ANNALS OF K-THEORY

no. 2 vol 1 2016

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Paul Baum, Erik Guentner and Rufus Willett

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A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 1, No. 2, 2016
dx.doi.org/10.2140/akt.2016.1.155

Expanders, exact crossed products, and the Baum–Connes conjecture

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We reformulate the Baum–Connes conjecture with coefficients by introducing a new crossed product functor for C^*-algebras. All confirming examples for the original Baum–Connes conjecture remain confirming examples for the reformulated conjecture, and at present there are no known counterexamples to the reformulated conjecture. Moreover, some of the known expander-based counterexamples to the original Baum–Connes conjecture become confirming examples for our reformulated conjecture.

1. Introduction

For a second-countable locally compact group G, the Baum–Connes conjecture (with coefficients) [Baum et al. 1994; Valette 2002] asserts that the Baum–Connes assembly map

$$K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_{\mathrm{red}} G) \tag{1.1}$$

is an isomorphism for all G-C^*-algebras A. Here the C^*-algebra A is equipped with a continuous action of G by C^*-algebra automorphisms and, as usual, $A \rtimes_{\mathrm{red}} G$ denotes the reduced crossed product. The conjecture has many deep and important connections to geometry, topology, representation theory and algebra. It is known to be true for large classes of groups: see for example [Higson and Kasparov 2001; Chabert et al. 2003; Lafforgue 2012].

Work of Higson, Lafforgue and Skandalis [Higson et al. 2002] has, however, shown the conjecture to be false in the generality stated above. The counterexamples to the Baum–Connes conjecture they discovered are closely connected to failures of *exactness* in the sense of Kirchberg and Wassermann [Brown and Ozawa

Baum was partially supported by NSF grant DMS-1200475. Guentner was partially supported by a grant from the Simons Foundation (#245398). Willett was partially supported by NSF grant DMS-1229939.
MSC2010: 22D25, 46L80, 46L85, 58B34.
Keywords: Gromov monster group, exotic crossed product, a-T-menable action, girth of graph.

2008, Chapter 5]. Recall that a locally compact group G is *exact* if for every short exact sequence of G-C^*-algebras

$$0 \to I \to A \to B \to 0$$

the corresponding sequence of reduced crossed products

$$0 \to I \rtimes_{\mathrm{red}} G \to A \rtimes_{\mathrm{red}} G \to B \rtimes_{\mathrm{red}} G \to 0$$

is still exact. All naturally occurring classes of locally compact groups[1] are known to be exact. For example, countable linear groups [Guentner et al. 2005], word hyperbolic groups [Roe 2005], and connected groups [Connes 1976, Corollary 6.9(c)] are all exact. Nonetheless, Gromov [2003] has indicated how to construct nonexact "monster" groups. (See [Arzhantseva and Delzant 2008; Coulon 2014; Osajda 2014] for detailed accounts of related constructions; the last of these is most relevant for this paper.) Higson, Lafforgue and Skandalis [Higson et al. 2002] used Gromov's groups to produce short exact sequences of G-C^*-algebras such that the resulting sequence of crossed products fails to be exact even on the level of K-theory. This produces a counterexample to the Baum–Connes conjecture with coefficients.

Furthermore, the Baum–Connes conjecture actually predicts that the functor associating to a G-C^*-algebra A the K-theory of the reduced crossed product $A \rtimes_{\mathrm{red}} G$ should send short exact sequences of G-C^*-algebras to six-term exact sequences of abelian groups. Thus any examples where exactness of the right-hand-side of the conjecture in (1.1) fails necessarily produce counterexamples; conversely, any attempt to reformulate the conjecture must take exactness into account.

Several results from the last five years show that some counterexamples can be obviated by using maximal completions, which are always exact. The first progress along these lines was work of Oyono-Oyono and Yu [2009] on the maximal coarse Baum–Connes conjecture for certain expanders. Developing these ideas, Yu and the third author [2012a; 2012b] showed that some of the counterexamples to the Baum–Connes conjecture coming from Gromov monster groups can be shown to be confirming examples if the maximal crossed product $A \rtimes_{\max} G$ is instead used to define the conjecture. Subsequently, the geometric input underlying these results was clarified by Chen, Wang and Yu [Chen et al. 2013], and the role of exactness, and also a-T-menability, in the main examples was made quite explicit by Finn-Sell and Wright [2014].

All this work suggests that the maximal crossed product sometimes has *better*

[1] Of course, what "naturally occurring" means is arguable! However, we think this can be reasonably justified.

properties than the reduced crossed product; however, there are well-known property (T) obstructions [Higson 1998] to the Baum–Connes conjecture being true for the maximal crossed product in general. The key idea of the current work is to study crossed products that combine the good properties of the maximal and reduced crossed products.

In this paper we shall study C^*-algebra crossed products that preserve short exact sequences. The Baum–Connes conjecture also predicts that a crossed product takes equivariantly Morita-equivalent G-C^*-algebras to Morita-equivalent C^*-algebras on the level of K-theory (this is true for the maximal and reduced crossed products, but not in general). We thus restrict attention to crossed products satisfying a *Morita compatibility* assumption that guarantees this.

We shall show that a minimal exact and Morita-compatible crossed product exists, and we shall use it to reformulate the Baum–Connes conjecture. Denoting the minimal exact and Morita-compatible crossed product by $A \rtimes_{\mathcal{E}} G$, we propose that the natural Baum–Connes assembly map

$$\mu : K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_{\mathcal{E}} G) \tag{1.2}$$

is an isomorphism for any second-countable locally compact group G and any G-C^*-algebra A.

This reformulation has the following four virtues:

(i) it agrees with the usual version of the conjecture for all exact groups and all a-T-menable groups;

(ii) the property (T) obstructions to surjectivity of the maximal Baum–Connes assembly map do not apply to it;

(iii) all known constructions of counterexamples to the usual version of the Baum–Connes conjecture (for groups, with coefficients) no longer apply;

(iv) there exist groups G and G-C^*-algebras A for which the old assembly map in (1.1) fails to be surjective, but for which the reformulated assembly map in (1.2) is an isomorphism.

Thanks to point (i) above, the reformulated assembly map is an isomorphism, or injective, in all situations where the usual version of the assembly map is known to have these properties.

Outline of the paper. In Section 2 we define what we mean by a general crossed product, and show that any such has an associated Baum–Connes assembly map. In Section 3 we define exact and Morita-compatible crossed products and show that there is a minimal crossed product with both of these properties. In Section 4 we show that the minimal exact and Morita-compatible crossed product has a descent functor in E-theory, and use this to state our reformulation of the Baum–Connes conjecture. In Section 5 we show that the property (T) obstructions to the maximal Baum–Connes assembly map being an isomorphism do not apply to our new conjecture. In Section 6 we show that our reformulated conjecture is true when an action is a-T-menable. In Section 7 we produce an example where the new conjecture is true, but the old version of the conjecture fails. Finally, in Section 8, we collect together some natural questions and remarks. In the Appendix we discuss some examples of exotic crossed products; this material is not used in the main body of the paper, but is useful for background and motivation.

2. Statement of the conjecture

Let G be a second-countable, locally compact group. Let C^* denote the category of C^*-algebras: an object in this category is a C^*-algebra, and a morphism is a $*$-homomorphism. Let G-C^* denote the category of G-C^*-algebras: an object in this category is C^*-algebra equipped with a continuous action of G and a morphism is a G-equivariant $*$-homomorphism.

We shall be interested in *crossed product functors* from G-C^* to C^*. The motivating examples are the usual maximal and reduced crossed product functors

$$A \mapsto A \rtimes_{\max} G, \quad A \mapsto A \rtimes_{\text{red}} G.$$

Recall that the maximal crossed product is the completion of the algebraic crossed product for the maximal norm. Here, the algebraic crossed product $A \rtimes_{\text{alg}} G$ is the space of continuous compactly supported functions from G to A, equipped with the usual twisted product and involution. Similarly, $A \rtimes_{\text{red}} G$ is the completion of the algebraic crossed product for the reduced norm. Further, the maximal norm dominates the reduced norm so that the identity on $A \rtimes_{\text{alg}} G$ extends to a (surjective) $*$-homomorphism $A \rtimes_{\max} G \to A \rtimes_{\text{red}} G$. Together, these $*$-homomorphisms comprise a natural transformation of functors.

With these examples in mind, we introduce the following definition.

Definition 2.1. A (C^**-algebra*) *crossed product* is a functor

$$A \mapsto A \rtimes_{\tau} G \quad : \quad G\text{-}C^* \to C^*,$$

such that each C^*-algebra $A \rtimes_{\tau} G$ contains $A \rtimes_{\text{alg}} G$ as a dense $*$-subalgebra, together with natural transformations

$$A \rtimes_{\max} G \to A \rtimes_{\tau} G \to A \rtimes_{\text{red}} G \tag{2.2}$$

which restrict to the identity on each $*$-subalgebra $A \rtimes_{\text{alg}} G$.

It follows that each C^*-algebra $A \rtimes_\tau G$ is a completion of the algebraic crossed product for a norm which necessarily satisfies

$$\|x\|_{\mathrm{red}} \le \|x\|_\tau \le \|x\|_{\max}$$

for every $x \in A \rtimes_{\mathrm{alg}} G$. Note also that the $*$-homomorphism $A \rtimes_\tau G \to B \rtimes_\tau G$ functorially induced by a G-equivariant $*$-homomorphism $A \to B$ is necessarily the extension by continuity of the standard $*$-homomorphism $A \rtimes_{\mathrm{alg}} G \to B \rtimes_{\mathrm{alg}} G$.

In the Appendix we shall see that there are in general many crossed products other than the reduced and maximal ones. Our immediate goal is to formulate a version of the Baum–Connes conjecture for a general crossed product. For reasons involving descent (that will become clear later), we shall formulate the Baum–Connes conjecture in the language of E-theory, as in [GHT 2000, §10].

We continue to let G be a second-countable, locally compact group and consider the τ-crossed product for G. The *τ-Baum–Connes assembly map* for G with coefficients in the G-C^*-algebra A is the composition

$$K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_{\max} G) \to K_*(A \rtimes_\tau G), \tag{2.3}$$

in which the first map is the usual *maximal* Baum–Connes assembly map and the second is induced by the $*$-homomorphism $A \rtimes_{\max} G \to A \rtimes_\tau G$. The domain of assembly is independent of the particular crossed product we are using. It is the *topological K-theory of G with coefficients in A*, defined as the direct limit of equivariant E-theory groups

$$K_*^{\mathrm{top}}(G; A) = \lim_{\substack{X \subseteq \underline{E}G \\ \text{cocompact}}} E^G(C_0(X), A),$$

where the direct limit is taken over cocompact subsets of $\underline{E}G$, a universal space for proper G-actions. The (maximal) assembly map is itself a direct limit of assembly maps for the individual cocompact subsets of $\underline{E}G$, each defined as a composition

$$E^G(C_0(X), A) \to E(C_0(X) \rtimes_{\max} G, A \rtimes_{\max} G) \to E(\mathbb{C}, A \rtimes_{\max} G), \tag{2.4}$$

in which the first map is the E-theoretic (*maximal*) *descent functor*, and the second map is composition with the class of the *basic projection* in $C_0(X) \rtimes_{\max} G$, viewed as an element of $E(\mathbb{C}, C_0(X) \rtimes_{\max} G)$. Compatibility of the assembly maps for the various cocompact subsets of $\underline{E}G$ indexing the direct limit follows from the uniqueness (up to homotopy) of the basic projection. For details see [GHT 2000, §10].

For the moment, we are interested in what validity of the τ-Baum–Connes conjecture — the assertion that the τ-Baum–Connes assembly map is an isomorphism — would predict about the τ-crossed product itself. The first prediction is concerned with *exactness*. Suppose

$$0 \to I \to A \to B \to 0$$

is a short exact sequence of G-C^*-algebras. Exactness properties of equivariant E-theory ensure that the sequence functorially induced on the left-hand side of assembly

$$K_*^{\mathrm{top}}(G; I) \to K_*^{\mathrm{top}}(G; A) \to K_*^{\mathrm{top}}(G; B)$$

is exact in the middle. (Precisely, this follows from the corresponding fact for each cocompact subset of $\underline{E}G$ upon passing to the limit.) Now, the assembly map is itself functorial for equivariant $*$-homomorphisms of the coefficient algebra. As a consequence, the functorially induced sequence on the right-hand side of assembly

$$K_*(I \rtimes_\tau G) \to K_*(A \rtimes_\tau G) \to K_*(B \rtimes_\tau G)$$

must be exact in the middle as well.

The second prediction concerns Morita invariance. To formulate it, let H be the countably infinite direct sum

$$H = L^2(G) \oplus L^2(G) \oplus \cdots$$

and denote the compact operators on H by $\mathcal{K}_G$, which we consider as a G-C^*-algebra in the natural way. Similarly, for any G-C^*-algebra A, we consider the spatial tensor product $A \otimes \mathcal{K}_G$ as a G-C^*-algebra via the diagonal action. Assume for simplicity that A and B are separable G-C^*-algebras. Then A and B are said to be *equivariantly Morita-equivalent* if $A \otimes \mathcal{K}_G$ is equivariantly $*$-isomorphic to $B \otimes \mathcal{K}_G$: results of [Curto et al. 1984] and [Mingo and Phillips 1984] show that this is equivalent to other, perhaps more usual, definitions (compare [GHT 2000, Proposition 6.11 and Theorem 6.12]). If A and B are equivariantly Morita-equivalent then $E^G(C, A) \cong E^G(C, B)$ for any G-C^*-algebra C [GHT 2000, Theorem 6.12]. There is thus an isomorphism

$$K_*^{\mathrm{top}}(G; A) \cong K_*^{\mathrm{top}}(G; B)$$

on the left-hand side of assembly. Assuming the τ-Baum–Connes conjecture is valid for G we must therefore also have an isomorphism

$$K_*(A \rtimes_\tau G) \cong K_*(B \rtimes_\tau G)$$

on the level of K-theory.

3. Crossed product functors

Motivated by the discussion in the previous section, we are led to study crossed product functors that have good properties with respect to exactness and Morita equivalence. The following two properties imply this "good behavior", and are particularly well-suited to our later needs.

Throughout this section, G is a second-countable, locally compact group.

Definition 3.1. The τ-crossed product is *exact* if for every short exact sequence

$$0 \to A \to B \to C \to 0$$

of G-C^*-algebras the corresponding sequence of C^*-algebras

$$0 \to A \rtimes_\tau G \to B \rtimes_\tau G \to C \rtimes_\tau G \to 0$$

is short exact.

Whereas the maximal crossed product functor is always exact in this sense (see Lemma A.6), the reduced crossed product functor is (by definition) exact precisely when G is an exact group [Kirchberg and Wassermann 1999, p. 170]. Note that if the τ-crossed product is exact, then the associated K-theory groups have the half-exactness property predicted by the τ-Baum–Connes conjecture and by half-exactness of K-theory.

For the second property, recall that $\mathcal{K}_G$ denotes the compact operators on the infinite sum Hilbert space $H = L^2(G) \oplus L^2(G) \oplus \cdots$, considered as a G-C^*-algebra with the natural action. Write Λ for the action of G on H. Recall that for any G-C^*-algebra A there are natural maps from A and G into the multiplier algebra $\mathcal{M}(A \rtimes_{\max} G)$, and we can identify A and G with their images under these maps. This gives rise to a covariant representation

$$(\pi, u) : (A \otimes \mathcal{K}_G, G) \to \mathcal{M}(A \rtimes_{\max} G) \otimes \mathcal{K}_G$$

defined by $\pi(a \otimes T) = a \otimes T$ and $u_g = g \otimes \Lambda_g$. The integrated form

$$\Phi : (A \otimes \mathcal{K}_G) \rtimes_{\max} G \to (A \rtimes_{\max} G) \otimes \mathcal{K}_G \tag{3.2}$$

of this covariant pair is well-known to be a $*$-isomorphism, which we call the *untwisting isomorphism.*

Definition 3.3. The τ-crossed product is *Morita-compatible* if the untwisting isomorphism descends to an isomorphism

$$\Phi : (A \otimes \mathcal{K}_G) \rtimes_\tau G \to (A \rtimes_\tau G) \otimes \mathcal{K}_G$$

of τ-crossed products.

Both the maximal and reduced crossed product functors are Morita-compatible: see Lemma A.7 in the Appendix. Note that, if $\rtimes_\tau$ is Morita-compatible, then it takes equivariantly Morita-equivalent (separable) G-C^*-algebras to Morita-equivalent C^*-algebras. Indeed, in this case we have

$$(A \rtimes_\tau G) \otimes \mathcal{K}_G \cong (A \otimes \mathcal{K}_G) \rtimes_\tau G \cong (B \otimes \mathcal{K}_G) \rtimes_\tau G \cong (B \rtimes_\tau G) \otimes \mathcal{K}_G,$$

where the middle isomorphism is Morita equivalence and the other two are Morita compatibility. Thus if τ is Morita-compatible then the associated K-theory groups have the Morita invariance property predicted by the τ-Baum–Connes conjecture.

