R-adic completion $\hat{R}$ satisfies the assumptions of case (3).

Suppose that R satisfies the assumptions of case (3) and R is m_R-adically complete. Let k be a coefficient field of R. The algebraic closure of k in $Q(R)$ is contained in R, so it is contained in $R/m_R = k$. Thus, k is algebraically closed in $Q(R)$. Suppose that k' is a finite Galois extension of k. $Q(R)\otimes_k k'$ is a field by [Zariski and Samuel 1958, Corollary 2, p. 198], and thus, $R' = R\otimes_k k'$ is a domain. R' is a local ring with residue field k' since $R'/m_R R' \cong R/m_R \otimes_k k' \cong k'$. R' is normal by [Grothendieck 1965, Corollary 6.14.2]. Thus, R' satisfies the assumptions of case (3).

Thus, in the reductions in the proof of Theorem 5.8 to Theorem 4.2, the only extensions that we need to consider are local homomorphisms $R\to R'$ that are either m_R-adic completion or a base extension by a Galois field extension. These extensions are all flat, and $m_R R' = m_{R'}$. Thus,

$$m_S^{nc}\cap I_n S = m_R^{nc} S\cap I_n S = (m_R^{nc}\cap I_n)S = (m_R^{nc}\cap J_n)S = m_R^{nc} S\cap J_n S = m_S^{nc}\cap J_n S$$

for all n. Thus, the condition (20) is preserved, so we reduce to the case (2) of this theorem and conclude that Theorem 6.1 is true in case (3). □

If R is a local ring and I is an ideal in R, then the saturation of I is

$$I^{\mathrm{sat}} = I : m_R^{\infty} = \bigcup_{k=1}^{\infty} I : m_R^k.$$

Corollary 6.3. *Suppose that R is a local ring of dimension $d>0$ such that one of the following holds*:

(1) *R is regular or*

(2) *R is analytically irreducible with algebraically closed residue field or*

(3) *R is normal, excellent and equicharacteristic with perfect residue field.*

Suppose that I is an ideal in R. Then the limit

$$\lim_{i\to\infty} \frac{\ell_R((I^i)^{\mathrm{sat}}/I^i)}{i^d}$$

exists.

Since $(I^n)^{\text{sat}}/I^n \cong H^0_{m_R}(R/I^n)$, the epsilon multiplicity of Ulrich and Validashti [2011]

$$\varepsilon(I) = \limsup \frac{\ell_R(H^0_{m_R}(R/I^n))}{n^d/d!}$$

exists as a limit under the assumptions of Corollary 6.3.

Corollary 6.3 is proven for more general families of modules when R is a local domain that is essentially of finite type over a perfect field k such that R/m_R is algebraic over k in [Cutkosky 2011]. The limit in Corollary 6.3 can be irrational as shown in [Cutkosky et al. 2005].

Proof of Corollary 6.3. By [Swanson 1997, Theorem 3.4], there exists $c \in \mathbb{Z}_+$ such that each power I^n of I has an irredundant primary decomposition

$$I^n = q_1(n) \cap \cdots \cap q_s(n),$$

where $q_1(n)$ is m_R-primary and $m_R^{nc} \subset q_1(n)$ for all n. As $(I^n)^{\text{sat}} = q_2(n) \cap \cdots \cap q_s(n)$, we have that

$$I^n \cap m_R^{nc} = m_R^{nc} \cap q_2(n) \cap \cdots \cap q_s(n) = m_R^{nc} \cap (I^n)^{\text{sat}}$$

for all $n \in \mathbb{Z}_+$. Thus, the corollary follows from Theorem 6.1, taking $I_i = I^i$ and $J_i = (I^i)^{\text{sat}}$. □

A stronger version of the previous corollary is true. The following corollary proves a formula proposed by Herzog et al. [2008, Introduction].

Suppose that R is a ring and I and J are ideals in R. Then the n-th symbolic power of I with respect to J is

$$I_n(J) = I^n : J^\infty = \bigcup_{i=1}^{\infty} I^n : J^i.$$

Corollary 6.4. *Suppose that R is a local domain of dimension d such that one of the following holds:*

(1) *R is regular or*

(2) *R is normal and excellent of equicharacteristic 0 or*

(3) *R is essentially of finite type over a field of characteristic 0.*

Suppose that I and J are ideals in R. Let s be the constant limit dimension of $I_n(J)/I^n$ for $n \gg 0$. Then

$$\lim_{n\to\infty} \frac{e_{m_R}(I_n(J)/I^n)}{n^{d-s}}$$

exists.

