



# Patching and admissibility over two-dimensional complete local domains

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We develop a patching machinery over the field  $E = K((X, Y))$  of formal power series in two variables over an infinite field  $K$ . We apply this machinery to prove that if  $K$  is separably closed and  $G$  is a finite group of order not divisible by  $\text{char } E$ , then there exists a  $G$ -crossed product algebra with center  $E$  if and only if the Sylow subgroups of  $G$  are abelian of rank at most 2.

## Introduction

Complete local domains play an important role in commutative algebra and algebraic geometry, and their algebraic properties were already described in 1946 by Cohen's structure theorem. The Galois theoretic properties of their quotient fields were extensively studied over the past two decades. The pioneering work in this line of research is due to Harbater [1987], who introduced the method of patching to prove that if  $R$  is a complete local domain with quotient field  $K$ , then every finite group occurs as a Galois group over  $K(x)$ . This result was strengthened by Pop [1996] and, in a different language, by Haran and Jarden [1998], who showed that if moreover  $R$  is of dimension 1, then every finite split embedding problem over  $K(x)$  is solvable.

The first step towards higher dimension was made by Harbater and Stevenson [2005], who essentially showed that if  $R$  is a complete local domain of dimension 2, then every finite split embedding problem over  $\text{Quot } R$  has  $|R|$  independent solutions. That is, the absolute Galois group of  $\text{Quot } R$  is semifree of rank  $|R|$  (see [Bary-Soroker et al. 2008] for details on this notion). This result was generalized in [Pop 2010; Paran 2010], where it is shown that if  $K$  is the quotient field of a complete local domain of dimension exceeding 1, then  $\text{Gal } K$  is semifree.

Despite the major progress made in the study of Galois theory over these fields, little is known about the structure of division algebras over them. A step in that

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Sylow subgroup of  $G$  by a Galois extension of  $E$  which is a maximal subfield in some  $E$ -central division algebra. We then patch these realizations into a Galois extension  $F/E$  with group  $G$ , in a way that also patches the division algebras into an  $E$ -central division algebra  $D$  containing  $F$  as a maximal subfield.

A key ingredient in our proof is a patching machinery over fields of the form  $K((X, Y))$ , where  $K$  may be an arbitrary infinite field. In [Pop 2010; Paron 2010], problems over  $K((X, Y))$  are lifted to  $K((X, Y))(Z)$ , solved there (via different methods), and then the solutions are specialized into solutions over  $K((X, Y))$  using Hilbertianity. This approach seems inapplicable to our current goal, since adequate extensions usually do not remain adequate under specialization. Instead we patch groups directly over  $K((X, Y))$ . To this end we define “analytic fields” over  $K((X, Y))$ , satisfying the axioms of algebraic patching (i.e., matrix factorization and intersection), as presented in [Haran and Jarden 1998] (a slightly different axiomatization from the “field patching” axiomatization of [Harbater and Hartmann 2007]). The construction of these analytic rings is of a rigid-geometric nature. In recent communication with David Harbater, we learned that a formal-geometric analogue of this form of patching was carried out by him in [2003, Theorem 5.3.9] in order to solve split embedding problems over the field  $\mathbb{C}((X, Y))$  of formal power series in two variables over the complex numbers. The core patching arguments in the proof of [Harbater 2003, Theorem 5.3.9] can be extended to replace  $\mathbb{C}$  by an arbitrary field and used to study admissible groups, in a similar fashion to our development here.

### 1. Analytic fields

In this section we establish our patching machinery. Fix an infinite field  $K$ , and let  $E = K((X, Y)) = \text{Quot}(K[[X, Y]])$  be the field of formal power series over  $K$  in the variables  $X$  and  $Y$ . Denote by  $v$  the order function of the maximal ideal  $\langle X, Y \rangle$  in  $K[[X, Y]]$ . Then  $v$  extends uniquely to a discrete rank-1 valuation of  $E$ . Note that  $K[[X, Y]]$  is strictly contained in the valuation ring of  $v$  in  $E$ .

**Construction 1.1** (analytic rings over  $E$ ). Let  $I$  be a finite set. For each  $i \in I$  let  $c_i \in K$ , such that  $c_i \neq c_j$  for  $i \neq j$  (such a choice is possible since  $K$  is infinite). For each  $i \in I$  denote  $z_i = Y/(X - c_i Y) \in E$ . For each  $J \subseteq I$ , consider the subring  $K[z_j \mid j \in J][X, Y]$  of  $E$ , and let  $D_J$  be the completion of this ring with respect to  $v$ . Note that for each  $J \subseteq I$ ,  $D_J \subseteq D_I$ , and that  $D_\emptyset = K[[X, Y]]$ , since  $K[[X, Y]]$  is complete. Let  $Q = \text{Quot } D_I$ , and for each  $i \in I$  let  $Q_i = E \cdot D_{I \setminus \{i\}}$  and  $Q'_i = \bigcap_{j \neq i} Q_j$ .

For the rest of this section, we fix the notation of **Construction 1.1**. A geometric interpretation of the rings defined in **Construction 1.1** appears in **Remark 1.12**. In order to present this interpretation, we need several lemmas.

**Lemma 1.2.** *Let  $i \in I$ . Then  $v$  is trivial on  $K(z_i)$ .*

*Proof.* It suffices to prove that  $v$  is trivial on  $K[z_i]$ . Let  $0 \neq f = \sum_{n=0}^d a_n z_i^n \in K[z_i]$ , with  $a_0, \dots, a_d \in K$ . Without loss of generality,  $a_0 \neq 0$ . We have

$$\sum_{n=0}^d a_n z_i^n = \frac{\sum_{n=0}^d a_n Y^n (X - c_i Y)^{d-n}}{(X - c_i Y)^d}.$$

By opening parentheses, the numerator in this expression can be written as a sum of monomials of degree  $d$ . For  $n = 0$  we get the summand  $a_0 X^d$ , while all other monomials in this presentation have a positive power of  $Y$ , so they do not cancel  $a_0 X^d$ . Thus the numerator has value  $d$ , and clearly so does the denominator, so  $v(f) = 0$ . □

**Corollary 1.3.** *The valuation  $v$  is trivial on  $K[z_i \mid i \in I]$ .*

*Proof.* Note that for each  $i, j \in I$  we have  $K(z_i) = K(z_j)$ . Thus by the preceding lemma,  $v$  is trivial on  $K(z_i) = K(z_j \mid j \in I)$ , and in particular  $v$  is trivial on the subring  $K[z_i \mid i \in I]$ . □

**Lemma 1.4.** *Let  $J \subseteq I$  and  $j \in J$ . Then the ring  $K[z_l \mid l \in J][X - c_j Y]$  is isomorphic to the ring of polynomials in one variable over  $K[z_l \mid l \in J]$ .*

*Proof.* To prove the claim we show that if  $\sum_{n=0}^d a_n (X - c_j Y)^n = 0$ , for  $a_0, \dots, a_d \in K[z_l \mid l \in J]$ , then  $a_0 = \dots = a_d = 0$ . If not, suppose (without loss of generality) that  $a_0 \neq 0$ . By [Corollary 1.3](#),  $v(a_0) = 0$  while  $v(a_k (X - c_j Y)^k) = v(a_k) + k = k > 0$  for each  $k > 0$ . Hence  $\infty = v(0) = v(\sum_{n=0}^d a_n (X - c_j Y)^n) = 0$ , a contradiction. □

By [Lemma 1.4](#), for each  $J \subseteq I, j \in J$ , each element of  $K[z_l \mid l \in J][X - c_j Y]$  has a unique presentation as a polynomial in  $X - c_j Y$ . Thus we have a natural valuation on this ring, given by  $v'(\sum_{n=0}^d a_n (X - c_j Y)^n) = \min(n \mid a_n \neq 0)$ , and we may form the completion  $K[z_l \mid l \in J][[X - c_j Y]]$  of this ring with respect to  $v'$ .