Our goal for the remainder of the section is to show that there is a "minimal" exact and Morita-compatible crossed product. To make sense of this, we introduce a partial ordering on the collection of crossed products for G: the σ-crossed product is *smaller* than the τ-crossed product if the natural transformation in (2.2) from the τ-crossed product to the reduced crossed product factors through the σ-crossed product, meaning that there exists a diagram

$$A \rtimes_\tau G \to A \rtimes_\sigma G \to A \rtimes_{\mathrm{red}} G$$

for every G-C^*-algebra A, where the maps from $A \rtimes_\tau G$ and $A \rtimes_\sigma G$ to $A \rtimes_{\mathrm{red}} G$ are the ones coming from the definition of a crossed product functor. Equivalently, for every $x \in A \rtimes_{\mathrm{alg}} G$ we have

$$\|x\|_{\mathrm{red}} \leq \|x\|_\sigma \leq \|x\|_\tau,$$

so that the identity on $A \rtimes_{\mathrm{alg}} G$ extends to a $*$-homomorphism $A \rtimes_\tau G \to A \rtimes_\sigma G$. In particular, the order relation on crossed products is induced by the obvious order relation on C^*-algebra norms on $A \rtimes_{\mathrm{alg}} G$.[2] The maximal crossed product is the maximal element for this ordering, and the reduced crossed product is the minimal element.

Recall that the *spectrum* of a C^*-algebra A is the set $\widehat{A}$ of equivalence classes of nonzero irreducible $*$-representations of A. We will conflate a representation ρ with the equivalence class it defines in $\widehat{A}$. For an ideal I in a C^*-algebra A, an irreducible representation of A restricts to a (possibly zero) irreducible representation of I, and conversely irreducible representations of I extend uniquely to irreducible representations of A. It follows that $\widehat{I}$ identifies canonically with

$$\{\rho \in \widehat{A} \mid I \nsubseteq \mathrm{Kernel}(\rho)\}.$$

Similarly, given a quotient $*$-homomorphism $\pi : A \to B$, the spectrum $\widehat{B}$ of B identifies canonically with the collection

$$\{\rho \in \widehat{A} \mid \mathrm{Kernel}(\pi) \subseteq \mathrm{Kernel}(\rho)\}$$

[2] Incidentally, this observation gets us around the set-theoretic technicalities inherent when considering the "collection of all crossed products".

of elements of $\widehat{A}$ that factor through π via the correspondence $\widehat{B} \ni \rho \leftrightarrow \rho \circ \pi \in \widehat{A}$. We will make these identifications in what follows without further comment; note that, having done this, a short exact sequence

$$0 \to I \to A \to B \to 0$$

gives rise to a canonical decomposition $\widehat{A} = \widehat{I} \sqcup \widehat{B}$.

We record the following basic fact as a lemma as we will refer back to it several times; for a proof, see for example [Dixmier 1977, Theorem 2.7.3].

Lemma 3.4. *For any nonzero element of a C^*-algebra, there is an irreducible representation that is nonzero on that element.* □

The next lemma is the last general fact we need about spectra.

Lemma 3.5. *Consider a diagram of C^*-algebras*

$$\begin{tikzcd} A_1 \arrow[r, "\phi"] \arrow[d, "\pi_1"] & A_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, dashed, "\psi"] & B_2 \end{tikzcd}$$

where ϕ is a $*$-homomorphism, and π_1 and π_2 are surjective $*$-homomorphisms. For each $\rho \in \widehat{A}_2$, define

$$\phi^* \rho := \{\rho' \in \widehat{A}_1 \mid \text{Kernel}(\rho \circ \phi) \subseteq \text{Kernel}(\rho')\}.$$

Then there exists a $*$-homomorphism $\psi : B_1 \to B_2$ making the diagram commute if and only if $\phi^* \rho$ is a subset of $\widehat{B}_1$ for all ρ in $\widehat{B}_2$ (where $\widehat{B}_2$ is considered as a subset of $\widehat{A}_2$).

Proof. Assume first that ψ exists. Let ρ be an element of $\widehat{B}_2$, and $\rho \circ \pi_2$ the corresponding element of $\widehat{A}_2$. Then

$$\begin{aligned} \phi^*(\rho \circ \pi_2) &= \{\rho' \in \widehat{A}_1 \mid \text{Kernel}(\rho \circ \pi_2 \circ \phi) \subseteq \text{Kernel}(\rho')\} \\ &= \{\rho' \in \widehat{A}_1 \mid \text{Kernel}(\rho \circ \psi \circ \pi_1) \subseteq \text{Kernel}(\rho')\} \\ &\subseteq \{\rho' \in \widehat{A}_1 \mid \text{Kernel}(\pi_1) \subseteq \text{Kernel}(\rho')\} \\ &= \widehat{B}_1. \end{aligned}$$

Conversely, assume that no such ψ exists. Then the kernel of π_1 is not a subset of the kernel of $\pi_2 \circ \phi$, so there exists $a \in A_1$ such that $\pi_1(a) = 0$, but $\pi_2(\phi(a)) \neq 0$. Lemma 3.4 implies that there exists $\rho \in \widehat{B}_2$ such that $\rho(\pi_2(\phi(a))) \neq 0$. Write $C = \rho(\pi_2(\phi(A_1)))$ and $c = \rho(\pi_2(\phi(a)))$. Then Lemma 3.4 again implies that there exists $\rho'' \in \widehat{C}$ such that $\rho''(c) \neq 0$. Let $\rho' = \rho'' \circ \rho \circ \pi_2 \circ \phi$, an element of $\widehat{A}_1$. Then

$$\text{Kernel}(\rho \circ \pi_2 \circ \phi) \subseteq \text{Kernel}(\rho') \tag{3.6}$$

and $\rho'(a) \neq 0$. Line (3.6) implies that ρ' is in $\phi^*\rho$, while the fact that $\rho'(a) \neq 0$ and $\pi_1(a) = 0$ implies that ρ' is not in the subset $\widehat{B}_1$ of $\widehat{A}_1$. Hence $\phi^*\rho \not\subseteq \widehat{B}_1$, as required. □

We now turn back to crossed products. Let A be a G-C^*-algebra and σ a crossed product. Let $S_\sigma(A)$ denote the subset of $\widehat{A \rtimes_{\max} G}$ consisting of representations of $A \rtimes_{\max} G$ that factor through the quotient $A \rtimes_\sigma G$; in other words, $S_\sigma(A)$ is the subset of $A \rtimes_{\max} G$ that identifies naturally with $\widehat{A \rtimes_\sigma G}$. In particular, $S_{\max}(A)$ denotes $\widehat{A \rtimes_{\max} G}$ and $S_{\mathrm{red}}(A)$ denotes $\widehat{A \rtimes_{\mathrm{red}} G}$, considered as a subset of $S_{\max}(A)$.

We will first characterize exactness in terms of the sets above. Let

$$0 \to I \to A \to B \to 0$$

be a short exact sequence of G-C^*-algebras. If σ is a crossed product, consider the corresponding commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \rtimes_{\max} G & \longrightarrow & A \rtimes_{\max} G & \longrightarrow & B \rtimes_{\max} G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_I} & & \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} & & \\ 0 & \longrightarrow & I \rtimes_\sigma G & \xrightarrow{\iota} & A \rtimes_\sigma G & \xrightarrow{\pi} & B \rtimes_\sigma G & \longrightarrow & 0 \end{array} \tag{3.7}$$

with exact top row. Note that the bottom row need not be exact, but we do have that the map π is surjective (by commutativity of the right-hand square and surjectivity of π_B), and that the kernel of π contains the image of ι (as σ is a functor).

We make the following identifications:

(i) $S_\sigma(A)$ is by definition a subset of $S_{\max}(A)$;

(ii) $S_{\max}(I)$ and $S_{\max}(B)$ identify canonically with subsets of $S_{\max}(A)$ as $I \rtimes_{\max} G$ and $B \rtimes_{\max} G$ are respectively an ideal and a quotient of $A \rtimes_{\max} G$;

(iii) $S_\sigma(I)$ and $S_\sigma(B)$ are by definition subsets of $S_{\max}(I)$ and $S_{\max}(B)$, respectively, and thus identify with subsets of $S_{\max}(A)$ by the identifications in the previous point.

Lemma 3.8. *Let the identifications above have been made.*

(i) *The map ι in (3.7) is injective if and only if*

$$S_{\max}(I) \cap S_\sigma(A) = S_\sigma(I).$$

(ii) *The kernel of π is equal to the image of ι in (3.7) if and only if*

$$S_{\max}(B) \cap S_\sigma(A) = S_\sigma(B).$$

Proof. For (i), as $\iota(I \rtimes_\sigma G)$ is an ideal in $A \rtimes_\sigma G$, we may identify its spectrum with a subset of $S_\sigma(A)$, and thus also of $S_{\max}(A)$. Commutativity of (3.7) identifies the spectrum of $\iota(I \rtimes_\sigma G)$ with

$$\{\rho \in S_{\max}(A) \mid \mathrm{Kernel}(\pi_A) \subseteq \mathrm{Kernel}(\rho), \rho(I \rtimes_{\max} G) \neq \{0\}\} = S_{\max}(I) \cap S_\sigma(A).$$

Lemma 3.4 implies the map ι is injective if and only if the spectrum of $\iota(I \rtimes_\sigma G)$ and $S_\sigma(I)$ are the same as subsets of $S_{\max}(A)$, so we are done.

For (ii), surjectivity of π canonically identifies $S_\sigma(B)$ with a subset of $S_\sigma(A)$. Part (i) and the fact that the image of ι is contained in the kernel of π imply that $S_\sigma(B)$ is disjoint from $S_{\max}(I) \cap S_\sigma(A)$ as subsets of $S_\sigma(A)$. Hence the kernel of π equals the image of ι if and only if

$$S_\sigma(A) = S_\sigma(B) \cup (S_{\max}(I) \cap S_\sigma(A)),$$

or, equivalently, if and only if

$$S_\sigma(B) = S_\sigma(A) \setminus S_{\max}(I). \tag{3.9}$$

As the top line of diagram (3.7) is exact, $S_{\max}(A)$ is equal to the disjoint union of $S_{\max}(I)$ and $S_{\max}(B)$, whence $S_\sigma(A) \setminus S_{\max}(I) = S_\sigma(A) \cap S_{\max}(B)$; the conclusion follows on combining this with the condition in (3.9). □

We now characterize Morita compatibility. Recall that there is a canonical "untwisting" $*$-isomorphism

$$\Phi : (A \otimes \mathcal{K}_G) \rtimes_{\max} G \to (A \rtimes_{\max} G) \otimes \mathcal{K}_G, \tag{3.10}$$

and that a crossed product σ is Morita-compatible if this descends to a $*$-isomorphism

$$(A \otimes \mathcal{K}_G) \rtimes_\sigma G \cong (A \rtimes_\sigma G) \otimes \mathcal{K}_G.$$

The following lemma is immediate from the fact that the spectrum of the right-hand-side of (3.10) identifies canonically with $S_{\max}(A)$.

Lemma 3.11. *A crossed product σ is Morita-compatible if and only if the bijection*

$$\widehat{\Phi} : S_{\max}(A \otimes \mathcal{K}_G) \to S_{\max}(A)$$

induced by Φ takes $S_\sigma(A \otimes \mathcal{K}_G)$ onto $S_\sigma(A)$. □

The following lemma is the final step in constructing a minimal exact and Morita-compatible crossed product.

Lemma 3.12. *Let Σ be a family of crossed products. Then there is a unique crossed product τ such that, for any G-C^*-algebra A,*

$$S_\tau(A) = \bigcap_{\sigma \in \Sigma} S_\sigma(A).$$

Proof. For each $\sigma \in \Sigma$, let I_σ denote the kernel of the canonical quotient map $A \rtimes_{\max} G \to A \rtimes_\sigma G$, and similarly for I_{red}. Note that I_{red} contains all the ideals I_σ. Let I denote the smallest ideal of $A \rtimes_{\max} G$ containing I_σ for all $\sigma \in \Sigma$. Define

$$A \rtimes_\tau G := (A \rtimes_{\max} G)/I;$$

as I is contained in I_{red}, this is a completion of $A \rtimes_{\text{alg}} G$ that sits between the maximal and reduced completions. The spectrum of $A \rtimes_\tau G$ is

$$S_\tau(A) = \{\rho \in S_{\max}(A) \mid I \subseteq \text{Kernel}(\rho)\}.$$

Lemma 3.4 implies that this is equal to

$$\{\rho \in S_{\max}(A) \mid I_\sigma \subseteq \text{Kernel}(\rho) \text{ for all } \sigma \in \Sigma\} = \bigcap_{\sigma \in \Sigma} S_\sigma(A),$$

as claimed. Uniqueness of the completion $A \rtimes_\tau G$ follows from Lemma 3.4 again.

Finally, we must check that $\rtimes_\tau$ defines a functor on G-C^*: if $\phi : A_1 \to A_2$ is an equivariant $*$-homomorphism, we must show that the dashed arrow in the diagram

$$\begin{tikzcd} A_1 \rtimes_{\max} G \arrow[r, "\phi \rtimes G"] \arrow[d] & A_2 \rtimes_{\max} G \arrow[d] \\ A_1 \rtimes_\tau G \arrow[r, dashed] & A_2 \rtimes_\tau G \end{tikzcd}$$

can be filled in. Fix $\sigma \in \Sigma$. Lemma 3.5 applied to the analogous diagram with τ replaced by σ implies that, for all $\pi \in S_\sigma(A_2)$, $(\phi \rtimes G)^*\pi$ is a subset of $S_\sigma(A_1)$. Hence for all $\pi \in S_\tau(A_2) = \bigcap_{\sigma \in \Sigma} S_\sigma(A_2)$ we have that $(\phi \rtimes G)^*\pi$ is a subset of $\bigcap_{\sigma \in \Sigma} S_\sigma(A_1) = S_\tau(A_1)$. Lemma 3.5 now implies that the dashed line can be filled in. $\square$

The part of the following theorem that deals with exactness is due to Eberhard Kirchberg.

Theorem 3.13. *There is a unique minimal exact and Morita-compatible crossed product.*

Proof. Let Σ be the collection of all exact and Morita-compatible crossed products, and let τ be the crossed product that satisfies $S_\tau(A) = \bigcap_{\sigma \in \Sigma} S_\sigma(A)$ as in Lemma 3.12. As τ is a lower bound for the set Σ, it suffices to show that τ is exact and Morita-compatible. The conditions in Lemmas 3.8 and 3.13 clearly pass to intersections, however, so we are done. $\square$

4. A reformulation of the conjecture

Continue with G a second-countable, locally compact group. We propose to reformulate the Baum–Connes conjecture, replacing the usual reduced crossed product with the minimal exact and Morita-compatible crossed product (the $\mathscr{E}$-crossed product). There is no change to the left side of the conjecture.

Definition 4.1. The *$\mathscr{E}$-Baum–Connes conjecture with coefficients* is the statement that the $\mathscr{E}$-Baum–Connes assembly map

$$\mu : K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_{\mathscr{E}} G)$$

is an isomorphism for every G-C^*-algebra A. Here $A \rtimes_{\mathscr{E}} G$ is the minimal exact and Morita-compatible crossed product.

When the group is exact, the reduced and $\mathscr{E}$-crossed products agree, and thus the original and reformulated Baum–Connes conjectures agree. Our main remaining goal is to show that known expander-based counterexamples to the original Baum–Connes conjecture are confirming examples for the reformulated conjecture. Indeed, our positive isomorphism results will hold in these examples for *every* exact and Morita-compatible crossed product, in particular for the reformulated conjecture involving the $\mathscr{E}$-crossed product. For the isomorphism results, we require an alternate description of the $\mathscr{E}$-Baum–Connes assembly map, amenable to the standard Dirac-dual Dirac method of proving the conjecture.

We recall the necessary background about E-theory. The *equivariant asymptotic category* is the category in which the objects are the G-C^*-algebras and in which the morphisms are homotopy classes of *equivariant asymptotic morphisms*. We shall denote the morphism sets in this category by $[\![A, B]\!]_G$. The *equivariant E-theory groups* are defined as particular morphism sets in this category:

$$E^G(A, B) = [\![\Sigma A \otimes \mathscr{K}_G, \Sigma B \otimes \mathscr{K}_G]\!]_G,$$

where $\Sigma A \otimes \mathscr{K}_G$ stands for $C_0(0,1) \otimes A \otimes \mathscr{K}_G$. The *equivariant E-theory category* is the category in which the objects are the G-C^*-algebras and in which the morphism sets are the equivariant E-theory groups.

The equivariant categories we have encountered are related by functors: there is a functor from the category of G-C^*-algebras to the equivariant asymptotic category which is the identity on objects, and which views an equivariant $*$-homomorphism as a “constant” asymptotic family; similarly there is a functor from the equivariant asymptotic category to the equivariant E-theory category which is the identity on objects and which “tensors” an asymptotic morphism by the identity maps on $C_0(0,1)$ and $\mathscr{K}_G$.

Finally, there is an ordinary (nonequivariant) theory parallel to the equivariant theory described above: the *asymptotic category* and *E-theory category* are categories in which the objects are C^*-algebras and the morphisms are appropriate homotopy classes of asymptotic morphisms; there are functors as above, which are the identity on objects. See [GHT 2000] for further background and details.