Proof. There exists a positive integer n_0 such that the set of associated primes of R/I^n stabilizes for $n \geq n_0$ by [Brodmann 1979]. Let $\{p_1, \ldots, p_t\}$ be this set of associated primes. We thus have irredundant primary decompositions for $n \geq n_0$

$$I^n = q_1(n) \cap \cdots \cap q_t(n), \tag{23}$$

where $q_i(n)$ are p_i-primary.

We further have that

$$I^n : J^\infty = \bigcap_{J \not\subset p_i} q_i(n). \tag{24}$$

Thus, $\dim I_n(J)/I^n$ is constant for $n \geq n_0$. Let s be this limit dimension. The set

$$A = \left\{ p \in \bigcup_{n \geq n_0} \mathrm{Ass}(I_n(J)/I^n) \;\middle|\; n \geq n_0 \text{ and } \dim R/p = s \right\}$$

is a finite set. Moreover, every such prime is in $\mathrm{Ass}(I_n(J)/I^n)$ for all $n \geq n_0$. For $n \geq n_0$, we have by the additivity formula [Serre 1965, p. V-2; Bruns and Herzog 1993, Corollary 4.6.8, p. 189] that

$$e_{m_R}(I_n(J)/I^n) = \sum_p \ell_{R_p}((I_n(J)/I^n)_p) e(m_{R/p}),$$

where the sum is over the finite set of primes $p \in \mathrm{Spec}(R)$ such that $\dim R/p = s$. This sum is thus over the finite set A.

Suppose that $p \in A$ and $n \geq n_0$. Then

$$I_p^n = \bigcap q_i(n)_p,$$

where the intersection is over the $q_i(n)$ such that $p_i \subset p$, and

$$I_n(J) = \bigcap q_i(n)_p,$$

where the intersection is over the $q_i(n)$ such that $J \not\subset p_i$ and $p_i \subset p$. Thus, there exists an index i_0 such that $p_{i_0} = p$ and

$$I_p^n = q_{i_0}(n)_p \cap I_n(J)_p.$$

By (23),

$$(I_p^n)^{\mathrm{sat}} = I_n(J)_p$$

for $n \geq n_0$. Since R_p satisfies one of the cases (1) or (3) of Theorem 6.1 or the conditions of [Cutkosky 2011, Corollary 1.5] and $\dim R_p = d - s$ (as R is universally catenary), the limit

$$\lim_{n \to \infty} \frac{\ell_R((I_n(J)/I_n)_p)}{n^{d-s}}$$

exists. □

Theorem 6.5. *Suppose that R is a d-dimensional local ring such that either*

(1) *R is regular or*

(2) *R is analytically unramified and equicharacteristic with perfect residue field.*

Suppose that $\{I_i\}$ is a graded family of m_R-primary ideals in R. Then

$$\lim_{n\to\infty}\frac{\ell_R(R/I_n)}{n^d/d!}=\lim_{p\to\infty}\frac{e(I_p)}{p^d}.$$

Here $e(I_p)$ is the multiplicity

$$e(I_p)=e_{I_p}(R)=\lim_{k\to\infty}\frac{\ell_R(R/I_p^k)}{k^d/d!}.$$

Theorem 6.5 is proven for valuation ideals associated to an Abhyankar valuation in a regular local ring that is essentially of finite type over a field in [Ein et al. 2003], for general families of m_R-primary ideals when R is a regular local ring containing a field in [Mustaţă 2002] and when R is a local domain that is essentially of finite type over an algebraically closed field k with $R/m_R=k$ in [Lazarsfeld and Mustaţă 2009, Theorem 3.8].

Proof of Theorem 6.5. There exists $c\in\mathbb{Z}_+$ such that $m_R^c\subset I_1$.

We first prove the theorem when R satisfies the assumptions of Theorem 4.2. Let ν be the valuation of $Q(R)$ constructed from S in the proof of Theorem 4.2 with associated valuation ideals K_λ in the valuation ring V_ν of ν.

Apply Lemma 4.3 if R is not regular to find $\alpha\in\mathbb{Z}_+$ such that

$$K_{\alpha n}\cap R\subset m_R^n$$

for all $n\in\mathbb{N}$. When R is regular so that $R=S$, the existence of such an α follows directly from (7). We will use the function $\varphi:R\setminus\{0\}\to\mathbb{N}^{d+1}$ of the proof of Theorem 4.2.

We have that

$$K_{\alpha cn}\cap R\subset m_R^{cn}\subset I_n$$

for all n.