**Proposition 1.5.** *Let  $J \subseteq I$  and  $j \in J$ . Then  $D_J = K[z_l \mid l \in J][[X - c_j Y]]$ , and  $v$  is given on  $D_J$  by  $v(\sum_{n=0}^\infty a_n (X - c_j Y)^n) = \min(n \mid a_n \neq 0)$ .*

*Proof.* By [Lemma 1.4](#),  $v$  coincides with  $v'$  (given in the paragraph preceding this proposition) on  $K[z_l \mid l \in J][X - c_j Y]$ ; hence  $K[z_l \mid l \in J][[X - c_j Y]]$  is the completion of  $K[z_l \mid l \in J][X - c_j Y]$  with respect to  $v$ , and  $v$  coincides with  $v'$  on the completion. Note that  $K[z_l \mid l \in J][X - c_j Y] = K[z_l \mid l \in J][X, Y]$  (since  $Y = z_j(X - c_j Y)$  and  $X = (1 + c_j z_j)(X - c_j Y)$ ); hence by the definition of  $D_J$  we are done. □

**Lemma 1.6.** *Let  $J \subseteq I$ . Then  $K[z_j \mid j \in J] = \sum_{j \in J} K[z_j]$ .*

*Proof.* For each  $i \neq j \in J$  we have

$$z_i \cdot z_j = \frac{Y^2}{(X - c_i Y) \cdot (X - c_j Y)} = \frac{1}{c_i - c_j} \cdot z_i + \frac{1}{c_j - c_i} z_j.$$

The claim now follows by induction on  $|I|$ . □

**Proposition 1.7.** *Let  $J, J' \subseteq I$ . Then for each  $f \in D_{J \cup J'}$ , there exist  $f_1 \in D_J$  and  $f_2 \in D_{J'}$  with  $v(f_1), v(f_2) \geq v(f)$ , such that  $f = f_1 + f_2$ .*

*Proof.* Replace  $J$  with  $J \setminus (J \cap J')$  to assume that  $J \cap J' = \emptyset$ . Moreover, without loss of generality  $J, J'$  are nonempty. Choose  $j \in J$  and  $j' \in J'$ , and let

$$A_J = K[z_l \mid l \in J], \quad A_{J'} = K[z_l \mid l \in J'], \quad A = K[z_l \mid l \in J \cup J'].$$

By [Proposition 1.5](#),  $D_J = A_J[[X - c_j Y]]$ ,  $D_{J'} = A_{J'}[[X - c_{j'} Y]]$ , and  $D_{J \cup J'} = A[[X - c_j Y]]$ . Let  $f = \sum_{n=m}^{\infty} a_n (X - c_j Y)^n \in D_{J \cup J'}$ , with  $a_m \neq 0$ . Then  $v(f) = m$  by [Proposition 1.5](#). By [Lemma 1.6](#),  $A = A_J + A_{J'}$ . For each  $n \geq m$ , let  $b_n \in A_J, b'_n \in A_{J'}$  such that  $a_n = b_n + b'_n$ . Let  $f_1 = \sum_{n=m}^{\infty} b_n (X - c_j Y)^n$  and  $f_2 = f - f_1 = \sum_{n=m}^{\infty} b'_n (X - c_j Y)^n$ . Then  $f_1 \in D_J$  and  $v(f_1) \geq m$ . It remains to prove that  $f_2 \in D_{J'}$  and that  $v(f_2) \geq m$ . This follows by the equality:

$$\begin{aligned} f_2 &= \sum_{n=m}^{\infty} b'_n (X - c_j Y)^n = \sum_{n=m}^{\infty} b'_n ((X - c_{j'} Y) + (c_{j'} - c_j) Y)^n \\ &= \sum_{n=m}^{\infty} (b'_n (1 + (c_{j'} - c_j) z_{j'})^n) (X - c_j Y)^n. \quad \square \end{aligned}$$

The next lemma is a variant of [[Harbater and Hartmann 2007](#), Lemma 3.3], allowing nonprincipal ideals.

**Lemma 1.8.** *Let  $R \subseteq R_1$  and  $R_2 \subseteq R_0$  be domains such that  $R_0 = R_1 + R_2$ . Let  $w$  be a nontrivial discrete valuation on  $\text{Quot } R_0$  such that  $R$  is complete with respect to  $w$  and  $w(x) \geq 0$  for all  $x \in R_0$ . Let  $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2$ , and  $\mathfrak{p}_0$  be the centers of  $w$  in  $R, R_1, R_2$ , and  $R_0$ , respectively. Suppose that  $\mathfrak{p}R_0 = \mathfrak{p}_0$  and  $R/\mathfrak{p} = R_1/\mathfrak{p}_1 \cap R_2/\mathfrak{p}_2$  (inside  $R_0/\mathfrak{p}_0$ ). Then  $R_1 \cap R_2 = R$ .*

*Proof.* First, note that  $\mathfrak{p}_0 = \mathfrak{p}_1 + \mathfrak{p}_2$ . Indeed, suppose  $x \in \mathfrak{p}_0 = \mathfrak{p}R_0$ . Then  $x = \sum_{i=1}^n a_i x_i$  for some  $a_1, \dots, a_n \in R_0$  and  $x_1, \dots, x_n \in \mathfrak{p}$ . For each  $1 \leq i \leq n$ , write  $a_i = b_i + b'_i$  with  $b_i \in R_1$  and  $b'_i \in R_2$ . Then  $\sum a_i x_i = \sum b_i x_i + \sum b'_i x_i \in \mathfrak{p}_1 + \mathfrak{p}_2$ , since  $\mathfrak{p} \subseteq \mathfrak{p}_1, \mathfrak{p}_2$ .

Let  $S = R_1 \cap R_2$  and  $\mathfrak{q}$  be the center of  $w$  at  $S$ . Then the sequence  $0 \rightarrow S \rightarrow R_1 \times R_2 \rightarrow R_0 \rightarrow 0$  is exact (where the second map is the diagonal map and the third map is subtraction). This sequence induces an exact sequence  $0 \rightarrow S/\mathfrak{q} \rightarrow (R_1/\mathfrak{p}_1) \times (R_2/\mathfrak{p}_2) \rightarrow R_0/\mathfrak{p}_0 \rightarrow 0$ . Indeed, the only nontrivial part in showing this

is to check that the kernel of the subtraction map is contained in the image of the diagonal map. Suppose  $(x_1 + \mathfrak{p}_1, x_2 + \mathfrak{p}_2) \in (R_1/\mathfrak{p}_1) \times (R_2/\mathfrak{p}_2)$  satisfies  $x_1 - x_2 \in \mathfrak{p}_0$ . Since  $\mathfrak{p}_0 = \mathfrak{p}_1 + \mathfrak{p}_2$  we may choose  $y_1 \in \mathfrak{p}_1, y_2 \in \mathfrak{p}_2$  such that  $x_1 - y_1 = x_2 - y_2$ . Then  $(x_1 + \mathfrak{p}_1, x_2 + \mathfrak{p}_2) = (x_1 - y_1 + \mathfrak{p}_1, x_2 - y_2 + \mathfrak{p}_2)$  belongs to the image of the diagonal map. Thus the sequence is exact.

Since  $\mathfrak{p}$  is the center of  $w$  on  $R$  and  $\mathfrak{p}_1$  the center of  $w$  on  $R_1$ , we have  $R \cap \mathfrak{p}_1 = \mathfrak{p}$ . In particular, the diagonal map  $R/\mathfrak{p} \rightarrow (R_1/\mathfrak{p}_1) \times (R_2/\mathfrak{p}_2)$  is injective. Since  $R_0 = R_1 + R_2$ , the subtraction map  $(R_1/\mathfrak{p}_1) \times (R_2/\mathfrak{p}_2) \rightarrow R_0/\mathfrak{p}_0$  is surjective. Thus, since  $R/\mathfrak{p} = R_1/\mathfrak{p}_1 \cap R_2/\mathfrak{p}_2$ , the sequence  $0 \rightarrow R/\mathfrak{p} \rightarrow (R_1/\mathfrak{p}_1) \times (R_2/\mathfrak{p}_2) \rightarrow R_0/\mathfrak{p}_0 \rightarrow 0$  is also exact. It follows that the natural map  $R/\mathfrak{p} \rightarrow S/\mathfrak{q}$  is an isomorphism. In particular,  $S = R + \mathfrak{p}S$ . By induction we have  $S = R + \mathfrak{p}^k S$  for each  $k \in \mathbb{N}$ . Since  $\mathfrak{p}_0 = \mathfrak{p}R_0, \mathfrak{p} \neq 0$ . Since  $w$  is discrete, there exists an integer  $m$  such that  $v(x) \geq m$  for each  $x \in \mathfrak{p}$ . Thus  $v(x) \geq mk$  for each  $x \in \mathfrak{p}^k$ ; hence  $R$  is  $w$ -dense in  $S$  and therefore the completion of  $R$  with respect to  $w$  contains  $S$ . By our assumptions,  $R$  is complete; hence  $R = S$ . □

