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Let X be a quasicompact algebraic stack with quasifinite and separated diagonal. We classify the thick $\otimes$-ideals of $\mathsf{D}_{\mathrm{qc}}(X)^c$. If X is tame, then we also compute the Balmer spectrum of the $\otimes$-triangulated category of perfect complexes on X. In addition, if X admits a coarse space X_{cs}, then we prove that the Balmer spectra of X and X_{cs} are naturally isomorphic.

1. Introduction

Let X be a quasicompact and quasiseparated scheme. Let $\mathsf{Perf}(X)$ be the $\otimes$-triangulated category of perfect complexes on X. A celebrated result of Thomason [1997, Theorem 3.15], extending the work of Hopkins [1987, Section 4] and Neeman [1992a, Theorem 1.5], is a classification of the thick $\otimes$-ideals of $\mathsf{Perf}(X)$ in terms of the *Thomason* subsets of $|X|$, which are those subsets $Y \subseteq |X|$ expressible as a union $\cup_\alpha Y_\alpha$ such that $|X| \setminus Y_\alpha$ is quasicompact and open.

If X is a quasicompact and quasiseparated algebraic space, Deligne–Mumford stack, or algebraic stack, then it is also natural to consider the $\otimes$-triangulated category $\mathsf{Perf}(X)$ of perfect complexes on X (see [Hall and Rydh 2014, Section 4] for precise definitions).

In general, Thomason's classification of thick $\otimes$-ideals of $\mathsf{Perf}(X)$ fails for algebraic stacks (Example 3.2). If one instead works with the $\otimes$-ideal $\mathsf{D}_{\mathrm{qc}}(X)^c \subseteq \mathsf{Perf}(X)$ of *compact* perfect complexes, then the first main result of this article is that the classification goes through without change.

Theorem 1.1 (classification of thick $\otimes$-ideals). *If X is a quasicompact algebraic stack with quasifinite and separated diagonal, then there is a bijective and inclusion preserving correspondence between the thick $\otimes$-ideals of $\mathsf{D}_{\mathrm{qc}}(X)^c$ and the Thomason subsets of $|X|$.*

Some special cases of Theorem 1.1 are the following:

- If k is a field and G is a finite group, then $\mathsf{D}^b(\mathsf{Proj}\, kG)$ has no nontrivial $\otimes$-ideals.

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- If Y is a quasiprojective scheme over a field k with a proper action of an affine group scheme G, then the thick $\otimes$-ideals of $\mathsf{D}(\mathsf{QCoh}^G(Y))^c$ are in bijective correspondence with the G-invariant Thomason subsets of X.

The first special case is easy to prove directly and is well-known (for example, [Benson et al. 2011, Proposition 2.1]). In some sense, this makes our results orthogonal to those of [Benson et al. 2011]. The second special case was only known in characteristic 0 when Y was normal or quasi-affine [Krishna 2009, Theorem 7.8] or in characteristic p when G is finite of order prime to p and X is smooth [Dubey and Mallick 2012, Theorem 1.2].

We prove Theorem 1.1 using tensor nilpotence with parameters (Theorem 2.3), which extends [Thomason 1997, Theorem 3.8] and [Hopkins 1987, Theorem 10ii] (compare [Neeman 1992a, 1.1]) to quasicompact algebraic stacks with quasifinite and separated diagonal. As should be expected, stacks of the form $[Y/G]$, where Y is an affine variety over a field k and G is a finite group with order divisible by the characteristic of k, are the most troublesome. This is dealt with in Lemma 2.6, which relies on some results developed in Appendix A.

If $\mathcal{T}$ is a $\otimes$-triangulated category, then Balmer [2005] has functorially constructed from $\mathcal{T}$ a locally ringed space $\mathrm{Sp}_{\mathrm{Bal}}(\mathcal{T})$, the *Balmer spectrum*. A fundamental result of Balmer [2005, Theorem 5.5], which was extended by Buan, Krause and Solberg [Buan et al. 2007, Theorem 9.5] to the non-noetherian setting, is that if X is a quasicompact and quasiseparated scheme, then there is a naturally induced isomorphism

$$X \to \mathrm{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X)).$$

An algebraic stack is *tame* if its stabilizer groups at geometric points are finite linearly reductive group schemes [Abramovich et al. 2008, Definition 2.2]. Every scheme and algebraic space is tame. Moreover, in characteristic zero, a stack is Deligne–Mumford if and only if is tame. In characteristic $p > 0$, there are nontame Deligne–Mumford stacks (e.g., $B_{\mathbb{F}_p}(\mathbb{Z}/p\mathbb{Z})$) and tame stacks that are not Deligne–Mumford (e.g., $B_{\mathbb{F}_p}\mu_p$). Nagata's theorem [Hall and Rydh 2015, Theorem 1.2] provides a classification of finite linearly reductive group schemes over fields, which allows one to determine whether a given algebraic stack is tame. Our definition of tame stack is substantially weaker than that what appears in [Abramovich et al. 2008, Definition. 3.1] (see Appendix A).

Tame stacks are precisely those stacks with quasifinite diagonal such that the compact objects of $\mathsf{D}_{\mathrm{qc}}(X)$ coincide with the perfect complexes. In particular, for tame stacks $\mathsf{D}_{\mathrm{qc}}(X)^c$ contains a monoidal unit and so becomes a $\otimes$-triangulated category. Using Theorem 1.1, we extend the result of [Buan et al. 2007] to tame stacks.

Theorem 1.2. *Let X be a quasicompact algebraic stack with quasifinite and separated diagonal. If X is tame, then there is a natural isomorphism of locally ringed spaces:*

$$(|X|, \mathcal{O}_{X_{\mathrm{Zar}}}) \to \mathsf{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X)),$$

where $\mathcal{O}_{X_{\mathrm{Zar}}}$ is the Zariski sheaf $U \mapsto \Gamma(U, \mathcal{O}_X)$.

Theorem 1.2 implies that the Balmer spectrum cannot be used to reconstruct locally separated algebraic spaces [Knutson 1971, Example 2]. Balmer [2013] has recently initiated the study of unramified monoids in ⊗-triangulated categories and Neeman [2015] has classified them in the case of a separated noetherian scheme. It is hoped that a refinement of the Balmer spectrum can be constructed from unramified monoids, which would — at least — permit the reconstruction of algebraic spaces.