Let

$$\Gamma(I_*)=\{(n_1,\dots,n_d,i)\mid(n_1,\dots,n_d)\in\varphi(I_i)\text{ and }n_1+\cdots+n_d\le\alpha ci\},$$
$$\Gamma(R)=\{(n_1,\dots,n_d,i)\mid(n_1,\dots,n_d)\in\varphi(R)\text{ and }n_1+\cdots+n_d\le\alpha ci\}.$$

As in the proof of Theorem 4.2, $\Gamma(I_*)$ and $\Gamma(R)$ satisfy the conditions (1)–(3). For fixed $p\in\mathbb{Z}_+$, let

$$\Gamma(I_*)(p)=\{(n_1,\dots,n_d,kp)\mid(n_1,\dots,n_d)\in\varphi(I_p^k)\text{ and }n_1+\cdots+n_d\le\alpha ckp\}.$$

We have inclusions of semigroups

$$k\Gamma(I_*)_p \subset \Gamma(I_*)(p)_{kp} \subset \Gamma(I_*)_{kp}$$

for all p and k.

By Theorem 3.2, given $\varepsilon > 0$, there exists p_0 such that $p \geq p_0$ implies

$$\operatorname{vol}(\Delta(\Gamma(I_*))) - \varepsilon \leq \lim_{k\to\infty} \frac{\#k\Gamma(I_*)_p}{k^d p^d}.$$

Thus,

$$\operatorname{vol}(\Delta(\Gamma(I_*))) - \varepsilon \leq \lim_{k\to\infty} \frac{\#\Gamma(I_*)(p)_{kp}}{k^d p^d} \leq \operatorname{vol}(\Delta(\Gamma(I_*))).$$

Again by Theorem 3.2, we can choose p_0 sufficiently large so that we also have

$$\operatorname{vol}(\Delta(\Gamma(R))) - \varepsilon \leq \lim_{k\to\infty} \frac{\#\Gamma(R)_{kp}}{k^d p^d} \leq \operatorname{vol}(\Delta(\Gamma)).$$

Now

$$\begin{aligned}\ell_R(R/I_p^k) &= \#\Gamma(R)_{pk} - \#\Gamma(I_*)(p)_{kp},\\ \ell_R(R/I_n) &= \#\Gamma(R)_n - \#\Gamma(I_*)_n.\end{aligned}$$

By Theorem 3.1,

$$\lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d} = \operatorname{vol}(\Delta(\Gamma(R))) - \operatorname{vol}(\Delta(\Gamma(I_*))).$$

Thus,

$$\lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d} - \varepsilon \leq \lim_{k\to\infty} \frac{\ell_R(R/I_p^k)}{k^d p^d} = \frac{e(I_p)}{d!\, p^d} \leq \lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d} + \varepsilon.$$

Taking the limit as $p \to \infty$, we obtain the conclusions of the theorem.

Now assume that R is general, satisfying the assumptions of the theorem. We reduce to the above case by a series of reductions, first taking the completion of R, then modding out by minimal primes and taking a base extension by a finite Galois extension.

The proof thus reduces to showing that

$$\lim_{p\to\infty} \frac{e_d(I_p, R)}{p^d} = \lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d/d!}$$

in each of the following cases:

(a) $$\lim_{p\to\infty} \frac{e_d(I_p\hat{R}, \hat{R})}{p^d} = \lim_{n\to\infty} \frac{\ell_{\hat{R}}(\hat{R}/I_n\hat{R})}{n^d/d!}.$$

(b) Suppose that the minimal primes of (the reduced ring) R are $\{q_1, \dots, q_s\}$. Let $R_i = R/q_i$, and suppose that

$$\lim_{p\to\infty} \frac{e_d(I_pR_i, R_i)}{p^d} = \lim_{n\to\infty} \frac{\ell_{R_i}(R_i/I_nR_i)}{n^d/d!}$$

for all i.

(c) Suppose that $k \subset R$ is a field and k' is a finite Galois extension of k containing R/m_R. Let $\{p_1, \dots, p_r\}$ be the maximal ideals of the semilocal ring $S = R \otimes_k k'$. Suppose that

$$\lim_{p\to\infty} \frac{e_d(I_pS_{p_i}, S_{p_i})}{p^d} = \lim_{n\to\infty} \frac{\ell_{S_{p_i}}(S_{p_i}/I_nS_{p_i})}{n^d/d!}$$

for all i.

Recall that

$$\frac{e_d(I_p, R)}{d!} = \lim_{k\to\infty} \frac{\ell_R(R/I_p^k)}{k^d}.$$

Case (a) follows since

$$\ell_R(R/I_p^k) = \ell_{\hat{R}}(\hat{R}/I_p^k\hat{R})$$

for all p and k.