**Lemma 1.9.** *The set  $\{z_i^n \mid i \in I, n \in \mathbb{N} \cup \{0\}\}$  is  $K$ -linearly independent.*

*Proof.* Suppose  $a_0 + \sum_{i \in I} \sum_{n=1}^{d_i} a_{i,n} z_i^n = 0$ , where  $d_i \in \mathbb{N}$  and  $a_0, a_{i,n} \in K$  for each  $i$  and  $n$ . We wish to show that  $a_0 = a_{i,n} = 0$  for each  $i$  and  $n$ . Suppose there exist  $i \in I$  and  $n \in \mathbb{N}$  such that  $a_{i,n} \neq 0$ . Without loss of generality,  $n = d_i$ . Since  $X - c_i Y$  is a prime element of  $K[X, Y]$ , it defines a discrete valuation on  $K(X, Y)$ , which we denote by  $w$ . We have  $w(Y) = w(Y - c_j X) = 0$  for each  $j \neq i$  in  $I$ . Thus  $w(a_0 + \sum_{j \neq i} \sum_{n=1}^{d_j} a_{j,n} z_j^n) \geq 0$ , while  $w(\sum_{n=1}^{d_i} a_{i,n} z_i^n) = -d_i$ . Thus  $w(0) = w(a_0 + \sum_{j \in I} \sum_{n=1}^{d_j} a_{j,n} z_j^n) = -d_i$ , a contradiction. □

**Proposition 1.10.** *Suppose  $J, J' \subseteq I$ . Then  $D_J \cap D_{J'} = D_{J \cap J'}$ .*

*Proof.* Clearly,  $D_{J \cap J'} \subseteq D_J \cap D_{J'}$ . For the converse inclusion, we distinguish between two cases. First suppose that  $J \cap J' \neq \emptyset$ , and fix  $j \in J \cap J'$ . Then  $D_J = K[z_k \mid k \in J][[X - c_j Y]]$  and  $D_{J'} = K[z_k \mid k \in J'][[X - c_j Y]]$ ; hence  $D_J \cap D_{J'} = (K[z_k \mid k \in J] \cap K[z_k \mid k \in J'])[[X - c_j Y]]$ . By Lemma 1.9  $K[z_k \mid k \in J] \cap K[z_k \mid k \in J'] = K[z_k \mid k \in J \cap J']$ ; hence  $y \in D_{J \cap J'}$ .

Now suppose that  $J \cap J' = \emptyset$  and let  $R = K[[X, Y]] = D_\emptyset, R_1 = D_J, R_2 = D_{J'}$ , and  $R_0 = D_{J \cup J'}$ . Since  $v(f) \geq 0$  for each  $f \in K[z_j \mid j \in J \cup J'][[X, Y]]$ , we also have  $v(f) \geq 0$  for each  $f$  in the completion  $R_0$ . The ring  $R$  is complete with respect to  $v$ , and  $R = R_1 + R_2$  by Proposition 1.7. Let  $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2$ , and  $\mathfrak{p}_0$  be the centers of  $v$  at  $R, R_1, R_2$ , and  $R_0$ , respectively. Then  $\mathfrak{p}$  is generated by  $X$  and  $Y$ , and  $\mathfrak{p}_0$  is generated by  $X - c_j Y$  for any  $j \in J$ , by Proposition 1.5. It follows that  $\mathfrak{p}R_0 = \mathfrak{p}_0$ . In order to apply Lemma 1.8, it remains to check that  $R_1/\mathfrak{p}_1 \cap R_2/\mathfrak{p}_2 = R/\mathfrak{p}$  in  $R_0/\mathfrak{p}_0$ . Indeed, we have  $R_1/\mathfrak{p}_1 = K[z_j \mid j \in J], R_2/\mathfrak{p}_2 = K[z_j \mid j \in J']$ , and  $R_0/\mathfrak{p}_0 = K$ . By Lemma 1.9, we are done. □





$D_J$  is the  $t$ -adic completion of  $R_U$ . In formal-geometric language, this means that  $D_J = \hat{R}_U$  is the ring of regular functions on the  $t$ -adic thickening of  $U$  [Harbater and Hartmann 2007, Notation 4.3].

**Corollary 1.13.** *Let  $J \subseteq I$ ,  $j \in J$ .*

(a) *For each  $0 \neq g \in D_J$ ,*

$$D_J[(X - c_j Y)^{-1}] = K((X - c_j Y))[z_k \mid k \in J] + g D_J[(X - c_j Y)^{-1}].$$

(b) *For each  $f \in D_J$  there are  $h \in K[[X - c_j Y]][z_j]$  and  $u \in D_J^\times$  such that  $f = hu$ .*

(c) *The ring  $Q_j$  is a field.*

*Proof.* In the notation of Remark 1.12, each element  $f \in D_J[t^{-1}]$  can be written in the form  $u \cdot h$  with  $u \in D_J[t^{-1}]^\times$  and  $h \in F[z_j]$ , by [Haran and Jarden 1998, Lemma 3.7]. If  $f \in D_J$  then we can multiply  $u$  and  $h$  by a power of  $t$  to assume that  $u \in D_J^\times$  and  $h \in K[[t]][z_j]$ . This proves (b). Part (a) is given by [Haran and Jarden 1998, Corollary 3.8]. By [Harbater and Hartmann 2007, Corollary 4.8] (now viewing the rings  $D_J$  in the formal-geometric context)  $\text{Quot } D_J$  is the compositum of  $K((t))(X/Y)$  and  $D_J$ . Since  $K((t))(X/Y) \subseteq E$ , we have  $\text{Quot } D_J = E D_J$ . Applying this argument for  $J \setminus \{j\}$  instead of  $J$  we have  $Q_j = \text{Quot}(D_{I \setminus \{j\}}) = E D_{I \setminus \{j\}}$ ; hence  $\text{Quot } Q_j = \text{Quot}(E D_{I \setminus \{j\}}) = E \text{Quot}(D_{I \setminus \{j\}}) = E D_{I \setminus \{j\}} = Q_j$  is a field.  $\square$

The proof of the following proposition is based on that of Corollary 4.4 of [Haran and Jarden 1998]. (We cannot use that corollary as it is, since condition (e') of that claim does not hold for  $D_I$  itself.)

**Proposition 1.14.** *Let  $i \in I$ ,  $n \in \mathbb{N}$ , and  $b \in \text{GL}_n(Q)$ . There exist  $b_1 \in \text{GL}_n(Q_i)$  and  $b_2 \in \text{GL}_n(Q'_i)$  such that  $b = b_1 \cdot b_2$ .*

*Proof.* Denote by  $|\cdot|$  the absolute value on  $Q$  that corresponds to  $v$ . Each of the rings  $A = D_I$ ,  $A_1 = D_{I \setminus \{i\}}$ , and  $A_2 = D_{\{i\}}$  is complete with respect to  $|\cdot|$  and Proposition 1.7 asserts that condition (d') of Example 4.3 of [Haran and Jarden 1998] holds for these rings. We extend  $|\cdot|$  to the maximum norm  $\|\cdot\|$  on  $M_n(Q)$ , as in the same example. Then  $M_n(A)$ ,  $M_n(A_1)$ , and  $M_n(A_2)$  are complete with respect to  $\|\cdot\|$  and condition (d) of [Haran and Jarden 1998, §4] holds. By Cartan's lemma [Haran and Jarden 1998, Lemma 4.2], for each  $a \in \text{GL}_n(A)$  with  $\|a - 1\| < 1$  there exist  $a_1 \in \text{GL}_1(A_1)$  and  $a_2 \in \text{GL}_1(A_2)$  such that  $a = a_1 \cdot a_2$ .

Let  $E_1 = \text{Quot } A_1 = Q_i$  and  $E_2 = \text{Quot } A_2 = Q'_i$ . In order to factor  $b$  (which need not be in  $\text{GL}_n(A)$ ), let  $t = X - c_i Y$ ,  $T = k[[t]]$ . By Remark 1.12 (for  $J = I$ )  $A_0 = T[z_k \mid k \in I]$  is a dense subring of  $A$ , and by Corollary 1.13(b) there exists  $h \in A_0$  such that  $hb \in M_n(A)$ . If  $hb = b_1 b'_2$  with  $b_1 \in \text{GL}_n(E_1)$  and  $b'_2 \in \text{GL}_n(E_2)$ , then  $b = b_1 b_2$  with  $b_2 = b'_2/h \in \text{GL}_n(E_2)$ . So we may assume that  $b \in M_n(A)$ .

Let  $0 \neq d = \det(b) \in A$ . By [Corollary 1.13\(b\)](#) there are  $0 \neq g \in A_0$  and  $u \in A^\times$  such that  $d = gu$ . Let  $b'' \in M_n(A)$  be the adjoint matrix of  $b$ , so that  $bb'' = d1$ . Let  $b' = u^{-1}b''$ . Then  $b' \in M_n(A)$  and  $bb' = g1$ . Put

$$V = \{a' \in M_n(A[t^{-1}]) \mid ba' \in gM_n(A[t^{-1}])\}, \quad V_0 = V \cap M_n(A_0[t^{-1}]).$$

$V$  is an additive subgroup of  $M_n(A[t^{-1}])$  and  $gM_n(A[t^{-1}]) \leq V$ . By [Corollary 1.13\(a\)](#),  $M_n(A[t^{-1}]) = M_n(A_0[t^{-1}]) + gM_n(A[t^{-1}])$ ; hence  $V = V_0 + gM_n(A[t^{-1}])$ .