We start with two technical lemmas. For a C^*-algebra B, let $\mathcal{M}(B)$ denote its multiplier algebra. If A is a G-C^*-algebra with G-action α, recall that elements of $A \rtimes_{\text{alg}} G$ are continuous compactly supported functions from G to A; we denote such a function by $(a_g)_{g\in G}$. Consider the canonical action of A on $A \rtimes_{\text{alg}} G$ by multipliers defined by setting

$$b \cdot (a_g)_{g\in G} := (ba_g)_{g\in G} \quad \text{and} \quad (a_g)_{g\in G} \cdot b := (a_g \alpha_g(b))_{g\in G} \tag{4.2}$$

for all $(a_g)_{g\in G} \in A \rtimes_{\text{alg}} G$ and $b \in A$. This action extends to actions of A on both $A \rtimes_{\max} G$ and $A \rtimes_{\text{red}} G$ by multipliers, i.e., there are $*$-homomorphisms $A \to \mathcal{M}(A \rtimes_{\max} G)$ and $A \to \mathcal{M}(A \rtimes_{\text{red}} G)$ such that the image of $b \in A$ is the extension of the multiplier defined in (4.2) to all of the relevant completion. Analogously, there is an action of G on $A \rtimes_{\text{alg}} G$ by multipliers defined for $h \in G$ by

$$u_h \cdot (a_g)_{g\in G} := (\alpha_h(a_{h^{-1}g}))_{g\in G} \quad \text{and} \quad (a_g)_{g\in G} \cdot u_h := \Delta(h^{-1})(a_{gh^{-1}})_{g\in G}, \tag{4.3}$$

where $\Delta : G \to \mathbb{R}_+$ is the modular function for a fixed choice of (left-invariant) Haar measure on G. This extends to a unitary representation

$$G \to \mathcal{U}(\mathcal{M}(A \rtimes_{\max} G)), \quad g \mapsto u_g$$

from G into the unitary group of $\mathcal{M}(A \rtimes_{\max} G)$, and similarly for $\mathcal{M}(A \rtimes_{\text{red}} G)$.

Lemma 4.4. *For any crossed product functor $\rtimes_\tau$ and any G-C^*-algebra A, the action of A on $A \rtimes_{\text{alg}} G$ in (4.2) extends to define an injective $*$-homomorphism*

$$A \to \mathcal{M}(A \rtimes_\tau G).$$

This in turn extends to a $*$-homomorphism

$$\mathcal{M}(A) \to \mathcal{M}(A \rtimes_\tau G)$$

from the multiplier algebra of A to that of $A \rtimes_\tau G$.

Moreover, the action of G on $A \rtimes_{\text{alg}} G$ in (4.3) extends to define an injective unitary representation

$$G \to \mathcal{U}(\mathcal{M}(A \rtimes_\tau G)).$$

Proof. The desired $*$-homomorphism $A \to \mathcal{M}(A \rtimes_\tau G)$ can be defined as the composition

$$A \to \mathcal{M}(A \rtimes_{\max} G) \to \mathcal{M}(A \rtimes_\tau G)$$

of the canonical action of A on the maximal crossed product by multipliers and the $*$-homomorphism on multiplier algebras induced by the surjective natural transformation between the maximal and τ-crossed products. Injectivity follows on considering the composition

$$A \to \mathcal{M}(A \rtimes_{\max} G) \to \mathcal{M}(A \rtimes_{\tau} G) \to \mathcal{M}(A \rtimes_{\mathrm{red}} G),$$

which is well known (and easily checked) to be injective. The $*$-homomorphism $A \to \mathcal{M}(A \rtimes_{\tau} G)$ is easily seen to be nondegenerate, so extends to the multiplier algebra of A as claimed. The existence and injectivity of the unitary representation $G \to \mathcal{U}(\mathcal{M}(A \rtimes_{\tau} G))$ can be shown analogously. □

Let now $\odot$ denote the algebraic tensor product (over $\mathbb{C}$) of two $*$-algebras, and as usual use $\otimes$ for the spatial tensor product of C^*-algebras. Recall that we denote elements of $A \rtimes_{\mathrm{alg}} G$ by $(a_g)_{g\in G}$. Equip $C[0,1]$ with the trivial G-action, and consider the function defined by

$$\phi : C[0,1] \odot (A \rtimes_{\mathrm{alg}} G) \to (C[0,1] \otimes A) \rtimes_{\mathrm{alg}} G, \quad f \odot (a_g)_{g\in G} \mapsto (f \otimes a_g)_{g\in G}. \tag{4.5}$$

It is not difficult to check that ϕ is a well-defined $*$-homomorphism.

Lemma 4.6. *Let A be a G-C^*-algebra, and $\rtimes_{\tau}$ be any crossed product. Then the $*$-homomorphism ϕ defined in (4.5) extends to a $*$-isomorphism*

$$\phi : C[0,1] \otimes (A \rtimes_{\tau} G) \cong (C[0,1] \otimes A) \rtimes_{\tau} G$$

on τ-crossed products. Moreover, if the τ-crossed product is exact, then the restriction of ϕ to $C_0(0,1) \odot (A \rtimes_{\mathrm{alg}} G)$ extends to a $*$-isomorphism

$$\phi : C_0(0,1) \otimes (A \rtimes_{\tau} G) \cong (C_0(0,1) \otimes A) \rtimes_{\tau} G.$$

Proof. The inclusion $A \to C[0,1] \otimes A$ defined by $a \mapsto 1 \otimes a$ is equivariant, so gives rise to a $*$-homomorphism

$$A \rtimes_{\tau} G \to (C[0,1] \otimes A) \rtimes_{\tau} G$$

by functoriality of the τ-crossed product. Composing this with the canonical inclusion of the right-hand side into its multiplier algebra gives a $*$-homomorphism

$$A \rtimes_{\tau} G \to \mathcal{M}((C[0,1] \otimes A) \rtimes_{\tau} G). \tag{4.7}$$

On the other hand, composing the canonical $*$-homomorphism

$$C[0,1] \to \mathcal{M}(C[0,1] \otimes A)$$

with the $*$-homomorphism on multiplier algebras from Lemma 4.4 gives a $*$-homomorphism

$$C[0,1] \to \mathcal{M}((C[0,1] \otimes A) \rtimes_{\tau} G). \tag{4.8}$$

Checking on the strictly dense $*$-subalgebra

$$(C[0,1]\otimes A)\rtimes_{\mathrm{alg}} G \quad \text{of} \quad \mathcal{M}((C[0,1]\otimes A)\rtimes_\tau G)$$

shows that the image of $C[0,1]$ under the $*$-homomorphism in (4.8) is central, whence combining it with the $*$-homomorphism in (4.7) defines a $*$-homomorphism

$$C[0,1]\odot(A\rtimes_\tau G)\to\mathcal{M}((C[0,1]\otimes A)\rtimes_\tau G),$$

and nuclearity of $C[0,1]$ implies that this extends to a $*$-homomorphism

$$C[0,1]\otimes(A\rtimes_\tau G)\to\mathcal{M}((C[0,1]\otimes A)\rtimes_\tau G).$$

It is not difficult to see that this $*$-homomorphism agrees with the map ϕ from (4.5) on the dense $*$-subalgebra $C[0,1]\odot(A\rtimes_{\mathrm{alg}} G)$ of the left-hand side and thus in particular has image in the C^*-subalgebra $(C[0,1]\otimes A)\rtimes_\tau G$ of the right-hand side. We have thus shown that the $*$-homomorphism ϕ from (4.5) extends to a $*$-homomorphism

$$\phi : C[0,1]\otimes(A\rtimes_\tau G)\to(C[0,1]\otimes A)\rtimes_\tau G.$$

It has dense image, and is thus surjective; in the $C[0,1]$ case it remains to show injectivity.

To this end, for each $t\in[0,1]$ let

$$\epsilon_t : (C[0,1]\otimes A)\rtimes_\tau G\to A\rtimes_\tau G$$

be the $*$-homomorphism induced by the G-equivariant $*$-homomorphism $C[0,1]\otimes A\to A$ defined by evaluation at t. Let F be an element of $C[0,1]\otimes(A\rtimes_\tau G)$, which we may think of as a function from $[0,1]$ to $A\rtimes_\tau G$ via the canonical isomorphism

$$C[0,1]\otimes(A\rtimes_\tau G)\cong C([0,1],A\rtimes_\tau G).$$

Checking directly on the dense $*$-subalgebra

$$C[0,1]\odot(A\rtimes_{\mathrm{alg}} G) \quad \text{of} \quad C[0,1]\otimes(A\rtimes_\tau G)$$

shows that $\epsilon_t(\phi(F))=F(t)$ for any $t\in[0,1]$. Hence, if F is in the kernel of ϕ, then $F(t)=0$ for all t in $[0,1]$, whence $F=0$. Hence ϕ is injective, as required.

Assume now that the τ-crossed product is exact, and look at the $C_0(0,1)$ case. The short exact sequence

$$0\to C_0(0,1]\to C[0,1]\to\mathbb{C}\to 0$$

combined with exactness of the maximal tensor product, nuclearity of commutative C^*-algebras and exactness of the τ-crossed product gives rise to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C_0(0,1]\otimes(A\rtimes_\tau G) \arrow[r] \arrow[d] & C[0,1]\otimes(A\rtimes_\tau G) \arrow[r] \arrow[d, "\phi"', "\cong"] & A\rtimes_\tau G \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & (C_0(0,1]\otimes A)\rtimes_\tau G \arrow[r] & (C[0,1]\otimes A)\rtimes_\tau G \arrow[r] & A\rtimes_\tau G \arrow[r] & 0 \end{tikzcd}$$

with exact rows, and where the leftmost vertical arrow is the restriction of ϕ. The restriction of ϕ to $C_0(0,1]\otimes(A\rtimes_\tau G)$ is thus an isomorphism onto $(C_0(0,1]\otimes A)\rtimes_\tau G$. Applying an analogous argument to the short exact sequence

$$0\to C_0(0,1)\to C_0(0,1]\to\mathbb{C}\to 0$$

completes the proof. □

Given this, the following result is an immediate generalization of [GHT 2000, Theorem 4.12], which treats the maximal crossed product. See also [GHT 2000, Theorem 4.16] for comments on the reduced crossed product.

Theorem 4.9. *If the τ-crossed product is both exact and Morita-compatible, then there is a "descent functor" from the equivariant E-theory category to the E-theory category which agrees with the τ-crossed product functor on objects and on those morphisms which are* (*represented by*) *equivariant $*$-homomorphisms.*

Proof. We follow the proof of [GHT 2000, Theorem 6.22]. It follows from Lemma 4.6 that a crossed product functor is always *continuous* in the sense of [GHT 2000, Definition 3.1]. Applying (an obvious analogue of) [GHT 2000, Theorem 3.5], an exact crossed product functor admits descent from the equivariant asymptotic category to the asymptotic category. Thus, we have maps on morphism sets in the asymptotic categories

$$E^G(A,B)=[\![\Sigma A\otimes\mathcal{K}_G,\Sigma B\otimes\mathcal{K}_G]\!]_G\to[\![(\Sigma A\otimes\mathcal{K}_G)\rtimes_\tau G,(\Sigma B\otimes\mathcal{K}_G)\rtimes_\tau G]\!]$$

which agree with the τ-crossed product on morphisms represented by equivariant $*$-homomorphisms. It remains to identify the right-hand side with the E-theory group $E(A\rtimes_\tau G,B\rtimes_\tau G)$. We do this by showing that

$$(C_0(0,1)\otimes A\otimes\mathcal{K}_G)\rtimes_\tau G\cong C_0(0,1)\otimes(A\rtimes_\tau G)\otimes\mathcal{K}_G.$$

This follows immediately from Morita compatibility and Lemma 4.6. □

We now have an alternate description of the τ-Baum–Connes assembly map in the case of an exact, Morita-compatible crossed product functor: we can descend directly to the τ-crossed products and compose with the basic projection. In detail, it follows from Definition 2.1 and the corresponding fact for the maximal and reduced crossed products that, if X is a proper, cocompact G-space, then all crossed products of $C_0(X)$ by G agree. We view the basic projection as an element of $C_0(X) \rtimes_\tau G$, giving a class in $E(\mathbb{C}, C_0(X) \rtimes_\tau G)$. We form the composition

$$E^G(C_0(X), A) \to E(C_0(X) \rtimes_\tau G, A \rtimes_\tau G) \to E(\mathbb{C}, A \rtimes_\tau G), \tag{4.10}$$

in which the first map is the E-theoretic τ-descent and the second is composition with the (class of the) basic projection. Taking the direct limit over the cocompact subsets of $\underline{E}G$ we obtain a map

$$K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_\tau G).$$

Proposition 4.11. *The map just defined is the τ-Baum–Connes assembly map.*

Proof. We must show that applying the maps (4.10) to an element $\theta \in E^G(C_0(X), A)$ gives the same result as applying those in (2.4) followed by the map on K-theory induced by the natural transformation $\psi_A : A \rtimes_{\max} G \to A \rtimes_\tau G$. Noting that $C_0(X) \rtimes_{\max} G = C_0(X) \rtimes_\tau G$ (as all crossed products applied to a proper algebra give the same result), we have the class of the basic projection

$$[p] \in E(\mathbb{C}, C_0(X) \rtimes_{\max} G) = E(\mathbb{C}, C_0(X) \rtimes_\tau G),$$

and the above amounts to saying that the morphisms

$$\psi_A \circ (\theta \rtimes_{\max} G) \circ [p],\ (\theta \rtimes_\tau G) \circ [p] : \mathbb{C} \to A \rtimes_\tau G \tag{4.12}$$

in the E-theory category are the same.

As the functors defined by the τ and maximal crossed products are continuous and exact, [GHT 2000, Proposition 3.6] shows that the natural transformation $A \rtimes_{\max} G \to A \rtimes_\tau G$ gives rise to a natural transformation between the corresponding functors on the asymptotic category. Hence if θ is any morphism in $[\![C_0(X), A]\!]_G$ the diagram

$$\begin{tikzcd} C_0(X) \rtimes_{\max} G \arrow[r, equal] \arrow[d, "\theta \rtimes_{\max} G"'] & C_0(X) \rtimes_\tau G \arrow[d, "\theta \rtimes_\tau G"] \\ A \rtimes_{\max} G \arrow[r, "\psi_A"] & A \rtimes_\tau G \end{tikzcd} \tag{4.13}$$

commutes in the asymptotic category. Hence by [GHT 2000, Theorem 4.6] the diagram

$$\begin{tikzcd} \Sigma(C_0(X) \rtimes_{\max} G) \otimes \mathcal{K} \arrow[r, equal] \arrow[d, "1 \otimes \theta \rtimes_{\max} G \otimes 1"'] & \Sigma(C_0(X) \rtimes_\tau G) \otimes \mathcal{K} \arrow[d, "1 \otimes \theta \rtimes_\tau G \otimes 1"] \\ \Sigma(A \rtimes_{\max} G) \otimes \mathcal{K} \arrow[r, "1 \otimes \psi_A \otimes 1"] & \Sigma(A \rtimes_\tau G) \otimes \mathcal{K} \end{tikzcd}$$

commutes in the asymptotic category, which says exactly that the diagram in (4.13) commutes in the E-theory category. In other words, as morphisms in the E-theory category,

$$\theta \rtimes_\tau G = \psi_A \circ (\theta \rtimes_{\max} G),$$

whence the morphisms in (4.12) are the same. □

We close the section with the following "two out of three" result, which will be our main tool for proving the $\mathscr{E}$-Baum–Connes conjecture in cases of interest.

Proposition 4.14. *Assume G is a countable discrete group. Let τ be an exact and Morita-compatible crossed product. Let*

$$0 \to I \to A \to B \to 0$$

be a short exact sequence of separable G-C^*-algebras. If the τ-Baum–Connes conjecture is valid with coefficients in two of I, A and B then it is valid with coefficients in the third.

In the case that G is exact (or just K-exact), the analogous result for the usual Baum–Connes conjecture was proved by Chabert and Echterhoff [2001, Proposition 4.2]. However, the result does *not* hold in general for the usual Baum–Connes conjecture due to possible failures of exactness on the right-hand side; indeed, its failure is the reason behind the known counterexamples.

We only prove Proposition 4.14 in the case of a discrete group as this is technically much simpler, and all we need for our results. As pointed out by the referee, one could adapt the proof of [Chabert and Echterhoff 2001, Proposition 4.2] to extend the result to the locally compact case; however, this would necessitate working in KK-theory. We give a direct E-theoretic proof here in order to keep our paper as self-contained as possible.

Before we start the proof, we recall the construction of the boundary map in equivariant E-theory associated to a short exact sequence

$$0 \to I \to A \to B \to 0$$

of G-C^*-algebras. See [GHT 2000, Chapter 5] for more details. Let $\{u_t\}$ be an approximate identity for I that is quasicentral for A and asymptotically G-invariant; such exists by [GHT 2000, Lemma 5.3]. Let $s : B \to A$ be a set-theoretic section. Then there is an asymptotic morphism

$$\sigma : C_0(0,1) \otimes B \to \mathfrak{A}(I) := \frac{C_b([1,\infty), I)}{C_0([1,\infty), I)}$$

which is asymptotic to the map defined on elementary tensors by

$$f \otimes b \mapsto (t \mapsto f(u_t)s(b))$$

(see [GHT 2000, Proposition 5.5]) such that the corresponding class

$$\sigma \in [\![C_0(0,1) \otimes B, I]\!]_G$$

does not depend on the choice of $\{u_t\}$ or s [GHT 2000, Lemma 5.7]. We then set

$$\gamma_I = 1 \otimes \sigma \otimes 1 \in [\![\Sigma(C_0(0,1) \otimes B) \otimes \mathcal{K}_G, \Sigma I \otimes \mathcal{K}_G]\!]_G = E_G(C_0(0,1) \otimes B, I)$$

to be the E-theory class associated to this extension. This construction works precisely analogously in the nonequivariant setting.