In case (b), we have that

$$\frac{e_d(I_p, R)}{p^d} = \sum_{i=1}^{s} \frac{e_d(I_pR_i, R_i)}{p^d}$$

by the additivity formula [Serre 1965, §V-3; Bruns and Herzog 1993, Corollary 4.6.8, p. 189] or directly from Lemma 5.1. Case (b) thus follows from the fact that

$$\lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d} = \sum_{i=1}^{s} \lim_{k\to\infty} \frac{\ell_R(R_i/I_nR_i)}{n^d}$$

by Lemma 5.1.

In case (c), we have that k' is Galois over k so that $S/p_i \cong k'$ for all i by Remark 5.5. Thus, Lemma 5.7 becomes

$$\ell_R(R/I_p^k) = \sum_{i=1}^{r} \ell_{S_{p_i}}(S_{p_i}/I_p^kS_{p_i})$$

for all p and k, from which this case follows. □

Suppose that R is a Noetherian ring and $\{I_i\}$ is a graded family of ideals in R. Let

$$s = s(I_*) = \limsup \dim R/I_i.$$

Let $i_0 \in \mathbb{Z}_+$ be the smallest integer such that

$$\dim R/I_i \le s \quad \text{for } i \ge i_0. \tag{25}$$

For $i \ge i_0$ and p a prime ideal in R such that $\dim R/p = s$, we have that $(I_i)_p = R_p$ or $(I_i)_p$ is p_p-primary.

In general, s is not a limit as is shown by the following simple example:

Example 6.6. Suppose that R is a Noetherian ring and $p \subset q \subset R$ are prime ideals. Let

$$I_i = \begin{cases} p & \text{if } i \text{ is odd,} \\ q & \text{if } i \text{ is even.} \end{cases}$$

We have that

$$I_i I_j = \begin{cases} p^2 \text{ or } q^2 & \text{if } i+j \text{ is even,} \\ pq & \text{if } i+j \text{ is odd.} \end{cases}$$

Thus, $I_i I_j \subset I_{i+j}$ for all i and j and

$$\dim R/I_i = \begin{cases} \dim R/p & \text{if } i \text{ is odd,} \\ \dim R/q & \text{if } i \text{ is even.} \end{cases}$$

Let

$$T = T(I_*) = \big\{p \in \operatorname{Spec}(R) \\ \mid \dim R/p = s \text{ and there exist arbitrarily large } j \text{ such that } (I_j)_p \ne R_p\big\}.$$

Lemma 6.7. *$T(I_*)$ is a finite set.*

Proof. Let U be the set of prime ideals p of R that are an associated prime of some I_i with $i_0 \le i \le 2i_0 - 1$ and ht $p = s$. Suppose that $q \in T$. There exists $j \ge i_0$ such that $(I_j)_q \ne R_q$. We can write $j = ai_0 + (i_0 + k)$ with $0 \le k \le i_0 - 1$ and $a \ge 0$. Thus, $I_{i_0}^a I_{i_0+k} \subset I_j$. Thus, $q \in U$ since $(I_{i_0}^a I_{i_0+k})_q \ne R_q$. □

Lemma 6.8. *There exist $c = c(I_*) \in \mathbb{Z}_+$ such that if $j \ge i_0$ and $p \in T(I_*)$, then*

$$p^{jc} R_p \subset I_j R_p.$$

Proof. There exists $a \in \mathbb{Z}_+$ such that for all $p \in T$, $p_p^a \subset (I_i)_p$ for $i_0 \le i \le 2i_0 - 1$.

Write $j = ti_0 + (i_0 + k)$ with $t \ge 0$ and $0 \le k \le i_0 - 1$. Then

$$p_p^{(t+1)a} \subset I_{i_0}^t I_{i_0+k} R_p \subset I_j R_p.$$

Let $c = \lceil a/i_0 \rceil + a$. Then

$$jc \ge a + j\frac{a}{i_0} = a + (ti_0 + i_0 + k)\frac{a}{i_0} \ge (t+1)a.$$

Thus, $p_p^{jc} \subset p_p^{(t+1)a} \subset (I_j)_p$. □

Let

$$A(I_*) = \{q \in T(I_*) \mid I_n R_q \text{ is } q_q\text{-primary for } n \geq i_0\}.$$

Lemma 6.9. *Suppose that $q \in T(I_*) \setminus A(I_*)$. Then there exists $b \in \mathbb{Z}_+$ such that $q_q^b \subset (I_n)_q$ for all $n \geq i_0$.*

Proof. There exists $n_0 \in \mathbb{Z}_+$ such that $n_0 \geq i_0$ and $(I_{n_0})_q = R_q$. Let $b \in \mathbb{Z}_+$ be such that $q_q^b \subset (I_n)_q$ for $0 \leq n < n_0$. Suppose that $n_0 \leq n$. Write $n = \beta n_0 + \alpha$ with $\beta \geq 0$ and $0 \leq \alpha < n_0$. Then

$$q_q^b \subset (I_{n_0}^{\beta} I_\alpha)_q \subset (I_n)_q. \qquad \square$$

We obtain the following asymptotic additivity formula:

Theorem 6.10. *Suppose that R is a d-dimensional local ring such that either*

(1) *R is regular or*

(2) *R is analytically unramified of equicharacteristic* 0.