Since  $A_0$  is dense in  $A$ ,  $gM_n(A_0[t^{-1}])$  is dense in  $gM_n(A[t^{-1}])$ . It follows that  $V_0 = V_0 + gM_n(A_0[t^{-1}])$  is dense in  $V = V_0 + gM_n(A[t^{-1}])$ . As  $b' \in V$ , there exists  $a_0 \in V_0$  such that  $\|b' - a_0\| < |g|/\|b\|$ . Put  $a = a_0/g \in M_n(Q)$ . Then  $ba \in M_n(A[t^{-1}])$  and  $\|1 - ba\| = \|(1/g)b(b' - a_0)\| \leq (1/|g|)\|b\| \cdot \|b' - a_0\| < 1$ . Hence  $\|ba\| = 1$ , so each entry in  $ba$  has a nonnegative value at  $v$ . By [Remark 1.12](#),  $v$  coincides with the  $t$ -adic valuation on  $A$ ; hence all the entries of  $ba$  belong to  $A$ . Thus  $ba \in M_n(A)$ , and since  $\|1 - ba\| < 1$  and  $M_n(A)$  is complete,  $ba \in \text{GL}_n(A)$ . In particular,  $\det a \neq 0$  and hence  $a \in \text{GL}_n(\text{Quot } A_0) \subseteq \text{GL}_n(E_2)$ . By the first paragraph of this proof, there exist  $b_1 \in \text{GL}_n(A_1) \subseteq \text{GL}_n(E_1)$  and  $b'_2 \in \text{GL}_n(A_2)$  such that  $ba = b_1b'_2$ . Then  $b_2 = b'_2a^{-1} \in \text{GL}_n(E_2)$  satisfies  $b = b_1b_2$ .  $\square$

**Corollary 1.15.** *Suppose  $G$  is a finite group. For each  $i \in I$  let  $F_i$  be a Galois extension of  $E$  with group  $G_i$  contained in  $G$ , such that  $F_i \subseteq Q'_i$ . If  $G = \langle G_i \mid i \in I \rangle$  then  $\mathcal{E} = (E, F_i, Q_i, Q; G_i, G)_{i \in I}$  is a patching datum [[Haran and Jarden 1998](#), Definition 1.1]. In particular,  $G$  occurs as a Galois group over  $E$ .*

*Proof.* By [Corollary 1.13\(c\)](#),  $Q_i$  is a field for each  $i \in I$ . Conditions (2a), (2b), and (2d) of [[Haran and Jarden 1998](#), Definition 1.1] are given in the hypothesis. Conditions (2c) and (2e) are given by [Propositions 1.11](#) and [1.14](#), respectively. Thus  $\mathcal{E}$  is a patching datum. By [[Haran and Jarden 1998](#), Lemma 1.3(a)], there exists a Galois extension  $F$  of  $E$  with group  $G$ .  $\square$

## 2. $p$ -groups

Fix the notation of [Section 1](#), including that of [Construction 1.1](#), and let  $p$  denote a prime number. In this section we realize  $p$ -groups of rank at most 2 by adequate extensions of  $E$ , and embed these extensions into the analytic fields.

**Lemma 2.1.** *Let  $J \subseteq I$ ,  $j \in J$ , and  $t = X - c_jY$ .*

- (a) *Suppose  $f = \sum_{l=0}^d f_l z_j^l \in K[[t]][[z_j]]$  is a polynomial such that  $v(f_1) = 0$  and  $v(f_l) > 0$  for each  $l > 1$ . Then  $f$  is prime in  $D_J[t^{-1}]$ .*
- (b) *The ring  $D_J[t^{-1}]$  is a unique factorization domain.*
- (c) *For each  $a, b, c \in K^\times$  with  $a \neq -b$  and  $2 \leq m \in \mathbb{N}$ , the elements  $1 + az_j + t^{m-1}z_j^m$ ,  $1 + bz_j - t^{m-1}z_j^m$ ,  $1 + cz_j$  are nonassociate primes of  $D_J[t^{-1}]$ .*

*Proof.* Set  $F = K((t))$ . Then  $D_J[t^{-1}] = F\{z_k \mid k \in J\}$  (see [Remark 1.12](#)). Viewed as an element of  $F\{z_j\}$ ,  $f$  is regular of pseudodegree 1; see Definition 1.4 of [\[Haran and Völklein 1996\]](#). Hence, by Corollary 1.7 of the same work, we have  $f = u \cdot q$ , where  $u \in F\{z_j\}^\times \subseteq D_J[t^{-1}]^\times$  and  $q = q_0 + z_j \in F[z_j]$  is a linear polynomial with  $v(q_0) \geq 0$ . Thus to prove (a), it suffices to show that  $q$  is prime in  $D_J[t^{-1}]$ . Without loss of generality  $q_0 \neq 0$ , and we set  $c = c_j - 1/q_0$ . Then  $q = z_j - 1/(c - c_j)$ ; hence, by [\[Paran 2008, Lemma 6.4\(a\)\]](#) (with  $D$ ,  $r$ , and 1 there replaced by  $F$ , 1, and  $j$  here),  $q$  generates the kernel of an epimorphism from  $D_J[t^{-1}]$  onto a domain (actually a field here); hence  $q$  is prime. This proves (a).

Since  $D_J[t^{-1}]$  is a principal ideal domain by [\[Haran and Jarden 1998, Proposition 3.9\]](#), part (b) follows.

By part (a),  $r = 1 + az_j + t^{m-1}z_j^m$ ,  $r' = 1 + bz_j - t^{m-1}z_j^m$ , and  $s = 1 + cz_j$  are primes of  $D_J[t^{-1}]$ . If  $s \mid r$ , then  $-1/c$  is a root of  $r$ , a contradiction. Thus  $r$  and  $s$  (and similarly,  $r'$  and  $s$ ) are nonassociates.

If  $r \mid r'$  then  $r \mid r + r'$ . By the argument of the preceding paragraph,  $r + r' = 2 + (a + b)z_j$  is a prime, nonassociate to  $r$ , a contradiction. This proves (c).  $\square$

**Lemma 2.2.** *Let  $K$  be a field that contains a primitive  $q$ -th root of unity, for some  $q \in \mathbb{N}$ . Let  $v$  be a discrete valuation on  $K$  which is trivial on the prime field of  $K$ , and let  $a \in K$  with  $v(a) = 0$ . Suppose  $L = K(a^{1/q})$  is a Kummer extension of  $K$ , and that  $L/K$  is unramified at  $v$ . Then  $v(x^\sigma) = v(x)$  for each  $x \in L$  and  $\sigma \in \text{Gal}(L/K)$ .*

*Proof.* Extend  $v$  arbitrarily to  $L$ , let  $O$  be the valuation ring of  $v$  in  $K$ , and  $O'$  the valuation ring of  $v$  in  $L$ . Since  $K$  contains a primitive  $q$ -th root of unity,  $q$  is not divisible by  $p = \text{char } K$ . Thus  $d = \text{disc}(T^q - a, K) = ka^{q-1}$ , where  $k \in \mathbb{Z}$  is not divisible by  $p$ . Hence  $v(d) = 0$ , and by [\[Fried and Jarden 2005, Lemma 6.1.2\]](#) we have  $O' = O[a^{1/q}]$ . Put  $\alpha = a^{1/q}$  and let  $x = \sum_{i=0}^{q-1} b_i \alpha^i \in K$ , with  $b_0, \dots, b_{q-1} \in K$ . We claim that  $v(x) = \min_i v(b_i)$ . Indeed, since  $L/K$  is unramified at  $v$ , we may multiply  $x$  by a power of a uniformizer of  $v$  in  $K$ , to assume that  $v(x) = 0$ . Since  $O' = O[\alpha]$ ,  $v(b_i) \geq 0$  for each  $0 \leq i \leq q-1$ . On the other hand  $v(x) \geq \min_i v(b_i \alpha^i) = \min_i v(b_i)$ , since  $v(\alpha) = (1/n)v(a) = 0$ . Thus  $v(b_i) = 0$  for some  $0 \leq i \leq q-1$ ; hence  $v(x) = \min_i v(b_i)$ .