If X is an algebraic stack with finite inertia (e.g., a separated Deligne–Mumford stack), then X admits a *coarse space* $\pi : X \to X_{\mathrm{cs}}$ [Keel and Mori 1997; Rydh 2013], which is the universal map from X to an algebraic space. If X has finite inertia, then X has separated diagonal. Thus we can also establish the following.

Theorem 1.3. *Let X be a quasicompact, quasiseparated algebraic stack with finite inertia and coarse space $\pi : X \to X_{\mathrm{cs}}$. If X is tame, then*

$$\mathsf{Sp}_{\mathrm{Bal}}(\mathsf{L}\pi^*) : \mathsf{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X)) \to \mathsf{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X_{\mathrm{cs}}))$$

is an isomorphism of ringed spaces.

Krishna [2009, Theorem 7.10] proved Theorem 1.3 when X is of the form $[W/G]$, where W is quasiprojective and normal or quasi-affine, and G is a linear algebraic group in characteristic 0 acting properly on W. Dubey and Mallick [2012, Theorem 1.2] proved a similar result in positive characteristic, but required W to be smooth and G a finite group with order not divisible by the characteristic of the ground field. In particular, Theorem 1.3 is stronger than all existing results and Theorems 1.1 and 1.2 are new.

Assumptions and conventions. A priori, we make no separation assumptions on our algebraic stacks. However, all stacks used in this article will be, at the least, quasicompact and quasiseparated. Usually, they will also have separated diagonal. If X is an algebraic stack, then let $|X|$ denote its associated Zariski topological space [Laumon and Moret-Bailly 2000, Section 5]. For derived categories of algebraic stacks, we use the conventions and notations of [Hall and Rydh 2014, Section 1]. In particular, if X is an algebraic stack, then $\mathsf{Mod}(X)$ is the abelian category of $\mathcal{O}_X$-modules on the lisse-étale site of X and $\mathsf{D}_{\mathrm{qc}}(X)$ denotes the unbounded derived category of $\mathcal{O}_X$-modules with quasicoherent cohomology sheaves. If $f : X \to Y$ is a morphism of algebraic stacks, then there is always an adjoint pair of unbounded derived functors

$$\mathsf{D}_{\mathrm{qc}}(X) \underset{\mathsf{L} f_{\mathrm{qc}}^*}{\overset{\mathsf{R}(f_{\mathrm{qc}})_*}{\rightleftarrows}} \mathsf{D}_{\mathrm{qc}}(Y).$$

If f is quasicompact, quasiseparated and representable, then $\mathsf{R}(f_{\mathrm{qc}})_*$ agrees with $\mathsf{R} f_*$, the unbounded derived functor of $f_* : \mathsf{Mod}(X) \to \mathsf{Mod}(Y)$ [Hall and Rydh 2014, Lemma 2.5(3) and Theorem 2.6(2)]. If f is smooth, then $\mathsf{L} f_{\mathrm{qc}}^*$ agrees with the unique extension of the exact functor $f^* : \mathsf{Mod}(Y) \to \mathsf{Mod}(X)$ to the unbounded derived category.

2. Tensor nilpotence with parameters

Definition 2.1. Let X be an algebraic stack and let $\xi : M \to N$ be a morphism in $\mathsf{D}_{\mathrm{qc}}(X)$. Let $Z \subseteq |X|$ be a subset. We say that ξ *vanishes at the points of* Z if for every algebraically closed field k and morphism $z : \operatorname{Spec} k \to X$ that factors through Z, then $\mathsf{L} z_{\mathrm{qc}}^* \xi$ is the zero map in $\mathsf{D}_{\mathrm{qc}}(\operatorname{Spec} k)$.

This definition is connected to a more familiar notion for schemes.

Lemma 2.2. *Let X be a scheme and let $\xi : M \to N$ be a morphism in $\mathsf{D}_{\mathrm{qc}}(X)$. If $Z \subseteq |X|$ is a subset, then ξ* vanishes at the points of Z *if and only if $\xi \otimes^{\mathsf{L}}_{\mathcal{O}_X} \kappa(z)$ is the zero map in $\mathsf{D}(\kappa(z))$ for every $z \in Z$, where $\kappa(z)$ denotes the residue field of z.*

Proof. We immediately reduce to the situation where $X = \operatorname{Spec} \kappa$ and κ is a field. It now suffices to prove that if $\kappa \subseteq k$ is a field extension, where k is algebraically closed, then $\xi \otimes k$ is the zero map in $\mathsf{D}(k)$ if and only if ξ is the zero map in $\mathsf{D}(\kappa)$. This is obvious. □

If $K \in \mathsf{D}_{\mathrm{qc}}(X)$, then the *cohomological support* of K is defined to be the subset

$$\operatorname{supph}(K) = \bigcup_{n \in \mathbb{Z}} \operatorname{supp}(\mathscr{H}^n(K)) \subseteq |X|.$$

For the basic properties of cohomological support, see [Hall and Rydh 2014, Lemma 4.8], which extends [Thomason 1997, Lemma 3.3] to algebraic stacks. The main result of this section is the following theorem.

Theorem 2.3 (tensor nilpotence with parameters). *Let X be a quasicompact algebraic stack with quasifinite and separated diagonal. Let $\psi : E \to F$ be a morphism in $\mathsf{D}_{\mathrm{qc}}(X)$, where $E \in \mathsf{D}_{\mathrm{qc}}(X)^c$. Let $K \in \mathsf{Perf}(X)$. If ψ vanishes at the points of* $\operatorname{supph}(K)$, *then there exists a positive integer n such that $K \otimes^{\mathsf{L}}_{\mathcal{O}_X} (\psi^{\otimes n}) = 0$ in* $\mathsf{D}_{\mathrm{qc}}(X)$.

The following example demonstrates that Theorem 2.3 cannot be weakened to the situation where $E \in \mathsf{Perf}(X)$.

Example 2.4. Let $X = B_{\mathbb{F}_2}(\mathbb{Z}/2\mathbb{Z})$, which is a quasicompact, nontame Deligne–Mumford stack with finite diagonal. Consider the adjunction morphism

$$\eta : \mathbb{O}_X \to x_*\mathbb{O}_{\mathbb{F}_2},$$

where $x : \operatorname{Spec} \mathbb{F}_2 \to X$ is the usual cover. Since $\operatorname{coker}(\eta) \cong \mathbb{O}_X$, the cone of η induces a natural map $\psi : \mathbb{O}_X \to \mathbb{O}_X[1]$. For all positive integers n, $\psi^{\otimes n} = \psi$. Clearly, ψ vanishes at the points of $|X|$ (because $x^*\eta$ is split). If $\psi = \psi^{\otimes n} = 0$ for some n, it is easily determined that this implies that $\mathbb{O}_X \in \mathsf{D}_{\mathrm{qc}}(X)^c$, which is false.