Suppose that $\{I_i\}$ is a graded family of ideals in R. Let $s = s(I_*) = \limsup \dim R/I_i$ and $A = A(I_*)$. Suppose that $s < d$. Then

$$\lim_{n\to\infty} \frac{e_s(m_R, R/I_n)}{n^{d-s}/(d-s)!} = \sum_{q \in A} \left(\lim_{k\to\infty} \frac{e((I_k)_q)}{k^{d-s}} \right) e(m_{R/q}).$$

Proof. Let i_0 be the (smallest) constant satisfying (25). By the additivity formula [Serre 1965, p. V-2; Bruns and Herzog 1993, Corollary 4.6.8, p. 189], for $i \geq i_0$,

$$e_s(m_R, R/I_i) = \sum_p \ell_{R_p}(R_p/(I_i)_p) e_{m_R}(R/p),$$

where the sum is over all prime ideals p of R with $\dim R/p = s$. By Lemma 6.7, for $i \geq i_0$, the sum is actually over the finite set $T(I_*)$ of prime ideals of R.

For $p \in T(I_*)$, R_p is a local ring of dimension at most $d - s$. Further, R_p is analytically unramified [Rees 1961; Huneke and Swanson 2006, Proposition 9.1.4]. By Lemma 6.8 and by Theorem 4.6 or 5.8, replacing $(I_i)_p$ with p_p^{ic} if $i < i_0$, we have that

$$\lim_{i\to\infty} \frac{\ell_{R_p}(R_p/(I_i)_p)}{i^{d-s}}$$

exists. Further, this limit is zero if $p \in T(I_*) \setminus A(I_*)$ by Lemma 6.9 and since $s < d$. Finally, we have

$$\lim_{i\to\infty} \frac{\ell_{R_q}(R_q/(I_i)_q)}{i^{d-s}/(d-s)!} = \lim_{k\to\infty} \frac{e_{(I_k)_q}(R_q)}{k^{d-s}}$$

for $q \in A(I_*)$ by Theorem 6.5. $\square$

7. An application to growth of valuation semigroups

As a consequence of our main result, we obtain the following theorem, which gives a positive answer to a question raised in [Cutkosky et al. 2010a]. This formula was established if R is a regular local ring of dimension 2 with algebraically closed residue field in [Cutkosky et al. 2010a] and if R is an arbitrary regular local ring of dimension 2 in [Cutkosky and Vinh 2011] using a detailed analysis of a generating sequence associated to the valuation. A valuation ω dominating a local domain R is zero-dimensional if the residue field of ω is algebraic over R/m_R.

Theorem 7.1. *Suppose that R is a regular local ring or an analytically unramified local domain. Further suppose that R has an algebraically closed residue field. Let $d > 0$ be the dimension of R. Let ω be a zero-dimensional rank-1 valuation of the quotient field of R that dominates R. Let $S^R(\omega)$ be the semigroup of values of elements of R, which can be regarded as an ordered subsemigroup of $\mathbb{R}_+$. For $n \in \mathbb{Z}_+$, define*

$$\varphi(n) = |S^R(\omega) \cap (0, n)|.$$

Then

$$\lim_{n\to\infty} \frac{\varphi(n)}{n^d}$$

exists.

Proof. Let $I_n = \{f \in R \mid \omega(f) \geq n\}$ and $\lambda = \omega(m_R) = \min\{\omega(f) \mid f \in m_R\}$. Let $c \in \mathbb{Z}_+$ be such that $c\lambda > 1$. Then $m_R^c \subset I_1$. By Theorem 4.6 or 5.2, we have that

$$\lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d}$$

exists.

Since R has an algebraically closed residue field, we have by [Cutkosky et al. 2010a; Cutkosky and Teissier 2010] that

$$\#\varphi(n) = \ell_R(R/I_n) - 1.$$

Thus, the theorem follows. □

In [Cutkosky et al. 2010a], it is shown that the real numbers β with $0 \leq \beta < \frac{1}{2}$ are the limits attained on a regular local ring R of dimension 2.

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