Now, let  $\sigma \in \text{Gal}(L/K)$  and let  $x = \sum_{i=0}^{q-1} b_i \alpha^i \in K$ , with  $b_0, \dots, b_{q-1} \in K$ , be an arbitrary element. We have  $\alpha^\sigma = \zeta \alpha$ , where  $\zeta$  is some  $q$ -th root of unity. Then  $v(x^\sigma) = v(\sum_{i=0}^{q-1} b_i \zeta^i \alpha^i) = \min_i v(b_i \zeta^i) = \min_i v(b_i) = v(x)$ .  $\square$

Recall that given a field  $K$ , any  $K$ -central simple algebra  $A$  is of the form  $M_n(D)$  for some  $K$ -division algebra  $D$ . The index of  $A$  is defined to be  $\text{ind } A = \sqrt{\dim_K D}$ . So,  $A$  is a division algebra if and only if  $\text{ind } A = \sqrt{\dim_K A}$ . Let us denote Brauer equivalence by  $\sim$  and the exponent of  $A$  (its order in the Brauer group) by  $\text{exp } A$ . A subfield  $F$  of  $A$  is a maximal subfield of  $A$  if and only if  $\dim_K A = [F : K]^2$ .

Furthermore, a field  $F$  is a maximal subfield of  $A$  if and only if  $\dim_K A = [F : K]^2$  and  $F$  splits  $A$  [Reiner 1975, Theorem 28.4 and Corollary 28.11].

The proof of the next proposition is partially based on that of [Harbater et al. 2009, Proposition 4.4].

**Proposition 2.3.** *Fix  $i \in I$ , and let  $H$  be an abelian  $p$ -group of rank at most 2, where  $p \neq \text{char } K$ . Suppose  $K$  contains an  $|H|$ -th primitive root of unity. Let  $E'$  be a finite extension of  $E$ . Then there exists an  $H$ -Galois extension  $F_i/E$  such that  $F_i \subseteq Q'_i$ ,  $F_i$  is contained as a maximal subfield in an  $E$ -division algebra  $D'_i$ , and  $D'_i \otimes_E E' Q_i$  remains a division algebra (where  $E' Q_i$  is the compositum of  $E'$  and  $Q_i$  in an algebraic closure of  $Q$ ).*

*Proof.* Let us start by constructing  $F_i$ . Write  $H = C_q \times C_{q'}$ , where  $q$  and  $q'$  are powers of  $p$ . For each  $k \in \mathbb{N}$ , the elements  $X - c_i Y + Y^k$  and  $X + c_i Y - Y^k$  are irreducible and hence prime in the unique factorization domain  $K[[X, Y]]$ . Only finitely many primes of  $K[[X, Y]]$  are ramified at  $E'/E$ ; hence for a sufficiently large  $2 \leq k \in \mathbb{N}$ ,  $f = X - c_i Y + Y^k$ , and  $g = X + c_i Y - Y^k$  are unramified at  $E'/E$ . That is, the corresponding valuations  $v_f$  and  $v_g$  are unramified. Let  $a = f/(X - c_i Y)$  and  $b = g/(X + c_i Y)$ . Clearly  $v_f(X - c_i Y) = v_f(g) = 0$ ; hence  $v_f(a) = 1$  and  $v_f(b) = 0$ . Similarly,  $v_g(a) = 0$  and  $v_g(b) = 1$ . Consider the polynomial  $h(U) = U^q - a$  over  $D_{\{i\}} = K[z_i][[X - c_i Y]]$ . Note that  $a = 1 + z_i^k (X - c_i Y)^{k-1}$ , hence  $h(1) \in (X - c_i Y)D_{\{i\}}$  and  $h'(1) = q \in K^\times \subseteq D_{\{i\}}^\times$ . By the ring version of Hensel's lemma (for the ideal  $(X - c_i Y)D_{\{i\}}$ )  $h(U)$  has a root  $s \in D_{\{i\}}$ . Note that  $v_f(s) = 1/q \notin \mathbb{Z}$ ; hence  $s \notin E$ . Since  $K$  contains a primitive  $|H|$ -th root of unity, it contains a primitive  $q$ -th root of unity. By Kummer theory  $E(s)/E$  is a Galois extension with group  $C_q$ . Similarly, there exists  $s' \in D_{\{i\}}$  satisfying  $(s')^{q'} = b$ , and  $E(s')/E$  is Galois with group  $C_{q'}$ . Let  $F_i = E(s, s') \subseteq Q'_i$ .

Since  $v_f(a) = 1$ ,  $h(U)$  is irreducible over  $E$ , by Eisenstein's criterion. Denoting the reduction modulo  $g$  by  $\bar{\cdot}$ ,  $\bar{h}(U) = U^q - \bar{a}$  is separable, since  $\bar{a} \neq 0$ . Thus by [Fried and Jarden 2005, Lemma 2.3.4],  $E(s)/E$  is unramified at  $v_g$ . Clearly,  $E(s')/E$  is totally ramified at  $v_g$ . Thus  $E(s)$  and  $E(s')$  are linearly disjoint over  $E$ ; hence  $\text{Gal}(F_i/E) = H$ .

Let  $D'_i$  be the quaternion algebra  $(a, b)_{qq'}$  [Pierce 1982, Section 15.4]. Note that  $D'_i$  can be also viewed as the cyclic algebra  $(E(a^{1/qq'})/E, \sigma, b)$ , for some generator  $\sigma$  of  $\text{Gal}(E(a^{1/qq'})/E)$ . We claim that  $F_i$  splits  $D'_i$ . By [Reiner 1975, Theorem 30.8], we have

$$D'_i \otimes_E E(s) \sim (E(s^{1/q'})/E(s), \sigma^q, b);$$

thus  $D'_i \otimes_E F_i \sim (F_i(s^{1/q'})/F_i, \sigma^q, b)$ . The cyclic algebra  $(F_i(s^{1/q'})/F_i, \sigma^q, b)$  is split if and only if  $b$  is a norm from  $F_i(s^{1/q'})$  (see for example [Reiner 1975, Theorem 30.4]), i.e., if and only if  $b \in N_{F_i(s^{1/q'})/F_i}(F_i(s^{1/q'}))$ . This holds since

$b = N_{F_i(s^{1/q'})/F_i}(s')$ . Thus  $F_i$  splits  $D'_i$ . As  $[F_i : E] = qq'$ ,  $F_i$  is a maximal subfield of the  $E$ -central simple algebra  $D'_i$ . We shall show that  $D'_i \otimes_E E'Q_i$  is a division algebra and from this it will follow that  $D'_i$  is a division algebra.

In order to show that  $D'_i \otimes_E E'Q_i$  is a division algebra we construct auxiliary valuations. Choose  $j \in I \setminus \{i\}$ , and let

$$t = X - c_j Y, \quad r = 1 + (c_j + c_i)z_j - t^{k-1}z_j^k, \quad r' = 1 + (c_j - c_i)z_j + t^{k-1}z_j^k.$$

By [Lemma 2.1\(c\)](#)  $r$  and  $r'$  are nonassociate prime elements in  $D_{I \setminus \{i\}}[t^{-1}]$ , so they define discrete valuations  $v_r$  and  $v_{r'}$  on  $Q_i = \text{Quot}(D_{I \setminus \{i\}}) = \text{Quot}(D_{I \setminus \{i\}}[t^{-1}])$  such that  $v_r(r') = v_{r'}(r) = 0$ . By [Lemma 2.1\(c\)](#) we also have

$$v_{r'}(1 + (c_j - c_i)z_j) = v_r(1 + (c_j - c_i)z_j) = 0.$$

Note that

$$b = \frac{X - c_j Y + (c_j + c_i)Y - Y^k}{X - c_j Y + (c_j - c_i)Y} = \frac{t + (c_j + c_i)t z_j - t^j z_j^k}{t + (c_j - c_i)t z_j} = \frac{r}{1 + (c_j - c_i)z_j}.$$

Similarly,  $a = r'/(1 + (c_j - c_i)z_j)$ . Thus  $v_r(b) = 1$ ,  $v_{r'}(b) = 0$ ,  $v_r(a) = 0$ , and  $v_{r'}(a) = 1$ . Then the polynomial  $U^{qq'} - a$  is irreducible over  $D_{I \setminus \{i\}}$ , by Eisenstein's criterion (using  $v_{r'}$ ). Thus  $Q_i(a^{1/qq'})/Q_i$  is unramified at  $v_r$  (again by [[Fried and Jarden 2005](#), Lemma 2.3.4]); hence so is  $E'Q_i(a^{1/qq'})/E'Q_i$ .