Lemma 4.15. *Let G be a countable discrete group. Given a short exact sequence of separable G-C^*-algebras*

$$0 \to I \to A \to B \to 0$$

there is an element $\gamma_I \in E^G(C_0(0,1) \otimes B, I)$ as above. There is also a short exact sequence of C^*-algebras

$$0 \to I \rtimes_\tau G \to A \rtimes_\tau G \to B \rtimes_\tau G \to 0$$

giving rise to $\gamma_{I \rtimes_\tau G} \in E(C_0(0,1) \otimes (B \rtimes_\tau G), I \rtimes_\tau G)$.

The descent functor associated to the τ-crossed product then takes γ_I to $\gamma_{I \rtimes_\tau G}$.

Proof. Identify A with the C^*-subalgebra $\{au_e \mid a \in A\}$ of $A \rtimes_\tau G$, and similarly for B and I. Choose any set-theoretic section $s : B \rtimes_\tau G \to A \rtimes_\tau G$, which we may assume has the property that $s(Bu_g) \subseteq Au_g$ for all $g \in G$. We then have that σ_I is asymptotic to the map

$$f \otimes b \mapsto (t \mapsto f(u_t)s(b)).$$

Checking directly, the image of σ_I under descent agrees with the formula

$$f \otimes \sum_{g \in G} bu_g \mapsto (t \mapsto f(u_t)s(b)u_g) \tag{4.16}$$

on elements of the algebraic tensor product $C_0(0,1) \odot (B \rtimes_{\text{alg}} G)$.

On the other hand, we may use s and $\{u_t\}$ (which identifies with a quasicentral approximate unit for $I \rtimes_\tau G$ under the canonical inclusion $I \to I \rtimes_\tau G$) to define $\sigma_{I \rtimes_\tau G}$, in which case the formula in (4.16) agrees with that for $\sigma_{I \rtimes_\tau G}$ on the dense $*$-subalgebra $(C_0(0,1) \otimes B) \rtimes_{\text{alg}} G$ of $(C_0(0,1) \otimes B) \rtimes_\tau G$. Thus, up to the identification

$$(C_0(0,1) \otimes B) \rtimes_\tau G \cong C_0(0,1) \otimes (B \rtimes_\tau G)$$

from Lemma 4.6, the image of $\sigma \in [\![C_0(0,1) \otimes B, I]\!]_G$ under descent is the same as $\sigma_{I \rtimes_\tau G} \in [\![C_0(0,1) \otimes (B \rtimes_\tau G), I \rtimes_\tau G]\!]$ and we are done. □

Proof of Proposition 4.14. Basic exactness properties of K-theory and exactness of the τ-crossed product give a six-term exact sequence on the right-hand side of the conjecture:

$$\begin{tikzcd} K_0(I \rtimes_\tau G) \arrow[r] & K_0(A \rtimes_\tau G) \arrow[r] & K_0(B \rtimes_\tau G) \arrow[d] \\ K_1(B \rtimes_\tau G) \arrow[u] & K_1(A \rtimes_\tau G) \arrow[l] & K_1(I \rtimes_\tau G) \arrow[l] \end{tikzcd} \tag{4.17}$$

Similarly, basic exactness properties of equivariant E-theory give a six-term sequence on the left-hand side:

$$\begin{tikzcd} K_0^{\mathrm{top}}(G; I) \arrow[r] & K_0^{\mathrm{top}}(G; A) \arrow[r] & K_0^{\mathrm{top}}(G; B) \arrow[d] \\ K_1^{\mathrm{top}}(G; B) \arrow[u] & K_1^{\mathrm{top}}(G; A) \arrow[l] & K_1^{\mathrm{top}}(G; I) \arrow[l] \end{tikzcd} \tag{4.18}$$

The corresponding maps in these diagrams are given by composition with elements of equivariant E-theory groups, and the corresponding descended elements of E-theory groups; for example, the left-hand vertical map in (4.18) is induced by the equivariant asymptotic morphism associated to the original short exact sequence of G-C^*-algebras, and the corresponding map in (4.17) is induced by its descended asymptotic morphism.

Further, the assignments

$$A \mapsto K_*(A \rtimes_\tau G), \quad A \mapsto K_*^{\mathrm{top}}(G; A)$$

define functors from E^G to abelian groups, and functoriality of descent together with associativity of E-theory compositions imply the assembly map is a natural transformation between these functors. Hence assembly induces compatible maps between the six-term exact sequences in (4.17) and (4.18). The result now follows from the five lemma. □

5. Some properties of the minimal exact and Morita-compatible crossed product

In this section, we study a natural class of exact and Morita-compatible crossed products, and use these to deduce some properties of the minimal exact and Morita-compatible crossed product. In particular, we show that the usual property (T) obstructions to surjectivity of the maximal Baum–Connes assembly map do not apply to our reformulated conjecture. We also give a concrete example of a crossed product that could be equal to the minimal one.

Throughout the section, G denotes a locally compact, second-countable group.

Definition 5.1. Let τ be a crossed product, and B a fixed unital G-C^*-algebra. For any G-C^*-algebra A, the *τ-B completion* of $A \rtimes_{\mathrm{alg}} G$, denoted $A \rtimes_{\tau,B} G$, is defined to be the image of the map

$$A \rtimes_\tau G \to (A \otimes_{\max} B) \rtimes_\tau G$$

induced by the equivariant inclusion

$$A \to A \otimes_{\max} B, \quad a \mapsto a \otimes 1.$$

Lemma 5.2. *For any G-C^*-algebra B and crossed product τ, the family of completions $A \rtimes_{\tau,B} G$ defined above is a crossed product functor.*

Proof. To see that $\rtimes_{\tau,B}$ dominates the reduced completion, note that as the τ completion dominates the reduced completion there is a commutative diagram

$$\begin{tikzcd} A \rtimes_{\tau,B} G \arrow[r, dashed] \arrow[d] & A \rtimes_{\mathrm{red}} G \arrow[d] \\ (A \otimes_{\max} B) \rtimes_\tau G \arrow[r] & (A \otimes_{\max} B) \rtimes_{\mathrm{red}} G \end{tikzcd}$$

where the vertical arrows are induced by the equivariant inclusion $a \mapsto a \otimes 1$, and the bottom horizontal arrow is the canonical natural transformation between the τ and reduced crossed products. We need to show the dashed horizontal arrow can be filled in. This follows as equivariant inclusions of G-C^*-algebras induce inclusions of reduced crossed products, whence the right vertical map is injective.

The fact that $\rtimes_{\tau,B}$ is a functor follows as the assignment $A \mapsto A \otimes_{\max} B$ defines a functor from the category of G-C^*-algebras to itself, and the τ-crossed product is a functor. $\square$

From now on, we refer to the construction in Definition 5.1 as the *τ-B-crossed product*.

Lemma 5.3. *Let τ be a crossed product, and B a unital G-C^*-algebra. If the τ-crossed product is Morita-compatible* (*respectively*, *exact*), *then the τ-B-crossed product is Morita-compatible* (*exact*).

Proof. To see Morita compatibility, consider the commutative diagram

$$\begin{tikzcd} (A \otimes \mathcal{K}_G) \rtimes_{\tau,B} G \arrow[rr, dashed] \arrow[d] & & (A \rtimes_{\tau,B} G) \otimes \mathcal{K}_G \arrow[d] \\ ((A \otimes \mathcal{K}_G) \otimes_{\max} B) \rtimes_\tau G \arrow[r, "\cong"] & ((A \otimes_{\max} B) \otimes \mathcal{K}_G) \rtimes_\tau G \arrow[r, "\cong"] & ((A \otimes_{\max} B) \rtimes_\tau G) \otimes \mathcal{K}_G \end{tikzcd}$$

where the left arrow on the bottom row comes from nuclearity of $\mathcal{K}_G$ and associativity of the maximal crossed product; the right arrow on the bottom row is the Morita-compatibility isomorphism; and the vertical arrows are by definition of the

τ-B crossed product. It suffices to show that the dashed arrow exists and is an isomorphism: this follows from the fact that the vertical arrows are injections.

To see exactness, consider a short exact sequence of G-C^*-algebras

$$0 \to I \to A \to Q \to 0$$

and the corresponding commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \rtimes_{\tau,B} G \arrow[r, "\iota"] \arrow[d] & A \rtimes_{\tau,B} G \arrow[r, "\pi"] \arrow[d] & Q \rtimes_{\tau,B} G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (I \otimes_{\max} B) \rtimes_\tau G \arrow[r] & (A \otimes_{\max} B) \rtimes_\tau G \arrow[r] & (Q \otimes_{\max} B) \rtimes_\tau G \arrow[r] & 0 \end{tikzcd}$$

Note that all the vertical maps are injections by definition. Moreover, the bottom row is exact by exactness of the maximal tensor product, and the assumed exactness of the τ-crossed product. The only issue is thus to show that the kernel of π is equal to the image of ι.

The kernel of π is $A \rtimes_{\tau,B} G \cap (I \otimes_{\max} B) \rtimes_\tau G$, so we must show that this is equal to $I \rtimes_{\tau,B} G$. The inclusion

$$I \rtimes_{\tau,B} G \subseteq A \rtimes_{\tau,B} G \cap (I \otimes_{\max} B) \rtimes_\tau G$$

is automatic, so it remains to show the reverse inclusion. Let x be an element of the right-hand side. Let $\{u_i\}$ be an approximate identity for I, and note that $\{v_i\} := \{u_i \otimes 1\}$ can be thought of as a net in the multiplier algebra of $(I \otimes_{\max} B) \rtimes_\tau G$ via Lemma 4.4. The net $\{v_i\}$ is an "approximate identity" in the sense that $v_i y$ converges to y for all $y \in (I \otimes_{\max} B) \rtimes_\tau G$. Let $\{x_i\}$ be a (bounded) net in $A \rtimes_{\mathrm{alg}} G$ converging to x in the $A \rtimes_{\tau,B} G$ norm, which we may assume has the same index set as $\{v_i\}$. Note that

$$\|v_i x_i - x\| \le \|v_i x_i - v_i x\| + \|v_i x - x\| \le \|v_i\| \|x_i - x\| + \|v_i x - x\|,$$

which tends to zero as i tends to infinity. Note, however, that $v_i x_i$ belongs to $I \rtimes_{\mathrm{alg}} G$ (considered as a $*$-subalgebra of $(I \otimes_{\max} B) \rtimes_\tau G$), so we are done. □

We now specialize to the case when τ is $\mathscr{E}$, the minimal exact crossed product.

Corollary 5.4. *For any unital G-C^*-algebra B, the $\mathscr{E}$-crossed product and $\mathscr{E}$-B-crossed product are equal.*

Proof. It is immediate from the definition that the $\mathscr{E}$-B-crossed product is no larger than the $\mathscr{E}$-crossed product. Lemma 5.3 implies that the $\mathscr{E}$-B-crossed product is exact and Morita-compatible, however, so they are equal by minimality of the $\mathscr{E}$-crossed product. □

Corollary 5.5. *For any unital G-C^*-algebra B and any G-C^*-algebra A, the map*

$$A \rtimes_{\mathscr{E}} G \to (A \otimes_{\max} B) \rtimes_{\mathscr{E}} G$$

induced by the inclusion $a \mapsto a \otimes 1$ is injective.

Proof. The image of the map $A \rtimes_{\mathscr{E}} G$ is (by definition) equal to $A \rtimes_{\mathscr{E},B} G$, so this is immediate from Corollary 5.4. □

The following result implies that the usual property (T) obstructions to surjectivity of the maximal Baum–Connes assembly map do not apply to the $\mathscr{E}$-Baum–Connes conjecture: see Corollary 5.7 below. The proof is inspired by [Brown and Ozawa 2008, Proof of Theorem 2.6.8, part (7) ⇒ (1)].

Proposition 5.6. *Say the C^*-algebra $C^*_{\mathscr{E}}(G) := \mathbb{C} \rtimes_{\mathscr{E}} G$ admits a nonzero finite-dimensional representation. Then G is amenable.*

Proof. Let $C_{\mathrm{ub}}(G)$ denote the C^*-algebra of bounded, (left) uniformly continuous functions on G, and let α denote the (left) action of G on this C^*-algebra, which is a continuous action by $*$-automorphisms. It will suffice (compare [Bekka et al. 2008, §G.1]) to show that if $C^*_{\mathscr{E}}(G)$ has a nonzero finite-dimensional representation then there exists an *invariant mean* on $C_{\mathrm{ub}}(G)$: a state ϕ on $C_{\mathrm{ub}}(G)$ such that $\phi(\alpha_g(f)) = \phi(f)$ for all $g \in G$ and $f \in C_{\mathrm{ub}}(G)$.

Assume then there is a nonzero representation $\pi : C^*_{\mathscr{E}}(G) \to \mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is finite-dimensional. Passing to a subrepresentation, we may assume π is nondegenerate, whence it comes from a unitary representation of G, which we also denote π. Applying Corollary 5.5 to the special case $A = \mathbb{C}$, $B = C_{\mathrm{ub}}(G)$, we have that $C^*_{\mathscr{E}}(G)$ identifies canonically with a C^*-subalgebra of $C_{\mathrm{ub}}(G) \rtimes_{\mathscr{E}} G$. Hence by Arveson's extension theorem (in the finite-dimensional case — see [Brown and Ozawa 2008, Corollary 1.5.16]) there exists a contractive completely positive map

$$\rho : C_{\mathrm{ub}}(G) \rtimes_{\mathscr{E}} G \to \mathscr{B}(\mathscr{H})$$

extending π. As π is nondegenerate, ρ is, whence [Lance 1995, Corollary 5.7] it extends to a strictly continuous unital completely positive map on the multiplier algebra, which we denote

$$\rho : \mathscr{M}(C_{\mathrm{ub}}(G) \rtimes_{\mathscr{E}} G) \to \mathscr{B}(\mathscr{H}).$$

Now, since π is a representation, the C^*-subalgebra $C^*_{\mathscr{E}}(G)$ of $\mathscr{M}(C_{\mathrm{ub}}(G) \rtimes_{\mathscr{E}} G)$ is in the multiplicative domain of ρ (compare [Brown and Ozawa 2008, p. 12]). Note that the image of G inside $\mathscr{M}(C_{\mathrm{ub}}(G) \rtimes_{\max} G)$ is in the strict closure of the $*$-subalgebra $C_c(G)$, whence the same is true in the image of G in $\mathscr{M}(C_{\mathrm{ub}}(G) \rtimes_{\mathscr{E}} G)$ given by Lemma 4.4; it follows from this and strict continuity of ρ that the image of G in $\mathcal{M}(C_{\mathrm{ub}}(G) \rtimes_{\mathscr{E}} G)$ is also in the multiplicative domain of ρ. Hence, for any $g \in G$ and $f \in C_{\mathrm{ub}}(G)$,

$$\rho(\alpha_g(f)) = \rho(u_g f u_g^*) = \pi(g)\rho(f)\pi(g)^*.$$

It follows that, if $\tau : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is the canonical tracial state, then $\tau \circ \rho$ is an invariant mean on $C_{\mathrm{ub}}(G)$, so G is amenable. □

We now discuss the relevance of this proposition to the property (T) obstructions to the maximal Baum–Connes conjecture. Recall that if G is a group with property (T) then for any finite-dimensional unitary representation π of G (for example, the trivial representation), there is a central *Kazhdan projection* p_π in $C^*_{\max}(G)$ that maps to the orthogonal projection onto the π-isotypical component in any unitary representation of G. When G is infinite and discrete,[3] it is well-known [Higson 1998, Discussion below 5.1] that the class of p_π in $K_0(C^*_{\max}(G))$ is not in the image of the maximal Baum–Connes assembly map. Thus, at least for infinite discrete groups, the projections p_π obstruct the maximal version of the Baum–Connes conjecture.

The following corollary, which is immediate from the above proposition, shows that these obstructions do not apply to the $\mathscr{E}$-crossed product.

Corollary 5.7. *Let G be a group with property (T), and π be a finite-dimensional representation of G. Write $C^*_{\mathscr{E}}(G) := \mathbb{C} \rtimes_{\mathscr{E}} G$. Then the canonical quotient map $C^*_{\max}(G) \to C^*_{\mathscr{E}}(G)$ sends p_π to zero.* □

Finally in this section, we specialize to the case of discrete groups and look at the particular example of the max-$l^\infty(G)$-crossed product. We show below that this crossed product is actually equal to the reduced one when G is exact. It is thus possible that the max-$l^\infty(G)$-crossed product actually is the $\mathscr{E}$-crossed product. As further evidence in this direction, note that for any *commutative* unital B, there is a unital equivariant map from B to $l^\infty(G)$ by restriction to any orbit. This shows that the max-$l^\infty(G)$-crossed product is the greatest lower bound of the max-B-crossed products as B ranges over commutative unital C^*-algebras. We do not know what happens when B is noncommutative: quite plausibly here one can get something strictly smaller. Of course, there could also be many other constructions of exact and Morita-compatible crossed products that do not arise as above.