Proof of Theorem 2.3. Let $\mathbf{E}$ be the category of representable, quasifinite, flat and separated morphisms of finite presentation over X. Let $\mathbf{D} \subseteq \mathbf{E}$ be the full subcategory whose objects are those $(U \xrightarrow{p} X)$ such that there exists an integer $n > 0$ with $p^*(K \otimes^{\mathsf{L}}_{\mathbb{O}_X} (\psi^{\otimes n})) = 0$. It suffices to prove that $\mathbf{D} = \mathbf{E}$. By the induction principle (Theorem B.1), it is sufficient to verify the following three conditions:

(I1) If $(U \to W) \in \mathbf{E}$ is an open immersion and $W \in \mathbf{D}$, then $U \in \mathbf{D}$.

(I2) If $(V \to W) \in \mathbf{E}$ is finite and surjective, where V is an affine scheme, then $W \in \mathbf{D}$.

(I3) If $(U \xrightarrow{j} W)$, $(W' \xrightarrow{f} W) \in \mathbf{E}$, where j is an open immersion and f is étale and an isomorphism over $W \setminus U$, then $W \in \mathbf{D}$ whenever U, $W' \in \mathbf{D}$.

Now condition (I1) is trivial and condition (I3) is Lemma 2.5. For condition (I2), by Lemma 2.6, it remains to prove that every affine scheme belongs to $\mathbf{D}$. By Lemma 2.2 and [Thomason 1997, Lemma 3.14] (or [Neeman 1992a, Lemma 1.2]), the result follows. ☐

Lemma 2.5. *Consider a* 2*-cartesian diagram of algebraic stacks*

$$\begin{array}{ccc} U' & \xhookrightarrow{j'} & W' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xhookrightarrow{j} & W \end{array}$$

where W is quasicompact and quasiseparated, j is a quasicompact open immersion and f is representable, étale, finitely presented and an isomorphism over $W \setminus U$. Let $\psi : E \to F$ be a morphism in $\mathsf{D}_{\mathrm{qc}}(W)$ and let $K \in \mathsf{D}_{\mathrm{qc}}(W)$. For each integer $n > 0$, let $\phi_n = K \otimes^{\mathsf{L}}_{\mathbb{O}_W} (\psi^{\otimes n})$. If $f^*\phi_n = 0$ and $j^*\phi_n = 0$, then $\phi_{2n} = 0$.

Proof. To simplify notation, we let $E_n = K \otimes^{\mathsf{L}}_{\mathbb{O}_W} E^{\otimes n}$ and $F_n = K \otimes^{\mathsf{L}}_{\mathbb{O}_W} F^{\otimes n}$. We will argue similarly to [Thomason 1997, Theorem 3.6], but using the Mayer–Vietoris triangle for étale neighbourhoods of stacks developed in [Hall and Rydh 2014, Lemma 5.7(1)] instead of [Thomason 1997, Lemma 3.5]. Let $k = f \circ j'$. By

[Hall and Rydh 2014, Lemma 5.7(1)], there is a distinguished triangle in $\mathsf{D}_{\mathrm{qc}}(W)$:

$$F_n \longrightarrow \mathsf{R}j_* j^* F_n \oplus \mathsf{R}f_* f^* F_n \longrightarrow \mathsf{R}k_* k^* F_n \xrightarrow{d} F_n[1].$$

By applying the homological functor $\mathrm{Hom}_{\mathcal{O}_W}(E_n, -)$ to the distinguished triangle above, we find that there exists a morphism $t : E_n \to \mathsf{R}k_* k^* F_n[-1]$ such that $\delta(t) = \phi_n$, where δ is the boundary map induced by d. But there is a commutative diagram

$$\begin{array}{ccccc} & & (\mathsf{R}k_* k^* F_n[-1]) \otimes^{\mathsf{L}}_{\mathcal{O}_W} E^{\otimes n} & & \\ & \overset{t\otimes \mathrm{Id}}{\nearrow} & \Big\downarrow{\scriptstyle \mathrm{Id}\,\otimes \psi^{\otimes n}} & \overset{\delta \otimes \psi^{\otimes n}}{\searrow} & \\ E_n \otimes^{\mathsf{L}}_{\mathcal{O}_W} E^{\otimes n} & & \xrightarrow{\qquad\qquad \phi_{2n} \qquad\qquad} & & F_n \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n} \\ & \underset{t\otimes \psi^{\otimes n}}{\searrow} & & \underset{\delta\otimes \mathrm{Id}}{\nearrow} & \\ & & (\mathsf{R}k_* k^* F_n[-1]) \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n} & & \end{array}$$

so it remains to prove that the vertical map above is zero. To see this, the projection formula [Hall and Rydh 2014, Corollary 4.12] implies that we have a commutative diagram

$$\begin{array}{ccc} (\mathsf{R}k_* k^* F_n[-1]) \otimes^{\mathsf{L}}_{\mathcal{O}_W} E^{\otimes n} & \xrightarrow{\sim} & \mathsf{R}k_* k^* (K \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n} \otimes^{\mathsf{L}}_{\mathcal{O}_W} E^{\otimes n}[-1]) \\ {\scriptstyle \mathrm{Id}\,\otimes \psi^{\otimes n}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathsf{R}k_* k^*(F^{\otimes n} \otimes \phi_n[-1])} \\ (\mathsf{R}k_* k^* F_n[-1]) \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n} & \xrightarrow{\sim} & \mathsf{R}k_* k^* (K \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n} \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n}[-1]) \end{array}$$

Since $k^* \phi_n = 0$, the result follows. □

The following lemma is similar to a special case of [Elagin 2011, Theorem 7.3 and Corollary 9.6]. Also, see [Krishna 2009, proof of Proposition 7.6; Dubey and Mallick 2012, Lemma 3.8].