Only finitely many primes of the unique factorization domain  $D_{I \setminus \{i\}}[t^{-1}]$  (see [Lemma 2.1\(b\)](#)) are ramified at the finite extension  $E'Q_i/Q_i$ ; hence, without loss of generality, we may assume that  $E'Q_i/Q_i$  is unramified at  $v_{r'}$  (by possibly choosing an even larger  $k$  beforehand). On the other hand,  $Q_i(a^{1/qq'})/Q_i$  is totally ramified at  $v_{r'}$ ; hence

$$[E'Q_i(a^{1/qq'}) : E'Q_i] = [Q_i(a^{1/qq'}) : Q_i] = qq'.$$

We can now show that  $D'_i \otimes_E E'Q_i$  is a division algebra. A sufficient condition for this to hold is that  $\exp(D'_i \otimes_E E'Q_i) = qq'$ . This happens if and only if for every  $1 \leq m \leq qq' - 1$  the algebra  $(E'Q_i(a^{1/qq'})/E'Q_i, \sigma, b^m) \sim (D'_i \otimes_E E'Q_i)^m$  does not split. Let  $N$  denote the norm  $N_{E'Q_i(a^{1/qq'})/E'Q_i}$ . For any  $1 \leq m \leq qq' - 1$ , the algebra  $(D'_i \otimes_E E'Q_i)^m$  splits if and only if  $b^m \in N(E'Q_i(a^{1/qq'})^\times)$  [[Reiner 1975](#), Theorem 30.4].

Since  $E'Q_i(a^{1/qq'})/E'Q_i$  is unramified at  $v_r$ , we have  $v_r(x) = v_r(x^\sigma)$  for each  $x \in E'Q_i(a^{1/qq'})$ , by [Lemma 2.2](#). Hence

$$v_r(N(x)) = \sum_{l=0}^{qq'-1} v_r(x^{\sigma^l}) = qq' v_r(x)$$

for all  $x \in E'Q_i(a^{1/qq'})$ . Since  $v_r(b) = 1$ ,  $b^m \notin N(E'Q_i(a^{1/qq'})^\times)$  for all  $1 \leq m \leq qq' - 1$ . Thus,  $\exp(D'_i \otimes_E E'Q_i) = qq'$  and  $D'_i \otimes_E E'Q_i$  is a division algebra.  $\square$

### 3. Patching and admissibility

We have established the patching machinery needed to prove our [Main Theorem](#). We first recall some general properties of induced algebras and Frobenius algebras.

**Remark 3.1** (induced algebras). Let  $G$  be a finite group and  $H \leq G$ . Let  $P/Q$  be a finite field extension with  $H = \text{Gal}(P/Q)$ . Let

$$N = \text{Ind}_H^G P = \left\{ \sum_{\theta \in G} a_\theta \theta \mid a_\theta \in P, a_\theta^\tau = a_{\theta\tau} \text{ for all } \theta \in G, \tau \in H \right\}$$

be a ring with respect to point-wise addition and multiplication. Then  $P$  can be embedded as a subring of  $N$  by choosing representatives  $\theta_1 = 1, \dots, \theta_k$  of  $H \backslash G$  and sending an element  $x \in P$  to  $\sum_{i=1, \tau \in H}^k x^\tau \theta_i \tau$ . Furthermore, by choosing different representatives  $N$  can be presented as a direct sum of copies of  $P$ .

If  $P$  splits a central simple  $Q$ -algebra  $A$  then

$$\text{Ind}_H^G P \otimes_Q A \cong \text{Ind}_H^G P \otimes_P (P \otimes_Q A) \sim \text{Ind}_H^G P \otimes_P P \cong \text{Ind}_H^G P;$$

hence  $A$  splits over  $\text{Ind}_H^G P$ . For a definition of a split separable (Azumaya) algebra over a ring, see [\[DeMeyer and Ingraham 1971, §5\]](#).

The next definition, remark, and proposition appear in Section 2.1 of [\[Jacobson 1996\]](#).

**Definition 3.2** (Frobenius algebras). Let  $F$  be a field. An  $F$ -algebra  $A$  is a Frobenius algebra if  $A$  contains a hyperplane that does not contain any nonzero one sided ideal of  $A$ .

**Remark 3.3.** An algebra  $A_1 \oplus \dots \oplus A_s$  is Frobenius if and only if  $A_i$  is Frobenius for each  $1 \leq i \leq s$ . Any algebra  $F[a]$  (with a single generator) is Frobenius. Let  $L/K$  be an  $H$ -extension. By [Remark 3.1](#),  $\text{Ind}_H^G L$  can be decomposed into a sum of copies of  $L$  and it follows that  $\text{Ind}_H^G L$  is a Frobenius algebra.

**Proposition 3.4** [\[Jacobson 1996, Theorem 2.2.3\]](#). *Let  $A$  be an  $F$ -central simple algebra and  $K$  a commutative Frobenius subalgebra of  $A$  such that  $\dim_F A = [K : F]^2$ . Then any embedding of  $K$  into  $A$  can be extended to an inner automorphism of  $A$ .*

**Lemma 3.5.** *Let  $R$  be an equicharacteristic complete local domain of dimension  $r$ . Suppose that the residue field of  $R$  contains a primitive root of unity of order  $k$ , for each  $k \in \mathbb{N}$  with  $\text{char } R \nmid k$ . Then  $R$  is a finite module over a subring of the form  $K[[X_1, \dots, X_r]]$ , where  $K$  is a field containing a primitive root of unity of order  $k$ , for each  $k \in \mathbb{N}$  with  $\text{char } K \nmid k$ .*

*Proof.* By Cohen's structure theorem [\[Matsumura 1986, §29\]](#),  $R$  is finitely generated over a subring of the form  $B = K_0[[X_1, \dots, X_n]]$ , for some field  $K_0$ . Since  $\dim B = \dim R = r$ , we have  $n = r$ .

Let  $K$  be the field obtained by adjoining all primitive roots of unity of order not divisible by  $\text{char } R$  to  $K_0$ . By our assumptions,  $K$  is contained in the residue field of  $R$ ; hence  $K/K_0$  is a finite (separable) extension. By Hensel's lemma,  $R$  contains  $K$ . Since  $[K : K_0]$  is finite,  $K(K_0[[X_1, \dots, X_r]]) = K[[X_1, \dots, X_r]]$ . Thus  $R$  is finite over  $K[[X_1, \dots, X_r]]$  (since it is finite over  $K_0[[X_1, \dots, X_r]]$ ).  $\square$

The final ingredient we need in order to prove our [Main Theorem](#) is patching of central simple algebras. The content of the next proposition is essentially given in [[Harbater et al. 2009](#)], but for specific fields  $Q_i$ , while here we present it for general fields satisfying a matrix factorization property. We note that [[Harbater et al. 2009](#), Theorem 4.1] uses the terminology of categories and equivalence of categories. Here we prefer a more explicit presentation, working with vector spaces and bases, as in [[Haran and Jarden 1998](#)]. The proof of the proposition combines the proof of [[Haran and Jarden 1998](#), Lemma 1.2] (where a more restricted assertion is made for specific types of algebras), and the proof of [[Harbater and Hartmann 2007](#), Theorem 7.1(vi)] (where the assertion is made for specific types of fields).

**Proposition 3.6.** *Let  $I$  be a finite set. For each  $i \in I$  let  $Q_i$  be a field contained in a  $Q_i$ -algebra  $A_i$ . Let  $Q$  be a field containing  $Q_i$  for each  $i \in I$ , and contained in a  $Q$ -algebra  $A_Q$  which contains  $A_i$  for each  $i \in I$ . Moreover, suppose that  $A_i Q = A_Q$  and  $\dim_{Q_i} A_i = \dim_Q A_Q$  for each  $i \in I$ . Finally, suppose that*

(\*) *for each  $B \in \text{GL}_n(Q)$  there exist  $B_i \in \text{GL}_n(Q_i)$  and  $B'_i \in \text{GL}_n(\bigcap_{j \neq i} Q_j)$  such that  $B = B_i B'_i$ .*

*Then, letting  $E = \bigcap_{i \in I} Q_i$ ,  $A = \bigcap_{i \in I} A_i$  is an  $E$ -algebra satisfying  $A Q_i = A_i$  for each  $i \in I$ . Moreover, if each  $A_i$  is central simple, then so is  $A$ .*

*Proof.* For each  $i \in I$ , let  $\mathcal{C}_i$  be a basis for  $A_i$  over  $Q_i$ . Since  $A_i Q = A_Q$ ,  $\text{Span}_Q(\mathcal{C}_i) = A_Q$ , and since  $\dim_{Q_i} A_i = \dim_Q A_Q$ ,  $\mathcal{C}_i$  is a basis for  $A_Q$  over  $Q$ , for each  $i \in I$ . We now construct a basis  $\mathcal{C}$  for  $A_Q$  over  $Q$ , which is also a basis for  $A_i$  over  $Q_i$ , for all  $i \in I$ .