Proposition 5.8. *Say G is exact. Then the* max-$l^\infty(G)$*-crossed product equals the reduced crossed product.*

Proof. Let A be a G-C^*-algebra. We will show that

$$(A \otimes l^\infty(G)) \rtimes_{\max} G = (A \otimes l^\infty(G)) \rtimes_{\mathrm{red}} G,$$

[3] It is suspected that this is true in general, but we do not know of a proof in the literature.

which will suffice to complete the proof. The main result of [Ozawa 2000] (compare also [Guentner and Kaminker 2002]) shows that the action of G on its Stone–Čech compactification βG is amenable. However, the Stone–Čech compactification of G is the spectrum of $l^\infty(G)$ and $A \otimes l^\infty(G)$ is a G-$l^\infty(G)$ algebra in the sense of [Anantharaman-Delaroche 2002, Definition 5.2], so [ibid., Theorem 5.3] (see also [Brown and Ozawa 2008, Theorem 4.4.3] for a slightly easier proof specific to the case that G is discrete) implies the desired result. □

We suspect a similar result holds for a general locally compact group (with $C_{\mathrm{ub}}(G)$ replacing $l^\infty(G)$). To adapt the proof above, one would need an analog of the equivalence of exactness and amenability of the action of G on the spectrum of $l^\infty(G)$ to hold in the nondiscrete case; this seems likely, but it does not appear to be known at present.

6. Proving the conjecture

In this section, we consider conditions under which the Baum–Connes conjecture with coefficients in a G-C^*-algebra A is true for exact and Morita-compatible crossed products and, in particular, when the $\mathscr{E}$-Baum–Connes conjecture is true. This is certainly the case when G is exact and the usual Baum–Connes conjecture for G with coefficients in A is valid. However, we are interested in the *nonexact* Gromov monster groups. We shall study actions of these groups with the Haagerup property as in the following definition (adapted from [Tu 1999, §3]).

Definition 6.1. Let G be a locally compact group acting on the right on a locally compact Hausdorff topological space X. A function $h : X \times G \to \mathbb{R}$ is of *conditionally negative type* if it satisfies the following conditions:

(i) the restriction of h to $X \times \{e\}$ is zero;

(ii) for every $x \in X$, $g \in G$, we have that $h(x, g) = h(xg, g^{-1})$;

(iii) for every x in X and any finite subsets $\{g_1, \dots, g_n\}$ of G and $\{t_1, \dots, t_n\}$ of $\mathbb{R}$ such that $\sum_i t_i = 0$ we have that

$$\sum_{i,j=1}^{n} t_i t_j h(xg_i, g_i^{-1} g_j) \le 0.$$

The action of G on X is *a-T-menable* if there exists a continuous conditionally negative type function h that is *locally proper*: for any compact $K \subseteq X$ the restriction of h to the set

$$\{(x, g) \in X \times G \mid x \in K, xg \in K\}$$

is a proper function.

In the precise form stated, the following theorem is essentially due to Tu [1999]. See also [Higson and Guentner 2004, Theorem 3.12; Higson 2000, Theorem 3.4; Yu 2000, Theorem 1.1] for closely related results.

Theorem 6.2. *Let G be a second-countable locally compact group acting a-T-menably on a second-countable locally compact space X. The τ-Baum–Connes assembly map*

$$K_*^{\mathrm{top}}(G; C_0(X)) \to K_*(C_0(X) \rtimes_\tau G)$$

is an isomorphism for every exact and Morita-compatible crossed product τ.

Proof. In the terminology of [Tu 1999, §3], Definition 6.1 says that the transformation groupoid $X \rtimes G$ admits a locally proper, negative type function, and therefore by [ibid., Proposition 3.8] acts properly by isometries on a field of Hilbert spaces. It then follows from [ibid., Théorème 1.1] and the discussion in [ibid., last paragraph of introduction] that there exists a proper $X \rtimes G$-algebra[4] $\mathscr{A}$ built from this action on a field of Hilbert spaces and equivariant E-theory elements

$$\alpha \in E^G(\mathscr{A}, C_0(X)), \quad \beta \in E^G(C_0(X), \mathscr{A})$$

such that

$$\alpha \circ \beta = 1 \quad \text{in } E^G(C_0(X), C_0(X)). \tag{6.3}$$

(Actually, Tu works in the framework of equivariant KK-theory in the reference used above. Using the natural transformation to equivariant E-theory, we obtain the result as stated here.)

Consider now the following diagram, where the vertical maps are induced by α, β above, E-theory compositions, and the descent functor from Theorem 4.9; and the horizontal maps are assembly maps:

$$\begin{tikzcd}
K_*^{\mathrm{top}}(G; C_0(X)) \arrow[r] \arrow[d, "\beta_*"] & K_*(C_0(X) \rtimes_\tau G) \arrow[d, "\beta_*"] \\
K_*^{\mathrm{top}}(G; \mathscr{A}) \arrow[r] \arrow[d, "\alpha_*"] & K_*(\mathscr{A} \rtimes_\tau G) \arrow[d, "\alpha_*"] \\
K_*^{\mathrm{top}}(G; C_0(X)) \arrow[r] & K_*(C_0(X) \rtimes_\tau G)
\end{tikzcd}$$

The diagram commutes as descent is a functor and E-theory compositions are associative. Moreover, the vertical compositions are isomorphisms by (6.3). Further, all crossed products are the same for a proper action, whence the central horizontal

[4] Precisely, this means that there is a locally compact proper G-space Z, an equivariant $*$-homomorphism from $C_0(Z)$ into the center of the multiplier algebra of A, and an equivariant, open, and continuous map $Z \to X$.

map identifies with the usual assembly map, and so is an isomorphism by [Chabert et al. 2001, Théorème 2.2]. Hence from a diagram chase the top and bottom maps are isomorphisms, which is the desired result. □

Remark 6.4. The Baum–Connes conjecture with coefficients is true for a-T-menable groups when defined with either the maximal or reduced crossed product [Higson and Kasparov 2001]. The argument above shows that this extends to any exact and Morita-compatible crossed product.

Based on this remark, it may be tempting to believe that for a-T-menable groups the Baum–Connes conjecture is true with values in *any* "intermediate completion" of the algebraic crossed product $A \rtimes_{\mathrm{alg}} G$. This is false (even if $A = \mathbb{C}$), as the following example shows.

Example 6.5. Let G be an a-T-menable group that is not amenable, for example a free group or $\mathrm{SL}(2, \mathbb{R})$. Let $C^*_S(G)$ denote the completion of $C_c(G)$ in the direct sum $\lambda \oplus 1$ of the regular and trivial representations.[5]

As G is not amenable the trivial representation is isolated in the spectrum of $C^*_S(G)$, whence this C^*-algebra splits as a direct sum

$$C^*_S(G) = C^*_{\mathrm{red}}(G) \oplus \mathbb{C}.$$

Let $p \in C^*_S(G)$ denote the unit of the copy of $\mathbb{C}$ in this decomposition, a so-called *Kazhdan projection*. The class $[p] \in K_0(C^*_r(G))$ generates a copy of $\mathbb{Z}$, which is precisely the kernel of the map on K-theory induced by the quotient map $C^*_S(G) \to C^*_{\mathrm{red}}(G)$.

The Baum–Connes conjecture is true for G by a-T-menability, whence $[p]$ is not in the image of the Baum–Connes assembly map

$$\mu : K^{\mathrm{top}}_*(G) \to K_*(C^*_S(G)),$$

and so the assembly map is not surjective. The discussion in Examples A.15 develops this a little further.

7. An example coming from Gromov monster groups

A Gromov monster group G is a discrete group whose Cayley graph contains an expanding sequence of graphs (an *expander*), in some weak sense. The geometric properties of expanders can be used to build a commutative G-C^*-algebra A for which the Baum–Connes conjecture with coefficients fails. In fact, Gromov monster groups are the only known source of such failures.

[5] $C^*_S(G)$ is the *Brown–Guentner crossed product* $\mathbb{C} \rtimes_{\mathrm{BG},S} G$ associated to the subset $S = \widehat{G}_r \cup \{1\}$ of the unitary dual: see the Appendix.

In this section we show that for some Gromov monster groups there is a separable commutative G-C^*-algebra B for which the $\mathscr{E}$-Baum–Connes conjecture is true, but the usual version using the reduced crossed product is false. The existence of such a B can be attributed to two properties: failure of exactness, and the presence of a-T-menability. The main result of this section is Theorem 7.9, which proves a-T-menability of a certain action.

The ideas in this section draw on many sources. The existence of Gromov monster groups was indicated by Gromov [2003]. More details were subsequently provided by Arzhantseva and Delzant [2008], and Coulon [2014]. The version of the construction we use in this paper is due to Osajda [2014]. The idea of using Gromov monsters to construct counterexamples to the Baum–Connes conjecture is due to Higson, Lafforgue and Skandalis [Higson et al. 2002, §7]. The construction of counterexamples we use in this section comes from [Willett and Yu 2012a, §8; 2012b]. The present exposition is inspired by subsequent work of Finn-Sell and Wright [2014], of Chen, Wang and Yu [Chen et al. 2013], and of Finn-Sell [2014a]. Note also that Finn-Sell [2014b] has obtained analogs of Theorem 7.9 below using a different method.

In order to discuss a-T-menability, we will be interested in kernels with the properties in the next definition.

Definition 7.1. Let X be a set, and $k : X \times X \to \mathbb{R}_+$ a function (a *kernel*).

The kernel k is *conditionally negative definite* if

(i) $k(x,x) = 0$, for every $x \in X$;

(ii) $k(x,y) = k(y,x)$, for every $x,\ y \in X$;

(iii) for every subset $\{x_1, \dots, x_n\}$ of X and every subset $\{t_1, \dots, t_n\}$ of $\mathbb{R}$ such that $\sum_{i=1}^n t_i = 0$ we have

$$\sum_{i,j=1}^{n} t_i t_j k(x_i, x_j) \le 0.$$

Assume now that X is a metric space. The kernel k is *asymptotically conditionally negative definite* if conditions (i) and (ii) above hold, and the following weak version of condition (iii) holds:

(iii)′ for every $r > 0$ there exists a bounded subset $K = K(r)$ of X such that for every subset $\{x_1, \dots, x_n\}$ of $X \setminus K$ of diameter at most r and every subset $\{t_1, \dots, t_n\}$ of $\mathbb{R}$ such that $\sum_{i=1}^n t_i = 0$ we have

$$\sum_{i,j=1}^{n} t_i t_j k(x_i, x_j) \le 0.$$

Continuing to assume that X is a metric space, a kernel k is *proper* if

$$\sup_{d(x,y)\leq r} k(x,y)$$

is finite for each $r > 0$ and if

$$\inf_{d(x,y)\geq r} k(x,y)$$

tends to infinity as r tends to infinity.

Remark 7.2. Using techniques similar to those in [Finn-Sell 2014a] (compare also [Willett 2015]), one can show that if X admits a *fibered coarse embedding into Hilbert space* as in [Chen et al. 2013, §2], then X admits a proper, asymptotically conditionally negative definite kernel. One can also show directly that if X admits a proper, asymptotically conditionally negative definite kernel, then the restriction to the boundary of the coarse groupoid of X has the Haagerup property as studied in [Finn-Sell and Wright 2014]. We will not need these properties, however, so do not pursue this further here.

Let now X and Y be metric spaces. A map $f : X \to Y$ is a *coarse embedding* if there exist nondecreasing functions ρ_- and ρ_+ from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that for all $x_1, x_2 \in X$,

$$\rho_-(d(x_1,x_2)) \leq d(f(x_1), f(x_2)) \leq \rho_+(d(x_1,x_2))$$

and such that $\rho_-(t)$ tends to infinity as t tends to infinity. A coarse embedding $f : X \to Y$ is a *coarse equivalence* if in addition there exists $C \geq 0$ such that every point of Y is distance at most C from a point of $f(X)$. Coarse equivalences have "approximate inverses": given a coarse equivalence $f : X \to Y$ there is a coarse equivalence $g : Y \to X$ such that $\sup_{x\in X} d(x, g(f(x)))$ and $\sup_{y\in Y} d(y, f(g(y)))$ are finite.

We record the following lemma for later use; the proof is a series of routine checks.

Lemma 7.3. *Let X and Y be metric spaces, and $f : X \to Y$ a coarse embedding. If k is a proper, asymptotically conditionally negative definite kernel on Y, then the pullback kernel $(f^*k)(x,y) := k(f(x), f(y))$ on X is proper and asymptotically conditionally negative definite.* □

We are mainly interested in metric spaces that are built from graphs. We identify a finite graph with its vertex set, and equip it with the edge metric: the distance between vertices x and y is the smallest number n for which there exists a sequence

$$x = x_0, x_1, \ldots, x_n = y$$

in which consecutive pairs span an edge.

Definition 7.4. Let (X_n) be a sequence of finite graphs such that

(i) each X_n is nonempty, finite, and connected;

(ii) there exists a D such that all vertices have degree at most D.

Equip the disjoint union $X = \bigsqcup_n X_n$ with a metric that restricts to the edge metric on each X_n and in addition satisfies

$$d\big(X_n, \textstyle\bigsqcup_{n \neq m} X_m\big) \to \infty \quad \text{as } n \to \infty.$$

The exact choice of metric does not matter for us: the identity map on X is a coarse equivalence between any two choices of metric satisfying these conditions. The metric space X is the *box space* associated to the sequence (X_n).

The *girth* of a graph X is the length of the shortest nontrivial cycle in X, and infinity if no nontrivial cycles exist. A box space X built from a sequence (X_n) as above has *large girth* if the girth of X_n tends to infinity as n tends to infinity.

A box space X associated to a sequence (X_n) is an *expander* if there exists $c > 1$ such that for all n and all subsets A of X_n with $|A| \leq |X_n|/2$, we have

$$\frac{\big|\{x \in X_n \mid d(x, A) \leq 1\}\big|}{|A|} \geq c.$$

Theorem 7.5. *Let X be a large-girth box space as in Definition 7.4. Then the distance function on X is a proper, asymptotically conditionally negative definite kernel.*

For the proof of this theorem, we shall require the following well-known lemma [Julg and Valette 1984, §2]. For convenience, we sketch a proof.

Lemma 7.6. *Let T be (the vertex set of) a tree. The edge metric is conditionally negative definite when viewed as a kernel $d : T \times T \to \mathbb{R}_+$.*

Proof. Let ℓ^2 denote the Hilbert space of square summable functions on the set of *edges* in T. Fix a base vertex x_0. For every vertex x let $\xi(x)$ be the characteristic function of those edges along the unique no-backtrack path from x_0 to x. The result is a routine calculation starting from the observation that

$$\|\xi(x) - \xi(y)\|^2 = d(x, y)$$

for every two vertices x and y. □

Proof of Theorem 7.5. Let $k(x, y) = d(x, y)$. Properness and conditions (i) and (ii) from the definition of asymptotically conditionally negative definite are trivially satisfied, so it remains to check condition (iii)$'$.

Given $r > 0$, let N be large enough that the following conditions are satisfied:

(a) if $n > N$ then $d\big(X_n, \bigsqcup_{m \neq n} X_m\big) > r$;

(b) if $n > N$ then the girth of X_n is at least $2r$.

The force of (b) is that if T_n is the universal cover of X_n then the covering map $T_n \to X_n$ is an isometry on sets of diameter r or less. Let $K = X_1 \sqcup \cdots \sqcup X_N$. It now suffices to show that d is conditionally negative definite when restricted to a finite subset F of $X \setminus K$ of diameter at most r. But such a subset necessarily belongs entirely to some X_n, and the covering map $T_n \to X_n$ admits an isometric splitting over F. Thus, restricted to $F \times F$, the metric d is the pullback of the distance function on T_n, which is conditionally negative definite by the previous lemma. $\square$

Let G be a finitely generated group. Fix a word length ℓ and associated left-invariant metric on G; the following definition is independent of the choice of length function.

Definition 7.7. The group G is a *special Gromov monster* if there exists a large-girth expander box space X as in Definition 7.4 and a coarse embedding from X to G.

Osajda [2014] has shown that special Gromov monsters in the sense above exist: in fact, he proves the existence of examples where the (large-girth, expander) box space X is isometrically embedded. Other constructions of Gromov monster groups, including Gromov's original one, produce maps of (expander, large-girth) box spaces into groups which are not (obviously) coarse embeddings: see the remarks in Section 8.4 below. The restriction to coarsely embedded box spaces is the reason for the terminology "special Gromov monster" above.

For the remainder of this section, let G be a special Gromov monster group, and let $f : X \to G$ be a coarse embedding of a large-girth, expander box space into G. Let $Z = f(X) \subset G$ be the image of f. For each natural number R, let $N_R(Z)$ be the R-neighborhood of Z in G.

Lemma 7.8. *There exists a kernel k on G such that for any $R \in \mathbb{N}$ the restriction of k to $N_R(Z)$ is proper and asymptotically conditionally negative definite.*

Proof. Let $p_0 : Z \to Z$ be the identity map. For $R \in \mathbb{N}$ inductively choose $p_R : N_R(Z) \to Z$ by stipulating that $p_{R+1} : N_{R+1}(Z) \to Z$ extends p_R, and satisfies $d(p_{R+1}(x), x) \leq R + 1$ for all $x \in N_{R+1}(Z)$. Note that each p_R is a coarse equivalence. Let $g : Z \to X$ be any choice of coarse equivalence, and let d be the distance function on X, so d has the properties in Theorem 7.5.