Lemma 2.6. *Let W be an algebraic stack and let $v : V \to W$ be a finite and faithfully flat morphism of finite presentation, where V is an affine scheme. Let $\psi : E \to F$ be a morphism in $\mathsf{D}_{\mathrm{qc}}(W)$, where $E \in \mathsf{D}_{\mathrm{qc}}(W)^c$. Let $K \in \mathsf{Perf}(W)$. If $v^*(K \otimes^{\mathsf{L}}_{\mathcal{O}_W} \psi) = 0$ in $\mathsf{D}_{\mathrm{qc}}(V)$, then $K \otimes^{\mathsf{L}}_{\mathcal{O}_W} \psi = 0$ in $\mathsf{D}_{\mathrm{qc}}(W)$.*

Proof. By [Hall and Rydh 2014, Corollary 4.15], $\mathsf{R}(v_{\mathrm{qc}})_*$ admits a right adjoint $v^\times$ and there is a functorial isomorphism $v^\times(\mathcal{O}_W) \otimes^{\mathsf{L}}_{\mathcal{O}_V} \mathsf{L}v^*_{\mathrm{qc}}(M) \simeq v^\times(M)$ for every $M \in \mathsf{D}_{\mathrm{qc}}(W)$. In particular, if $v^*(K \otimes^{\mathsf{L}}_{\mathcal{O}_W} \psi) = 0$ in $\mathsf{D}_{\mathrm{qc}}(V)$, then $v^\times(K \otimes^{\mathsf{L}}_{\mathcal{O}_W} \psi) = 0$ in $\mathsf{D}_{\mathrm{qc}}(V)$. By adjunction, it follows that the induced composition

$$\mathsf{R}(v_{\mathrm{qc}})_* v^\times (K \otimes^{\mathsf{L}}_{\mathcal{O}_W} E) \to K \otimes^{\mathsf{L}}_{\mathcal{O}_W} E \to K \otimes^{\mathsf{L}}_{\mathcal{O}_W} F$$

vanishes in $\mathsf{D}_{\mathrm{qc}}(W)$. Thus it suffices to prove that

$$\mathsf{R}(v_{\mathrm{qc}})_* v^{\times}(K \otimes^{\mathsf{L}}_{\mathbb{O}_W} E) \to K \otimes^{\mathsf{L}}_{\mathbb{O}_W} E$$

admits a section. Since $E \in \mathsf{D}_{\mathrm{qc}}(W)^c$ and $K \in \mathsf{Perf}(W)$, it follows that $K \otimes^{\mathsf{L}}_{\mathbb{O}_W} E$ lies in $\mathsf{D}_{\mathrm{qc}}(W)^c$. Hence, we need only prove that if $M \in \mathsf{D}_{\mathrm{qc}}(W)^c$, the trace morphism $\mathrm{Tr}_M : \mathsf{R}(v_{\mathrm{qc}})_* v^{\times}(M) \to M$ admits a section. By Lemma A.1, M is quasi-isomorphic to a direct summand of $\mathsf{R}(v_{\mathrm{qc}})_* P$, where $P \in \mathsf{Perf}(V)$. Thus we are reduced to proving that $\mathrm{Tr}_{\mathsf{R}(v_{\mathrm{qc}})_* P}$ admits a section. This is trivial and the result follows. □

3. The classification of thick ⊗-ideals

If $\mathcal{T}$ is a ⊗-triangulated category and $S \subseteq \mathcal{T}$ is a subset, then define $\langle S \rangle_{\otimes} \subseteq \mathcal{T}$ to be the smallest thick ⊗-ideal of $\mathcal{T}$ containing S.

To prove Theorem 1.1, we require this analogue of [Thomason 1997, Lemma 3.14]:

Lemma 3.1. *Let X be a quasicompact algebraic stack with quasifinite and separated diagonal. If $P, Q \in \mathsf{D}_{\mathrm{qc}}(X)^c$ and* $\mathrm{supph}(P) \subseteq \mathrm{supph}(Q)$, *then* $\langle P \rangle_{\otimes} \subseteq \langle Q \rangle_{\otimes}$.

Proof. Argue exactly as in [Thomason 1997, Lemma 3.14] (cf. [Neeman 1992a, Lemma 1.2]), but using Theorem 2.3 instead of Thomason's Theorem 3.8. □

The following example shows Lemma 3.1 cannot be extended to $P, Q \in \mathsf{Perf}(X)$ when X is nontame. It also shows that Thomason's classification (Theorem 1.1) does not hold for $\mathsf{Perf}(X)$ in this case too.

Example 3.2. Let $x : \operatorname{Spec} \mathbb{F}_2 \to X$ be as in Example 2.4. Let $P = \mathbb{O}_X$ and let $Q = x_* \mathbb{O}_{\operatorname{Spec} \mathbb{F}_2}$. Then $P, Q \in \mathsf{Perf}(X)$ and $\mathrm{supph}(P) = \mathrm{supph}(Q)$. Note that $Q \in \mathsf{D}_{\mathrm{qc}}(X)^c$ and $P \notin \mathsf{D}_{\mathrm{qc}}(X)^c$. Since $\mathsf{D}_{\mathrm{qc}}(X)^c$ is a thick ⊗-ideal of $\mathsf{Perf}(X)$, it follows that $\langle Q \rangle_{\otimes} \subseteq \mathsf{D}_{\mathrm{qc}}(X)^c$. But if $\langle P \rangle_{\otimes} = \langle Q \rangle_{\otimes}$, then $P \in \mathsf{D}_{\mathrm{qc}}(X)^c$. But $P \notin \mathsf{D}_{\mathrm{qc}}(X)^c$; thus we have a contradiction.

Following Thomason [1997, Theorem 3.15] (or Neeman [1992a, Theorem 1.5]), given Lemma 3.1, we can prove Theorem 1.1.

Proof of Theorem 1.1. If $Y \subseteq |X|$ is a Thomason subset, then define

$$\mathcal{I}_Y = \{P \in \mathsf{D}_{\mathrm{qc}}(X)^c : \mathrm{supph}(P) \subseteq Y\}.$$

Clearly, $\mathcal{I}_Y$ is a thick ⊗-ideal of $\mathsf{D}_{\mathrm{qc}}(X)^c$. If $\mathcal{T}$ is a thick ⊗-ideal of $\mathsf{D}_{\mathrm{qc}}(X)^c$, then define

$$\varphi(\mathcal{T}) = \bigcup_{Q \in \mathcal{T}} \mathrm{supph}(Q).$$

By [Hall and Rydh 2014, Lemma 4.8(3)], $\varphi(\mathcal{T})$ is a Thomason subset of $|X|$. It suffices to prove that $\mathcal{I}_{\varphi(\mathcal{T})} = \mathcal{T}$ and $\varphi(\mathcal{I}_Y) = Y$.