For each subset  $J$  of  $I$  we find, by induction on  $|J|$ , a basis  $\mathcal{V}_J$  for  $A_Q$  over  $Q$  which is also a basis for  $A_j$  over  $Q_j$ , for each  $j \in J$ . Then for  $I = J$  we will get the basis  $\mathcal{C}$ .

If  $J = \emptyset$  there is nothing to prove. Suppose that  $|J| \geq 1$ , choose  $k \in J$  and let  $J' = J \setminus \{k\}$ . By assumption there is a basis  $\mathcal{V}_{J'}$  for  $A_Q$  over  $Q$  which is a basis for  $A_i$  over  $Q_i$  for each  $i \in J'$ . Since  $\mathcal{C}_i$  is a common basis for  $A_Q$  and  $A_i$ , there is a matrix  $B \in \text{GL}_n(Q)$  such that  $\mathcal{C}_k B = \mathcal{V}_{J'}$ . By condition (\*) in the statement of the proposition, there exist  $B_k \in \text{GL}_n(Q_k)$  and  $M \in \text{GL}_n(\bigcap_{k \neq j \in I} Q_j) \subseteq \bigcap_{j \in J'} \text{GL}_n(Q_j)$  such that  $B = B_k M$ . Put  $\mathcal{V}_J = \mathcal{V}_{J'} M^{-1}$ . Then  $\mathcal{V}_J$  is a basis for  $A_Q$  over  $Q$  which is also a basis for  $A_j$  over  $Q_j$  for each  $j \in J'$ . Moreover,  $\mathcal{V}_J$  is also a basis for  $A_k$  over  $Q_k$ , since  $\mathcal{V}_J = \mathcal{V}_{J'} B_k M^{-1} = \mathcal{V}_{J'} B_k$ . This completes the induction.

The existence of the common basis  $C$  implies that  $AQ_i = A_i$  for each  $i \in I$ . As  $A_i$  is a  $Q_i$ -central simple algebra for any  $i \in I$  (a single  $i$  suffices),  $A$  is an  $E$ -central simple algebra. This follows, for example, from [Saltman 1999, Theorem 2.2.c].  $\square$

**Proposition 3.7.** *Let  $R$  be an equicharacteristic complete local domain of dimension 2, with residue field containing a primitive root of unity of order  $k$  for each  $k \in \mathbb{N}$  with  $\text{char } R \nmid k$ . Let  $G$  be a finite group of order not divisible by  $\text{char } R$ , whose Sylow subgroups are abelian of rank at most 2. Then  $G$  is admissible over  $\text{Quot } R$ .*

*Proof.* By Lemma 3.5,  $R$  is a finite module over a subring of the form  $B = K[[X, Y]]$ , where  $K$  contains a primitive root of unity of order  $k$  for each  $k \in \mathbb{N}$  not divisible by  $p = \text{char } R$ . Let  $E = \text{Quot } B = K((X, Y))$  and  $E' = \text{Quot } R$ . Then  $E'$  is a finite extension of  $E$ .

Part A: A patching datum. Let  $(p_i)_{i \in I}$  be the prime factors of  $n = |G|$ , for some index set  $I$ . For each  $i \in I$ , let  $G_i$  be a  $p_i$ -Sylow subgroup of  $G$ . Apply Construction 1.1 to obtain rings  $Q_i$ ,  $i \in I$ , contained in the common field  $Q$ . For each  $i \in I$ , we may apply Proposition 2.3 to obtain a Galois extension  $F_i/E$  with group  $G_i$ , such that  $F_i \subseteq Q'_i$  and  $F_i$  is contained as a maximal subfield in a division  $E$ -algebra  $D'_i$ . Moreover,  $D'_i \otimes_E E' Q_i$  remains a division algebra. Thus  $D_i := D'_i \otimes_E Q_i$  is also a division algebra. By Corollary 1.13(c),  $Q_i$  is a field for each  $i \in I$ . Put  $P_i = F_i Q_i$ . Since  $F_i$  splits  $D'_i$ ,  $P_i$  splits  $D_i$ , and since  $[P_i : Q_i] = [F_i : E] = \text{ind}(D'_i) = \text{ind}(D_i)$ ,  $P_i$  is a maximal subfield of  $D_i$ . Let  $\mathcal{C} = (E, F_i, Q_i, Q; G_i, G)_{i \in I}$ . By Corollary 1.15,  $\mathcal{C}$  is a patching datum.

Part B: Induced algebras [Haran and Jarden 1998, §1]. Consider the induced algebra  $N = \text{Ind}_1^G Q$  of dimension  $n$  over  $Q$ , and the  $Q_i$ -subalgebra  $N_i = \text{Ind}_{G_i}^G P_i$  for each  $i \in I$  (see Remark 3.1). Then  $G$  acts on  $N$  by  $(\sum_{\theta \in G} a_\theta \theta)^\sigma = \sum_{\theta \in G} a_\theta \sigma^{-1} \theta = \sum_{\theta \in G} a_{\sigma\theta} \theta$  for each  $\sigma \in G$ . The field  $Q$  is embedded diagonally in  $N$ , which induces an embedding of  $Q_i$  in  $N_i$ , for each  $i \in I$ . We view these embeddings as containments. By Lemma 1.2 of [Haran and Jarden 1998] there is a basis for  $N$  over  $Q$ , which is also a basis for  $N_i$  over  $Q_i$ , for each  $i \in I$ . In particular, we have  $N_i Q = N$  for each  $i \in I$ . By Lemma 1.3 of the same paper,  $F = \bigcap_{i \in I} N_i$  is a Galois field extension of  $E$  with group  $G$ , and there exists an  $E$ -embedding of  $F$  into  $Q$ . Denote the image of  $F$  under this embedding by  $F'$ .

Part C: Division algebras. It remains to prove that the extension  $F'/E$  is adequate. Let  $A_Q = M_n(Q)$ , and for each  $i \in I$  let  $n_i = [G : G_i]$ . As  $A_Q$  is split of dimension  $n^2$  and  $N$  is of dimension  $n$  over  $Q$ , we also have an embedding of  $N$  into  $A_Q$ . We view  $N$  as a subalgebra of  $A_Q$  via this embedding.

Fix  $i \in I$ . Since  $P_i = F_i Q_i$  splits  $D_i$ , it follows by Remark 3.1 that  $N_i$  also splits  $D_i$ . Moreover, by [DeMeyer and Ingraham 1971, Theorem 5.5] there is a central



simple  $Q_i$ -algebra  $A_i$  which is Brauer equivalent to  $D_i$ , in which  $N_i$  embeds as a maximal commutative separable  $Q_i$ -subalgebra so that  $\dim_{Q_i}(A_i) = \dim_{Q_i}(N_i)^2 = n^2$ . We view  $N_i$  as contained in  $A_i$  via this embedding.

Since  $P_i$  splits  $D_i$ , we have

$$D_i \otimes_{Q_i} Q \cong (D_i \otimes_{Q_i} P_i) \otimes_{P_i} Q \cong M_{n/n_i}(P_i) \otimes_{P_i} Q \cong M_{n/n_i}(Q).$$

Since  $\text{ind}(D_i) = n/n_i$  and  $\dim_Q A_Q = n^2$  we get that  $A_i \cong M_{n_i}(D_i)$ . Thus we have  $A_i \otimes_{Q_i} Q \cong M_{n_i}(D_i) \otimes_{Q_i} Q \cong M_n(Q) = A_Q$ , and we denote the induced  $Q$ -algebras isomorphism  $A_i \otimes_{Q_i} Q \rightarrow A_Q$  by  $\psi_i$ . We cannot identify these two algebras via this isomorphism, since it might not be compatible with the containments  $N_i \subseteq A_i$  and  $N \subseteq A_Q$ . This compatibility problem can be settled similarly to [Harbater et al. 2009, Lemma 4.2]:

By Part B we have  $N = N_i Q$  and  $\dim_{Q_i} N_i = \dim_Q N = n$ . Thus we have an isomorphism  $\delta_i : N = N_i Q \rightarrow N_i \otimes_{Q_i} Q$  for which the following diagram commutes:

$$\begin{array}{ccc} A_i & \xrightarrow{\quad\quad\quad} & A_i \otimes_{Q_i} Q \\ \downarrow & & \uparrow \text{id} \otimes_{Q_i} Q \\ N_i & \xrightarrow{\delta_i} & N_i \otimes_{Q_i} Q. \end{array} \tag{2}$$