For each R, let k_R be the pullback kernel $(g \circ p_R)^* d$, which Lemma 7.3 implies is proper and asymptotically conditionally negative definite. The choice of the functions p_R implies that for $R > S$ the kernel k_R extends k_S, and so these functions piece together to define a kernel k on $\bigcup_R N_R(Z) = G$. $\square$

We will now construct an a-T-menable action of G.

For each natural number R, let $\overline{N_R(Z)}$ be the closure of $N_R(Z)$ in the Stone–Čech compactification βG of G. Let

$$Y = \left(\bigcup_{R \in \mathbb{N}} \overline{N_R(Z)} \right) \cap \partial G,$$

i.e., Y is the intersection of the open subset $\bigcup_{R\in\mathbb{N}} \overline{N_R(Z)} \subset \beta G$ with the Stone–Čech corona ∂G.

Next we define an action of G on Y. This is best done by considering the associated C^*-algebras of continuous functions. The C^*-algebra of continuous functions on $\bigcup_{R\in\mathbb{N}} \overline{N_R(Z)}$ naturally identifies with

$$A = \overline{\bigcup_{R \in \mathbb{N}} \ell^\infty(N_R(Z))},$$

the C^*-subalgebra of $\ell^\infty(G)$ generated by all the bounded functions on the R-neighborhoods of Z. For every x and g in G we have

$$d(x, xg) = \ell(g),$$

so that the right action of G on itself gives rise to an action on $\ell^\infty(G)$ that preserves A. In this way A is a G-C^*-algebra. Note that A contains $C_0(G)$ as a G-invariant ideal, and Y identifies naturally with the maximal ideal space of the G-C^*-algebra $A/C_0(G)$.

Theorem 7.9. *The action of G on Y is a-T-menable.*

Proof. Let k be as in Lemma 7.8. Say g is an element of G and y is an element of Y, so contained in some $\overline{N_R(Z)}$. Note that the set $\{k(x, xg)\}_{x \in N_R(Z)}$ is bounded by properness of the restriction of k to $N_{R+\ell(g)}(Z)$. Hence, thinking of y as an ultrafilter on $N_R(Z)$, we may define

$$h(y, g) = \lim_y k(x, xg).$$

This definition does not depend on the choice of R. We claim that the function

$$h : Y \times G \to \mathbb{R}_+$$

thus defined has the properties from Definition 6.1.

Indeed, condition (i) follows as

$$h(y, e) = \lim_y k(x, x) = 0$$

for any y. For condition (ii), note that

$$h(y, g) = \lim_y k(x, xg) = \lim_y k(xg, x) = h(xg, g^{-1}).$$

For condition (iii), let y be fixed, $\{g_1, \ldots, g_n\}$ be a subset of G and $\{t_1, \ldots, t_n\}$ a subset of $\mathbb{R}$ such that $\sum t_i = 0$. Then

$$\sum_{i,j=1}^{n} t_i t_j h(y g_i, g_i^{-1} g_j) = \lim_y \sum_{i,j=1}^{n} k(x g_i, x g_i g_i^{-1} g_j) = \lim_y \sum_{i,j=1}^{n} k(x g_i, x g_j).$$

Let r be larger than the diameter of $\{x g_1, \ldots, x g_n\}$, and note that removing the finite set $K(r)$ as in the definition of asymptotic conditionally negative definite kernel from $N_R(Z)$ does not affect the ultralimit $\lim_y \sum k(x g_i, x g_j)$. We may thus think of this as an ultralimit over a set of nonpositive numbers, and thus nonpositive.

Finally, we check local properness. Let K be a compact subset of Y. As $\{\overline{N_R(Z)} \cap Y \mid R \in \mathbb{N}\}$ is an open cover of Y, the set K must be contained in some $\overline{N_R(Z)}$. Assume that y and yg are both in K. Choose any net (x_i) in $N_R(Z)$ converging to y and, passing to a subnet, assume that the elements $x_i g$ are all contained in $N_R(Z)$. Passing to another subnet, assume that $\lim_i k(x_i, x_i g)$ exists. We then have that

$$\begin{aligned} h(y, g) &= \lim_y k(x, xg) = \lim_i k(x_i, x_i g) \\ &\geq \inf\{k(x, y) \mid x, y \in N_R(Z),\ d(x, y) \geq \ell(g)\}, \end{aligned}$$

which tends to infinity as $\ell(g)$ tends to infinity (at a rate depending only on R, whence only on K) by properness of the restriction of k to $N_R(Z)$. This completes the proof. $\square$

We are now ready to produce our example of a C^*-algebra B for which the usual Baum–Connes assembly map

$$\mu : K_*^{\mathrm{top}}(G; B) \to K_*(B \rtimes_{\mathrm{red}} G)$$

fails to be surjective, but for which the $\mathscr{E}$-Baum–Connes assembly map

$$\mu : K_*^{\mathrm{top}}(G; B) \to K_*(B \rtimes_{\mathscr{E}} G)$$

is an isomorphism.

Assume that G is a special Gromov monster group. Then there exists a *Kazhdan projection* p in some matrix algebra $M_n(A \rtimes_{\mathrm{red}} G)$ over $A \rtimes_{\mathrm{red}} G$ such that the corresponding class $[p] \in K_0(A \rtimes_{\mathrm{red}} G)$ is not in the image of the assembly map: see [Willett and Yu 2012a, §8]. We may write

$$p = \lim_{n \to \infty} \sum_{g \in F_n} \sum_{i,j=1}^{n} f_{ijg}^{(n)} e_{ij}[g],$$

where F_n is a finite subset of G, $\{e_{ij}\}_{i,j=1}^n$ are the standard matrix units for $M_n(\mathbb{C})$, and each $f_{gij}^{(n)}$ is an element of A.

Let $h : Y \times G \to \mathbb{R}_+$ be a function as in Definition 6.1, and let $C_0(W)$ be the C^*-subalgebra of $C_0(Y)$ generated by the countably many functions $\{x \mapsto h(x,g)\}_{g\in G}$, the restriction of the countably many functions $f_{gij}^{(n)}$ to Y, and all translates of these elements by G. Let B be the preimage of $C_0(W)$ in A. Then the following hold (compare [Higson and Guentner 2004, Lemma 4.2]):

(i) B is separable;

(ii) the action of G on W is a-T-menable;

(iii) the Kazhdan projection is contained in a matrix algebra over the reduced crossed product $B \rtimes_{\mathrm{red}} G$.

Corollary 7.10. *The $\mathscr{E}$-Baum–Connes assembly map with coefficients in the algebra B is an isomorphism. On the other hand, the usual Baum–Connes assembly map for G with coefficients in B is not surjective.*

Proof. The C^*-algebra B sits in a G-equivariant short exact sequence of the form

$$0 \to C_0(G) \to B \to C_0(W) \to 0.$$

The action of G on the space W is a-T-menable, so the $\mathscr{E}$-Baum–Connes conjecture with coefficients in $C_0(W)$ is true by Theorem 6.2. The $\mathscr{E}$-Baum–Connes conjecture with coefficients in $C_0(G)$ is true by properness of this algebra (which also forces $C_0(G) \rtimes_{\mathscr{E}} G = C_0(G) \rtimes_{\mathrm{red}} G$). The result for the $\mathscr{E}$-Baum–Connes conjecture now follows from Proposition 4.14.

On the other hand, the results of [Willett and Yu 2012a] show that the class $[p] \in K_0(A \rtimes_{\mathrm{red}} G)$ is not in the image of the assembly map; by naturality of the assembly map in the coefficient algebra, the corresponding class $[p] \in K_0(B \rtimes_{\mathrm{red}} G)$ is not in the image of the assembly map either. □

Remark 7.11. It seems very likely that an analogous statement holds for A itself. However, here we pass to a separable C^*-subalgebra to avoid technicalities that arise in the nonseparable case.

8. Concluding remarks and questions

8.1. ***The role of exactness.*** Given the current state of knowledge on exactness and the Baum–Connes conjecture, we do not know which of the following (vague) statements is closer to the truth:

(i) Failures of exactness are the fundamental reason for failure of the Baum–Connes conjecture (with coefficients, for groups).

(ii) Failures of exactness are a convenient way to detect counterexamples to the Baum–Connes conjecture, but counterexamples arise for more fundamental reasons.

The statement that the $\mathscr{E}$-Baum–Connes conjecture is true is a precise version of statement (i), and the material in this paper provides some evidence for its validity. Playing devil's advocate, we outline some evidence for statement (ii) below.

8.1.1. *Groupoid counterexamples.* As well as the counterexamples to the Baum–Connes conjecture with coefficients for groups that we have discussed, Higson, Lafforgue and Skandalis [Higson et al. 2002] also use failures of exactness to produce counterexamples to the Baum–Connes conjecture for *groupoids*.

One can use the precise analog of Definition 2.1 to define general groupoid crossed products, and then for a particular crossed product τ define the τ-Baum–Connes assembly map as the composition of the maximal groupoid Baum–Connes assembly map and the map on K-theory induced by the quotient map from the maximal crossed product to the τ-crossed product. It seems (we did not check all the details) that the program of this paper can also be carried out in this context: there is a minimal groupoid crossed product with good properties, and one can reformulate the groupoid Baum–Connes conjecture with coefficients accordingly. The work of Popescu [2004] on groupoid-equivariant E-theory is relevant here.

However, in the case of groupoids this method will *not* obviate all known counterexamples. In fact, the following result is not difficult to extrapolate from [Higson et al. 2002, §2, first counterexample]. For any groupoid G and groupoid crossed product τ, let $C^*_\tau(G)$ denote $C_0(G^{(0)}) \rtimes_\tau G$, a completion of the groupoid convolution algebra $C_c(G)$.

Proposition. *There exists a* (*locally compact*, *Hausdorff*, *second-countable*, *étale*) *groupoid G such that for any groupoid crossed product τ*, *there exists a projection $p_\tau \in C^*_\tau(G)$ whose K-theory class is not in the image of the τ-assembly map.*

Proof. Let Γ_∞ be the discrete group $\mathrm{SL}(3,\mathbb{Z})$ and for each n let $\Gamma_n = \mathrm{SL}(3,\mathbb{Z}/n\mathbb{Z})$ and let $\pi_n : \Gamma_\infty \to \Gamma_n$ be the obvious quotient map. In [Higson et al. 2002, §2], the authors show how to construct a locally compact, Hausdorff second-countable groupoid G out of this data: roughly, the base space of G is $\mathbb{N} \cup \{\infty\}$, and G is the bundle of groups with Γ_n over the point n in $\mathbb{N} \cup \{\infty\}$.

As explained in [Higson et al. 2002, §2, first counterexample], there is a projection p_{red} in $C^*_{\mathrm{red}}(G)$ whose K-theory class is not in the image of the reduced assembly map; roughly, p_{red} exists since the trivial representation of $\mathrm{SL}(3,\mathbb{Z})$ is isolated among the congruence representations. However, as $\mathrm{SL}(3,\mathbb{Z})$ has property (T), the trivial representation is isolated among *all* unitary representations of this group, and therefore there is a projection $p_{\max}$ in $C^*_{\max}(G)$ that maps to p_{red} under the canonical quotient map. Let p_τ denote the image of $p_{\max}$ under the canonical quotient map from the maximal crossed product to the τ-crossed product. As the reduced assembly map factors through the τ-assembly map, the fact that the class of p_{red} is not in the image of the reduced assembly map implies that the class of p_τ is not in the image of the τ-assembly map. □

For groupoids, then, statement (ii) above seems the more reasonable one. Having said this, we think the methods of this paper can be used to obviate some of the other groupoid counterexamples in [Higson et al. 2002], and it is natural to try to describe the groupoids for which this can be done. This question seems interesting in its own right, and it might also suggest phenomena that could occur in the less directly accessible group case.

8.1.2. *Geometric property (T) for expanders.* As mentioned above, all current evidence suggests that statement (i) above might be the correct one for groups and group actions. It is crucial here that the only expanders anyone knows how to coarsely embed into a group are those with "large girth", as we exploited in Section 7.

Yu and the third author [2012b, §7; 2014] studied a property of expanders called *geometric property (T)*, which is a strong negation of the Haagerup-type properties used in Section 7. Say G is a group containing a coarsely embedded expander with geometric property (T) (it is not known whether such a group exists!). Then we may construct the analogue of the C^*-algebra B used in Corollary 7.10. For this B and any crossed product $\rtimes_\tau$ the C^*-algebra $B \rtimes_\tau G$ will contain a Kazhdan projection that (modulo a minor technical condition, which should be easy to check) will not be in the image of the τ-assembly map. In particular, this would imply that the $\mathscr{E}$-Baum–Connes conjecture fails for the group G and coefficient C^*-algebra B.

It is thus very natural to ask if one can embed an expander with geometric property (T) into a group. We currently do not know enough to speculate on this either way.

8.2. ***Other exact crossed products.*** We use the crossed product $\rtimes_{\mathscr{E}}$ for our reformulation of the Baum–Connes conjecture as it has the following two properties:

(i) It is exact and Morita-compatible.

(ii) It is equal to the reduced crossed product when the group is exact.

However, the results of Theorem 6.2 and Corollary 7.10 are true for any exact and Morita-compatible crossed product. It is thus reasonable to consider other crossed products with properties (i) and (ii) above.

For example, consider the family of crossed products introduced by Kaliszewski, Landstad and Quigg [Kaliszewski et al. 2013] that we discuss in the Appendix. These are all Morita-compatible, and one can consider the minimal exact crossed product from this smaller class. This minimal Kaliszewski–Landstad–Quigg crossed product would have particularly good properties: for example, it would be a functor on a natural Morita category of correspondences [Buss and Echterhoff 2015, §2]. It is not clear to us if $\rtimes_{\mathscr{E}}$ has similarly good properties, or if it is equal to the "minimal exact Kaliszewski–Landstad–Quigg crossed product".

Another natural example is the max-$l^\infty(G)$-crossed product that we looked at in Proposition 5.8 above: it is possible that this is equal to the $\mathscr{E}$-crossed product. If it is not equal to the $\mathscr{E}$-crossed product, it would be interesting to know why.

8.3. *Consequences of the reformulated conjecture.* Most of the applications of the Baum–Connes conjecture to topology and geometry, for example to the Novikov and Gromov–Lawson conjectures (see [Baum et al. 1994, §7]), follow from the *strong Novikov conjecture*:[6] the statement that the maximal assembly map with trivial coefficients

$$\mu : K_*^{\mathrm{top}}(G) \to K_*(C^*_{\max}(G)) \tag{8.1}$$

is injective. This is implied by injectivity of the $\mathscr{E}$-assembly map, so the reformulated conjecture still has these same consequences. Moreover, isomorphism of the $\mathscr{E}$-assembly map implies that the assembly map in (8.1) is *split* injective.

On the other hand, the Kadison–Kaplansky conjecture states that if G is a torsion-free discrete group then there are no nontrivial projections in the reduced group C^*-algebra $C^*_{\mathrm{red}}(G)$. It is predicted by the classical form of the Baum–Connes conjecture. However, it is *not* predicted by our reformulated conjecture for nonexact groups. The reformulated conjecture does not even predict that there are no nontrivial projections in the exotic group C^*-algebra $\mathbb{C} \rtimes_{\mathscr{E}} G$, essentially as this C^*-algebra does not (obviously) have a faithful trace.

It is thus natural to look for counterexamples to the Kadison–Kaplansky conjecture among nonexact groups.

8.4. *Weak coarse embeddings.* Let $X = \bigsqcup X_n$ be a box space as in Definition 7.4 and G be a finitely generated group equipped with a word metric. A collection of functions $f_n : X_n \to G$ is a *weak coarse embedding* if

(i) there is a constant $c > 0$ such that

$$d_G(f_n(x), f_n(y)) \le c\, d_{X_n}(x, y)$$

for all n and all $x, y \in X_n$;

(ii) the limit

$$\lim_{n\to\infty} \max\left\{ \frac{|f_n^{-1}(x)|}{|X_n|} \mid x \in G \right\}$$

is zero.

[6] Some authors use "strong Novikov conjecture" to refer to the stronger statement that the reduced assembly map with trivial coefficients is injective.

If (X_n) is a sequence of graphs, and $f : X \to G$ is a coarse embedding from the associated box space into a group G, then the sequence of maps $(f_n : X_n \to G)$ defined by restricting f is a weak coarse embedding. Some versions of the Gromov monster construction (for example, [Gromov 2000; Arzhantseva and Delzant 2008]) show that weak coarse embeddings of large-girth, expander box spaces into groups exist,[7] but it is not clear from these constructions that coarse embeddings are possible.

In their original construction of counterexamples to the Baum–Connes conjecture with coefficients [Higson et al. 2002, §7], Higson, Lafforgue and Skandalis used the existence of a group G and a weak coarse embedding of an expander $(f_n : X_n \to G)$. They use this data to construct G-spaces Y and Z, and show that the Baum–Connes assembly map fails to be an isomorphism either with coefficients in $C_0(Y)$ or with coefficients in $C_0(Z)$. Their techniques do not show that the reformulated conjecture will fail for one of these coefficients, but we do not know if the reformulated conjecture is true under these assumptions either.

On the other hand, to produce our examples where the reformulated conjecture is true but the old conjecture fails (compare Corollary 7.10) we need to know the existence of a group G and a coarse embedding $f : X \to G$ of a large-girth, expander box space; such groups are the *special Gromov monsters* of Definition 7.7. We appeal to recent results of Osajda [2014] to see that appropriate examples exist.