Obviously, $\mathcal{T} \subseteq \mathcal{I}_{\varphi(\mathcal{T})}$. For the reverse inclusion, if $P \in \mathcal{I}_{\varphi(\mathcal{T})}$, then

$$\operatorname{supph}(P) \subseteq \bigcup_{Q\in\mathcal{T}} \operatorname{supph}(Q).$$

Since $\operatorname{supph}(P)$ and $\operatorname{supph}(Q)$ are constructible for every $Q \in \mathcal{T}$, it follows that there is a finite subset $J \subseteq \mathcal{T}$ such that

$$\operatorname{supph}(P) \subseteq \bigcup_{Q\in J} \operatorname{supph}(Q) = \operatorname{supph}(\oplus_{Q\in J} Q).$$

By Lemma 3.1, $\langle P\rangle_\otimes \subseteq \langle \oplus_{Q\in J} Q\rangle_\otimes \subseteq \mathcal{T}$. Thus $P \in \mathcal{T}$ and $\mathcal{I}_{\varphi(\mathcal{T})} = \mathcal{T}$.

Obviously, $Y \supseteq \varphi(\mathcal{I}_Y)$. Since Y is Thomason, it is expressible as a union $\cup_\alpha Y_\alpha$ such that $|X| \setminus Y_\alpha$ is quasicompact and open. By [Hall and Rydh 2014, Theorem A], for every α there is a compact complex Q_α with support Y_α. It follows that if $y \in Y$, then $y \in \operatorname{supph}(Q_\alpha) \subseteq Y$ for some α. In other words, $y \in \varphi(\mathcal{I}_Y)$, so $Y = \varphi(\mathcal{I}_Y)$. □

4. The Balmer spectrum of a tame stack

We will prove Theorem 1.2 using [Buan et al. 2007, Proposition 6.1].

Proof of Theorem 1.2. Let $s : (|X|, \operatorname{supph}) \to \big(|\operatorname{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X))|, \sigma_X\big)$ be the uniquely induced morphism of support data, where σ_X denotes the universal support datum. Since X is tame, it has finite cohomological dimension [Hall and Rydh 2015, Theorem 2.1(2)]; hence, $\mathsf{D}_{\mathrm{qc}}(X)^c = \mathsf{Perf}(X)$ [Hall and Rydh 2014, Remark 4.6]. By Theorem 1.1, $(|X|, \operatorname{supph})$ is classifying and by [Laumon and Moret-Bailly 2000, Corollaries 5.6.1 and 5.7.2] we know that $|X|$ is spectral. By [Buan et al. 2007, Proposition 6.1], s is a homeomorphism. By definition, $\mathcal{O}_{\operatorname{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X))}$ is the sheafification of the presheaf

$$(j : U \subseteq X) \mapsto \operatorname{End}_{\mathsf{Perf}(X)/\ker(j^*)\cap\mathsf{Perf}(X)}(j^*\mathcal{O}_X).$$

Since $|X|$ has a basis consisting of quasicompact open subsets, it is sufficient to identify $\operatorname{End}_{\mathsf{Perf}(X)/\ker(j^*)\cap\mathsf{Perf}(X)}(j^*\mathcal{O}_X)$ when j is a quasicompact open immersion. By [Hall and Rydh 2014, Lemma 6.7(2)], $\ker(j^*)$ is the localising envelope of a set of objects with compact image in $\mathsf{D}_{\mathrm{qc}}(X)$. By Thomason's localisation theorem (e.g., [Hall and Rydh 2014, Theorem 3.10] or [Neeman 1992b, Theorem 2.1]), $\mathsf{Perf}(U)$ is the thick closure of $\mathsf{Perf}(X)/\ker(j^*)\cap\mathsf{Perf}(X)$. Since there are natural isomorphisms

$$\operatorname{End}_{\mathsf{Perf}(X)/\ker(j^*)\cap\mathsf{Perf}(X)}(j^*\mathcal{O}_X) \cong \operatorname{End}_{\mathsf{Perf}(U)}(\mathcal{O}_U) \cong \operatorname{End}_{\mathcal{O}_U}(\mathcal{O}_U) = \Gamma(U, \mathcal{O}_X),$$

the result follows. □

Proof of Theorem 1.3. Since X has finite inertia, it has separated diagonal. By [Rydh 2013, Theorem 6.12], π is a separated universal homeomorphism, so X_{cs} is a quasicompact and quasiseparated algebraic space. By [Rydh 2013, Theorem 6.12], the natural map $(|X|, \mathcal{O}_{X_{\mathrm{Zar}}}) \to (|X_{\mathrm{cs}}|, \mathcal{O}_{(X_{\mathrm{cs}})_{\mathrm{Zar}}})$ is an isomorphism of locally ringed spaces. By Theorem 1.2, the result follows. □

Appendix A: Tame stacks and coarse spaces

We establish here some basic results about $\mathsf{R}(\pi_{\mathrm{qc}})_*$, where $\pi : X \to X_{\mathrm{cs}}$ is the coarse space of a quasiseparated algebraic stack X with finite inertia. Our first result, however, is a useful lemma that characterises the compact objects on a certain class of algebraic stacks, which includes BG for all finite groups G. This is likely known, though we are unaware of a reference for this result in the generality required.

Lemma A.1. *Let W be an algebraic stack and let $v : V \to W$ be a finite and faithfully flat morphism of finite presentation, where V is an affine scheme. If $M \in \mathsf{D}_{\mathrm{qc}}(W)^c$, then M is quasi-isomorphic to a direct summand of $\mathsf{R}(v_{\mathrm{qc}})_* P$ for some $P \in \mathsf{Perf}(V)$.*