By Remark 3.3,  $N = \text{Ind}_1^G Q$  is a Frobenius (commutative) subalgebra of  $A_Q$ . By Proposition 3.4, the embedding  $\psi_i(\text{id} \otimes_{Q_i} Q)\delta_i : N \rightarrow A_Q$  extends to an inner automorphism  $\alpha_i$  of  $A_Q$ . Let  $\psi'_i = \alpha_i^{-1} \psi_i$ . Then  $\alpha_i^{-1} \psi_i(\text{id} \otimes_{Q_i} Q)\delta_i$  is the identity map on  $N = N_i Q$ , so we have the commutative diagram

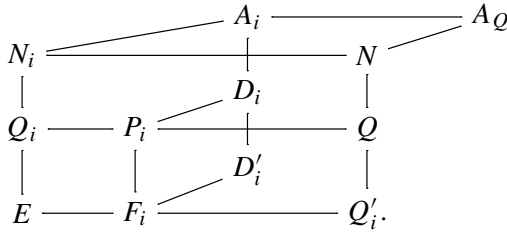
$$\begin{array}{ccc} A_i \otimes_{Q_i} Q & \xrightarrow{\psi'_i} & A_Q \\ (\text{id} \otimes_{Q_i} Q)\delta_i \uparrow & & \uparrow \\ N_i Q & \xlongequal{\quad\quad\quad} & N. \end{array} \tag{3}$$

Combining (2) and (3), we get the following commutative diagram:

$$\begin{array}{ccc} A_i & \xrightarrow{\quad\quad\quad} & A_i \otimes_{Q_i} Q \xrightarrow{\psi'_i} & A_Q \\ \downarrow & & \uparrow (\text{id} \otimes_{Q_i} Q)\delta_i & \uparrow \\ N_i & \xrightarrow{\quad\quad\quad} & N_i Q \xlongequal{\quad\quad\quad} & N. \end{array}$$

This diagram gives an embedding  $A_i \rightarrow A_Q$  which is compatible with the containments  $N_i \subseteq A_i$  and  $N \subseteq A_Q$ , so we may now identify  $A_i$  as a subring of  $A_Q$ , via this embedding. Moreover, since  $\psi'_i$  is an isomorphism, we have  $A_i \otimes_{Q_i} Q = A_i Q = A_Q$

by this identification. The following diagram explains the containment relations:



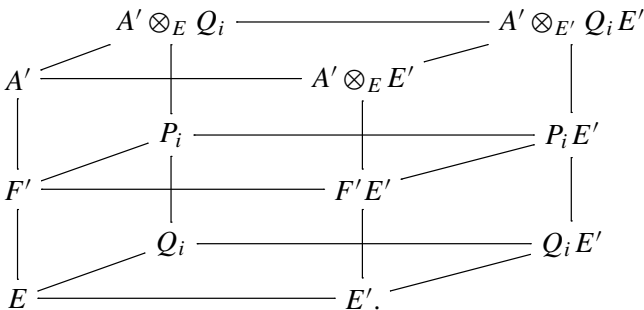
Let  $A = \bigcap_{i \in I} A_i$ . By Proposition 3.6,  $A$  is a central simple  $E$ -algebra for which  $AQ_i = A_i$  for each  $i \in I$ . In particular,  $A = M_k(D)$  for some division algebra  $D$  of index  $n/k$ .

Now,  $D \otimes_E Q_i$  is Brauer equivalent to  $A \otimes_E Q_i \cong A_i$ , which is Brauer equivalent to  $D_i$ . Thus,  $n/n_i = \text{ind}(D_i) \mid \text{ind } D$  for each  $i \in I$  and  $n = \text{lcm}_i(n/n_i) \mid \text{ind } D$ . It follows that  $k = 1$  and  $A$  is a division algebra. Naturally,  $F$  is a subfield of  $A$  and  $\text{ind } A = [F : E]$ . It follows that  $F$  is a maximal subfield of the division algebra  $A$ .

By choosing a basis for  $A/F$  and considering the corresponding structure constants one can form an  $E$ -division algebra  $A'$  which is  $E$ -isomorphic to  $A$  such that  $F'$  is a maximal subfield of  $A'$ .

We will show that  $A' \otimes_E E'$  is an  $E'$ -division algebra, but first let us show that this implies that  $F'E'/E'$  is an adequate  $G$ -extension (and hence  $G$  is  $E'$ -admissible). Indeed, if  $A' \otimes_E E'$  is a division algebra, then  $F' \otimes_E E'$  is a field. It follows that  $F' \otimes_E E' \cong F'E'$ , since  $F' \otimes_E E'$  is  $G$ -Galois over  $E'$  [Saltman 1999, Theorem 6.3]. Thus,  $[F'E' : E'] = [F' : E]$  and  $F' \cap E' = E$ . Since  $F'E'$  splits  $A' \otimes_E E'$  and as  $\text{ind}(A' \otimes_E E') = [F'E' : E']$ ,  $F'E'$  is a maximal subfield of  $A' \otimes_E E'$  and hence an adequate  $G$ -extension.

In order to show that  $A' \otimes_E E'$  is an  $E'$ -division algebra, we first note that for each  $i \in I$ ,  $P_i = F_i Q_i = F' Q_i$ , by [Haran and Völklein 1996, Lemma 3.6(b)]. Thus, we have the diagram



As mentioned above,  $A' \otimes_E Q_i$  is Brauer equivalent to  $D_i = D'_i \otimes_E Q_i$ . Thus  $A' \otimes_E Q_i E'$  is Brauer equivalent to  $D'_i \otimes_{Q_i} Q_i E'$ , which by the choice of  $D'_i$  is a

division algebra. Then, for all  $i \in I$ ,

$$\frac{n}{n_i} = \text{ind}(D'_i) = \text{ind}(D'_i \otimes_{Q_i} Q_i E') \mid \text{ind}(A' \otimes_E Q_i E') \mid \text{ind}(A' \otimes_E E').$$

It follows that  $n = \text{lcm}_{i \in I}(n/n_i) \mid \text{ind}(A' \otimes_E E')$ . Hence  $n = \text{ind}(A' \otimes_E E')$ , which shows that  $A' \otimes_E E'$  is a division algebra. □

As a corollary, we get our [Main Theorem](#), which we restate for convenience:

**Theorem 3.8.** *Let  $R$  be an equicharacteristic complete local domain of dimension 2, with a separably closed residue field. Let  $E = \text{Quot } R$  and let  $G$  be a finite group of order not divisible by  $\text{char } E$ . Then  $G$  is  $E$ -admissible if and only if all the Sylow subgroups of  $G$  are abelian of rank at most 2.*

*Proof.* By [Proposition 3.7](#), if the Sylow subgroups of  $G$  are abelian of rank at most 2 then  $G$  is  $E$ -admissible. For the converse, assume  $G$  is  $E$ -admissible. For a prime  $v$  of  $E$ , let  $\text{ram}_v$  denote the ramification map  $\text{ram}_v : \text{Br } E \rightarrow H^1(G_{E_v}, Q/Z)$  [[Saltman 1999](#)]. Following [[Harbater et al. 2009](#)], we say that an  $\alpha \in \text{Br } E$  is determined by ramification with respect to a set of primes  $\Omega$  if there is a prime  $v \in \Omega$  for which  $\exp(\alpha) = \exp(\text{ram}_v(\alpha))$ . Let  $D$  be an  $E$ -division algebra with maximal subfield  $L$  that has Galois group  $G = \text{Gal}(L/E)$ . Let  $p = \text{char } E$  (possibly  $p = 0$ ). By [[Harbater et al. 2009](#), Theorem 3.3], if  $D$  satisfies:

- (1) the order of  $D$  is prime to  $p$  and  $\text{ind } D = \exp D$ , and
- (2)  $D$  is determined by ramification with respect to some set of discrete valuations whose residue characteristic is prime to  $|G|$ ,

then  $G$  has Sylow subgroups that are abelian of rank at most 2. Condition (1) is satisfied for any  $\alpha$  of order prime to  $p$  by Theorem 2.1 of [[Colliot-Thélène et al. 2002](#)], while condition (2) is satisfied by Corollary 1.9(c) of the same paper with respect to the set of codimension 1 primes of  $R$ . □

**Remark 3.9.** Let  $E$  be as above. By [[Colliot-Thélène et al. 2002](#), Theorem 2.1], any Brauer class  $\alpha \in \text{Br } E$  of order prime to  $\text{char } E$  has  $\text{ind}(\alpha) = \exp(\alpha)$ . Thus by [[Schacher 1968](#), Proposition 2.2], a subfield of an  $E$ -division algebra is also a maximal subfield of some  $E$ -division algebra.

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