8.5. ***Further questions.*** The following (related) questions seem natural; we do not currently know the answer to any of them. Unfortunately, nonexact groups are quite poorly understood (for example, there are no concrete countable[8] examples), so many of these questions might be difficult to approach directly.

Questions. (i) Can one coarsely embed an expander with geometric property (T) into a (finitely generated) discrete group?

(ii) Can one characterize exact crossed products in a natural way, e.g., by a "slice map property"?

(iii) It is shown in [Roe and Willett 2014] that for G countable and discrete the reduced crossed product is exact if and only if it preserves exactness of the

[7] Arzhantseva and Delzant [2008] show something much stronger than this: very roughly, they prove the existence of maps $f_n : X_n \to G$ that are "almost a quasi-isometry", and where the deviation from being a quasi-isometry is "small" relative to the girth. See [ibid., §7] for detailed statements. There is no implication either way between the condition that a sequence of maps $(f : X_n \to G)$ be a coarse embedding, and the condition that it satisfy the "almost quasi-isometry" properties of [ibid., §7]. We do not know if the existence of an "almost quasi-isometric" embedding of a box space into a group implies the existence of a coarse embedding.

[8] Exactness passes to closed subgroups, so finding concrete uncountable examples — like permutation groups on infinitely many letters — is easy given that some countable nonexact group exists at all.

sequence

$$0 \to C_0(G) \to l^\infty(G) \to l^\infty(G)/C_0(G) \to 0.$$

Is this true for more general crossed products? Is there another natural "universal short exact sequence" that works for a general crossed product?

(iv) Say G is a nonexact group, and let $C^*_{\mathscr{E}}(G)$ denote $\mathbb{C} \rtimes_{\mathscr{E}} G$, a completion of the group algebra. Can this completion be equal to $C^*_{\mathrm{red}}(G)$?

(v) Is the $\mathscr{E}$-crossed product equal to the minimal exact Kaliszewski–Landstad–Quigg crossed product?

(vi) Is the $\mathscr{E}$-crossed product equal to the max-$l^\infty(G)$ crossed product described in Proposition 5.8?

(vii) Does the $\mathscr{E}$-crossed product give rise to a descent functor on KK-theory?[9]

(viii) Is the reformulated conjecture true for the counterexamples originally constructed by Higson, Lafforgue and Skandalis?

Appendix: Some examples of crossed products

In this appendix we discuss some examples of crossed products. These examples are not necessary for the development in the main piece. However, they are important as motivation and to show the sort of examples that can arise (and contradict overly optimistic conjectures).

We will look at two families of exotic crossed products, which were introduced in [Brown and Guentner 2013] and [Kaliszewski et al. 2013]. For many groups, both families contain uncountably many natural examples that are distinct from the reduced and maximal crossed products; thus there is a rich theory of exotic crossed products. We will show this and that one family is always exact, the other always Morita-compatible. We conclude with two examples showing that the Baum–Connes conjecture fails for many exact crossed products.

The material draws on work of Brown and Guentner [2013], of Kaliszewski, Landstad and Quigg [Kaliszewski et al. 2013] and of Buss and Echterhoff [2014; 2015]. The third author is grateful to Alcides Buss and Siegfried Echterhoff for some very illuminating discussions of these papers.

Let G be a locally compact group. We will write $u : G \to \mathcal{U}(\mathcal{H})$, $g \mapsto u_g$ for a unitary representation of G, and use the same notation for the integrated forms

$$u : C_c(G) \mapsto \mathcal{B}(\mathcal{H}), \quad u : C^*_{\max}(G) \to \mathcal{B}(\mathcal{H})$$

[9] Added in proof: the answer to this is "yes": see [Buss et al. 2014, §5 and §7].

as for the representation itself. If A is a G-C^*-algebra, we will write a covariant pair of representations for (A, G) in the form

$$(\pi, u) : (A, G) \to \mathcal{B}(\mathcal{H}),$$

where $\pi : A \to \mathcal{B}(\mathcal{H})$ is a $*$-representation and $u : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation satisfying the covariance relation

$$u_g \pi(a) u_g^* = \pi(g(a)), \quad g \in G, \, a \in A.$$

Recall from Section 2 that $A \rtimes_{\mathrm{alg}} G$ denotes the space of compactly supported continuous functions from G to A equipped with the usual twisted product and involution, and that $A \rtimes_{\max} G$ denotes the maximal crossed product. Write

$$\pi \rtimes u : A \rtimes_{\mathrm{alg}} G \to \mathcal{B}(\mathcal{H}), \quad \pi \rtimes u : A \rtimes_{\max} G \to \mathcal{B}(\mathcal{H})$$

for the integrated forms of (π, u).

Recall that if S is a collection of unitary representations of G, and u is a unitary representation of G, then u is said to be *weakly contained in* S if

$$\|u(f)\| \le \sup_{v \in S} \|v(f)\| \tag{A.1}$$

for all $f \in C_c(G)$.

Let $\widehat{G}$ denote the unitary dual of G, i.e., the set of unitary equivalence classes of irreducible unitary representations of G. We will identify each class in $\widehat{G}$ with a choice of representative when this causes no confusion. The unitary dual is topologized by the following closure operation: if S is a subset of $\widehat{G}$, then the closure $\overline{S}$ consists of all those elements of $\widehat{G}$ that are weakly contained in S. Let $\widehat{G}_r$ denote the closed subset of $\widehat{G}$ consisting of all (equivalence classes of) irreducible unitary representations that are weakly contained in the (left) regular representation.

Definition A.2. A subset S of $\widehat{G}$ is *admissible* if its closure contains $\widehat{G}_r$.

Note that $\widehat{G}$ and $\widehat{G}_r$ identify canonically with the spectra of the maximal and reduced group C^*-algebras $C^*_{\max}(G)$ and $C^*_{\mathrm{red}}(G)$, respectively. If S is an admissible subset of $\widehat{G}$, define a C^*-norm on $C_c(G)$ by

$$\|f\|_S := \sup_{u \in S} \|u(f)\|$$

and let $C^*_S(G)$ denote the corresponding completion. Note that, as $\overline{S}$ contains $\widehat{G}_r$, the identity map on $C_c(G)$ extends to a quotient map

$$C^*_S(G) \to C^*_{\mathrm{red}}(G).$$

We will now associate two crossed products to each admissible $S \subseteq \widehat{G}$. The first was introduced by Brown and Guentner [2013, §5] (at least in a special case), and the second by Kaliszewski, Landstad and Quigg [Kaliszewski et al. 2013, §6] (it was subsequently shown to define a functor by Buss and Echterhoff [2014, §7]).

Definition A.3. Let S be an admissible subset of $\widehat{G}$. A covariant pair (π, u) for a G-C^*-algebra A is an *S-representation* if u is weakly contained in S. Define the *Brown–Guentner S-crossed-product* (or "BG S-crossed-product") of A by G, denoted $A \rtimes_{\mathrm{BG},S} G$, to be the completion of $A \rtimes_{\mathrm{alg}} G$ for the norm

$$\|x\| := \sup\{\|(\pi \rtimes u)(x)\|_{\mathcal{B}(\mathcal{H})} \mid (\pi, u) : (A, G) \to \mathcal{B}(\mathcal{H}) \text{ an } S\text{-representation}\}.$$

If S is unambiguous, we will often write $A \rtimes_{\mathrm{BG}} G$.

Definition A.4. Let A be a G-C^*-algebra, and let

$$A \rtimes_{\max} G \otimes C^*_S(G)$$

denote the spatial tensor product of the maximal crossed product $A \rtimes_{\max} G$ and $C^*_S(G)$; let $\mathcal{M}(A \rtimes_{\max} G \otimes C^*_S(G))$ denote its multiplier algebra. Let

$$(\pi, u) : (A, G) \to \mathcal{M}(A \rtimes_{\max} G) \otimes \mathcal{M}(C^*_S(G)) \subseteq \mathcal{M}(A \rtimes_{\max} G \otimes C^*_S(G))$$

be the covariant representation defined by

$$\pi : a \mapsto a \otimes 1, \quad u : g \mapsto g \otimes g.$$

Note that this integrates to an injective $*$-homomorphism

$$\pi \rtimes u : A \rtimes_{\mathrm{alg}} G \to \mathcal{M}(A \rtimes_{\max} G \otimes C^*_S(G)).$$

Define the *Kaliszewski–Landstad–Quigg S-crossed-product* (or "KLQ S-crossed-product") of A by G, denoted $A \rtimes_{\mathrm{KLQ},S} G$, to be the completion of

$$(\pi \rtimes u)(A \rtimes_{\mathrm{alg}} G)$$

inside $\mathcal{M}(A \rtimes_{\max} G \otimes C^*_S(G))$. If S is unambiguous, we will often write $A \rtimes_{\mathrm{KLQ}} G$.

For the reader comparing this to [Brown and Guentner 2013] and [Kaliszewski et al. 2013], we note that the constructions in those papers use spaces of matrix coefficients rather than subsets of $\widehat{G}$ to build crossed products. Standard duality arguments show that the two points of view are equivalent: we use subsets of $\widehat{G}$ here simply as this seemed to lead more directly to the results we want.

We now show that both the Brown–Guentner and Kaliszewski–Landstad–Quigg crossed products are crossed product functors.

Proposition A.5. *Let S be an admissible subset of $\widehat{G}$. Let $\phi : A \to B$ be a G-equivariant $*$-homomorphism. Let*

$$\phi \rtimes G : A \rtimes_{\mathrm{alg}} G \to B \rtimes_{\mathrm{alg}} G$$

denote its integrated form. Then $\phi \rtimes G$ extends to $*$-homomorphisms on both the BG and KLQ S-crossed-products. In particular, $\rtimes_{\mathrm{BG}}$ and $\rtimes_{\mathrm{KLQ}}$ are crossed product functors in the sense of Definition 2.1.

Proof. We first consider the BG crossed product. Let x be an element of $A \rtimes_{\mathrm{alg}} G$ and note that

$$\|(\phi\rtimes G)(x)\|_{B\rtimes_{\mathrm{BG}}G} = \sup\{\|((\pi\circ\phi)\rtimes u)(x)\| \mid (\pi,u) \text{ an } S\text{-representation of } (B,G)\}.$$

However, the set that we are taking the supremum over on the right-hand side is a subset of

$$\{\|(\pi \rtimes u)(x)\| \mid (\pi, u) \text{ an } S\text{-representation of } (A, G)\},$$

and the $A \rtimes_{\mathrm{BG}} G$ norm of x is defined to the supremum over this larger set. This shows that

$$\|(\phi \rtimes G)(x)\|_{B\rtimes_{\mathrm{BG}}G} \leq \|x\|_{A\rtimes_{\mathrm{BG}}G}$$

and thus that $\phi \rtimes G$ extends to the BG crossed product.

The argument for the KLQ crossed product is essentially as in [Buss and Echterhoff 2014, Proposition 5.2].[10] Define $\mathcal{M}_0(A \rtimes_{\max} G \otimes C^*_S(G))$ to be the C^*-subalgebra of $\mathcal{M}(A \rtimes_{\max} G \otimes C^*_S(G))$ consisting of all those m such that $m(1 \otimes b)$ and $(1 \otimes b)m$ are in $A \rtimes_{\max} G \otimes C^*_S(G)$ for all $b \in C^*_S(G)$, and note that there is a unique extension of the $*$-homomorphism

$$(\phi \rtimes G) \otimes \mathrm{id} : A \rtimes_{\max} G \otimes C^*_S(G) \to B \rtimes_{\max} G \otimes C^*_S(G)$$

to a $*$-homomorphism

$$(\phi \rtimes G) \otimes \mathrm{id} : \mathcal{M}_0(A \rtimes_{\max} G \otimes C^*_S(G)) \to \mathcal{M}(B \rtimes_{\max} G \otimes C^*_S(G))$$

(whether or not ϕ is nondegenerate) by [Echterhoff et al. 2006, Proposition A.6(i)]. Hence there is a commutative diagram

$$\begin{tikzcd}
A \rtimes_{\mathrm{alg}} G \arrow[r] \arrow[d, "\phi\rtimes G"] & \mathcal{M}_0(A \rtimes_{\max} G \otimes C^*_S(G)) \arrow[d, "(\phi\rtimes G)\otimes\mathrm{id}"] \\
B \rtimes_{\mathrm{alg}} G \arrow[r] & \mathcal{M}(B \rtimes_{\max} G \otimes C^*_S(G))
\end{tikzcd}$$

where the horizontal arrows are the injective $*$-homomorphisms used to define $A \rtimes_{\mathrm{KLQ}} G$ and $B \rtimes_{\mathrm{KLQ}} G$ (it is clear that the image of the former actually lies in $\mathcal{M}_0(A \rtimes_{\max} G \otimes C^*_S(G))$). In particular, $\phi \rtimes G$ extends to a $*$-homomorphism between the closures

$$\phi \rtimes G : \overline{A \rtimes_{\mathrm{alg}} G} \to \overline{B \rtimes_{\mathrm{alg}} G}$$

[10] Our thanks to the referee for pointing out that there was a gap in our original argument and suggesting this reference.

of the algebraic crossed products $A \rtimes_{\mathrm{alg}} G$ and $B \rtimes_{\mathrm{alg}} G$ inside $\mathcal{M}(A \rtimes_{\max} G \otimes C^*_S(G))$ and $\mathcal{M}(B \rtimes_{\max} G \otimes C^*_S(G))$, respectively, and thus by definition to a map between the KLQ crossed products. $\square$

Note that if S is dense in $\widehat{G}$ then both the BG and KLQ crossed products associated to S are equal to the maximal crossed product. On the other hand, if the closure of S is just $\widehat{G}_r$, then the KLQ crossed product is equal to the reduced crossed product [Kaliszewski et al. 2013, p. 18, point (4)], but the analog of this is not true in general for the BG crossed product, as follows for example from Lemma A.8 below.

We now look at exactness and Morita compatibility (Definitions 3.1 and 3.3, respectively). We will prove the following results:

(i) BG crossed products are always exact;

(ii) KLQ crossed products are always Morita-compatible;

(iii) BG crossed products are Morita-compatible only in the trivial case when $\overline{S} = \widehat{G}$.

We do not know anything about exactness of KLQ crossed products, other than in the special cases when $\overline{S} = \widehat{G}$ and $\overline{S} = \widehat{G}_r$; this seems to be a very interesting question in general.

Lemma A.6. *For any admissible S, the BG S-crossed-product is exact.*

Proof. Let

$$0 \to I \xrightarrow{\iota} A \xrightarrow{\rho} B \to 0$$

be a short exact sequence of G-C^*-algebras, and consider its "image"

$$0 \to I \rtimes_{\mathrm{BG}} G \xrightarrow{\iota \rtimes G} A \rtimes_{\mathrm{BG}} G \xrightarrow{\rho \rtimes G} B \rtimes_{\mathrm{BG}} G \to 0$$

under the functor $\rtimes_{\mathrm{BG}}$

It follows from the fact that $\rtimes_{\mathrm{BG}}$ is a functor that $(\rho \rtimes G) \circ (\iota \rtimes G)$ is zero. Moreover, $\rho \rtimes G$ has dense image and is thus surjective.

To see that $\iota \rtimes G$ is injective, note that if

$$(\pi, u) : (I, G) \to \mathcal{B}(\mathcal{H})$$

is an S-representation, then the representation $\tilde{\pi} : A \to \mathcal{B}(\mathcal{H})$ defined on $\pi(I) \cdot \mathcal{H}$ by

$$\tilde{\pi}(a)(\pi(i)v) = \pi(ai)v$$

fits together with u.