Proof. If $P \in \mathsf{Perf}(V)$, then $\mathsf{R}(v_{\mathrm{qc}})_* P \in \mathsf{D}_{\mathrm{qc}}(W)^c$ [Hall and Rydh 2014, Corollary 4.15 and Example 3.8]. Thus, let $\mathcal{T} \subseteq \mathsf{D}_{\mathrm{qc}}(W)^c$ be the subcategory with objects those $N \in \mathsf{D}_{\mathrm{qc}}(W)^c$ that are quasi-isomorphic to direct summands of $\mathsf{R}(v_{\mathrm{qc}})_* P$ for some $P \in \mathsf{Perf}(V)$. Clearly, $\mathcal{T}$ is closed under shifts and direct summands. We now prove that $\mathcal{T}$ is triangulated. Thus let $f : N' \to N$ be a morphism in $\mathcal{T}$ and complete it to a distinguished triangle

$$N' \xrightarrow{f} N \xrightarrow{c} N'' \xrightarrow{\partial} N'[1].$$

We now prove that $N'' \in \mathcal{T}$. By assumption, there are $P, P' \in \mathsf{Perf}(V)$ and C, $C' \in \mathsf{D}_{\mathrm{qc}}(W)^c$ and quasi-isomorphisms $N \oplus C \simeq \mathsf{R}(v_{\mathrm{qc}})_* P$, $N' \oplus C' \simeq \mathsf{R}(v_{\mathrm{qc}})_* P'$. It follows that there is a distinguished triangle

$$N' \oplus C' \xrightarrow{f \oplus 0} N \oplus C \xrightarrow{c \oplus \mathrm{id}_C \oplus 0} N'' \oplus C \oplus C'[1] \xrightarrow{\partial \oplus p_{C'[1]}} N' \oplus C'[1],$$

where $p_{C'[1]} : C \oplus C'[1] \to C'[1]$ is the natural projection. In particular, we are reduced to the situation where $N' = \mathsf{R}(v_{\mathrm{qc}})_* P'$ and $N = \mathsf{R}(v_{\mathrm{qc}})_* P$. In this case, the morphism $f : N' \to N$ by duality induces a morphism $\tilde{f} : P' \to v^\times \mathsf{R}(v_{\mathrm{qc}})_* P$. It follows that the composition $\mathsf{R}(v_{\mathrm{qc}})_* P' \xrightarrow{f} \mathsf{R}(v_{\mathrm{qc}})_* P \to \mathsf{R}(v_{\mathrm{qc}})_* v^\times \mathsf{R}(v_{\mathrm{qc}})_* P$ is the map $\mathsf{R}(v_{\mathrm{qc}})_* \tilde{f}$. Now form a distinguished triangle

$$P' \xrightarrow{\tilde{f}} v^\times \mathsf{R}(v_{\mathrm{qc}})_* P \xrightarrow{k} K \xrightarrow{\delta} P'[1].$$

Since the morphism $\mathsf{R}(v_{\mathrm{qc}})_* P \to \mathsf{R}(v_{\mathrm{qc}})_* v^\times \mathsf{R}(v_{\mathrm{qc}})_* P$ admits a retraction, there exist a $Q \in \mathsf{D}_{\mathrm{qc}}(W)^c$ and a quasi-isomorphism $\mathsf{R}(v_{\mathrm{qc}})_* v^\times \mathsf{R}(v_{\mathrm{qc}})_* P \simeq \mathsf{R}(v_{\mathrm{qc}})_* P \oplus Q$.

There is an induced morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathsf{R}(v_{qc})_*P' & \xrightarrow{\mathsf{R}(v_{qc})_*\tilde{f}} & \mathsf{R}(v_{qc})_*v^{\times}\mathsf{R}(v_{qc})_*P & \xrightarrow{\mathsf{R}(v_{qc})_*k} & \mathsf{R}(v_{qc})_*K & \xrightarrow{\mathsf{R}(v_{qc})_*\delta} & \mathsf{R}(v_{qc})_*P'[1] \\
\| & & \downarrow \wr & & \downarrow & & \downarrow \\
\mathsf{R}(v_{qc})_*P' & \xrightarrow{f\oplus 0} & \mathsf{R}(v_{qc})_*P \oplus Q & \xrightarrow{c\oplus \mathrm{id}_Q} & N'' \oplus Q & \xrightarrow{\partial + 0} & \mathsf{R}(v_{qc})_*P'[1].
\end{array}$$

It follows that $\mathsf{R}(v_{qc})_*K \simeq N'' \oplus Q$ and so $N'' \in \mathcal{T}$. By [Hall and Rydh 2014, Example 6.5 and Proposition 6.6], $\mathsf{D}_{qc}(W)$ is compactly generated by $v_*\mathcal{O}_V$. But Thomason's Theorem [Neeman 1992b, Theorem 2.1] implies that $\mathsf{D}_{qc}(W)^c$ is the smallest thick subcategory containing $v_*\mathcal{O}_V$. The result follows. □

Let $F : \mathcal{S} \to \mathcal{T}$ be a triangulated functor between triangulated categories. Assume that $\mathcal{S}$ and $\mathcal{T}$ admit t-structures. We say that F is *left* (resp. *right*) *t-exact* if $F(\mathcal{S}^{\geq 0}) \subseteq \mathcal{T}^{\geq 0}$ (resp. $F(\mathcal{S}^{\leq 0}) \subseteq \mathcal{T}^{\leq 0}$). We say that F is *t-exact* if it is both left and right t-exact. The following result was suggested to us by David Rydh.

Theorem A.2. *If X be a quasiseparated algebraic stack with finite inertia and coarse space $\pi : X \to X_{cs}$, then the restriction of $\mathsf{R}(\pi_{qc})_*$ to $\mathsf{D}_{qc}(X)^c$ is t-exact.*

Proof. By [Hall and Rydh 2014, Lemma 1.2(4)], this may be checked étale-locally on X_{cs}. Thus, we may assume that X_{cs} is an affine scheme. Since π is a universal homeomorphism, it follows that X is quasicompact. Also, since X has finite inertia, it has quasifinite and separated diagonal. By Theorem B.5, there exist morphisms of algebraic stacks $V \xrightarrow{v} W \xrightarrow{p} X$ such that V is an affine scheme, v is finite, faithfully flat and finitely represented and p is a representable, separated and finitely presented Nisnevich covering. By [Rydh 2013, Proposition 6.5], we may further assume that p is fixed-point reflecting. We now apply [Rydh 2013, Theorem 6.10] to conclude that the diagram

$$\begin{array}{ccc}
W & \xrightarrow{p} & X \\
\omega \downarrow & & \downarrow \pi \\
W_{cs} & \xrightarrow{p_{cs}} & X_{cs}
\end{array}$$

is cartesian and p_{cs} is representable, separated, étale and of finite presentation. Thus, it suffices to prove the result on W.

Clearly $\mathsf{R}(\pi_{qc})_*$ is left t-exact, so it remains to address the right t-exactness. Take $M \in \mathsf{D}_{qc}(W)^c \cap \mathsf{D}_{qc}^{\leq 0}(W)$. By Lemma A.1, we may assume that there exists a map $i : M \to \mathsf{R}(v_{qc})_*P$, where $P \in \mathsf{Perf}(V)$, that admits a retraction r. It follows that the composition $M \xrightarrow{i} \mathsf{R}(v_{qc})_*P \to \tau^{>0}\mathsf{R}(v_{qc})_*P$ is the zero map. Therefore the induced map $\mathsf{R}(\omega_{qc})_*M \to \mathsf{R}(\omega_{qc})_*\tau^{>0}\mathsf{R}(v_{qc})_*P$ is the zero map. But v and

$\omega \circ v$ are affine, so there is a natural quasi-isomorphism $\tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*\mathsf{R}(v_{\mathrm{qc}})_*P \simeq \mathsf{R}(\omega_{\mathrm{qc}})_*\tau^{>0}\mathsf{R}(v_{\mathrm{qc}})_*P$. The resulting map

$$\tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*M \to \tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*\mathsf{R}(v_{\mathrm{qc}})_*P$$

is 0 and also coincides with $\tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*(i)$, which admits a retraction $\tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*(r)$. In particular, $\tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*M \simeq 0$ and the result follows. □

Abramovich et al. [2008] work with a more restrictive definition of tame, rendering the following corollary a tautology. Indeed, they assume that X has finite inertia and is locally of finite presentation over a base scheme S and that $\pi : X \to X_{\mathrm{cs}}$ is such that π_* is exact on quasicoherent sheaves. In our case, we make none of these assumptions, rendering it nontrivial.

Corollary A.3. *Let X be a quasiseparated algebraic stack with finite inertia and coarse space $\pi : X \to X_{\mathrm{cs}}$. The following are equivalent*:

1. (1) *X is tame*;
2. (2) $\pi_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(X_{\mathrm{cs}})$ *is exact*;
3. (3) $\mathsf{R}\pi_* : \mathsf{D}^+_{\mathrm{qc}}(X) \to \mathsf{D}^+_{\mathrm{qc}}(X_{\mathrm{cs}})$ *is t-exact*;
4. (4) $\mathsf{R}(\pi_{\mathrm{qc}})_* : \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X_{\mathrm{cs}})$ *is t-exact.*

Proof. We begin with some preliminary reductions. The morphism π is a separated universal homeomorphism [Rydh 2013, Theorem 6.12], so X_{cs} is a quasiseparated algebraic space and π is quasicompact and quasiseparated. Thus by Lemma 1.2(2) of [Hall and Rydh 2014] we get the implication (3) ⇒ (4), and by Theorem 2.6(2) of the same reference we have that (4) ⇒ (3). Clearly, item (1) may be verified after passing to an affine étale presentation of X_{cs}, and similarly for items (2) and (3) [Hall and Rydh 2014, Lemma 1.2(4) and Lemma 2.2(6)]. We may consequently assume that X_{cs} is an affine scheme. Since π has finite diagonal, it has affine diagonal, so we have (2)⇔(3) [Hall et al. 2014, Proposition 2.1]. By [Hall and Rydh 2015, Theorem C, (1) ⇒ (3)], we now obtain that (2) ⇒ (1). It remains to address (1) ⇒ (2).

Arguing exactly as in the proof of Theorem A.2, we may further assume that X admits a finite, faithfully flat and finitely presented cover $v : V \to X$, where V is an affine scheme. Since X is tame, $\mathcal{O}_X \in \mathsf{D}_{\mathrm{qc}}(X)^c$. By Theorem A.2, it follows that the induced morphism $\mathcal{O}_X \to v_*\mathcal{O}_V$ admits a retraction. If $M \in \mathsf{QCoh}(X)$, then it follows immediately that the natural map $M \to v_*v^*M$ admits a retraction. Thus, if $f : M \to N$ is a surjection in $\mathsf{QCoh}(X)$, then f is a retraction of the surjection v_*v^*f. Since $\pi \circ v$ is affine, $\pi_*v_*v^*f$ is surjective. In particular, π_*f is a retraction of a surjection, thus is surjective. The result follows. □

Appendix B: The induction principle

The *induction principle* [Stacks 2015, Tag 08GL] for algebraic spaces is closely related to the étale dévissage results of [Rydh 2011a]. When working with derived categories, where locality results are often quite subtle, it is often advantageous to have the strongest possible criteria at your disposal. In this appendix, we will prove the following induction principle for stacks with quasifinite and separated diagonal.

Before stating this result, we require some notation. Fix an algebraic stack S. If $P_1, \dots, P_r$ is a list of properties of morphisms of algebraic stacks over S, let $\mathbf{Stack}_{P_1,\dots,P_r/S}$ denote the full 2-subcategory of the category of algebraic stacks over S whose objects are those $(x : X \to S)$ such that x has properties $P_1, \dots, P_r$. The following abbreviations will be used: ét (étale), qff (quasifinite flat), sep (separated), fp (finitely presented) and rep (representable).

For example, $\mathbf{Stack}_{\mathrm{rep,sep,qff,fp}/S}$ consists of those algebraic stacks $x : X \to S$ such that x is representable, separated, quasifinite flat, and finitely presented. In a similar way, $\mathbf{Stack}_{\mathrm{rep,sep,ét,fp}/S}$ consists of those algebraic stacks over S, $x : X \to S$, such that x is representable, separated, étale, and finitely presented. Note that while every morphism $(X' \to X)$ in $\mathbf{Stack}_{\mathrm{rep,sep,ét,fp}/S}$ is representable, separated, étale, and finitely presented; in $\mathbf{Stack}_{\mathrm{rep,sep,qff,fp}/S}$ they can only be assumed to be representable, separated, quasifinite, and finitely presented (i.e., there are nonflat morphisms between objects).