Finally, note that as $(\rho \rtimes G) \circ (\iota \rtimes G) = 0$, there is a surjective $*$-homomorphism

$$\frac{A \rtimes_{\mathrm{BG}} G}{I \rtimes_{\mathrm{BG}} G} \to B \rtimes_{\mathrm{BG}} G;$$

we must show that this is injective. Let $\phi : A \rtimes_{\mathrm{BG}} G \to \mathcal{B}(\mathcal{H})$ be a nondegenerate $*$-representation containing $I \rtimes_{\mathrm{BG}} G$ in its kernel; it will suffice to show that ϕ descends to a $*$-representation of $B \rtimes_{\mathrm{BG}} G$. As ϕ is nondegenerate, it is the integrated form of some S-representation

$$(\pi, u) : (A, G) \to \mathcal{B}(\mathcal{H}).$$

As $I \rtimes_{\mathrm{alg}} G$ is contained in the kernel of ϕ, I is contained in the kernel of π. Hence (π, u) descends to a covariant pair for (B, G), which is of course still an S-representation. Its integrated form thus extends to $B \rtimes_{\mathrm{BG}} G$. □

Lemma A.7. *The KLQ S-crossed-product is Morita-compatible for any admissible S.*

Proof. Let $\mathcal{K}_G$ denote the algebra of compact operators on the infinite amplification $\bigoplus_{n\in\mathbb{N}} L^2(G)$ of the regular representation equipped with the natural conjugation action. Let A be a G-C^*-algebra, and let

$$\Phi : (A \otimes \mathcal{K}_G) \rtimes_{\max} G \to (A \rtimes_{\max} G) \otimes \mathcal{K}_G$$

denote the untwisting isomorphism from line (3.2). Consider the isomorphism

$$\Phi \otimes 1 : (A \otimes \mathcal{K}_G) \rtimes_{\max} G \otimes C^*_S(G) \to (A \rtimes_{\max} G) \otimes \mathcal{K}_G \otimes C^*_S(G)$$

and its extension

$$\Phi \otimes 1 : \mathcal{M}((A \otimes \mathcal{K}_G) \rtimes_{\max} G \otimes C^*_S(G)) \to \mathcal{M}((A \rtimes_{\max} G) \otimes \mathcal{K}_G \otimes C^*_S(G))$$

to multiplier algebras. Up to the canonical identification

$$A \rtimes_{\max} G \otimes \mathcal{K}_G \otimes C^*_S(G) \cong (A \rtimes_{\max} G) \otimes C^*_S(G) \otimes \mathcal{K}_G,$$

the restriction of $\Phi \otimes 1$ to

$$(A \otimes \mathcal{K}_G) \rtimes_{\mathrm{KLQ}} G \subseteq \mathcal{M}((A \otimes \mathcal{K}_G) \rtimes_{\max} G \otimes C^*_S(G))$$

identifies with the untwisting isomorphism from this C^*-algebra to

$$\begin{aligned}(A \rtimes_{\mathrm{KLQ}} G) \otimes \mathcal{K}_G &\subseteq \mathcal{M}(A \rtimes_{\max} G \otimes C^*_S(G)) \otimes \mathcal{K}_G \\ &\subseteq \mathcal{M}(A \rtimes_{\max} G \otimes C^*_S(G) \otimes \mathcal{K}_G).\end{aligned}$$ □

Lemma A.8. *The BG S-crossed-product is Morita-compatible for an admissible S if and only if S is dense in $\widehat{G}$.*

Proof. If S is dense in $\widehat{G}$, then $\rtimes_{\mathrm{BG}}$ is equal to the maximal crossed product and well-known to be Morita-compatible.

For the converse, let $\mathcal{K}_G$ be as in the definition of Morita compatibility. Let $U : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation that extends faithfully to $C^*_{\max}(G)$. Consider now the covariant pair

$$(\pi, u) : (\mathcal{K}_G, G) \to \mathcal{B}\big(\textstyle\bigoplus_{n\in\mathbb{N}} L^2(G) \otimes \mathcal{H}\big),$$

defined by

$$\pi : T \mapsto T \otimes 1, \quad u : g \mapsto \big(\textstyle\bigoplus \lambda_g\big) \otimes U_g,$$

which by the explicit form of the untwisting isomorphism is a faithful representation (with image $\mathcal{K}_G \otimes C^*_{\max}(G)$). On the other hand, the representation u is weakly contained in the regular representation by Fell's trick. Hence, by admissibility of S, (π, u) is an S-representation, and thus extends to $\mathcal{K}_G \rtimes_{\mathrm{BG}} G$. We conclude that the canonical quotient map

$$\mathcal{K}_G \rtimes_{\max} G \to \mathcal{K}_G \rtimes_{\mathrm{BG}} G$$

is an isomorphism.

On the other hand, consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{K}_G \rtimes_{\max} G & \xrightarrow{\Phi,\cong} & \mathcal{K}_G \otimes C^*_{\max}(G) \\
\Big\| & & \Big\downarrow \mathrm{id} \otimes \rho \\
\mathcal{K}_G \rtimes_{\mathrm{BG}} G & \longrightarrow & \mathcal{K}_G \otimes C^*_S(G)
\end{array}$$

where Φ is the untwisting isomorphism, $\rho : C^*_{\max}(G) \to C^*_S(G)$ is the canonical quotient map, and the bottom line is defined to make the diagram commute. To say that $\rtimes_{\mathrm{BG}}$ is Morita-compatible means by definition that the surjection on the bottom line is an isomorphism. This implies that the right-hand vertical map is an isomorphism, whence ρ is an isomorphism and so $\overline{S} = \widehat{G}$. $\square$

We now characterize when the various BG and KLQ crossed products are the same. The characterizations imply that for nonamenable G the families of BG and KLQ crossed products both tend to be fairly large (Proposition A.10 and Examples A.13), and that the only crossed product common to both is the maximal crossed product (Lemma A.14). Note that the second part of Proposition A.10 also appears in [Buss and Echterhoff 2015, Proposition 2.2] (in different language).

Definition A.9. A subset S of $\widehat{G}$ is an *ideal* if for any unitary representation u and any $v \in S$ the tensor product representation $u \otimes v$ is weakly contained in S.

Proposition A.10. *Let S, R be admissible subsets of $\widehat{G}$.*

(i) *The BG crossed products defined by S and R are the same if and only if the closures of R and S in $\widehat{G}$ are the same. In particular, BG crossed products are in one-to-one correspondence with closed subsets of $\widehat{G}$ that contain $\widehat{G}_r$.*

(ii) *The KLQ crossed products defined by S and R are the same if and only if the closed ideals in $\widehat{G}$ generated by R and S are the same. In particular, KLQ crossed products are in one-to-one correspondence with closed ideals of $\widehat{G}$ that contain $\widehat{G}_r$.*

Proof. We look first at the BG crossed products. Note that a covariant pair $(\pi, u) : (A, G) \to \mathcal{B}(\mathcal{H})$ is an S-representation if and only if it is an $\overline{S}$-representation. This shows that the BG crossed products associated to S and $\overline{S}$ are the same, and thus that if $\overline{R} = \overline{S}$ then their BG crossed products are the same.

Conversely, note that if R and S have the same BG crossed products, then considering the trivial action on $\mathbb{C}$ shows that $C^*_S(G) = C^*_R(G)$. This happens (if and) only if $\overline{R} = \overline{S}$.

Look now at the KLQ crossed products. If S is an admissible subset of $\widehat{G}$, denote by $\langle S \rangle$ the closed ideal generated by S. Let A be a G-C^*-algebra, and consider the covariant representation of (A, G) into

$$\mathcal{M}(A \rtimes_{\max} G) \otimes \mathcal{M}(C^*_{\max}(G)) \subseteq \mathcal{M}(A \rtimes_{\max} G \otimes C^*_{\max}(G))$$

defined by

$$\pi : a \mapsto a \otimes 1, \quad u : g \mapsto g \otimes g.$$

The integrated form of this representation defines a $*$-homomorphism

$$A \rtimes_{\mathrm{alg}} G \to \mathcal{M}(A \rtimes_{\max} G \otimes C^*_{\max}(G))$$

and the closure of its image is isomorphic to $A \rtimes_{\max} G$ by [Kaliszewski et al. 2013, p. 18, point (3)]. It follows that to define $A \rtimes_{\mathrm{KLQ},S} G$ we may take the closure of the image of $A \rtimes_{\mathrm{alg}} G$ under the integrated form of the covariant pair of (A, G) with image in

$$\mathcal{M}(A \rtimes_{\max} G) \otimes \mathcal{M}(C^*_{\max}(G)) \otimes \mathcal{M}(C^*_S(G)) \subseteq \mathcal{M}(A \rtimes_{\max} G \otimes C^*_{\max}(G) \otimes C^*_S(G))$$

defined by

$$\pi : a \mapsto a \otimes 1 \otimes 1, \quad u : g \mapsto g \otimes g \otimes g. \tag{A.11}$$

However, the closure of the image of the integrated form of the representation

$$u : G \to \mathcal{U}(C^*_{\max}(G) \otimes C^*_S(G)), \quad g \mapsto g \otimes g \tag{A.12}$$

is easily seen to be $C^*_{\langle S\rangle}(G)$. Therefore the integrated form of the representation in (A.11) identifies with the integrated form of the covariant pair of (A, G) with image in

$$\mathcal{M}(A\rtimes_{\max} G)\otimes\mathcal{M}(C^*_{\langle S\rangle}(G))\subseteq\mathcal{M}(A\rtimes_{\max} G\otimes C^*_{\langle S\rangle}(G))$$

defined by

$$\pi : a\mapsto a\otimes 1,\quad u : g\mapsto g\otimes g.$$

This discussion shows that S and $\langle S\rangle$ give rise to the same KLQ crossed product, and thus that if $\langle S\rangle=\langle R\rangle$ then S and R define the same KLQ crossed product.

Conversely, note that $\mathbb{C}\rtimes_S G$ is (by definition) the C^*-algebra generated by the integrated form of the unitary representation in (A.12) and, as already noted, this is $C^*_{\langle S\rangle}(G)$. In particular, if R and S have the same KLQ crossed product then $C^*_{\langle S\rangle}(G)$ and $C^*_{\langle R\rangle}(G)$ are the same, and this forces $\langle R\rangle=\langle S\rangle$. $\square$

Examples A.13. Let G be a locally compact group. For any $p\in[1,\infty)$, let S_p denote those (equivalences classes of) irreducible unitary representations for which there is a dense set of matrix coefficients in $L^p(G)$. Then S_p is an ideal in $\widehat{G}$ containing $\widehat{G}_r$. Building on seminal work of Haagerup [1978/79], Okayasu [2014] has shown that for $G=F_2$, the free group on two generators, the completions $C^*_{S_p}(G)$ are all different as p varies through $[2,\infty)$. It follows by an induction argument that the same is true for any discrete G containing F_2 as a subgroup.[11]

Hence, in particular, for "many" nonamenable G there is an uncountable family of distinct closed ideals $\{\overline{S_p}\mid p\in[2,\infty)\}$ in $\widehat{G}$, and thus an uncountable family of distinct KLQ and BG completions.

The next lemma discusses the relationship between the BG and KLQ crossed products associated to the same S. Considering the trivial crossed products of $\mathbb{C}$ with respect to the trivial action as in the proof of Proposition A.10 shows that the question is only interesting when S is a closed ideal in $\widehat{G}$, so we only look at this case. Compare [Kaliszewski et al. 2013, Example 6.6] and also [Quigg and Spielberg 1992] for a more detailed discussion of similar phenomena.

Lemma A.14. *Let S be a closed ideal in $\widehat{G}$ containing $\widehat{G}_r$. Then for any G-C^*-algebra A the identity on $A\rtimes_{\mathrm{alg}} G$ extends to a quotient $*$-homomorphism*

$$A\rtimes_{\mathrm{BG}} G\to A\rtimes_{\mathrm{KLQ}} G$$

from the BG S-crossed-product to the KLQ S-crossed-product.

Moreover, this quotient map is an isomorphism for $A=\mathcal{K}_G$ if and only if $S=\widehat{G}$ (in which case we have $\rtimes_{\mathrm{BG}}=\rtimes_{\mathrm{KLQ}}=\rtimes_{\max}$).

[11] This is also true more generally; whether it is true for *any* nonamenable locally compact G seems to be an interesting question.

Proof. Let $\mathcal{H}$ and $\mathcal{H}_A$ be faithful representation spaces for $C^*_S(G)$ and $A \rtimes_{\max} G$ respectively. As S is an ideal, the representation

$$A \rtimes_{\mathrm{alg}} G \to \mathcal{M}(A \rtimes_{\max} G \otimes C^*_S(G)) \subseteq \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H})$$

defining $A \rtimes_{\mathrm{KLQ}} G$ is the integrated form of an S-representation of (A, G), and thus extends to $A \rtimes_{\mathrm{BG}} G$. This shows the existence of the claimed quotient map.

For the second part, note that the arguments of Lemmas A.7 and A.8 show that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{K}_G \rtimes_{\mathrm{BG}} G \arrow[r, "\cong"] \arrow[d] & \mathcal{K}_G \otimes C^*_{\max}(G) \arrow[d, "\mathrm{id} \otimes \rho"] \\
\mathcal{K}_G \rtimes_{\mathrm{KLQ}} G \arrow[r, "\cong"] & \mathcal{K}_G \otimes C^*_S(G)
\end{tikzcd}$$

where the left-hand vertical map is the quotient extending the identity map on $\mathcal{K}_G \rtimes_{\mathrm{alg}} G$, and the right-hand vertical map is the quotient extending the identity on the algebraic tensor product $\mathcal{K}_G \odot C_c(G)$. Hence if

$$\mathcal{K}_G \rtimes_{\mathrm{BG}} G = \mathcal{K}_G \rtimes_{\mathrm{KLQ}} G$$

then we must have that $\rho : C^*_{\max}(G) \to C^*_S(G)$ is an isomorphism; as S is closed, this forces $S = \widehat{G}$. $\square$

We conclude this appendix with two examples showing that one should not in general expect exact crossed products to satisfy the Baum–Connes conjecture.

Examples A.15. Let G be a nonamenable group, and let $S = \widehat{G}_r \cup \{1\}$, where 1 is the class of the trivial representation (compare Example 6.5). As 1 is a finite-dimensional representation it is a closed point in $\widehat{G}$, and thus S is a closed subset of $\widehat{G}$. Moreover, nonamenability of G implies that 1 is an isolated point of S. It follows as in Example 6.5 that there is a *Kazhdan projection* p in $C^*_S(G)$ whose image in any representation maps onto the G-fixed vectors. The class of this projection $[p] \in K_0(C^*_S(G))$ cannot be in the image of the Baum–Connes assembly map in many cases:[12] for example, if G is discrete (see [Higson 1998, discussion below 5.1]), or if the Baum–Connes conjecture is true for $C^*_r(G)$ (for example if G is almost connected [Chabert et al. 2003]). Hence the Baum–Connes conjecture fails for the BG crossed product associated to S in this case.

In particular, for any nonamenable discrete or almost connected G, there is an exact crossed product for which the Baum–Connes conjecture fails. Note that this is true even for a-T-menable groups, where the Baum–Connes conjecture is true for both the maximal and reduced crossed products.

[12] We would guess it can never be in the image, but we do not know how to prove this.

A similar, perhaps more natural, example can be arrived at by starting with $G = \mathrm{SL}(2,\mathbb{Z})$, which is a nonamenable, a-T-menable group. Let

$$u_n : \mathrm{SL}(2,\mathbb{Z}) \to \mathcal{B}(l^2(\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})))$$

be the n-th congruence representation, and define a norm on $C_c(G)$ by

$$\|x\|_{\mathrm{cong}} := \sup_n\{\|u_n(x)\|\}.$$

Note that this norm dominates the reduced norm. To see this, let $\lambda : G \to \mathcal{U}(l^2(G))$ be the regular representation. Let $x \in C_c(G)$ and $\xi \in l^2(G)$ have finite support. As the supports of ξ and $\lambda(x)\xi$ are finite subsets of G, they are mapped injectively to $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for all suitably large n. It follows that for all suitably large n we may find ξ_n in $l^2(\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z}))$ with $\|\xi_n\| = \|\xi\|$ and $\|u_n(x)\xi_n\| = \|\lambda(x)\xi\|$: indeed, ξ_n can be taken to be the pushforward of ξ. As $\xi \in l^2(G)$ was an arbitrary element of finite support, the desired inequality $\|x\|_{\mathrm{cong}} \geq \|\lambda(x)\|$ follows from this.

Isolation of the trivial representation in the spectrum of $C^*_{\mathrm{cong}}(\mathrm{SL}(2,\mathbb{Z}))$ is a consequence of Selberg's theorem [1965] (see also [Lubotzky 1994, §4.4]), and the same construction of a Kazhdan projection goes through.

As our second class of examples, let G be any locally compact group and S an admissible subset of $\widehat{G}$. Consider the commutative diagram coming from the Baum–Connes conjecture for the BG crossed product associated to S:

$$\begin{tikzcd} K_*^{\mathrm{top}}(G) \arrow[r] \arrow[dr] & K_*(C^*_{\max}(G)) \arrow[d] \\ & K_*(C^*_S(G)) \end{tikzcd}$$

Assuming the Baum–Connes conjecture for the BG S-crossed product, the diagonal map is an isomorphism, and Lemma A.8 (together with the Baum–Connes conjecture for this crossed product and coefficients in $\mathcal{K}_G$) implies that the vertical map is an isomorphism. Hence the horizontal map (the maximal Baum–Connes assembly map) is an isomorphism.

However, for discrete property (T) groups (see [Higson 1998, discussion below 5.1] again) for example, the maximal assembly map is definitely not an isomorphism. Hence for discrete property (T) groups, the Baum–Connes conjecture will fail for *all* BG crossed products.

Acknowledgements

We thank Goulnara Arzhantseva, Nate Brown, Alcides Buss, Siegfried Echterhoff, Nigel Higson, Eberhard Kirchberg, Ralf Meyer, Damian Osajda, John Quigg, and

Dana Williams for illuminating discussions on aspects of this paper. The first author thanks the University of Hawai'i at Mānoa for the generous hospitality extended to him during his visits to the university. The second and third authors thank the Erwin Schrödinger Institute in Vienna for its support and hospitality during part of the work on this paper. We would also like to thank the anonymous referee for many helpful comments.

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Received 14 Jan 2015. Revised 21 Apr 2015. Accepted 14 May 2015.

PAUL BAUM: baum@math.psu.edu
Department of Mathematics, The Pennsylvania State University, 206 McAllister Building, University Park, PA 16802, United States

ERIK GUENTNER: erik@math.hawaii.edu
Department of Mathematics, University of Hawai'i at Mānoa, 2565 McCarthy Mall, Honolulu, HI 96822-2273, United States

RUFUS WILLETT: rufus@math.hawaii.edu
Department of Mathematics, University of Hawai'i at Mānoa, 2565 McCarthy Mall, Honolulu, HI 96822-2273, United States