Theorem B.1 (induction principle). *Let S be a quasicompact algebraic stack with quasicompact and separated diagonal. If S has quasifinite diagonal, let*

$$\mathbf{E} = \mathbf{Stack}_{\mathrm{rep,sep,qff,fp}/S};$$

or if S is Deligne–Mumford, let

$$\mathbf{E} = \mathbf{Stack}_{\mathrm{rep,sep,ét,fp}/S}.$$

Let $\mathbf{D} \subseteq \mathbf{E}$ be a full subcategory satisfying the following properties:

(I1) *if $(X' \to X) \in \mathbf{E}$ is an open immersion and $X \in \mathbf{D}$, then $X' \in \mathbf{D}$;*

(I2) *if $(X' \to X) \in \mathbf{E}$ is finite, flat, and surjective, where X' is an affine scheme, then $X \in \mathbf{D}$;*

(I3) *if $(U \xrightarrow{j} X)$, $(X' \xrightarrow{f} X) \in \mathbf{E}$, where j is an open immersion and f is étale and an isomorphism over $X \setminus U$, then $X \in \mathbf{D}$ whenever $U, X' \in \mathbf{D}$.*

Then $\mathbf{D} = \mathbf{E}$. In particular, $S \in \mathbf{D}$.

Proof. Combine Lemma B.3 with Theorem B.5. □

We wish to point out that Theorem B.1 relies on the existence of coarse spaces for stacks with finite inertia (i.e., the Keel–Mori theorem [Keel and Mori 1997; Rydh 2013]).

Nisnevich coverings. It will be useful to consider some variants and refinements of [Krishna and Østvær 2012, Sections 7–8].

If $p : W \to X$ is a representable morphism of algebraic stacks, then a *splitting sequence* for p is a sequence of quasicompact open immersions

$$\varnothing = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X,$$

such that p restricted to $X_i \setminus X_{i-1}$, when given the induced reduced structure, admits a section for each $i = 1, \dots, r$. In this situation, we say that p has a splitting sequence of length r. An étale and representable morphism of algebraic stacks $p : W \to X$ is a *Nisnevich covering* if it admits a splitting sequence.

Example B.2. Let X be a quasicompact and quasiseparated scheme. Then there exists an affine scheme W and a Nisnevich covering $p : W \to X$. Indeed, taking $W = \amalg_{i=1}^{n} U_i$, where the $\{U_i\}$ form a finite affine open covering of X gives the claim.

The following lemma is proved by a straightforward induction on the length of the splitting sequence.

Lemma B.3 (Nisnevich dévissage). *Let S be a quasicompact and quasiseparated algebraic stack. Let* $\mathbf{E}$ *be* $\mathbf{Stack}_{\mathrm{rep,ét,fp}\,/S}$ *or* $\mathbf{Stack}_{\mathrm{rep,sep,ét,fp}\,/S}$. *Let* $\mathbf{D} \subseteq \mathbf{E}$ *be a full 2-subcategory with the following properties*:

(N1) *if $(X' \to X) \in \mathbf{E}$ is an open immersion and $X \in \mathbf{D}$, then $X' \in \mathbf{D}$*;

(N2) *if $(U \xrightarrow{j} X)$, $(X' \xrightarrow{f} X) \in \mathbf{E}$, where j is an open immersion and f is an isomorphism over $X \setminus U$, then $X \in \mathbf{D}$ whenever $U, X' \in \mathbf{D}$.*

If $p : W \to X$ is a Nisnevich covering in $\mathbf{E}$ and $W \in \mathbf{D}$, then $X \in \mathbf{D}$.

The following lemma will also be useful.

Lemma B.4. *Let $p : W \to X$ be a Nisnevich covering of algebraic stacks.*

(1) *If $f : X' \to X$ is a morphism of algebraic stacks, then the pull back $p' : W' \to X'$ of p along f is a Nisnevich covering.*

(2) *Let $w : W' \to W$ be a Nisnevich covering of finite presentation. If p is of finite presentation and X is quasicompact and quasiseparated, then $p \circ w : W' \to X$ is a Nisnevich covering.*

Presentations. The following theorem refines [Rydh 2011a, Theorem 7.2] and will be crucial for the proof of Theorem B.1.

Theorem B.5. *Let X be a quasicompact algebraic stack with quasifinite and separated diagonal. Then there exist morphisms of algebraic stacks*

$$V \xrightarrow{v} W \xrightarrow{p} X$$

such that

- *V is an affine scheme*;
- *v is finite*, *flat*, *surjective and of finite presentation*;
- *p is a separated Nisnevich covering of finite presentation.*

In addition, *if S is a Deligne–Mumford stack*, *it can be arranged that v is also étale.*

Proof. The proof is similar to [Rydh 2013, Proposition 6.11; 2011a, Theorem 7.3].

By [Rydh 2011a, Theorem 7.1], there is an affine scheme U and a representable, separated, quasifinite, flat, and surjective morphism $u : U \to X$ of finite presentation. Let $W = \underline{\mathrm{Hilb}}^{\mathrm{open}}_{U/X} \to X$ be the subfunctor of the relative Hilbert scheme parametrising open and closed immersions to U over X. It follows that $p : W \to X$ is étale, representable and separated [Rydh 2011b, Corollary 6.2].

We now prove that p is a Nisnevich covering. To see this, we note that there exists a sequence of quasicompact open immersions

$$\varnothing = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X,$$

such that the restriction of u to $Z_i = (X_i \setminus X_{i-1})_{\mathrm{red}}$ for $i = 1, \dots, r$ is finite, flat and finitely presented. By definition of $p : W \to X$, it follows immediately that $p|_{Z_i}$ admits a section corresponding to $u|_{Z_i}$ and so p is a separated Nisnevich covering.

Let $v : V \to W$ be the universal family, which is finite, flat, surjective and of finite presentation. Also, $V \to U$ is representable, étale and separated [Rydh 2011b, Corollary 6.2]. Suitably shrinking W, we obtain a separated Nisnevich covering $p : W \to X$ of finite presentation fitting into a 2-commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{q} & U \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ W & \xrightarrow{p} & X \end{array} \tag{B.1}$$

and q is étale, separated and surjective. By Zariski's Main Theorem [Laumon and Moret-Bailly 2000, Theorem A.2], q is quasi-affine. By [Rydh 2013, Theorem 5.3], W has a coarse space $\pi : W \to W_{\mathrm{cs}}$ such that W_{cs} is a quasi-affine scheme and $\pi \circ v$

is affine. By Example B.2 and Lemma B.4, we may further reduce to the situation where W_{cs} is an affine scheme. Since $\pi \circ v$ is affine, the result follows. □

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ANNALS OF K-THEORY

2016 vol. 1 no. 3

Multiplicative differential algebraic K-theory and applications 227
Ulrich Bunke and Georg Tamme

The Balmer spectrum of a tame stack 259
Jack Hall

Algebraic Kasparov K-theory, II 275
Grigory Garkusha

The local symbol complex of a reciprocity functor 317
Evangelia Gazaki